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Nonparametric estimation for uncertain differential equations

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Abstract
In recent years, the researches on parameter estimation of uncertain differential equations have developed significantly. However, when we deal with some nonparametric uncertain differential equations, the parameter estimation may not be used directly. To deal with these uncertain differential equations, it is important to consider the nonparametric estimation with the help of the observations. As an important branch of uncertain differential equation, autonomous uncertain differential equation may be properly applied to model some uncertain autonomous dynamic systems. In this paper, we propose a Legendre polynomial based method for the nonparametric estimation of autonomous uncertain differential equations. After that, some numerical examples are given and the residuals as well as uncertain hypothesis tests are used to prove the acceptability of these estimations. In application, we consider an atmospheric carbon dioxide model by the proposed method of nonparametric estimation.

Keywords Uncertainty theory · Uncertain differential equation · Nonparametric estimation · Uncertain hypothesis test · Atmospheric carbon dioxide

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1 Introduction

Since uncertainty theory was founded by Liu (2007) and perfected by Liu (2009), it has been widely applied into many fields, such as physics, engineering and finance. As an important part of uncertainty theory, uncertain differential equation driven by Liu process was introduced by Liu (2008). After that, uncertain differential equation quickly became a good tool to describe uncertain dynamic systems.

With a lot of good properties, uncertain differential equations attracted the attentions of many researchers. In 2010, a sufficient condition for the existence and uniqueness of the solution of an uncertain differential equation was proposed by Chen and Liu (2010). After that, a crucial Yao-Chen formula was proposed by Yao and Chen (2013) to connect uncertain differential equations with ordinary differential equations. Since the analytic solutions of uncertain differential equations may usually be unavailable, based on Yao-Chen formula, many numerical methods for uncertain differential equations were proposed, such as Euler method (Yao, 2013) and Runge–Kutta method (Yang and Shen, 2015). Uncertain differential equations had been widely applied to many fields, such as financial markets (Liu, 2009) and optimal control (Zhu, 2010, 2019).

As mentioned above, uncertain differential equations may properly describe uncertain dynamic systems. In applications, we usually construct a model based on uncertain differential equation to fit the corresponding uncertain dynamic system. Thus, it is crucial to work on the parameter estimation of uncertain differential equations. On that purpose, moment estimation (Yao and Liu, 2020), least squares estimation (Sheng et al., 2020), generalized moment estimation (Liu, 2021), maximum likelihood estimation (Liu and Liu, 2022a) and the method of moments with the help of residuals (Liu and Liu, 2022b) were proposed. In 2022, Ye and Liu (2022a) proposed the concept of uncertain hypothesis test and then applied it to uncertain differential equations (Ye and Liu 2022b). In recent years, parameter estimation in uncertain differential equations made a contribution to the epidemic model of COVID-19 (Chen et al., 2021; Jia and Chen, 2021; Lio and Liu, 2021). Moreover, for the sake of coping with uncertain dynamic systems with memory characteristic, Zhu (2015) gave the definition of uncertain fractional differential equation. Then, He et al. (2022) introduced an algorithm of parameter estimation for uncertain fractional differential equations based on method of moments.

Actually, many uncertain dynamic systems in our lives are autonomous, whose states at the next time are only related to the previous states. To cope with this kind of autonomous uncertain dynamic systems, we propose the definition of autonomous uncertain differential equation. For some cases, the information of an autonomous uncertain dynamic system is not enough to construct a parametric model. In this situation, only a nonparametric model is available. To solve the problem of nonparametric estimation, we need to obtain a parametric model as the approximation of the nonparametric model. Legendre polynomials have many outstanding properties such as orthogonality, symmetry and recursiveness. In 2020, Gu et al. (2020) proposed a Legendre polynomials based numerical method for the solutions of optimal control problems, which suggested that the Legendre polynomials have good property of arbitrary approximation for continuous functions. Thus, we may utilize the Legendre polynomials sequence to approximate the nonparametric functions in uncertain differential equations. In this paper, we discuss the method of nonparametric estimation for autonomous uncertain differential equations.

The rest of this paper is organized as follows. In Sect. 2, we introduce some definitions and properties of Legendre polynomials, uncertainty differential equations and uncertain hypothesis tests. Then, a method of nonparametric estimation for uncertain differential equations is proposed in Sect. 3. In Sect. 4, we give three numerical examples to show the effectiveness of our method. Finally in Sect. 5, we apply the nonparametric estimation into the atmospheric carbon dioxide model.

2 Preliminaries

Let C_t be a Liu process. Suppose that $f, g : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are two functions. Uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t,\ \ t \in [0, T] \tag{1}$$

may be used to describe uncertain dynamic systems. If functions f and g are parametric forms, then there are many researches for the estimations of unknown parameters. However, in some cases, we may only construct the nonparametric models. To cope with these models, we consider to utilize an orthogonal function sequence with unknown weights to approximate the nonparametric functions f and g in (1). Then, the method of parameter estimation may be used to estimate the weights.

In this paper, with many outstanding properties, the Legendre polynomials are used for the approximation of nonparametric function. For $x \in \mathbb{R}$, the Legendre polynomials $p_n(x)$ are the polynomial solutions to Legendre's differential equation

$$\frac{d}{dx}\left(\left(1 - x^2\right)\frac{d}{dx}p_n(x)\right) + n(n+1)p_n(x) = 0,\ \ n = 0, 1, 2, \dots$$

Thus, Legendre polynomials are given in the following form

$$p_0(x) = 1,\ \ p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n,\ \ n = 1, 2, \dots \tag{2}$$

Based on Eq. (2), we have

$$\begin{aligned} p_1(x) &= \frac{1}{2}\frac{d}{dx}\left(x^2 - 1\right) = x,\\ p_2(x) &= \frac{1}{2^2 \times 2!}\frac{d^2}{dx^2}\left(x^2 - 1\right)^2 = \frac{1}{2}\left(3x^2 - 1\right),\\ p_3(x) &= \frac{1}{2^3 \times 3!}\frac{d^3}{dx^3}\left(x^2 - 1\right)^3 = \frac{1}{2}\left(5x^3 - 3x\right),\\ &\dots \end{aligned}$$

The following three properties of Legendre polynomials are useful.

Property 1 [*Orthogonality* (*Kashin and Saakian*, 1984)] *Legendre polynomials are orthogonal polynomials with respect to the* L^2 *on the interval* $[-1, 1]$, *i.e.*,

$$\int_{-1}^{1} p_m(x)p_n(x)dx = \frac{2}{2n+1}\delta_{mn},$$

where δ_{mn} *denotes the Kronecker symbol*

$$\delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

Property 2 [*Symmetric and antisymmetric* (*Kashin and Saakian*, 1984)] *Each Legendre polynomial is symmetric or antisymmetric on the interval* $[-1, 1]$, *i.e.*,

$$\begin{cases} p_n(-x) = -p_n(x), \ n \text{ is odd}, \\ p_n(-x) = p_n(x), \ n \text{ is even}. \end{cases}$$

Property 3 [*Recursiveness* (*Kashin and Saakian*, 1984)] *On the interval* $[-1, 1]$, *the Legendre polynomials follow the three-term recurrence relation*, *which is known as Bonnets recursion formula*

$$(n+1)p_{n+1}(x) = (2n+1)xp_n(x) - np_{n-1}(x)$$

and

$$\frac{x^2-1}{n}\frac{d}{dx}p_n(x) = xp_n(x) - p_{n-1}(x).$$

A continuous function $\varphi(x)$ ($x \in [-1, 1]$) may be expressed in terms of Legendre series as

$$\varphi(x) = \sum_{i=0}^{\infty} c_i p_i(x),$$

where $c_i \in \mathbb{R}$ for $i = 0, 1, 2, \ldots$ In calculation, $\varphi(x)$ is approximated by the partial sum of Legendre series. Then, we have

$$\varphi(x) \approx \sum_{i=0}^{K} c_i p_i(x), \tag{3}$$

where $c_i \in \mathbb{R}$ for $i = 0, 1, 2, \ldots, K$ and K is an appropriate positive integer.

For some uncertain dynamic systems, the variations of the current states are only directly affected by the states of themselves, such as the stock price and the atmospheric greenhouse gas. The uncertain dynamic systems with this phenomenon are called autonomous uncertain dynamic systems. To cope with this kind of systems, we give the definition of autonomous uncertain differential equation.

Definition 1 Let C_t be a Liu process. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are two functions. Then

$$dX_t = f(X_t)dt + g(X_t)dC_t \tag{4}$$

is called an autonomous uncertain differential equation. A solution of (4) is an uncertain process X_t such that

$$X_t = X_0 + \int_0^t f(X_s)ds + \int_0^t g(X_s)dC_s$$

holds almost surely.

With the help of the Legendre polynomials, we may obtain an approximation of the autonomous uncertain differential Eq. (4) with weights. Then, the unknown weights in the approximation may be estimated by the method of parameter estimation. Ye and Liu (2022b) discussed uncertain hypothesis test for uncertain differential equations, which proposed a standard to test whether the estimated parameters are acceptable.

Theorem 1 (Ye and Liu, 2022b) *Let ξ be a population with regular uncertainty distribution $\mathcal{L}(a, b)$ with unknown parameters a and b. Then the test for the hypotheses*

$$H_0 : a = a_0 \text{ and } b = b_0 \text{ versus } H_1 : a \neq a_0 \text{ or } b \neq b_0$$

at significance level β is

$$W = \left\{ (z_1, z_2, \ldots, z_n) : \text{ there are at least } \lfloor \beta n \rfloor + 1 \text{ of indexes } i\text{'s with} \right.$$
$$\left. 1 \leq i \leq n \text{ such that } z_i < \Phi_0^{-1}\left(\frac{\beta}{2}\right) \text{ or } z_i > \Phi_0^{-1}\left(1 - \frac{\beta}{2}\right) \right\}.$$

where $\Phi_0^{-1}(\beta)$ is the inverse uncertainty distribution of $\mathcal{L}(a_0, b_0)$, i.e.,

$$\Phi_0^{-1}(\beta) = (1 - \beta)a_0 + \beta b_0.$$

Remark 1 For uncertain differential Eq. (1) with observations $x_{t_1}, x_{t_2}, \ldots, x_{t_n}$ of X_t at $t_1, t_2, \ldots, t_n$, respectively, residuals of the estimation of (1) should be obtained as $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n$ with the help of Liu and Liu (2022b). According to Liu and Liu (2022b), the residual ε_i may be regraded as a sample of the linear uncertainty distribution $\mathcal{L}(0, 1)$ if estimated uncertain differential equation is appropriate.

Based on the conception of uncertain hypothesis test for uncertain differential equations proposed by Ye and Liu (2022b), to test whether the estimation may properly fit the observations x_{t_1}, x_{t_2}, $\cdots$, x_{t_n}, the test at a given significance level $\beta = 0.05$ is

$$H_0 : a = 0 \text{ and } b = 1 \text{ versus } H_1 : a \neq 0 \text{ or } b \neq 1$$

and the reject set is

$$W = \Bigg\{ (\varepsilon_2, \varepsilon_3, \dots, \varepsilon_n) : \text{ there are at least } \lfloor 0.05(n-1) \rfloor + 1 \text{ of indexes } i\text{'s} \\ \text{with } 2 \le i \le n \text{ such that } \varepsilon_i < 0.025 \text{ or } \varepsilon_i > 0.975 \Bigg\}.$$

If the vector of the $n-1$ residuals belongs to the test W, i.e.,

$$\left(\varepsilon_2, \varepsilon_3, \dots, \varepsilon_n\right) \in W,$$

then the estimation of uncertain differential Eq. (1) is not a good fit to the observed data $x_{t_1}, x_{t_2}, \dots, x_{t_n}$. If

$$\left(\varepsilon_2, \varepsilon_3, \dots, \varepsilon_n\right) \notin W,$$

then the estimation of uncertain differential Eq. (1) is an applicable fit to the observed data $x_{t_1}, x_{t_2}, \dots, x_{t_n}$.

3 Nonparametric estimation

Consider an autonomous uncertain differential equation in the following form

$$dX_t = f(X_t)dt + \sigma g(X_t)dC_t,\ t \in [0, T], \tag{5}$$

where $f : \mathbb{R} \to \mathbb{R}$ is an unknown continuous function, $g : \mathbb{R} \to \mathbb{R}$ is a known continuous function and σ is an unknown parameter. Let $0 < t_1 < t_2 < \cdots < t_n = T$. Assume that there are n observations $x_{t_1}, x_{t_2}, \dots, x_{t_n}$ of solution X_t at the times $t_1, t_2, \dots, t_n$, respectively.

According to Eq. (3), for a fixed $K \in \mathbb{N}$, we have

$$f(X_t) \approx \sum_{i=0}^{K} c_i p_i(X_t), \tag{6}$$

where $c_i \in \mathbb{R}$ for $i = 0, 1, 2, \dots, K$. According to (6), the approximation of model (5) may be obtained in the following form

$$dX_t = \sum_{i=0}^{K} c_i p_i(X_t)dt + \sigma g(X_t)dC_t,\ t \in [0, T]. \tag{7}$$

As suggested by Yao and Liu (2020), for $j = 1, 2, \dots, n-1$, we have

$$\frac{X_{t_{j+1}} - X_{t_j} - \left(\sum_{i=0}^{K} c_i p_i(X_{t_j})\right)\left(t_{j+1} - t_j\right)}{g(X_{t_j})(t_{j+1} - t_j)} = \sigma \frac{C_{t_{j+1}} - C_{t_j}}{t_{j+1} - t_j},$$

i.e.,

$$\frac{X_{t_{j+1}} - X_{t_j}}{g(X_{t_j})(t_{j+1} - t_j)} = \sum_{i=0}^{K} \frac{c_i p_i(X_{t_j})}{g(X_{t_j})} + \sigma \frac{C_{t_{j+1}} - C_{t_j}}{t_{j+1} - t_j}. \tag{8}$$

According to the definition of Liu process C_t,

$$\frac{C_{t_{j+1}} - C_{t_j}}{t_{j+1} - t_j} \sim \mathcal{N}(0,1), \quad j = 1, 2, \dots, n-1$$

are independent identically distributed. Thus, we denote that

$$\epsilon_0 = \sigma \frac{C_{t_{j+1}} - C_{t_j}}{t_{j+1} - t_j} \sim \mathcal{N}(0, \sigma)$$

is the noise. Denote that $\boldsymbol{c}_K = (c_0, c_1, \dots, c_K)$. We consider the following uncertain linear regression model

$$y = \boldsymbol{c}_K \boldsymbol{\eta}^{\mathrm{T}} + \epsilon_0, \tag{9}$$

where $\boldsymbol{\eta} = \left(\eta_0, \eta_1, \dots, \eta_K\right)$ is a vector of explanatory variables and y is a response variable. According to Eq. (8), we may assume that the observations of $\boldsymbol{\eta}$ and y are

$$\boldsymbol{\eta}_j = \left(\frac{p_0(x_{t_j})}{g(x_{t_j})}, \frac{p_1(x_{t_j})}{g(x_{t_j})}, \dots, \frac{p_K(x_{t_j})}{g(x_{t_j})} \right), \quad j = 1, 2, \dots, n-1 \tag{10}$$

and

$$y_j = \frac{x_{t_{j+1}} - x_{t_j}}{g(x_{t_j})(t_{j+1} - t_j)}, \quad j = 1, 2, \dots, n-1, \tag{11}$$

respectively. Apparently, the regression model (9) is equivalent to Eq. (8). Suppose that $\tilde{\boldsymbol{c}}_K = \left(c_0^*, c_1^*, c_2^*, \dots, c_K^*\right)$ is the estimation of $\boldsymbol{c}_K$ and σ^* is the estimation of σ. Based on the method of uncertain maximum likelihood estimation proposed by Lio and Liu (2020), $\tilde{\boldsymbol{c}}_K$ solves the minimization problem

$$\min_{\boldsymbol{c}_K} \bigvee_{j=1}^{n-1} |y_j - \boldsymbol{c}_K \boldsymbol{\eta}_j^{\mathrm{T}}| \tag{12}$$

and σ^* solves the maximization problem

$$\max_{\sigma > 0} \frac{\dfrac{\pi}{\sqrt{3}\sigma} \exp\left(\dfrac{\pi}{\sqrt{3}\sigma} \bigvee_{j=1}^{n-1} |y_j - \tilde{\boldsymbol{c}}_K \boldsymbol{\eta}_j^{\mathrm{T}}| \right)}{\left(1 + \exp\left(\dfrac{\pi}{\sqrt{3}\sigma} \bigvee_{j=1}^{n-1} |y_j - \tilde{\boldsymbol{c}}_K \boldsymbol{\eta}_j^{\mathrm{T}}| \right) \right)^2}.$$

According to Liu and Liu (2022a), we have

$$\sigma^* = \frac{\pi}{\sqrt{3}\lambda} \bigvee_{j=1}^{n-1} |y_j - \tilde{\boldsymbol{c}}_K \boldsymbol{\eta}_j^{\mathrm{T}}|, \tag{13}$$

where λ is the root of equation

$$1 + x + \exp(x) - x\exp(x) = 0$$

and may be taken as 1.5434 approximately in numerical solution.

It is important to first find an appropriate estimation of K (denoted as K^*). Based on the property of Legendre polynomial approximation, the right side of (6) would converge to the left side of (6) with $K \to +\infty$, i.e., for $j = 1, 2, \ldots, n-1$, we have

$$\lim_{K\to+\infty} \left| f(x_{t_j}) - \boldsymbol{c}_K \boldsymbol{\eta}_j^{\mathrm{T}} \right| = 0.$$

Since

$$\left| y_j - \boldsymbol{c}_K \boldsymbol{\eta}_j^{\mathrm{T}} \right| \le \left| \left(y_j - f(x_{t_j}) \right) \right| + \left| \left(f(x_{t_j}) - \boldsymbol{c}_K \boldsymbol{\eta}_j^{\mathrm{T}} \right) \right|$$

and

$$\left| y_j - \boldsymbol{c}_K \boldsymbol{\eta}_j^{\mathrm{T}} \right| \ge \left| \left(y_j - f(x_{t_j}) \right) \right| - \left| \left(f(x_{t_j}) - \boldsymbol{c}_K \boldsymbol{\eta}_j^{\mathrm{T}} \right) \right|,$$

based on the Squeeze theorem, we have

$$\lim_{K\to+\infty} \left| y_j - \boldsymbol{c}_K \boldsymbol{\eta}_j^{\mathrm{T}} \right| = \left| y_j - f(x_{t_j}) \right|.$$

Thus, we have

$$\lim_{K\to+\infty} \bigvee_{j=1}^{n-1} |y_j - \boldsymbol{c}_K \boldsymbol{\eta}_j^{\mathrm{T}}| = \bigvee_{j=1}^{n-1} \left| y_j - f(x_{t_j}) \right|.$$

According to the objective function of optimization problem (12), we denote that

$$\phi(K) = \bigvee_{j=1}^{n-1} \left| \frac{x_{t_{j+1}} - x_{t_j}}{g(x_{t_j})(t_{j+1} - t_j)} - \sum_{i=0}^{K} \frac{c_i^* p_i(x_{t_j})}{g(x_{t_j})} \right|, \ K \in \mathbb{N}.$$

In order to avoid too much computation, we suppose that Δ is the acceptable error and the smallest K that satisfies

$$\left| \phi(K) - \phi(K-1) \right| < \Delta$$

is the value of K^*. Then the value of $\tilde{\boldsymbol{c}}_{K^*}$ may be obtained. Now we give the Algorithm 1 to calculate the values of $\tilde{\boldsymbol{c}}_{K^*}$ and K^*.

Algorithm 1 Estimation for uncertain differential equation (7)

Require: $n > 0$; $K \in \mathbb{N}$
Set:$K = 0$; $\Delta > 0$; $Opt_0 = 0$
While $Error \geq \Delta$ **do**
  $Opt = 0$
  Obtain $\tilde{\boldsymbol{c}}_K$ from (12)
  For $j = 1$ **to** $n-1$ **do**
    $Opt = Opt \bigvee \left| \frac{x_{t_{j+1}} - x_{t_j} - \tilde{\boldsymbol{c}}_K \boldsymbol{p}_K^{\mathrm{T}}(x_{t_j})(t_{j+1}-t_j)}{g(x_{t_j})(t_{j+1}-t_j)} \right|$
  End For
  $Error = |Opt - Opt_0|$
  $Opt_0 = Opt$
  $K^* = K$
  $\tilde{\boldsymbol{c}}_{K^*} = \tilde{\boldsymbol{c}}_K$
  Obtain σ^* from (13)
  $K = K + 1$
End While

Remark 2 For uncertain differential Eq. (5), if $g : \mathbb{R} \to \mathbb{R}$ is also an unknown continuous function, with the help of Legendre polynomials, similar to (7), we have the approximation of (5) in the following form

$$dX_t = \sum_{i=0}^{K_1} c_i p_i(X_t)dt + \sum_{i=0}^{K_2} d_i p_i(X_t)(X_t)dC_t,\ t \in [0, T], \tag{14}$$

where $c_0, c_1, \dots, c_{K_1}, d_0, d_1, \dots, d_{K_2} \in \mathbb{R}$. Then we have

$$\frac{X_{t_{j+1}} - X_{t_j}}{\left(\sum_{i=0}^{K_2} d_i p_i(X_{t_j})\right)(t_{j+1}-t_j)} - \frac{\sum_{i=0}^{K_1} c_i p_i(X_{t_j})}{\sum_{i=0}^{K_2} d_i p_i(X_{t_j})} = \frac{C_{t_{j+1}} - C_{t_j}}{t_{j+1}-t_j} \sim \mathcal{N}(0,1). \tag{15}$$

Based on the method of maximum likelihood estimation, the estimations of c_0, c_1, $\cdots$, c_{K_1}, d_0, d_1, $\cdots$, d_{K_2} are the solutions of the following optimization problem

$$\min_{c_0,\dots,c_{K_1},d_0,\dots,d_{K_2}} \bigvee_{j=1}^{n-1} \left| \frac{x_{t_{j+1}} - x_{t_j}}{\left(\sum_{i=0}^{K_2} d_i p_i(x_{t_j})\right)(t_{j+1}-t_j)} - \frac{\sum_{i=0}^{K_1} c_i p_i(x_{t_j})}{\sum_{i=0}^{K_2} d_i p_i(x_{t_j})} \right|. \tag{16}$$

Apparently, the absolute values of $d_0, d_1, \dots, d_{K_2}$ in the solutions of optimization problem (16) will be very large. Since the aim of nonparametric estimation is to find a properly estimated uncertain differential Eq. (14) as the estimation of uncertain differential Eq. (5), estimations of $d_0, d_1, \dots, d_{K_2}$ with large absolute values will give the solution of (14) a strong disturbance, which is obviously unsatisfactory.

In fact, we may easily obtain the estimation of f in a uncertain differential Eq. (5), which is the most important thing in nonparametric estimation if g is already known. Thus, in application, we tend to define g in a known and acceptable form for observations. Then Algorithm 1 may be employed for the estimation of this uncertain differential equation.

4 Numerical experiments

Example 1 Consider uncertain differential equation

$$dX_t = f(X_t)dt + \sigma dC_t \tag{17}$$

with some observations of X_t which are shown in Table 1.

Utilizing (6), we have the approximation of uncertain differential Eq. (17) in the following form

$$dX_t = \sum_{i=0}^{K} c_i p_i(X_t)dt + \sigma dC_t.$$

With the help of Algorithm 1 (where $\Delta = 10^{-4}$), we have $K^* = 2$ and

$$c_0^* = -0.1073, \quad c_1^* = 1.7032, \quad c_2^* = -1.8488, \quad \sigma^* = 0.6787.$$

Thus the estimated uncertain differential equation of (17) is

$$dX_t = \big(-0.1073p_0(X_t) + 1.7032p_1(X_t) - 1.8488p_2(X_t)\big)dt + 0.6787dC_t. \tag{18}$$

According to Liu and Liu (2022b), the residuals of uncertain differential Eq. (18) may be obtained and given in Table 2. If uncertain differential Eq. (18) does fit the observed data in Table 1 well, then the residuals in Table 2 should follow the linear uncertainty distribution $\mathcal{L}(0, 1)$, i.e.,

Table 1 Observed data in Example 1

j	t_j	x_{t_j}	j	t_j	x_{t_j}	j	t_j	x_{t_j}
1	0.04	0.0608	10	0.40	0.3690	19	0.76	0.7343
2	0.08	0.0746	11	0.44	0.3886	20	0.80	0.7700
3	0.12	0.1101	12	0.48	0.4495	21	0.84	0.7824
4	0.16	0.1385	13	0.52	0.4930	22	0.88	0.8097
5	0.20	0.1938	14	0.56	0.5352	23	0.92	0.8281
6	0.24	0.2308	15	0.60	0.5557	24	0.96	0.8516
7	0.28	0.2964	16	0.64	0.6150	25	1.00	0.8789
8	0.32	0.3198	17	0.68	0.6605			
9	0.36	0.3485	18	0.72	0.6938			

Table 2 Residuals of uncertain differential Eq. (18)

j	ε_j	j	ε_j	j	ε_j
2	0.1800	10	0.1822	18	0.5653
3	0.4728	11	0.1761	19	0.7185
4	0.3332	12	0.7747	20	0.7005
5	0.7354	13	0.5441	21	0.3863
6	0.4211	14	0.5482	22	0.6490
7	0.8239	15	0.2439	23	0.5541
8	0.2113	16	0.8239	24	0.6690
9	0.2758	17	0.7027	25	0.7574

$$\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{25} \sim \mathcal{L}(0, 1).$$

Apparently, all the residuals are included in [0.025, 0.975], we have

$$\left(\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{25}\right) \notin W.$$

It follows from Remark 1 that the uncertain differential Eq. (18) is an applicable fit to the observed data in Table 1.

Example 2 Consider uncertain differential equation

$$dX_t = f(X_t)dt + \sigma X_t dC_t \tag{19}$$

with some observations of X_t which are shown in Table 3.

Similarly to Example 1, we have the estimation of uncertain differential Eq. (19) in the following form

Table 3 Observed data in Example 2

j	t_j	x_{t_j}	j	t_j	x_{t_j}	j	t_j	x_{t_j}
1	0.08	0.1109	10	0.80	0.2901	19	1.52	0.5011
2	0.16	0.1261	11	0.88	0.3137	20	1.60	0.5397
3	0.24	0.1390	12	0.96	0.3281	21	1.68	0.5902
4	0.32	0.1533	13	1.04	0.3514	22	1.76	0.5898
5	0.40	0.1564	14	1.12	0.3651	23	1.84	0.6588
6	0.48	0.1752	15	1.20	0.3665	24	1.92	0.7149
7	0.56	0.1995	16	1.28	0.4180	25	2.00	0.8000
8	0.64	0.2337	17	1.36	0.4437			
9	0.72	0.2617	18	1.44	0.4706			

Table 4 Residuals of uncertain differential Eq. (20)

j	ε_j	j	ε_j	j	ε_j
2	0.7953	10	0.5794	18	0.4793
3	0.6068	11	0.4718	19	0.5050
4	0.5739	12	0.3230	20	0.5669
5	0.1808	13	0.4550	21	0.6341
6	0.6208	14	0.3183	22	0.1786
7	0.6817	15	0.1856	23	0.7051
8	0.7887	16	0.7760	24	0.4936
9	0.6109	17	0.4716	25	0.5630

Table 5 Observed data in Example 3

j	t_j	x_{t_j}	j	t_j	x_{t_j}	j	t_j	x_{t_j}
1	0.04	0.0412	10	0.40	0.6413	19	0.76	0.6795
2	0.08	0.1083	11	0.44	0.6401	20	0.80	0.6789
3	0.12	0.1983	12	0.48	0.6517	21	0.84	0.6352
4	0.16	0.3162	13	0.52	0.6645	22	0.88	0.6162
5	0.20	0.4185	14	0.56	0.6360	23	0.92	0.6086
6	0.24	0.5236	15	0.60	0.6236	24	0.96	0.6455
7	0.28	0.5861	16	0.64	0.6404	25	1.00	0.6806
8	0.32	0.6138	17	0.68	0.6463			
9	0.36	0.6030	18	0.72	0.6732			

$$\begin{aligned} dX_t = &\big(-3.8379p_0(X_t) + 10.2704p_1(X_t) - 6.7872p_2(X_t) \\ &+3.7387p_3(X_t) + 0.3933p_4(X_t)\big)dt + 1.0668X_t dC_t. \end{aligned} \tag{20}$$

The residuals of uncertain differential Eq. (20) may be obtained and given in Table 4. Since all the residuals are included in [0.025, 0.975], it follows from Remark 1 that the uncertain differential Eq. (20) is an applicable fit to the observed data in Table 3.

Example 3 Consider the uncertain differential equation with initial value

$$\begin{cases} dX_t = f(X_t)dt + \sigma X_t dC_t, \ t \in [0, 1], \\ X_0 = 0. \end{cases} \tag{21}$$

Let us give the function $f(x)$ by

$$f(x) = 2\sin 5x + \cos 2x.$$

and the true value of σ by $\sigma = 1$. Then the corresponding α-path X_t^α of (21) satisfies

$$\begin{cases} dX_t^\alpha = (2\sin 5X_t^\alpha + \cos 2X_t^\alpha)dt + \frac{\sqrt{3}}{\pi}X_t^\alpha \ln \frac{\alpha}{1-\alpha}dt, \ t \in [0,1], \\ X_0^\alpha = 0. \end{cases} \tag{22}$$

Note that the inverse uncertainty distribution of solution X_t to (21) is solution X_t^α to (22). That is, for $\alpha \in (0,1)$, X_t^α is a sample point of X_t at time t. Observed data in Table 5 are produced by solving (22) for arbitrary $\alpha \in (0,1)$ at time t_i.

Based on the observed data, similarly to Example 1, we have the estimation of uncertain differential Eq. (21) in the following form

$$\begin{cases} dX_t = \big(44.5335p_0(X_t) - 111.0393p_1(X_t) + 120.2136p_2(X_t) - 95.3485p_3(X_t) \\ \qquad +44.9572p_4(X_t) - 12.4951p_5(X_t)\big)dt + 1.3989X_t dC_t, \ t \in [0,1], \\ X_0 = 0. \end{cases} \tag{23}$$

The residuals of uncertain differential Eq. (23) may be obtained and given in Table 6. Since all the residuals are included in [0.025, 0.975], it follows from Remark 1 that the uncertain differential Eq. (23) is an applicable fit to the observed data in Table 5.

As may be seen in (23), the estimated function of $f(x)$ is

$$\begin{aligned} f^*(x) = {} & 44.5335p_0(x) - 111.0393p_1(x) + 120.2136p_2(x) \\ & - 95.3485p_3(x) + 44.9572p_4(x) - 12.4951p_5(x). \end{aligned}$$

Since all the observations of X_t are located in interval [0, 0.8], in Fig. 1, we give the image of true function $f(x)$ and its estimation $f^*(x)$ in $x \in [0, 0.8]$. Obviously, $f^*(x)$ may properly approximate $f(x)$.

All these three numerical examples suggest that the estimated uncertain differential equations may fit the corresponding observations well.

Table 6 Residuals of uncertain differential Eq. (23)

j	ε_j	j	ε_j	j	ε_j
2	0.5885	10	0.6839	18	0.7506
3	0.5638	11	0.4054	19	0.6709
4	0.7247	12	0.5586	20	0.6291
5	0.3625	13	0.6343	21	0.1612
6	0.7267	14	0.2272	22	0.1864
7	0.6057	15	0.2497	23	0.2090
8	0.4671	16	0.5331	24	0.6929
9	0.1721	17	0.4906	25	0.8120

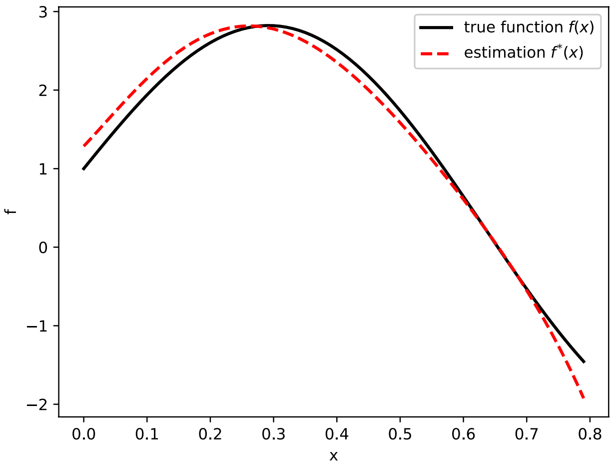

Fig. 1 Image of true function $f(x)$ and the estimation $f^*(x)$

5 Application to atmospheric carbon dioxide problem

In this section, we apply the method of nonparametric estimation into the modeling of the atmospheric carbon dioxide problem. The data of monthly mean atmospheric carbon dioxide measured at Mauna Loa Observatory (Hawaii) from February 2018 to September 2022 are shown in Table 7. (All the data may be found at https://www.co2.earth/. The unit of these data is *parts per million*)

Table 7 Data of atmospheric carbon dioxide

Time	Value	Time	Value	Time	Value	Time	Value
2018.2	408.52	2019.4	413.54	2020.6	416.60	2021.8	414.47
2018.3	409.59	2019.5	414.86	2020.7	414.62	2021.9	413.30
2018.4	410.45	2019.6	414.16	2020.8	412.78	2021.1	413.93
2018.5	411.44	2019.7	411.97	2020.9	411.52	2021.11	415.01
2018.6	410.99	2019.8	410.18	2020.1	411.51	2021.12	416.71
2018.7	408.90	2019.9	408.76	2020.11	413.12	2022.1	418.19
2018.8	407.16	2019.1	408.75	2020.12	414.26	2022.2	419.28
2018.9	405.71	2019.11	410.48	2021.1	415.52	2022.3	418.81
2018.1	406.19	2019.12	411.98	2021.2	416.75	2022.4	420.23
2018.11	408.21	2020.1	413.61	2021.3	417.64	2022.5	420.99
2018.12	409.27	2020.2	414.34	2021.4	419.05	2022.6	420.99
2019.1	411.03	2020.3	414.74	2021.5	419.13	2022.7	418.90
2019.2	411.96	2020.4	416.45	2021.6	418.94	2022.8	417.19
2019.3	412.18	2020.5	417.31	2021.7	416.96	2022.9	415.95

5.1 Modeling of atmospheric carbon dioxide problem

Denote that the observed times are $t_1 = 0.1, t_2 = 0.2, \ldots, t_{56} = 5.6$. Assume that z is the vector consisting of all values in Table 7. Since the Legendre polynomials may only approximate the continuous functions in domain $[-1, 1]$, we multiply all values by $\frac{0.8}{\max z}$, i.e., 0.0019. Then the processed data are denoted as the observed data $x_{t_1}, x_{t_2}, \ldots, x_{t_{56}}$ at times $t_1, t_2, \ldots, t_{56}$ and given in Table 8.

Now we construct the following atmospheric carbon dioxide model based on the processed data in Table 8.

$$dX_t = f(X_t)dt + \sigma X_t dC_t, \tag{24}$$

where $f : \mathbb{R} \to \mathbb{R}$ is an unknown continuous function and $\sigma > 0$ is an unknown parameter. Approximating the uncertain differential Eq. (24) by the following form

$$dX_t = \sum_{i=0}^{K} c_i p_i(X_t)dt + \sigma X_t dC_t.$$

Then, Algorithm 1 may be used to obtain $K^* = 2$ and

$$c_0^* = -19.0413, \quad c_1^* = 31.6427, \quad c_2^* = -13.6719, \quad \sigma^* = 0.0577.$$

Thus the estimated uncertain differential equation of (24) is

$$dX_t = \left(-19.0413 p_0(X_t) + 31.6427 p_1(X_t) - 13.6719 p_2(X_t)\right)dt + 0.0577 X_t dC_t. \tag{25}$$

The residuals of the uncertain differential Eq. (25) may be obtained and given in Table 9. Since all the residuals are included in [0.025, 0.975], the uncertain differential Eq. (25) is an applicable fit to the observed data in Table 8. Therefore, we

Table 8 Processed data

1	0.1	0.7763	15	1.5	0.7858	29	2.9	0.7917	43	4.3	0.7876
2	0.2	0.7783	16	1.6	0.7884	30	3	0.7879	44	4.4	0.7854
3	0.3	0.7800	17	1.7	0.7870	31	3.1	0.7844	45	4.5	0.7866
4	0.4	0.7819	18	1.8	0.7829	32	3.2	0.7820	46	4.6	0.7886
5	0.5	0.7810	19	1.9	0.7795	33	3.3	0.7820	47	4.7	0.7919
6	0.6	0.7770	20	2	0.7768	34	3.4	0.7850	48	4.8	0.7947
7	0.7	0.7737	21	2.1	0.7767	35	3.5	0.7872	49	4.9	0.7968
8	0.8	0.7710	22	2.2	0.7800	36	3.6	0.7896	50	5	0.7959
9	0.9	0.7719	23	2.3	0.7829	37	3.7	0.7919	51	5.1	0.7986
10	1	0.7757	24	2.4	0.7860	38	3.8	0.7936	52	5.2	0.8000
11	1.1	0.7777	25	2.5	0.7874	39	3.9	0.7963	53	5.3	0.8000
12	1.2	0.7811	26	2.6	0.7881	40	4	0.7965	54	5.4	0.7960
13	1.3	0.7828	27	2.7	0.7914	41	4.1	0.7961	55	5.5	0.7928
14	1.4	0.7833	28	2.8	0.7930	42	4.2	0.7923	56	5.6	0.7904

Table 9 Residuals of uncertain differential Eq. (25)

j	ε_j	j	ε_j	j	ε_j	j	ε_j
2	0.6945	16	0.7595	30	0.2340	44	0.3330
3	0.6633	17	0.4215	31	0.2318	45	0.6497
4	0.6892	18	0.1839	32	0.3009	46	0.7282
5	0.4319	19	0.2171	33	0.5162	47	0.8180
6	0.1751	20	0.2578	34	0.7852	48	0.8070
7	0.2082	21	0.4992	35	0.7304	49	0.7744
8	0.2428	22	0.7905	36	0.7569	50	0.5364
9	0.5872	23	0.7659	37	0.7642	51	0.8210
10	0.8228	24	0.7901	38	0.7293	52	0.7575
11	0.6921	25	0.6698	39	0.8078	53	0.6540
12	0.7956	26	0.6203	40	0.6319	54	0.2834
13	0.6826	27	0.8171	41	0.5852	55	0.3065
14	0.5635	28	0.7212	42	0.2656	56	0.3576
15	0.7555	29	0.4558	43	0.1751		

say that the dynamic process of atmospheric carbon dioxide measured at Mauna Loa Observatory from February 2018 to September 2022 follows the uncertain differential Eq. (25).

5.2 Completion of lost data

Due to the unexpected accidents, some data may be lost in management or storage. Let us take this atmospheric carbon dioxide problem for example. Referring to Table 7, we assume that the data in March 2018, June 2019 are lost. That is, the proceed data at $j = 2$ and $j = 17$ in Table 8 are lost. For the parametric form

$$dX_t = \sum_{i=0}^{K} c_i p_i(X_t)dt + \sigma X_t dC_t,$$

similarly to Sect. 5.1, the estimated values may be obtained as $K^* = 2$ and

$$c_0^* = -17.8493, \quad c_1^* = 29.6757, \quad c_2^* = -12.8425, \quad \sigma^* = 0.0576.$$

Then the estimation of model (24) is

$$dX_t = \left(-17.8493p_0(X_t) + 29.6757p_1(X_t) - 12.8425p_2(X_t)\right) + 0.0576X_t dC_t. \tag{26}$$

The residuals of uncertain differential Eq. (26) may be obtained and given in Table 10. Since all the residuals are included in [0.025, 0.975], uncertain differential Eq. (26) is an applicable fit to the existing observed data in Table 8.

According to the definitions of forecast value and confidence interval given by Lio and Liu (2018), we may obtain the forecast value of y in uncertain regression model (9) as

Table 10 Residuals of uncertain differential Eq. (26)

j	ε_j	j	ε_j	j	ε_j	j	ε_j
2	–	16	0.7570	30	0.7232	44	0.2661
3	0.6009	17	–	31	0.4576	45	0.1754
4	0.6639	18	0.7612	32	0.2346	46	0.3341
5	0.6902	19	0.4890	33	0.2325	47	0.6515
6	0.4327	20	0.1842	34	0.3015	48	0.7300
7	0.1749	21	0.2171	35	0.5172	49	0.8197
8	0.2073	22	0.2575	36	0.7865	50	0.8087
9	0.2408	23	0.4986	37	0.7321	51	0.7763
10	0.5846	24	0.7909	38	0.7588	52	0.5379
11	0.8221	25	0.7669	39	0.7661	53	0.8225
12	0.6917	26	0.7916	40	0.7312	54	0.7589
13	0.7961	27	0.6716	41	0.8095	55	0.6551
14	0.6839	28	0.6223	42	0.6337	56	0.2835
15	0.5647	29	0.8189	43	0.5868		

$$\hat{y} = \tilde{\boldsymbol{c}}_{K^*}\, \boldsymbol{\eta}^{\mathrm{T}} = \sum_{i=0}^{K^*} c_i^* \eta_i \tag{27}$$

and γ-confidence interval (e.g., $\gamma = 0.95$) of y as

$$\left[\hat{y} - \frac{\sigma^* \sqrt{3}}{\pi} \ln \frac{1+\gamma}{1-\gamma}, \hat{y} + \frac{\sigma^* \sqrt{3}}{\pi} \ln \frac{1+\gamma}{1-\gamma}\right]. \tag{28}$$

For a fixed j, we suppose that the observed value of $X_{t_{j+1}}$ is missing. Thus, we denote the forecast value of $X_{t_{j+1}}$ as $\hat{X}_{t_{j+1}}$. According to (10) and (11), we may obtain the j-th observations of $\boldsymbol{\eta}$ as

$$\boldsymbol{\eta}_j = \left(\frac{p_0(x_{t_j})}{x_{t_j}}, \frac{p_1(x_{t_j})}{x_{t_j}}, \ldots, \frac{p_K(x_{t_j})}{x_{t_j}}\right)$$

and the j-th observations of y as

$$y_j = \frac{\hat{X}_{t_{j+1}} - x_{t_j}}{x_{t_j}(t_{j+1} - t_j)}.$$

Substituting $\boldsymbol{\eta} = \boldsymbol{\eta}_j$ and $\hat{y} = y_j$ into (27), we have the forecast value of $X_{t_{j+1}}$ as

$$\hat{X}_{t_{j+1}} = x_{t_j} + \left(\sum_{i=0}^{K^*} c_i^* p_i(x_{t_j})\right)(t_{j+1} - t_j). \tag{29}$$

Substituting $\boldsymbol{\eta} = \boldsymbol{\eta}_j$ and $\hat{y} = y_j$ into (28), we have the γ-confidence interval of $X_{t_{j+1}}$ as

$$\left[\hat{X}_{t_{j+1}} - x_{t_j}(t_{j+1} - t_j)\frac{\sigma^*\sqrt{3}}{\pi}\ln\frac{1+\gamma}{1-\gamma}, \hat{X}_{t_{j+1}} + x_{t_j}(t_{j+1} - t_j)\frac{\sigma^*\sqrt{3}}{\pi}\ln\frac{1+\gamma}{1-\gamma}\right]. \quad (30)$$

With (29), we have the forecast values of X_{t_2} and $X_{t_{17}}$ as $\hat{X}_{t_2} = 0.7763$ and $\hat{X}_{t_{17}} = 0.7878$, respectively. Then with (30), we have the 95%-confidence interval of X_{t_2} and $X_{t_{17}}$ as [0.7673, 0.7853] and [0.7786, 0.7970], respectively.

Restoring the processed data to the original data, we have the forecast value of the lost data in March 2018 as 408.52 with 95%-confidence interval [403.77, 413.27] and the forecast value of the lost data in June 2019 as 414.57 with 95%-confidence interval [409.74, 419.39]. Referring to Table 7, we may clearly see that the actual value in March 2018 is 409.59 which is in the interval [403.77, 413.27] and the actual value in June 2019 is 414.16 which is in the interval [409.74, 419.39]. Thus we claim that model (26) may complete the lost data well.

In general, the estimated model obtained by nonparametric estimation may properly fit the observations and complete the lost data. Thus we claim that our method of nonparametric estimation is effective for this atmospheric carbon dioxide model.

6 Conclusion

In this paper, we proposed a method of nonparametric estimation for autonomous uncertain differential equations. An algorithm was introduced and illustrated with three numerical examples. With the help of residuals and uncertain hypothesis test, we proved that the estimated uncertain differential equations may fit their observations well. Finally, we applied the nonparametric estimation into the atmospheric carbon dioxide problem. With the data of monthly mean atmospheric carbon dioxide from February 2018 to September 2022, we obtained an uncertain differential equation based model by using the method of nonparametric estimation. Then, the model was proved to be an applicable fit to the observations and an effective tool to complete the lost data. In the future, we will further research on the method of nonparametric estimation for nonautonomous uncertain differential equations whose variations of the current states are directly affected by both the states and the time.

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