# Church Synthesis on Register Automata over Linearly Ordered Data Domains* 

Léo Exibard ${ }^{1}$, Emmanuel Filiot ${ }^{2}$ and Ayrat Khalimov ${ }^{2}$<br>${ }^{1}$ Reykjavik University, Iceland. ${ }^{2}$ Université libre de Bruxelles, Belgium.


#### Abstract

In a Church synthesis game, two players, Adam and Eve, alternately pick some element in a finite alphabet, for an infinite number of rounds. The game is won by Eve if the $\boldsymbol{\omega}$-word formed by this infinite interaction belongs to a given language $\boldsymbol{S}$, called the specification. It is well-known that for $\boldsymbol{\omega}$ regular specifications, it is decidable whether Eve has a strategy to enforce the specification no matter what Adam does. We study the extension of Church synthesis games to the linearly ordered data domains ( $\mathbb{Q}, \leq$ ) and $(\mathbb{N}, \leq)$. In this setting, the infinite interaction between Adam and Eve results in an $\boldsymbol{\omega}$-data word, i.e., an infinite sequence of elements in the domain. We study this problem when specifications are given as register automata. Those automata consist in finite automata equipped with a finite set of registers in which they can store data values, that they can then compare with incoming data values with respect to the linear order. Church games over ( $\mathbb{N}, \leq$ ) are however undecidable, even for deterministic register automata. Thus, we introduce one-sided Church games, where Eve instead operates over a finite alphabet, while Adam still manipulates data. We show that they are determined, and that deciding the existence of a winning strategy is in ExpTime, both for $\mathbb{Q}$ and $\mathbb{N}$. This follows from a study of constraint sequences, which abstract the behaviour of register automata, and allow us to reduce Church games to $\boldsymbol{\omega}$-regular games. We present an application of one-sided Church games to a transducer synthesis problem. In this application, a transducer models a reactive system (Eve) which outputs data stored in its registers, depending on its interaction with an environment (Adam) which inputs data to the system.


Keywords: Synthesis, Church Game, Register Automata, Register Transducers, Ordered Data Words

[^0]
## 2012 ACM Subject Classification:

| Theory of computation | $\rightarrow$ | Logic and verification |
| :--- | :--- | :--- |
| Theory of computation | $\rightarrow$ | Automata over infinite objects |
| Theory of computation | $\rightarrow$ | Transducers |

## 1 Introduction

Church synthesis. Reactive synthesis is the problem of automatically constructing a reactive system from a specification of correct executions, i.e. a non-terminating system which interacts with an environment, and whose executions all comply with the specification, no matter how the environment behaves. The earliest formulation of synthesis dates back to Church, who proposed to formalize it as a game problem: two players, Adam in the role of the environment and Eve in the role of the system, alternately pick the elements from two finite alphabets $I$ and $O$ respectively. Adam starts with $i_{0} \in I$, Eve responds with $o_{0} \in O$, ad infinitum. Their interaction results in the $\omega$-word $w=i_{0} o_{0} i_{1} o_{1} \ldots \in(I \cdot O)^{\omega}$. The winner is decided by a winning condition, represented as a language $S \subseteq(I \cdot O)^{\omega}$ called specification: if $w \in S$, the play is won by Eve, otherwise by Adam. Eve wins the game if she has a strategy $\lambda_{\exists}: I^{+} \rightarrow O$ to pick elements in $O$, depending on what has been played so far, so that no matter the input sequence $i_{0} i_{1} \ldots$ chosen by Adam, the resulting $\omega$-word $i_{0} \lambda\left(i_{0}\right) i_{1} \lambda\left(i_{0} i_{1}\right) \ldots$ belongs to $S$. Similarly, Adam wins the game if he has a strategy $\lambda_{\forall}: O^{*} \rightarrow I$ to win against any strategy Eve uses. In the original Church problem, specifications are $\omega$-regular languages, i.e. languages definable in monadic second-order logic with one successor or equivalently, deterministic parity automata. The seminal papers [14, 44] have shown that Church games (for $\omega$-regular specification) are determined: either Eve wins or otherwise Adam wins. Moreover, given a Church game, the winner of the game is computable. Finally, justifying the use of Church games as a formulation of reactive synthesis, finite-memory strategies are sufficient to win (both for Eve and Adam). This implies that if Eve wins a Church game, one can effectively construct a finite-state machine (e.g. a Mealy machine) implementing a winning strategy.

Church synthesis and games on graphs have been extensively studied for specifications given in linear-time temporal logic (LTL) [43] - recently supported by a tool competition [49] -, as well as in many other settings, for example, quantitative, distributed, non-competitive (see [5, 13] and the references therein). Yet, those works focus on control, sometimes with complex interactions between the synthesized systems, rather than on data. This is reflected already in the original formulation by Church: Adam and Eve interact via finite alphabets $I$ and $O$, intended to model control actions rather than proper pieces of data. But real-life systems often operate values from a large to infinite data domain. Examples include data-independent programs [53, 35, 41], software with integer parameters [11], communication protocols with message parameters [19], and more [10, 51, 18]. The goal of this paper is to study extensions of reactive synthesis, and its formulation as Church games, to infinite data domains: $(\mathbb{Q}, \leq)$ and $(\mathbb{N}, \leq)$ in particular.

Church synthesis over infinite data domains. Church games naturally extend to an infinite data domain $\mathscr{D}$ : Adam and Eve alternately pick data in $\mathscr{D}$, and their infinite interaction results in an $\omega$-data word $d_{0} d_{0}^{\prime} d_{1} d_{1}^{\prime} \cdots \in \mathscr{D}^{\omega}$. The game is won by Eve if it belongs to a given specification $S \subseteq \mathscr{D}^{\omega}$. Accordingly, strategies for Eve have type $\mathscr{D}^{+} \rightarrow \mathscr{D}$, while strategies for Adam have type $\mathscr{D}^{*} \rightarrow \mathscr{D}$. In this paper, we study specifications given by a standard extension of finite-state automata to infinite data domains called register automata [36]: they use a finite set of registers to store data values, and a finite set of predicates over the data domain to test those values. In each step, the automaton reads a data value from $\mathscr{D}$ and compares it with the values held in its registers using the predicates (and possibly constants). Depending on this comparison, it decides to store the value in some of the registers, and then moves to a successor state. This way, it builds a sequence of configurations (pairs of state and register values) representing its run on reading a data word from $\mathscr{D}^{\omega}$ : it is accepted if the visited states satisfy a certain parity condition. In this paper, we study specifications given by deterministic register automata over $\mathbb{Q}$ or $\mathbb{N}$, which can use the predicate $\leq$ and the constant 0 to test data values.

Contributions. Our first result is an impossibility result: deciding the winner of a Church game for specifications given by deterministic register automata over ( $\mathbb{N}, \leq$ ) is an undecidable problem (Theorem 1). We introduce the one-sided restriction on Church games: Adam still has the full power of picking data values, but Eve's behaviour is restricted to picking elements from a finite alphabet only. Despite being asymmetric, one-sided Church games are quite expressive. For example, they model synthesis scenarios for runtime data monitors that monitor the input data stream and raise a Boolean flag when a critical trend happens (like oscillations above a certain amplitude), and for systems that need to take control actions depending on sensor measurements (a heating controller for instance). Formally, in one-sided Church games, there is a finite set of elements $\Sigma$ in which Eve picks her successive choices. Accordingly, specifications are languages $S \subseteq(\mathscr{D} \Sigma)^{\omega}$, in this paper defined by deterministic one-sided register automata (defined naturally by alternating between register automata transitions and finite-state automata transitions). Eve's strategies have type $\lambda_{\exists}: \mathscr{D}^{+} \rightarrow \Sigma$ while Adam's strategies have type $\lambda_{\forall}: \Sigma^{*} \rightarrow \mathscr{D}$. We prove the following about one-sided Church games whose specifications are given by one-sided deterministic register automata over $(\mathbb{Q}, \leq)$ and $(\mathbb{N}, \leq)$ :

1. they are determined: every game is either won by Eve or Adam
2. they are decidable: the winner can be computed in time exponential in the number of registers of the specification,
3. if Eve wins, then she has a winning strategy which can be implemented by a transducer with registers (which can be effectively constructed).

Transducers with registers extend Mealy machines with a finite set of registers: they have finitely many states, and given any state and a test over the input data value, deterministically, they assign the current value to some registers (or none), output an element of $\Sigma$, and update their state. Therefore, the last result echoes the similar result in the $\omega$-regular setting (finite-memory strategies can be effectively constructed for the winner), and supports the fact that one-sided Church games on register
automata are an adequate framework for effective synthesis of machines processing streams of data.

Example 1. Figure 1 illustrates a specification given by a deterministic one-sided register automaton, alternating between square and circle states, depending on whether their outgoing transitions read data values or elements in a finite alphabet $\Sigma=\{a, b\}$. It can be seen as a game arena where Adam controls the square states while Eve controls the circle states. To simplify the presentation, two parts of the automaton are not depicted and have been summarised as "Eve wins" and "Eve loses": any run going in the former part is non-accepting and any run going in the latter part is accepting (this can be modelled by a parity condition). So, Eve's objective is to force executions into "Eve wins", whatever input data values are issued by Adam. There are two registers, $r_{M}$ and $r_{l}$. The test $\top$ (true) means that the transition can be taken irrespective of the value played, the test $r_{l}<*<r_{M}$ means that the value should be between the values of registers $r_{l}$ and $r_{M}$, and the test 'else' means the opposite. The writing $\downarrow r$ means that the value is stored into the register $r$. At first, Adam provides some data value $\ell_{M}$, serving as a maximal value stored in $r_{M}$. Register $r_{l}$, initially 0 , holds the last data value $d_{l}$ played by Adam. Consider state $C$ : if Adam provides a value outside of the interval $] \ell_{l}, \ell_{M}\left[\right.$, he loses; if it is strictly between $\ell_{l}$ and $\ell_{M}$, it is stored into register $r_{l}$ and the game proceeds to state $D$. There, Eve can either respond with label $b$ and move to state $E$, or with $a$ to state $C$. In state $E$, Adam wins if he can provide a data value strictly between $\ell_{l}$ and $\ell_{M}$, otherwise he loses. Eve wins this game in $\mathbb{N}$ : for example, she could always respond with label $a$, looping in states $C-D$. After a finite number of steps, Adam is forced to provide a data value $\geq \ell_{M}$, losing the game. An alternative Eve winning strategy, that does depend on Adam data, is to loop in $C-D$ until $\ell_{M}-d_{l}=1$ (thus, she has to memorise the first Adam value $\ell_{M}$ ), then move to state $E$, where Adam loses. In the dense domain $(\mathbb{Q}, \leq)$, however, the game is won by Adam, because he can always provide a value within $] d_{l}, \ell_{M}\left[\right.$ for any $d_{l}<d_{M}$, so the game either loops in $C-D$ forever or reaches "Eve loses".


Figure 1: Eve wins this game in $\mathbb{N}$ but loses in $\mathbb{Q}$.

Proof overview. We give intuitions about the main ingredients to show decidability. The key idea used to solve problems about register automata is to forget the precise values of input data and registers, and track instead the constraints (sometimes called types) describing the relations between them. In our example, all registers start in

0 so the initial constraint is $r_{l}^{1}=r_{M}^{1}$, where $r^{i}$ abstracts the value of register $r$ at step $i$. Then, if Adam provides a data above the value of $r_{l}$, the constraint becomes $r_{l}^{2}<r_{M}^{2}$ in state $B$. Otherwise, if Adam had provided a data equal to the value in $r_{l}$, the constraint would be $r_{l}^{2}=r_{M}^{2}$. In this way the constraints evolve during the play, forming an infinite sequence. Looping in states $C-D$ induces the constraint sequence $\left(r_{l}^{i}<r_{l}^{i+1}<r_{M}^{i}=r_{M}^{i+1}\right)_{i>2}$. It forms an infinite chain $r_{l}^{3}<r_{l}^{4}<\ldots$ bounded by constant $r_{M}^{3}=r_{M}^{4}=\ldots$ from above. In $\mathbb{N}$, as it is a well-founded order, it is not possible to assign values to the registers at every step to satisfy all constraints, so the sequence is not satisfiable. Before elaborating on how this information can be used to solve Church games, we describe our results on satisfiability of constraint sequences. This topic was inspired by the work [47] which studies, among others, the nonemptiness problem of constraint automata, whose states and transitions are described by constraints. In particular, they show [47, Appendix C] that satisfiability of constraint sequences can be checked by nondeterministic $\omega \mathrm{B}$-automata [6]. Nondeterminism however poses a challenge in synthesis, and it is not known whether games with a winning objective given as a nondeterministic $\omega \mathrm{B}$-automaton are decidable. In contrast, we describe a deterministic max-automaton [8] characterising the satisfiable constraint sequences in $\mathbb{N}$. As a consequence of [9], games over such automata are decidable. Then we study two kinds of constraint sequences inspired by Church games with register automata. First, we show that the satisfiable lasso-shaped ${ }^{1}$ constraint sequences, of the form $u v^{\omega}$, are recognisable by deterministic parity automata. Second, we show how to assign values to registers on-the-fly in order to satisfy a constraint sequence induced by a play in the Church game.

To solve one-sided Church games with a specification given as a register automaton $S$ for $(\mathbb{N}, \leq)$ and $(\mathbb{Q}, \leq)$, we reduce them to certain finite-arena zero-sum games, which we call automata games. The states and transitions of the game are those of the specification automaton $S$. The winning condition requires Eve to satisfy the original objective of $S$ only on feasible plays, i.e. those that induce satisfiable constraint sequences. In our example, the play $A \cdot B \cdot(C \cdot D)^{\omega}$ does not satisfy the parity condition, yet it is won by Eve in the automaton game since it is not satisfiable in $\mathbb{N}$, and therefore there is no corresponding play in the Church game. We show that if Eve wins the automaton game, then she wins the Church game, using a strategy that simulates the register automaton $S$ and simply picks one of its transitions. It is also sufficient: if Adam wins the automaton game then he wins the Church game. To prove this, we construct, from a winning strategy of Adam in the automaton game, a winning strategy of Adam (that manipulates data) in the Church game. This step uses the previously mentioned results on satisfiability of constraint sequences. Over $(\mathbb{N}, \leq)$, we cannot solve the automaton game directly, as it is not $\omega$-regular. We instead reduce it to an $\omega$-regular approximation of it which considers quasi-feasible sequences, a notion which is more liberal than feasibility but coincides with it on lasso-shaped words.

Related works. This paper is an extended version of the conference paper [25]. It follows a line of works about synthesis from register automata specifications [23,

[^1]$37,38,27]$, which focused on register automata over data domains $(\mathcal{D},=)$ equipped with equality tests only. The synthesis of data systems has also been investigated in $[31,40]$. They do not rely on register automata and are also limited to equality tests or do not study data comparison. Thus, systems that output the largest value seen so far, grant a resource to a process with the lowest ID, or raise an alert when a heart sensor reads values forming a dangerous curve, are out of reach of those synthesis methods. These systems require $\leq$.

In this paper, we consider specifications given by deterministic register automata. Already in the case of infinite alphabets $(\mathscr{D},=)$, dropping the determinism requirement leads to undecidability: finding a winner of a Church game is undecidable when specifications are given as nondeterministic or universal register automata [23, 27]. To recover decidability, in the case of universal register automata, those works restrict Eve strategies to register transducers with an a priori fixed number of registers. This problem is called register-bounded synthesis. Recently in [26], register-bounded synthesis have been extended to various data domains such as $(\mathbb{N}, \leq),(\mathbb{Z}, \leq)$, or $\left(\Sigma^{*}, \preceq\right)$ where $\Sigma$ is an arbitrary finite alphabet and $\preceq$ is the prefix relation. The results of [26] are orthogonal to the results of this paper, although they rely on the study of constraint sequences we conduct here.

The paper [28] studies synthesis from variable automata with arithmetic. Those automata are incomparable with register automata: on the one hand, they allow addition on top of a dense order predicate, but on the other hand they do not allow updating the content of the registers along the run. Note that they do not consider the case of a discrete order. The paper [29] studies strategy synthesis but, again, mainly over a dense domain. A one-sided setting similar to ours was studied in [30] for Church games whose winning condition is given by formulas of the Logic of Repeating Values (a fragment of LTL with the freeze quantifier [20]), but only for $(\mathscr{D},=)$. That work was extended to domain $(\mathbb{Z}, \leq)$ in [4]. There, the authors show that the realisability problem in one-sided setting on $(\mathbb{Z}, \leq)$ for Constraint LTL and its prompt variant are 2EXPTIME-complete. Deterministic register automata are more expressive than Constraint LTL, so our work subsumes their decidability result, yet the lower expressivity of Constraint LTL enables simpler arguments. We note that our proof ideas - abstracting data words by finite-alphabet words and utilising regularity of abstracted words - are somewhat similar to those in papers on Constraint LTL [21, $4]$. The work on automata with atoms [39] implies our decidability result for $(\mathbb{Q}, \leq)$, even in the two-sided setting, but not the complexity result, and it does not apply to ( $\mathbb{N}, \leq$ ). Our setting in $\mathbb{N}$ is loosely related to monotonic games [2]: they both forbid infinite descending behaviours, but the direct conversion is unclear. Games on infinite arenas induced by pushdown automata [52, 12, 1] or one-counter systems [48, 32] are orthogonal to our games.

Outline. In Section 2, we introduce preliminary notions. Section 3 introduces Church synthesis games along with the main tools and results (with proofs postponed). Section 4 presents the postponed proofs for Church synthesis, relying on results about satisfiability of constraint sequences over $(\mathbb{N}, \leq)$ described in Section 5 .

## 2 Preliminaries

In this paper, $\mathbb{N}=\{0,1, \ldots\}$ is the set of natural numbers (including 0 ). We assume some knowledge of $\omega$-regular languages and $\omega$-automata, and refer to e.g. [15] for an introduction.
$\omega$-data words. In this paper, an ordered data domain, or simply data domain, $\mathscr{D}$ is an infinite countable set of elements called data, linearly ordered by some order denoted $<$. We consider two data domains, $\mathbb{N}$ and $\mathbb{Q}$, with their usual order. An $\omega$ data word over $\mathscr{D}$ is an infinite sequence $d_{0} d_{1} \ldots$ of data in $\mathscr{D}$. We denote by $\mathscr{D}^{\omega}$ the set of $\omega$-data words. Similarly, we denote by $\mathscr{D}^{*}$ the set of finite sequences (possibly empty) of elements in $\mathscr{D}$.
Registers. Let $R$ be a finite set of elements called registers, intended to contain data values, i.e. values in $\mathscr{D}$. A register valuation is a mapping $\nu: R \rightarrow \mathscr{D}$ (also written $\nu \in \mathscr{D}^{R}$ ). For any data $\ell \in \mathscr{D}$, we write $d^{R}$ to denote the constant valuation $\nu_{d}(r)=\ell$ for all $r \in R$.

A test is a maximally consistent set of atoms of the form $* \bowtie r$ for $r \in R$ and $\bowtie \in\{=,<,>\}$. We may represent tests as conjunctions of atoms instead of sets. The symbol ' $*$ ' is used as a placeholder for incoming data. For example, for $R=\left\{r_{1}, r_{2}\right\}$, the expression $r_{1}<*$ is not a test because it is not maximal, but $\left(r_{1}<*\right) \wedge\left(*<r_{2}\right)$ is a test. We denote $\mathrm{Tst}_{R}$ the set of all tests and just Tst if $R$ is clear from the context. A register valuation $\nu \in \mathscr{D}^{R}$ and data $d \in \mathscr{D}$ satisfy a test tst $\in$ Tst, written $(\nu, d) \models$ tst, if all atoms of tst get satisfied when we replace the placeholder $*$ by $d$ and every register $r \in R$ by $\nu(r)$. An assignment is a subset asgn $\subseteq R$. Given an assignment asgn, a data $d \in \mathscr{D}$, and a valuation $\nu$, we define update ( $\nu, d$, asgn) to be the valuation $\nu^{\prime}$ s.t. $\forall r \in$ asgn: $\nu^{\prime}(r)=\mathbb{d}$ and $\forall r \notin$ asgn: $\nu^{\prime}(r)=\nu(r)$.

Register automata. A specification deterministic register automaton, or simply deterministic register automaton is a tuple $S=\left(Q, q_{\iota}, R, \delta, \alpha\right)$ where $Q=Q_{A} \uplus Q_{E}$ is a set of states partitioned into Adam and Eve states, the state $q_{\iota} \in Q_{A}$ is initial, $R$ is a set of registers, $\delta=\delta_{A} \uplus \delta_{E}$ is a (total and deterministic) transition function where, for $P \in\{A, E\}$, we have, by setting $\bar{A}=E$ and $\bar{E}=A: \delta_{P}:\left(Q_{P} \times\right.$ Tst $\rightarrow$ Asgn $\left.\times Q_{\bar{P}}\right)$; and $\alpha: Q \rightarrow\{1, \ldots, c\}$ is a priority function where $c$ is the priority index.

A configuration of $A$ is a pair $(q, \nu) \in Q \times \mathscr{D}^{R}$, describing the state and register content; the initial configuration is $\left(q_{\iota}, 0^{R}\right)$. A run of $S$ on a word $w=\ell_{0} \ell_{1} \ldots \in \mathscr{D}^{\omega}$ is a sequence of configurations $\rho=\left(q_{0}, \nu_{0}\right)\left(q_{1}, \nu_{1}\right) \ldots \in\left(\left(Q_{A} \times \mathscr{D}^{R}\right)\left(Q_{E} \times \mathscr{D}^{R}\right)\right)^{\omega}$ starting in the initial configuration $\left(\left(q_{0}, \nu_{0}\right)=\left(q_{\iota}, 0^{R}\right)\right)$ and such that for every $i \geq 0$ : by letting tst $_{i}$ be a unique test for which $\left(\nu_{i}, d_{i}\right)=\operatorname{tst}_{i}$, we have $\delta\left(q_{i}\right.$, tst $\left._{i}\right)=\left(\operatorname{asgn}_{i}, q_{i+1}\right)$ for some $\operatorname{asgn}_{i}$ and $\nu_{i+1}=\operatorname{update}\left(\nu_{i}, d_{i}, \operatorname{asgn}_{i}\right)$. Because the transition function $\delta$ is deterministic and total, every word induces a unique run in $S$. The run $\rho$ is accepting if the maximal priority visited infinitely often is even. A word is accepted by $S$ if it induces an accepting run. The language $L(S)$ of $S$ is the set of all words it accepts.

Interleavings. Specification register automata are meant to recognise interleavings of inputs (provided by Adam) and output (provided by Eve), hence the partitioning of states. Often, we need to combine them or conversely tell them apart. Thus, given
two words $u=u_{0} u_{1} \cdots \in \mathscr{D}^{\omega}$ and $v=v_{0} v_{1} \cdots \in \mathscr{D}^{\omega}$, we formally define their interleaving $u \otimes v=u_{0} v_{0} u_{1} v_{1} \cdots \in \mathscr{D}^{\omega}$. We note that given a word $w=w_{0} w_{1} \cdots \in$ $\mathscr{D}^{\omega}$, it can be uniquely decomposed into $w=u \otimes v$, where $u=w_{0} w_{2} \cdots \in \mathscr{D}^{\omega}$ and $v=w_{1} w_{3} \cdots \in \mathscr{D}^{\omega}$.

Games. A two-player zero-sum game, or simply a game, is a tuple $G=$ $\left(V_{\forall}, V_{\exists}, v_{0}, E, W\right)$ where $V_{\forall}$ and $V_{\exists}$ are disjoint sets of vertices controlled by Adam and Eve, $v_{0} \in V_{\forall}$ is initial, $E \subseteq\left(V_{\forall} \times V_{\exists}\right) \cup\left(V_{\exists} \times V_{\forall}\right)$ is a turn-based transition relation, and $W \subseteq\left(V_{\forall} \cup V_{\exists}\right)^{\omega}$ is a winning objective. An Eve strategy is a mapping $\lambda_{\exists}:\left(V_{\forall} V_{\exists}\right)^{+} \rightarrow V_{\forall}$ such that $\left(v_{\exists}, \lambda\left(v_{\forall}^{0} v_{\exists}^{0} \ldots v_{\forall}^{k} v_{\exists}^{k}\right)\right) \in E$ for all paths $v_{\forall}^{0} v_{\exists}^{0} \ldots v_{\forall}^{k} v_{\exists}^{k}$ of $G$ starting in $v_{\forall}^{0}=v_{0}$ and ending in $v_{\exists}^{k} \in V_{\exists}$ (where $k \geq 0$ ). Note that $\lambda_{\exists}$ only depends on the $V_{\exists}$ component, since the $V_{\forall}$ part is determined by the $V_{\exists}$ part, so we sometimes define it as $\lambda_{\exists}: V_{\exists}^{+} \rightarrow V_{\forall}$. Adam strategies are defined similarly, by inverting the roles of $\exists$ and $\forall$. A strategy is finite-memory if it can be computed by a finite-state machine, and positional if it only depends on the current vertex. A play is a sequence of vertices starting in $v_{0}$ and satisfying the edge relation $E$. It is won by Eve if it belongs to $W$ (otherwise it is won by Adam). An infinite play $\pi=v_{0} v_{1} \ldots$ is compatible with an Eve strategy $\lambda$ when for all $i \geq 0$ s.t. $v_{i} \in V_{\exists}$ : $v_{i+1}=\lambda\left(v_{0} \ldots v_{i}\right)$. An Eve strategy is winning if all infinite plays compatible with it are winning. A game is determined (respectively, finite-memory determined, positionally determined) if either Adam or Eve has a winning strategy (resp., a finite-memory winning strategy, a positional winning strategy).

A finite-arena game is a game whose arena is finite, i.e. where $V_{\forall}$ and $V_{\exists}$ are finite. Among them, we distinguish $\omega$-regular games, where the winning condition is an $\omega$ regular language. In particular, a parity game is a game whose winning condition is defined through a parity function $\alpha: V_{\forall} \uplus V_{\exists} \rightarrow\{1, \ldots, c\}$, where a play $v_{0} v_{1} \ldots$ is winning for Eve if and only if the maximal priority seen infinitely often is even. It is well-known that $\omega$-regular games are finite-memory determined and reduce to parity games, which are positionally determined and can be solved in $n^{c}$ [33] (see also [16]), where $n$ is the size of the game and $c$ the priority index.

Note that in register automata, Adam is represented as $A$ and Eve as $E$, while in games he is $\forall$ and she is $\exists$. This is to visually distinguish automata from games.

## 3 Church Synthesis Games

A Church synthesis game is given as a tuple $G=(I, O, S)$, where $I$ is an input alphabet, $O$ is an output alphabet, and $S \subseteq(I \cdot O)^{\omega}$ is a specification. Its semantics is provided by the game $\left(\left\{v_{0}\right\} \cup O, I, v_{0}, E, S\right)$, where $E=\left(\left(\left\{v_{0}\right\} \cup O\right) \times I\right) \cup(I \times O)$, but we rephrase it to provide a stronger intuition. In particular, it is at first counterintuitive that Adam owns $O$ vertices, and Eve $I$ vertices; this is because both players choose their move by targeting a specific vertex.

Thus, in a Church synthesis game, two players, Adam (the environment, who provides inputs) and Eve (the system, who controls outputs), interact. Their strategies are respectively represented as mappings $\lambda_{\forall}: v_{0} \cdot O^{*} \rightarrow I$ (often simply represented as $\lambda_{\forall}: O^{*} \rightarrow I$ for symmetry) and $\lambda_{\exists}: I^{+} \rightarrow O$. Given $\lambda_{\forall}$ and $\lambda_{\exists}$, the outcome $\lambda_{\forall} \| \lambda_{\exists}$ is the infinite sequence $i_{0} o_{0} i_{1} o_{1} \ldots$ such that for all $j \geq 0: i_{j}=\lambda_{\forall}\left(o_{0} \ldots o_{j-1}\right)$
and $o_{j}=\lambda_{\exists}\left(i_{0} \ldots i_{j}\right)$. If $\lambda_{\forall} \|_{\exists} \in S$, the outcome is won by Eve, otherwise by Adam. Eve wins the game if she has a strategy $\lambda_{\exists}$ such that for every Adam strategy $\lambda_{\forall}$, the outcome $\lambda_{\forall} \| \lambda_{\exists}$ is won by Eve. Solving a synthesis game amounts to finding whether Eve has a winning strategy. Synthesis games are parameterised by classes of alphabets and specifications. A game class is determined if every game in the class is either won by Eve or by Adam.

The class of synthesis games where $I$ and $O$ are finite and where $S$ is an $\omega$ regular language is known as Church games; they are decidable and determined. They also enjoy the finite-memoriness property: if Eve wins a game then she can win it with a strategy that is represented as a finite-state machine [14] (see also [50] for a game-theoretic presentation of those results).

We study synthesis games where $I=O=\mathscr{D}$ is an ordered data domain and the specifications are described by deterministic register automata. In the following, we let $G_{S}^{\mathscr{D}}=(\mathscr{D}, \mathscr{D}, S)$ be the Church synthesis game with input and output alphabet $\mathscr{D}$ and specification $S$, and simply write $G_{S}$ when $\mathscr{D}$ is clear from the context.

### 3.1 Church games on register automata

We start our study with a negative result, that highlights the difficulty of the problem: over the data domain $(\mathbb{N}, \leq)$, Church games are undecidable. Indeed, if the two players pick data values, one can simulate a two-counter machine as follows: one player provides the values of the counters, while the other checks that no cheating happens on the increments and decrements. This can be done using the fact that $c^{\prime}=c+1$ whenever there does not exist any $\ell$ such that $c<\ell<c^{\prime}$.

Theorem 1. Deciding the existence of a winning strategy for Eve in a Church game whose specification is a deterministic register automaton over $(\mathbb{N}, \leq)$ is undecidable.

Proof idea. We reduce from the halting problem of 2-counter machines, which is undecidable [42]. We define a specification with 4 registers $r_{1}, r_{2}, z$ and $t$. Registers $r_{1}$ and $r_{2}$ each store the value of one counter; $z$ stores 0 to conduct zero tests and $t$ is used as a buffer. We now describe how to increment $c_{1}$ (see Figure 2a); the cases of $c_{2}$ and of decrementing are similar. Eve suggests a value $d>r_{1}$, which is stored in $t$. Then, Adam checks that the increment was done correctly: Eve cheated if and only if Adam can provide a data $d^{\prime}$ such that $r_{1}<d^{\prime}<d$. If he cannot, $d$ is stored in $r_{1}$, thus updating the value of the counter. The acceptance condition is then a reachability one, asking that a halting instruction is eventually met. Now, if $M$ halts, then its run is finite and the values of the counters are bounded by some $B$. As a consequence, there exists a strategy of Eve which simulates the run by providing the values of the counters along the run. Conversely, if $M$ does not halt, then no halting instruction is reachable by simulating $M$ correctly, and Adam is able to check that Eve does not cheat during its simulation.

Proof. We reduce from the halting problem of deterministic 2-counter machines, which is undecidable [42]. Among multiple formalisations of counter machines, we pick the following one: a 2 -counter machine has two counters which contain integers,

(a) Gadget for instruction inc ${ }_{1}$.

(b) Gadget for instruction ifz $_{1}\left(k^{\prime}, k^{\prime \prime}\right)$.

Figure 2: Gadgets for 2CM instructions. The instruction number $k$ is stored in the state of the automaton. The state $\dot{z}$ (resp. $\xi$ ) is a rejecting sink (resp. accepting sink). Non-depicted transitions go to the sink state that is losing for the player that takes them.
initially valued 0 . It is composed of a finite set of instructions $M=\left(I_{1}, \ldots, I_{m}\right)$, each instruction being of the form $\operatorname{inc}_{j}, \operatorname{dec}_{j}, \operatorname{ifz}_{j}\left(k^{\prime}, k^{\prime \prime}\right)$ for $j=1,2$ and $k^{\prime}, k^{\prime \prime} \in$ $\{1, \ldots, m\}$, or halt. The semantics are defined as follows: a configuration of $M$ is a triple $\left(k, c_{1}, c_{2}\right)$, where $1 \leq k \leq m$ and $c_{1}, c_{2} \in \mathbb{N}$. The transition relation (which is actually a function, as $M$ is deterministic) is then, from a configuration ( $k, c_{1}, c_{2}$ ):

- If $I_{k}=\operatorname{inc}_{1}$, then the machine increments $c_{1}$ and jumps to the next instruction $I_{k+1}:\left(k, c_{1}, c_{2}\right) \rightarrow\left(k+1, c_{1}+1, c_{2}\right)$. Similarly for $\mathrm{inc}_{2}$.
- If $I_{k}=\operatorname{dec}_{1}$ and $c_{1}>0$, then $\left(k, c_{1}, c_{2}\right) \rightarrow\left(k+1, c_{1}-1, c_{2}\right)$. If $c_{1}=0$, then the computation fails and there is no successor configuration. Similarly for $\operatorname{dec}_{2}$.
- If $I_{k}=\mathrm{ifz} z_{1}\left(k^{\prime}, k^{\prime \prime}\right)$, then $M$ jumps to $k^{\prime}$ or $k^{\prime \prime}$ according to a zero-test on $c_{1}$ : if $c_{1}=0$, then $\left(k, c_{1}, c_{2}\right) \rightarrow\left(k^{\prime}, c_{1}, c_{2}\right)$, otherwise $\left(k, c_{1}, c_{2}\right) \rightarrow\left(k^{\prime \prime}, c_{1}, c_{2}\right)$. Similarly for ifz $_{2}$.

A run of the machine is then a finite or infinite sequence of successive configurations, starting at $(1,0,0)$. We say that $M$ halts whenever it admits a finite run which ends in a configuration $\left(k, c_{1}, c_{2}\right)$ such that $I_{k}=$ halt.

Let $M=\left(I_{1}, \ldots, I_{m}\right)$ be a 2 -counter machine. We associate to it the following specification deterministic register automaton: $S$ has states $Q=Q_{A} \uplus Q_{E}$, where, for $P \in\{A, E\}, Q_{P}=(\{0, \ldots, m+1\} \cup(\{0, \ldots, m+1\} \times\{y, n\}) \cup\{z, \xi\}) \times\{P\}$. The letters $y$ and $n$ are used to remember whether an ifz test evaluated to true or false; they are only used by $A$, but we included them in $Q_{E}$ for symmetry. The initial state of $S$ is $(0, A)$. The automaton has four registers $r_{1}, r_{2}, t, z$. The acceptance is defined by the reachability condition $F=\{(\xi, A)\}$, while $z$ signals rejecting sink states. The transitions of $S$ are defined by the following procedure:

- Initially, there is a transition $(0, A) \xrightarrow{\top}(1, E)$ so that the implementation can start the simulation.
- Then, for each $k \in\{1, \ldots, m\}$ :
- If $I_{k}=\operatorname{inc}_{j}$ for $j=1,2$, then we add to the transitions of $S$ the gadget from Figure 2a, i.e. output transition $(k, E) \xrightarrow{*>r_{1}, \downarrow t}(k, A)$ and input transitions $(k, A) \xrightarrow{r_{1}<*<t}(\xi, E),(k, A) \xrightarrow{*=t, \downarrow r_{1}}(k+1, E)$ and $(k, A) \xrightarrow{* \leq r_{1}}(\xi, E)$, $(k, A) \xrightarrow{*>t}(\jmath, E)$.
- The case $I_{k}=\operatorname{dec}_{j}$ for $j=1,2$ is similar: we add output transition $(k, E) \xrightarrow{*<r_{1}, \downarrow t}$ $(k, A)$ and input transitions $(k, A) \xrightarrow{t<*<r_{1}}(k, E),(k, A) \xrightarrow{*=t, \downarrow r_{1}}(k+1, E)$ and $(k, A) \xrightarrow{* \geq r_{1}}(\xi, E),(k, A) \xrightarrow{*<t}(\xi, E)$. Note that in our definition, if $c_{j}=0$, then the instruction $\operatorname{dec}_{j}$ should be blocking, i.e. the computation should fail, which is consistent with the fact that in that case, the implementation cannot provide $d<r_{1}$.
- If $I_{k}=\operatorname{ifz}_{j}\left(k^{\prime}, k^{\prime \prime}\right)$, then we add the gadget of Figure 2b, i.e. output transitions $(k, E) \xrightarrow{*=r_{1} \wedge *=z}(k, y, A),(k, E) \xrightarrow{*=r_{1} \wedge *>z}(k, n, A)$ and input transitions $(k, y, A) \xrightarrow{\top}\left(k^{\prime}, E\right)$ and $(k, n, A) \xrightarrow{\top}\left(k^{\prime \prime}, E\right)$.
- If $I_{k}=$ halt, we add a transition $(k, E) \xrightarrow{\top}(\leftrightharpoons, A)$.
- Finally, $(\xi, P) \xrightarrow{\top}(\grave{\jmath}, \bar{P})$ and $(\mathfrak{k}, P) \xrightarrow{\top}(\downarrow, \bar{P})$ for $P \in\{A, E\}$, so that both $\xi$ and $\dot{z}$ are sink states alternating between the players. In the following, we sometimes write $\xi$ for $(\xi, P)$ and $\xi$ for $(\xi, P)$, since the owner of the state does not matter.
Now, assume that $M$ admits an accepting run $\rho=\left(k_{1}, c_{1}^{1}, c_{2}^{1}\right) \rightarrow \cdots \rightarrow\left(k_{n}, c_{1}^{n}, c_{2}^{n}\right)$, where $n \in \mathbb{N}, k_{1}=1, c_{1}^{1}=c_{2}^{1}=0$ and $I_{k_{n}}=$ halt. The values of the counters are bounded by some $B \leq n$. Then, let $\lambda^{\rho}$ be the strategy of Eve which ignores the input provided by Adam and plays the output $w_{\rho}=c_{0}^{j_{0}} \ldots c_{n-1}^{j_{n-1}} 0^{\omega}$, where for $1 \leq l<n$, $j_{l}$ is the index of the counter modified or tested at step $l$ (i.e. $j_{l}=1,2$ is such that $I_{k_{l}}=\operatorname{inc}_{j_{l}}, \operatorname{dec}_{j_{l}}$ of $\operatorname{ifz}_{j_{l}}\left(k^{\prime}, k^{\prime \prime}\right)$ ). Formally, for all $u \in \mathbb{N}^{+}$of length $l \geq 0$, we let $\lambda^{\rho}(u)=c_{l}^{j_{l}}$ if $l \leq n-1$ and $\lambda^{\rho}(u)=0$ otherwise.

Let us show that $\lambda^{\rho}$ is a winning strategy for Eve. Let $u \in \mathbb{N}^{\omega}$ be an input word provided by Adam. We show by induction on $l$ that in $S$ the partial run over $(u \otimes w)[: 2 l+1]$ is either in state $\hat{\xi}$ or $S$ is in configuration $\left(\left(k_{l}, E\right), \tau_{l}\right)$, where $\tau_{l}\left(r_{1}\right)=c_{l}^{1}$ and $\tau_{l}\left(r_{2}\right)=c_{l}^{2}$.

Initially, $S$ is in configuration $\left((0, A), \tau_{R}^{0}\right)$. Then, whatever Adam plays, it transitions to $\left((1, E), \tau_{R}^{0}\right)$, so the invariant holds. Now, assume it holds up to step $l$. If $S$ is in $(\jmath, E)$, the only available transition is $(\jmath, E) \xrightarrow{\top}(\jmath, A)$, and then $(\jmath, A) \xrightarrow{\top}(\jmath, E)$, so the invariant holds at step $l+2$ ( $\zeta$ is a sink state). Otherwise, necessarily $l<n$, $S$ is in configuration $\left(\left(k_{l}, E\right), \tau_{l}\right)$ and there are four cases:

- $I_{k_{l}}=\mathrm{inc}_{j}$. By definition, $j=j_{l}$. We treat the case $j=1$, the other case is similar. Then, Eve plays $c_{l}^{1}=c_{l-1}^{1}+1$, which is such that $c_{l}^{1}>\tau_{l}\left(r_{1}\right)$. Then, there does not exist $d$ such that $\tau_{l}\left(r_{1}\right)<d<\tau_{l}(t)$ since $\tau_{l}\left(r_{1}\right)=c_{l-1}^{1}$ and $\tau_{l}(t)=c_{l-1}^{1}+1$, so the play cannot transition to $(\xi, E)$. Now, either Adam plays $u_{l+1}=\tau_{l}(t)=c_{l-1}^{1}+1$, in which case $S$ evolves to configuration $\left(\left(k_{l+1}, E\right), c_{l+1}^{1}, c_{l+1}^{2}\right)$, and the invariant holds. Otherwise, $u_{l+1} \neq \tau_{l}(t)$ and $S$ goes to $(\jmath, E)$ and the invariant holds as well.
- The case of $I_{k_{l}}=\operatorname{dec}_{j}$ is similar. Let us just mention that the computation does not block at this step, otherwise $\rho$ is not a run of $M$, so the transition $d<r_{j}$ can indeed be taken by Eve.
- $I_{k_{l}}=\operatorname{ifz}_{j}\left(k^{\prime}, k^{\prime \prime}\right)$. Again, $j=j_{l}$, and we treat the case $j=1$. Eve plays $c_{l}^{1}$; there are two cases. If $c_{l}^{1}=0$, the transition $*=r_{1} \wedge *=z$ is taken in $S$, since at every step, $\tau_{l}(z)=0$ (this register is never modified). If $c_{l}^{1} \neq 0$, then the transition $*=r_{1} \wedge *>z$ is taken. In both cases, whatever Adam plays, $S$ then evolves to $\left(\left(k_{l+1}, E\right), \tau_{l+1}\right)$ (where $\left.\tau_{l+1}=\tau_{l}\right)$ and the invariant holds.
- Finally, if $I_{k_{l}}=$ halt, then whatever Eve plays, $S$ transitions to $(\xi, A)$, and whatever Adam plays, the automaton transitions to $(今, E)$.

As a consequence, $\xi$ is eventually reached whatever the input, which means that for all $u \in \mathbb{N}^{\omega}, u \otimes I(u) \in S$, i.e. $I$ is indeed an implementation of $S$.

Conversely, assume that Eve has a winning strategy $\lambda_{\exists}$ in $G_{S}$. Let $\rho$ be the maximal run of $M$ (i.e. either $\rho$ ends in a configuration with no successor, or it is infinite). It is unique since $M$ is deterministic. Let $n=\|\rho\|$, with the convention that $n=\infty$ if $\rho$ is infinite. Let us build by induction a play of a strategy ${ }^{2}$ of Adam $\lambda_{\forall}$ such that for all $l<n,\left(\lambda_{\forall} \| \lambda_{\exists}\right)[: 2 l]=c_{l}^{j_{l}}$. and the configuration reached by $S$ over $\left(\lambda_{\forall} \otimes \lambda_{\exists}\right)[: 2 l]$ is $\left(\left(k_{l}, E\right), \tau_{l}\right)$. Initially, let $u_{0}=0$. As the initial test is $\top, S$ anyway evolves to state $(1, E)$, with $\tau\left(r_{1}\right)=\tau\left(r_{2}\right)=0$.

Now, assume we built such input $u$ up to $l$. There are again four cases:

- $I_{k_{l}}=$ inc $_{j}$. Then, Eve provides some output data $d_{E}>\tau_{l}\left(r_{j}\right)$. Assume by contradiction that $d_{E}>\tau_{l}\left(r_{j}\right)+1$. Then, $\lambda_{\exists}$ is not winning because if Adam plays $d_{A}=\tau_{l}\left(r_{j}\right)+1, S$ goes to state $(\xi, E)$, which is a sink rejecting state, so the play is losing irrelevant of what both players play after this move. So, necessarily, $d_{E}=\tau_{l}\left(r_{j}\right)+1=c_{l}^{j_{l}}$, and $S$ evolves to configuration $\left(k_{l+1}, \tau_{l+1}\right)$.
- The case $I_{k_{l}}=\operatorname{dec}_{j}$ is similar. Necessarily, $c_{j}^{l}>0$, otherwise Eve cannot provide any output data and the play is losing for Eve, which contradicts the fact that $\lambda_{\exists}$ is winning. Thus, the computation does not block here.
- $I_{k_{l}}=\mathrm{ifz}_{j}\left(k^{\prime}, k^{\prime \prime}\right)$. The output transitions of the gadget constrain Eve to output $d_{E}=\tau_{l}\left(r_{j}\right)=c_{l}^{j_{l}}$, and irrelevant of what Adam plays $S$ then evolves to configuration $\left(\left(k_{l+1}, E\right), \tau_{l+1}\right)$.
- $I_{k_{l}}=$ halt. Then, it means that $n<\infty$ and $l=n$, so the invariant vacuously holds.

Now, $\rho$ cannot be infinite, otherwise $\lambda_{\forall} \|_{\exists} \lambda_{\exists}$ is not accepted by $S$ because $Я$ is never reached and Eve would not win. It moreover cannot block on some dec ${ }_{j}$ instruction, as demonstrated in the induction. Thus, a halt instruction is eventually reached, which means that $\rho$ is a halting run of $M: M$ halts.

### 3.2 Church games on one-sided register automata

In light of this undecidability result, we consider one-sided synthesis games, where Adam provides data but Eve reacts with labels from a finite alphabet (a similar restriction was studied in [30] for domain $(\mathscr{D},=)$ ). Specifications are now given as a language $S \subseteq(\mathscr{D} \cdot \Sigma)^{\omega}$, recognised by a one-sided deterministic register automaton.

[^2]Definition 1. A one-sided deterministic register automaton, or simply one-sided register automaton $S=\left(\Sigma, Q, q_{\iota}, R, \delta, \alpha\right)$ is a deterministic register automaton that additionally has a finite alphabet $\Sigma$ of Eve labels. Its states are again partitioned into Adam and Eve states $Q=Q_{A} \uplus Q_{E}$, and it has an initial state $q_{\iota} \in Q_{A}$. Its transition function $\delta=\delta_{A} \uplus \delta_{E}$ is again total, but now has $\delta_{E}: Q_{E} \times \Sigma \rightarrow Q_{A}$. The rest is defined as for deterministic register automata: $\delta_{A}: Q_{A} \times \mathrm{Tst} \rightarrow \operatorname{Asgn} \times Q_{E}$; $R$ is a set of registers, and finally $\alpha: Q \rightarrow\{1, \ldots, c\}$ is a priority function where $c$ is the priority index.

The notions of configurations and runs are defined analogously, except for the asymmetry between input and output: a configuration of $A$ is a pair $(q, \nu) \in Q \times \mathscr{D}^{R}$, describing the state and register content; the initial configuration is $\left(q_{\nu}, 0^{R}\right)$. A run of $S$ on a word $w=\ell_{0} a_{0} d_{1} a_{1} \ldots \in(\mathscr{D} \Sigma)^{\omega}$ (note the interleaving of $\mathscr{D}$ and $\Sigma$ ) is a sequence of configurations $\rho=\left(q_{0}, \nu_{0}\right)\left(p_{0}, \nu_{1}\right)\left(q_{1}, \nu_{1}\right)\left(p_{0}, \nu_{2}\right) \ldots \in\left(\left(Q_{A} \times \mathscr{D}^{R}\right)\left(Q_{E} \times \mathscr{D}^{R}\right)\right)^{\omega}$ starting in the initial configuration (i.e. $\left.\left(q_{0}, v_{0}\right)=\left(q_{\nu}, 0^{R}\right)\right)$ and such that for every $i \geq 0$ :

- (reading an input data value) by letting tst $_{i}$ be a unique test for which $\left(\nu_{i}, d_{i}\right) \models$ tst $_{i}$, we have $\delta\left(q_{i}, \operatorname{tst}_{i}\right)=\left(\operatorname{asgn}_{i}, p_{i}\right)$ for some $\operatorname{asgn}_{i}$ and $\nu_{i+1}=\operatorname{update}\left(\nu_{i}, d_{i}, \operatorname{asgn}_{i}\right)$, as for deterministic register automata;
- (reading an output letter from $\Sigma) \delta\left(p_{i}, a_{i}\right)=q_{i+1}$, as for finite-state automata.

Again, because the transition function $\delta$ is deterministic and total, every word induces a unique run in $S$. The run $\rho$ is accepting if the maximal priority visited infinitely often is even. A word is accepted by $S$ if it induces an accepting run. The language $L(S)$ of $S$ is the set of all words it accepts.

Figure 1 shows an example of a one-sided automaton. For instance, it rejects the words $3 a 1 b 2(\Sigma \mathscr{D})^{\omega}$ and accepts the words $3 a 1 a 2 b(D D)^{\omega}$.

The rest of this paper is dedicated to showing that Church games whose specification are defined by one-sided register automata over $(\mathbb{Q}, \leq)$ or $(\mathbb{N}, \leq)$ are decidable in exponential time, and that those games are determined. Formally,

Theorem 2. Let $S=\left(\Sigma, Q, q_{\iota}, R, \delta, \alpha\right)$ be a one-sided register automaton over $(\mathbb{N}, \leq)$ or $(\mathbb{Q}, \leq)$.

1. The problem of determining the winner of the Church synthesis game $G=$ $(\mathscr{D}, \mathscr{D}, S)$ is decidable in time polynomial in $|Q|$ and exponential in $c$ and $|R|$.
2. $G_{S}$ is determined, i.e. either Eve or Adam has a winning strategy in $G_{S}$.

The above is a wrapper theorem, that aggregates Theorems 9 for $(\mathbb{Q}, \leq)$ and 18 for $(\mathbb{N}, \leq)$. We defer the proof to Section 4 . The result for $(\mathbb{Q}, \leq)$ can be derived from [21] or [39, Section 7], but we include it for pedagogical reasons, as it allows us to introduce the main tools in a simple setting and to highlight the difficulties that creep up when we shift to ( $\mathbb{N}, \leq$ ).

In the case of a finite alphabet, the game-theoretic approach to solving Church games whose specification is given by a deterministic finite-state automaton consists in playing on the automaton, in the following sense: the arena consists of the automaton, and Adam and Eve alternately choose an input (respectively, output) letter, or
equivalently (since the automaton is deterministic) an input (resp., output) transition of the automaton. Then, Eve wins whenever the word they jointly produced is accepted by the automaton.

Here, we follow the same approach, with the additional difficulty that the players manipulate data values from an infinite alphabet. Thus, it is not immediate to relate the data values they choose with the corresponding transitions of the automaton. To that end, we study the link between the automaton game (where players pick transitions in the automaton) and the corresponding Church game. This is done through the key notion of feasible action words: a sequence of transition labels is feasible whenever it labels a run over some data word. Adam is then asked to provide feasible action words, otherwise he loses. To show that the automaton game is equivalent with the Church game, it remains to show that a strategy of Adam in the automaton game can be translated to a strategy in the Church game. The key ingredient is to be able to instantiate a given action by a data value on-the-fly, while the play unfolds.

Over $(\mathbb{Q}, \leq)$, as we demonstrate, the set of feasible action words is $\omega$-regular, so the automaton game is $\omega$-regular as well. Moreover, from a given configuration, one can locally determine whether an action can be instantiated with a data value, and pick it accordingly, which yields the sought strategy translation. Thus, both games are equivalent, and we get decidability since $\omega$-regular games are decidable. The case of $(\mathbb{N}, \leq)$ is much more involved and requires further developments, so we start the presentation with $(\mathbb{Q}, \leq)$ to sharpen our tools.

### 3.3 The automaton game

For the rest of this section, fix a one-sided register automaton $S=\left(\Sigma, Q, q_{\iota}, R, \delta, \alpha\right)$ over an ordered data domain $\mathscr{D}$ (it can be either $(\mathbb{Q}, \leq)$ or $(\mathbb{N}, \leq)$ ).

Before introducing the game itself, we define the main technical notion, which relates the syntax and semantics of register automata.

Definition 2. An action word is a sequence ( $\left.\mathrm{tst}_{0}, \operatorname{asgn}_{0}\right)\left(\mathrm{tst}_{1}, \operatorname{asgn}{ }_{1}\right) \ldots$ from (Tst $\times$ Asgn) ${ }^{*, \omega}$. It is $\mathscr{D}$-feasible (or simply feasible when $\mathscr{D}$ is clear from the context) if there exists a sequence $\nu_{0} d_{0} \nu_{1} d_{1} \ldots$ of register valuations $\nu_{i}$ and data $d_{i}$ over $\mathscr{D}$ such that $\nu_{0}=0^{R}$ and for all $i: \nu_{i+1}=\operatorname{update}\left(\nu_{i}, d_{i}, \operatorname{asgn}_{i}\right)$ and $\left(\nu_{i}, d_{i}\right) \models \operatorname{tst}_{i}$.

We denote by Feasible $(R)$ the set of action words over $R$ feasible in $\mathscr{D}$.
With the Church game $(\mathscr{D}, \mathscr{D}, S)$, we associate the following automaton game, which is a finite-arena game $G_{S}^{f}=\left(V_{\forall}, V_{\exists}, v_{0}, E, W_{S}^{f}\right)$. Essentially, it memorises the transitions taken by the automaton $S$ during the play of Adam and Eve. It has $V_{\forall}=\left\{q_{\iota}\right\} \cup\left(\Sigma \times Q_{A}\right), V_{\exists}=\mathrm{Tst} \times \operatorname{Asgn} \times Q_{E}, v_{0}=q_{\iota}, E=E_{0} \cup E_{\forall} \cup E_{\exists}$ where:

- $E_{0}=\left\{\left(v_{0},\left(\right.\right.\right.$ tst, asgn,$\left.\left.u_{0}\right)\right) \mid \delta\left(v_{0}\right.$, tst $)=\left(\right.$ asgn,$\left.\left.u_{0}\right)\right\}$,
- $E_{\forall}=\{((\sigma, v),(\mathrm{tst}$, asgn, $u)) \mid \delta(v$, tst $)=($ asgn,$u)\}$, and
- $E_{\exists}=\{((\mathrm{tst}, \operatorname{asgn}, u),(\sigma, v)) \mid \delta(u, \sigma)=v\}$.

We let:

$$
W_{S}^{f}=\left\{\begin{array}{l|l}
v_{0}\left(\operatorname{tst}_{0}, \operatorname{asgn}_{0}, u_{0}\right)\left(\sigma_{0}, v_{1}\right) \ldots & \begin{array}{l}
\left(\operatorname{tst}_{0} \operatorname{asgn}_{0}\right) \ldots \in \text { Feasible }_{\mathscr{D}}(R) \\
\Rightarrow v_{0} u_{0} v_{1} u_{1} \ldots=\alpha
\end{array}
\end{array}\right\}
$$

The strategies of Adam and Eve in the automaton game are of the form $\lambda_{\forall}^{f}$ : $V_{\forall}\left(V_{\exists} V_{\forall}\right)^{*} \rightarrow V_{\exists}$ and $\lambda_{\exists}^{f}:\left(V_{\forall} V_{\exists}\right)^{+} \rightarrow V_{\forall}$. Since the automaton $S$ is deterministic, they can equivalently be expressed as $\lambda_{\forall}^{f}: \Sigma^{*} \rightarrow$ Tst and $\lambda_{\exists}^{f}: \mathrm{Tst}^{+} \rightarrow \Sigma$.

Let us show that $G_{S}^{f}$ is a sound abstraction of $G_{S}$, in the sense that a winning strategy of Eve in $G_{S}^{f}$ can be translated to a winning strategy of Eve in $G_{S}$, for both $(\mathbb{Q}, \leq)$ and $(\mathbb{N}, \leq)$ :

Proposition 3. Let $S$ be a deterministic register automaton. If Eve has a winning strategy in $G_{S}^{f}$, then she has a winning strategy in the Church game $G_{S}$.

Proof. The main idea of the proof is that is $G_{S}$, Eve has more information than in $G_{S}^{f}$, since she knows what data values Adam played, while in $G_{S}^{f}$ she can only access the corresponding tests.

Formally, let $\lambda_{\exists}^{f}:\left(V_{\forall} V_{\exists}\right)^{+} \rightarrow V_{\forall}$ be a winning Eve strategy in $G_{S}^{f}$. We construct a winning Eve strategy $\lambda_{\exists}: \mathrm{Tst}^{+} \rightarrow \Sigma$ in $G_{S}$ as follows ${ }^{3}$. Fix an arbitrary sequence tst $_{0} \ldots$ tst $_{k}$; we define $\lambda_{\exists}\left(\right.$ tst $_{0} \ldots$ tst $\left._{k}\right)$. First, for all $0 \leq i \leq k-1$, we inductively define $v_{0}, u_{0}, v_{1}, u_{1}, \ldots, v_{k} \in\left(Q_{A} \cup Q_{E}\right), \operatorname{asgn}_{0}, \ldots, \operatorname{asgn}_{k}$, and $\sigma_{1}, \ldots, \sigma_{k} \in \Sigma$ :

- The state $v_{0}=q_{\iota}$ is the initial state of $S$.
- For all $0 \leq i \leq k$, define $u_{i} \in Q_{E}$ and $\operatorname{asgn}_{i}$ to be such that $\left(\operatorname{asgn}_{i}, u_{i}\right)=\delta\left(v_{i}\right.$, tst $\left._{i}\right)$, $\sigma_{i+1}=\lambda_{\exists}^{\bar{f}}\left(v_{0}\left(\operatorname{tst}_{0}, \operatorname{asgn}_{0}, u_{0}\right)\left(\sigma_{1}, v_{1}\right) \ldots\left(\operatorname{tst}_{i}, \operatorname{asgn}_{i}, u_{i}\right)\right)$, and $v_{i+1}=\delta\left(u_{i}, \sigma_{i}\right)$.
We then set $\lambda_{\exists}\left(\operatorname{tst}_{0} \ldots \mathrm{tst}_{k}\right)=\sigma_{k+1}$. We now show that the constructed Eve strategy $\lambda_{\exists}$ is winning in $G_{S}$. Consider an arbitrary Adam data strategy $\lambda_{\forall}^{\mathscr{D}}$, and let $\left(v_{0}, \nu_{0}\right)\left(u_{0}, \nu_{1}\right)\left(v_{1}, \nu_{1}\right)\left(u_{1}, \nu_{2}\right) \ldots$ be an infinite run in $G_{S}$ on reading the outcome $\lambda_{\forall}^{\mathscr{D}} \| \lambda_{\exists}$; it is enough to show that $v_{0} u_{0} v_{1} u_{1} \ldots$ satisfies the parity condition. Let $d_{0} d_{1} \ldots$ be the sequence of data produced by Adam during the play, let $\sigma_{0} \sigma_{1} \ldots$ be the labels produced by Eve strategy $\lambda_{\exists}$, and let $\bar{a}=\left(\operatorname{tst}_{0}, \operatorname{asgn}_{0}\right)\left(\operatorname{tst}_{1}, \operatorname{asgn}_{1}\right) \ldots$ be the tests and assignments performed by the automaton during the run. Then, the sequence $v_{0}\left(\mathrm{tst}_{0}, \operatorname{asgn}_{0}, u_{0}\right)\left(\sigma_{0}, v_{1}\right)\left(\mathrm{tst}_{1}, \operatorname{asgn}_{1}, u_{1}\right) \ldots$ constitutes a play in $G_{S}^{f}$, which is compatible with $\lambda_{\exists}^{f}$. Moreover, as witnessed by $\nu_{0} d_{0} \nu_{1} \ell_{1} \ldots$, the action word $\bar{a}$ is feasible. Therefore, since $\lambda_{\exists}^{f}$ is winning, the sequence $v_{0} u_{0} v_{1} u_{1} \ldots$ satisfies the parity condition.

The converse direction of the above proposition is in general harder, as it amounts to showing that the information provided by tests is enough. For the case of $(\mathbb{Q}, \leq)$, the density of the domain allows to instantiate tests on-the-fly, in a way that does

[^3]not jeopardise the feasibility of the overall sequence (Section 4.1). The case of ( $\mathbb{N}, \leq$ ) is much harder, and is the subject of most of Section 4.

### 3.4 Application to transducer synthesis

The Church synthesis game models the reactive synthesis problem: $S$ is a specification, and a winning strategy in $G$ corresponds to a reactive program which implements $S$, i.e. whose set of behaviours abides by $S$.

In the finite alphabet case, Church synthesis games are $\omega$-regular. Since those games are finite-memory determined, it means that if a specification admits an implementation, then it admits a finite-state one [14], that can be modelled as a finite-state transducer (i.e., a Mealy machine). In this section, we study at which conditions we can get an analogue of this result for specifications defined by input-driven register automata [27]. Those specifications consist in two-sided automata where the output data values are restricted to be the content of some register (in other words, the implementation is not allowed to generate data). Input-driven automata can be simulated by one-sided automata, in that output registers can be seen as finite labels. Correspondingly, we target register transducers, which generalise finite-state transducers to data domains in the same way as register automata generalise finite-state automata. We then show that finite-memory strategies in the automaton game induce register transducer implementations. Indeed, a finite-memory strategy corresponds to a sub-automaton of $S$, which picks output transitions in $S$ with the help of its memory. This sub-automaton can then be interpreted as a register transducer with $R$ registers. Note that this result is reminiscent of Proposition 5 in [27].

We now define input-driven register automata, register transducers, and then define the synthesis problem and show that it is decidable.
Input-driven register automata. An input-driven deterministic register automaton is a two-sided register automaton whose output data are required to be the content of some register. Formally, it is a tuple $S=\left(Q, q_{\iota}, R, \delta, \alpha\right)$ where $Q=$ $Q_{A} \uplus Q_{E}, q_{\iota} \in Q_{A}$ and the transition function is

$$
\delta:\left(Q_{A} \times \mathrm{Tst} \rightarrow \mathrm{Asgn} \times Q_{E}\right) \cup\left(Q_{E} \times \mathrm{Tst}=\rightarrow \operatorname{Asgn}_{\emptyset} \times Q_{A}\right)
$$

where $\mathrm{Tst}_{=}$consists of tests which contain at least one atom of the form $*=r$ for some $r \in R$, i.e. the output data value must be equal to some specification register, and $\operatorname{Asgn}_{\emptyset}=\{\emptyset\}$ meaning that output data values are never assigned to any register. This is without loss of generality, given that the output value has to be equal to the content of some register.

Correspondence with one-sided register automata. To an input-driven register automaton specification, we associate a one-sided register automaton by treating output registers as finite labels. Formally, let $S=\left(Q, q_{\iota}, R, \delta, \alpha\right)$ be an input-driven register automaton. Its associated one-sided automaton is $S^{\prime}=\left(\mathrm{Tst}_{=}, Q, q_{\iota}, R, \delta^{\prime}, \alpha\right)$ (note that the finite output alphabet is $\mathrm{Tst}=$ ). Up to remembering equality relations between registers, we can assume that from an output state, all outgoing transitions can be taken, independently of the registers' configuration, i.e. that from a reachable
output configuration $\left(q_{E}, \tau\right)$, for all transitions $t=q_{E} \xrightarrow{\text { tst }=, \varnothing} q_{A}^{\prime}$, there exists $d$ such that $q_{E} \xrightarrow[t]{d} q_{A}^{\prime}$. This however induces a blowup of $Q$ exponential in $|R|$.

The transition function is $\delta_{A}^{\prime}=\delta_{A}$, and $\delta_{E}^{\prime}\left(q_{E}\right.$, tst $)=q_{A}^{\prime}$ if and only if $\delta_{E}\left(q_{E}, \mathrm{tst}\right)=\left(\varnothing, q_{A}^{\prime}\right)$. Overall, the size of $S^{\prime}$ is exponential in $|R|$ (because of the assumption we made on output transitions) and polynomial in $|Q|$.

Register transducers. A register transducer (RT) is a tuple $T=\left(Q, q_{\iota}, R, \delta\right)$, where $Q$ is a set of states and $q_{\iota} \in Q$ is initial, $R$ is a finite set of registers. The transition function $\delta$ is a (total) function $\delta: Q \times \mathrm{Tst} \rightarrow$ Asgn $\times R \times Q$.

The semantics of $T$ are provided by the associated register automaton $S_{T}$. It has states $Q^{\prime}=\left(Q_{A} \cup\left\{z_{A}\right\}\right) \uplus\left(Q_{E} \cup\left\{z_{E}\right\}\right)$, where $Q_{A}$ and $Q_{E}$ are two disjoint copies of $Q$ and $z_{A},\left\{_{E}\right.$ jointly form a rejecting sink. It has initial state $q_{\iota}$ and set of registers $R$. Its transition function is defined as $q_{A} \xrightarrow[S_{T}]{\mathrm{tst}, \text { asgn }} q_{E} \xrightarrow[S_{T}]{r^{=}, \varnothing} q_{A}^{\prime}$ and $q_{E} \xrightarrow[A_{T}]{r^{\neq}, \varnothing}$ \& $A$ whenever $q \xrightarrow[T]{\mathrm{tst} \mid \text { asgn }, r} q^{\prime}$, where $q \xrightarrow[T]{\mathrm{tst} \mid \text { asgn }, r} q^{\prime}$ stands for $\delta(q, \mathrm{tst})=\left(\operatorname{asgn}, r, q^{\prime}\right)\left(\right.$ similarly for $\left.A_{T}\right)$. Additionally, we let $\boldsymbol{\psi}_{A} \xrightarrow[A_{T}]{\mathrm{T}, \varnothing} \dot{y}_{E} \xrightarrow[A_{T}]{\mathrm{T}, \varnothing} \dot{\eta}_{A}$. The priority function is defined as $\alpha: q \in$ $Q^{\prime} \mapsto 2$ and $z_{A}, z_{E} \mapsto 1$, i.e. all states but $z_{A}, z_{E}$ are accepting. Then, $T$ recognises the (total) function $f_{T}: d_{0}^{A} \ell_{1}^{A} \cdots \mapsto q_{0}^{E} d_{1}^{E} \ldots$ such that $\ell_{0}^{A} d_{0}^{E} d_{1}^{A} d_{1}^{E} \cdots \in L\left(A_{T}\right)$. For each input $\omega$-data word, the associated output $\omega$-data word exists since all states but
 the sink state are determined by the input ones, and they only contain equality tests so the corresponding output data values are unique.

## Synthesis for input-driven output specifications

Given a specification $S$, we say that a function $f$ realises $S$ if they have the same domain and its graph is included in $S$, i.e. $\operatorname{dom}(f)=\operatorname{dom}(S)$ and for all input $x \in \operatorname{dom}(S),(x, f(x)) \in S$. We then say that a register transducer $T$ realises the register automaton specification $S$ if $f_{T}$ does, i.e. $L(T) \subseteq L(S)$.

The register transducer synthesis problem then asks to produce a $T$ that realises $S$ when such $T$ exists, otherwise output "unrealisable". Note that $T$ and $S$ can have different sets of registers.

Proposition 4. Let $S=\left(Q, q_{\iota}, R, \delta, \alpha\right)$ be an input-driven register automaton, and $S^{\prime}$ its associated one-sided register automaton. If $S$ admits a register transducer implementation, then Eve has a winning strategy in the Church game $G_{S^{\prime}}$ associated with $S^{\prime}$.

Proof. Assume that there exists a register transducer $T$ which realises $S$. From $T$, we define a strategy $\lambda^{T}$ in $G$, which simulates $T$ and $S$ in parallel. Given a history $d_{0}^{\mathrm{i}} \ldots d_{n}^{\mathrm{i}}$, let $d_{n}^{\mathrm{D}}$ be the data output by $T$. As $S$ is deterministic, there exists a unique run over the history $\ell_{0}^{\mathrm{i}} \ell_{0}^{\infty} \ldots d_{n}^{\mathrm{i}} \alpha_{n}^{\oplus}$; let $t=q_{E} \xrightarrow{\text { tst }=\varnothing} q_{A}^{\prime}$ be the transition taken by $S$ on reading $\ell_{n}^{\triangleright}$. Then, define $\lambda^{T}\left(\ell_{0}^{\mathrm{i}} \ldots \ell_{n}^{\mathrm{i}}\right)=$ tst $_{=}$. Now, for a play in $G$ consistent with $\lambda^{T}$, consider the associated run in $S^{\prime}$. As $T$ is an implementation and the sequence of
transitions is feasible (as witnessed by the data given as input), this run is necessarily accepting, so $\lambda^{T}$ is indeed a winning strategy in $G$.

Proposition 5. Let $S=\left(Q, q_{\iota}, R, \delta, \alpha\right)$ be an input-driven register automaton, and $S^{\prime}$ its associated one-sided register automaton. If Eve wins $G_{S^{\prime}}^{f}$ with a finite-memory strategy, then $S$ admits a register transducer implementation.

Proof. Let $S=\left(Q, q_{\iota}, R, \delta, \alpha\right)$ be an input-driven register automaton, and $S^{\prime}$ its associated one-sided register automaton. Assume that Eve has a finite-memory winning strategy in $G_{S}^{f}$ that is computed by a finite-state automaton $M$ with states $P$, initial memory $p_{0}$, transition function $\mu: P \times V_{\exists} \rightarrow P$ and move selection $s: P \rightarrow V_{\forall}$. Thus, given a history $h=v_{0} \ldots v_{n} \in V_{\exists}^{+}, \lambda_{\exists}(h): V_{\exists}^{+} \rightarrow \mathrm{Tst}=$ is defined as $s(p)$, where $p_{0} \xrightarrow[M]{h} p$. Then, consider $T=\left(Q \times P,\left(q_{\iota}, p_{0}\right), R, \delta^{\prime}\right)$. We define $\delta^{\prime}$ as follows: assume the transducer is in state $(q, p)$. Then, the transducer receives input satisfying some test tst. In $S$, it corresponds to some input transition $\delta(q$, tst $)=\left(\right.$ asgn, $\left.q^{\prime}\right)$. The memory is updated to $\mu(p,(\mathrm{tst}$, asgn $))=p^{\prime}$, and $s\left(p^{\prime}\right)=\mathrm{tst}_{=}$. Let $r$ be such that $\mathrm{tst}_{=} \Rightarrow r^{=}$(such $r$ necessarily exists by definition of $\mathrm{Tst}_{=}$). Then, we let $\delta((q, p)$, tst $)=\left(\operatorname{asgn}, r,\left(q^{\prime}, p^{\prime}\right)\right)$. Now, let $w=\alpha_{0}^{A} \ell_{1}^{A} \ldots$ be an input data word, and $T(w)=\ell_{0}^{E} \ell_{1}^{E} \ldots$ By construction, the run of $S$ over $w \otimes T(w)=q_{0}^{A} d_{0}^{E} \ell_{1}^{A} d_{1}^{E} \ldots$ corresponds to a play consistent with $\lambda_{\exists}$, so it is accepting (since it is feasible, as witnessed by $w \otimes T(w)$ ). As a consequence, $w \otimes T(w) \in L(S)$, which means that $T$ is indeed a register transducer implementation of $S$.

In the proof of Theorem 1, Eve's strategy consists in outputting a finite data word with $B \geq 0$ distinct data values, and then only zeroes. Thus, it can be implemented with a register transducer with $B$ registers, provided that its registers can be initialised with non-zero data values (in our setting, we assume all registers are initialised to 0 ). As a consequence, we get:

Theorem 6. For specifications defined by two-sided deterministic register automata over data domains $(\mathbb{Q}, \leq)$, the register transducer synthesis problem is undecidable, provided that registers can be initialised to an arbitrary valuation.

Remark 1. The decidability status of the synthesis problem for register transducers with a fixed initial valuation $0^{R}$ is open.

## 4 Solving Church Synthesis Games on ( $\mathbb{N}, \leq$ )

We now have the main tools in hand to solve Church synthesis games over ordered data domains. As an introduction, before the case of $(\mathbb{N}, \leq)$, we apply those tools to $(\mathbb{Q}, \leq)$.

### 4.1 Warm-up: the case of $(\mathbb{Q}, \leq)$

First, let us observe that in that case, the automaton game is $\omega$-regular:
Proposition 7. Let $S$ be a one-sided register automaton over $(\mathbb{Q}, \leq)$. Then $G_{S}^{f}$ is an $\omega$-regular game.

Proof. Let $S=\left(\Sigma, Q, q_{\iota}, R, \delta, \alpha\right)$ be a one-sided register automaton over $(\mathbb{Q}, \leq)$, and let $G_{S}^{f}=\left(V_{\forall}, V_{\exists}, v_{0}, E, W_{S}^{f}\right)$ be its associated automaton game. $G_{S}^{f}$ is a finitearena game; it remains to show that it is $\omega$-regular, i.e. that $W_{S}^{f}$ is $\omega$-regular. Recall that $W_{S}^{f}=\left\{v_{0}\left(\operatorname{tst}_{0}, \operatorname{asgn}_{0}, u_{0}\right)\left(\sigma_{0}, v_{1}\right) \ldots \mid\left(\operatorname{tst}_{0} \operatorname{asgn}_{0}\right) \ldots \in \operatorname{Feasible}_{\mathscr{D}}(R) \Rightarrow\right.$ $\left.v_{0} u_{0} v_{1} u_{1} \ldots=\alpha\right\}$. By Theorem 20 (on page 30), we know that Feasible $_{\mathscr{D}}(R)$ is $\omega$ regular; since $\alpha$ is a parity condition, one can then build an $\omega$-regular automaton recognising $W_{S}^{f}$ using standard automata constructions.

From Proposition 3, we already know that for all one-sided register automata $S$ (over $(\mathbb{Q}, \leq)$ or $(\mathbb{N}, \leq)$ ), $G_{S}^{f}$ soundly abstracts $G_{S}$. We now show the converse for $(\mathbb{Q}, \leq):$

Proposition 8. Let $S$ be a one-sided register automaton over $(\mathbb{Q}, \leq)$. If Eve has a winning strategy in $G_{S}$, then she has a winning strategy in the Church game $G_{S}^{f}$.

Proof. We show the result by contraposition. Assume that Eve does not win $G_{S}^{f}$. As $G_{S}^{f}$ is $\omega$-regular (Proposition 7), it is determined, so Adam has a winning strategy $\lambda_{\forall}^{f}: V_{\forall}\left(V_{\forall} V_{\exists}\right)^{*} \rightarrow V_{\exists}$ in $G_{S}^{f}$. We construct the winning Adam data strategy $\lambda_{\forall}^{\mathbb{Q}}$ in $G_{S}$ step-by-step, by instantiating the tests on-the-fly. When the test is an equality, pick the corresponding data, and when it is of the form $r<*<r^{\prime}$, take some rational number strictly in the interval.

Formally, suppose we are in the middle of a play: $d_{0} \ldots d_{k-1}$ has been played by Adam $\lambda_{\forall}^{\mathbb{Q}}$ and $\sigma_{0} \ldots \sigma_{k-1}$ has been played by Eve; both sequences are empty initially. We want to know the value $d_{k}$ for $\lambda_{\forall}^{\mathbb{Q}}\left(\sigma_{0} \ldots \sigma_{k-1}\right)$. Let $\left(v_{0}, \nu_{0}\right)\left(u_{0}, \nu_{1}\right)\left(v_{1}, \nu_{1}\right)\left(u_{1}, \nu_{2}\right) \ldots\left(v_{k}, \nu_{k}\right)$ be the current run prefix of the register automaton $G_{S}$ (initially $\left(v_{0}, \nu_{0}\right)$ ). We construct the corresponding play prefix $v_{0}\left(\operatorname{tst}_{0}, \operatorname{asgn}_{0}, u_{0}\right)\left(\sigma_{0}, v_{1}\right)\left(\mathrm{tst}_{1}, \operatorname{asgn}_{1}, u_{1}\right)\left(\sigma_{1}, v_{2}\right) \ldots\left(\sigma_{k-1}, v_{k}\right)$ of $G_{f}$ (initially $\left.v_{0}\right)$. We assume that this play prefix adheres to $\lambda_{\forall}^{f}$ (this holds initially). We now consult $\lambda_{\forall}^{f}$ : let $\left(\operatorname{tst}_{k}, \operatorname{asgn}_{k}, u_{k}\right)=\lambda_{\forall}^{f}\left(\sigma_{k-1}, v_{k}\right)$. Using $\operatorname{tst}_{k}$ and $\nu_{k}$, we construct $\ell_{k}$ as follows.

- If tst ${ }_{k}$ contains $*=r$ for some $r \in R$, we set $\mathscr{d}_{k}=\nu_{k}(r)$.
- If $\mathrm{tst}_{k}$ is of the form $r<*$ for all $r \in R$, then set $d_{k}=\max \left(\nu_{k}\right)+1$, i.e. take the largest value held in the registers plus 1.
- Similarly, if $\mathrm{tst}_{k}$ is of the form $*<r$ for all $r \in R$, then set $d_{k}=\min \left(\nu_{k}\right)-1$.
- Otherwise, for every $r \in R$, the test $\operatorname{tst}_{k}$ has either $r<*$ or $*<r$. We now pick two registers $r, s$ such that the test contains $r<*$ and $*<s$ and no register holds a value between $\nu_{k}(r)$ and $\nu_{k}(s)$. Then we set $\mathcal{d}_{k}=\frac{\nu_{k}(r)+\nu_{k}(s)}{2}$.

It is easy to see that $d_{k}$ satisfies $\mathrm{tst}_{k}$, i.e. $\left(\nu_{k}, d_{k}\right) \models \mathrm{tst}_{k}$. Finally, define $\nu_{k+1}=$ update ( $\nu_{k}, \ell_{k}, \operatorname{asgn}_{k}$ ). Thus, the next configuration of the run in the register automaton is $\left(u_{k}, \nu_{k+1}\right)$. In $G_{f}$, the play is extended by $\left(\operatorname{tst}_{k}, \operatorname{asgn}_{k}, u_{k}\right)$; notice that the resulting extended play again adheres to the winning Adam strategy $\lambda_{\forall}^{f}$. Therefore, starting from the empty sequences of Adam data choices and Eve label choices, step-by-step we construct the values for $\lambda_{\forall}^{\mathbb{Q}}$.

Then, each play consistent with this strategy in $G_{S}$ corresponds to a unique run in $S$, which is also a play in $G_{f}$. As $\lambda_{\forall}^{f}$ is winning, such a run is accepting, so $\lambda_{\forall}$ is winning: Eve does not win $G_{S}$.

We are now ready to show:
Theorem 9. Let $S=\left(\Sigma, Q, q_{\iota}, R, \delta, \alpha\right)$ be a one-sided register automaton over $(\mathbb{Q}, \leq)$.

1. The problem of determining if Eve wins the Church synthesis game $G=(\mathscr{D}, \mathscr{D}, S)$ is decidable in time polynomial in $|Q|$ and exponential in $c$ and $|R|$.
2. $G_{S}$ is determined, i.e. either Eve or Adam has a winning strategy in $G_{S}$.

Proof of Theorem 9. First, by Propositions 3 and 8, we know that Eve $G_{S}$ iff she wins $G_{S}^{f}$.

By analysing the constructions of Propositions 7 and Theorem 20, we get that the automaton game $G_{S}^{f}$ is of size polynomial in $|Q|$ and exponential in $|R|$, and has a number of priorities linear in $c$, so it can be solved in $O\left(\left(\operatorname{poly}(|Q|) 2^{\text {poly }(|R|)}\right)^{c}\right)$, which yields item 1 of the theorem.

Then, determinacy (item 2) follows from the determinacy of $G_{S}^{f}$, since it is equivalent with $G_{S}$.

As a consequence of Propositions 4 and 5, we also get:
Proposition 10. Let $S$ be an input-driven register automaton, and $S^{\prime}$ its associated one-sided register automaton. The following are equivalent:

- Eve has a winning strategy in $G_{S^{\prime}}$
- Eve has a winning strategy in $G_{S^{\prime}}^{f}$
- Eve has a finite-memory winning strategy in $G_{S^{\prime}}^{f}$
- $S$ admits a register transducer implementation
- $S$ admits an implementation

Thus, we have:
Theorem 11. For specifications defined by deterministic input-driven output register automata over data domains $(\mathbb{Q}, \leq)$, the register transducer synthesis problem is equivalent with the synthesis problem (for arbitrary implementations) and can be solved in time polynomial in $|Q|$ and exponential in $c$ and $|R|$.

Remark 2. For data domain $(\mathbb{Q}, \leq)$, the synthesis problem for specifications defined by two-sided register automata is also decidable, if the target implementation is any program, as the Church game again reduces to a parity game: checking feasibility is still doable using a parity automaton. However, in general, register transducers might not suffice; e.g. the environment can ask the system to produce an infinite sequence of data values in increasing order. Yet, it can be shown that implementations can be restricted to simple programs, which can be modelled by register transducers which have the additional ability to pick a data between two others, e.g. by computing $\frac{d_{1}+d_{2}}{2}$ : such ability suffices to translate a finite-memory strategy in the automaton game to an implementation.

We now shift to the main result of the paper, namely that Church synthesis games are decidable over ( $\mathbb{N}, \leq$ ). We start by providing some results on actions sequences over $(\mathbb{N}, \leq)$ that highlight the difficulties and hint at how to overcome them (Section 4.2). We then use those results to define an $\omega$-regular approximation of the automaton game that we show to be sound and complete (Section 4.3).

### 4.2 Action sequences over ( $\mathbb{N}, \leq$ )

## Action sequences over $(\mathbb{N}, \leq)$ are not $\omega$-regular

First, contrary to $(\mathbb{Q}, \leq)$, one needs a global condition on action sequences to check whether they are feasible. To get an intuition, consider the action sequence $(T\{r\})((r>*) r)^{\omega}$, that asks for an initial data value (stored in $r$ ), and then repeatedly asks to provide smaller and smaller data values. While feasible in $(\mathbb{Q}, \leq)$, such a sequence is not feasible in $(\mathbb{N}, \leq)$, as it would yield an infinite descending chain in $\mathbb{N}$. And, actually, the discreteness of $(\mathbb{N}, \leq)$ implies that the set of feasible action sequences is not $\omega$-regular in $(\mathbb{N}, \leq)$ (see, e.g., $[21$, Corollary 6.5] or [47, Appendix C]). We provide an example, for self-containedness.

Example 2. consider the automaton of Figure 3, which essentially consists in that of Figure 1 (on page 4) where we allow Adam to repeatedly try his luck by taking the transition from $C$ to $B$. Note that the priorities (written above the states) ensure that if he does so, he loses. Then, consider sequences of states in $A\left(B C(D C)^{*}\right)^{\omega}$,


Figure 3: Eve wins this game in $\mathbb{N}$ (but loses in $\mathbb{Q}$ ).
where Adam initially picks a value, the game transitions to $B$ then $C$, then Adam and Eve loop between $B$ and $C$ for some time, until at some point Adam transitions back to $B$, and so on. To check whether such a sequence actually corresponds to a play, one needs to check that there exists a uniform bound (the content of $r_{M}$ ) over the iterations of $D C$. Formally, plays in $A\left(B C(D C)^{*}\right)^{\omega}$ are of the form $A\left(B C(D C)^{n_{0}}\right)\left(B C(D C)^{n_{1}}\right) \ldots$ where there exists $\mathrm{B} \geq 0$ such that for all $i \geq 0$, $n_{i} \leq \mathrm{B}$. By an elementary pumping argument, one can show that this language is not $\omega$-regular [6].

This implies that Feasible $\mathbb{N}^{( }(R)$ is not $\omega$-regular whenever $|R| \geq 2$, and neither is the automaton game. We thus consider an $\omega$-regular over-approximation of the automaton game, and show that both games are actually equivalent.

## Constraint sequences, consistency and satisfiability

To introduce the said approximation, we first require a further study of Feasible ${ }_{\mathbb{N}}(R)$, that we conduct through the notion of constraint sequences. To ease the comparison between $(\mathbb{Q}, \leq)$ and $(\mathbb{N}, \leq)$, we define them for both domains. Thus, in this section, fix an ordered domain $\mathscr{D}$.

Given a set of registers $R$ (which can also be thought of as variables), we let $R^{\prime}=\left\{r^{\prime} \mid r \in R\right\}$ be the set of their primed versions. Given a valuation $\nu \in \mathscr{D}^{R}$, define $\nu^{\prime} \in \mathscr{D}^{R^{\prime}}$ to be the valuation that maps $\nu^{\prime}\left(r^{\prime}\right)=\nu(r)$ for every $r \in R$.

Definition 3. A constraint over $R$ is a total non-strict preorder over $R \cup R^{\prime}$, i.e. a total order with ties allowed. It can be represented as a maximally consistent set of atoms of the form $t_{1} \bowtie t_{2}$ where $t_{1}, t_{2} \in R \cup R^{\prime}$, where the symbol $\bowtie$ denotes one of $>,<$, or $=$.

Given a constraint $C$, the writing $C_{\mid R}$ denotes the subset of its atoms $r \bowtie s$ for $r, s \in R$, and $C_{\mid R^{\prime}}$ denotes the subset of atoms over primed registers. Given a set $S$ of atoms $r^{\prime} \bowtie s^{\prime}$ over $r^{\prime}, s^{\prime} \in R^{\prime}$, let unprime $(S)$ be the set of atoms derived by replacing every $r^{\prime} \in R^{\prime}$ by $r$.

A state constraint relates registers in the current moment only: it contains atoms over non-primed registers, so it has no atoms over primed registers. Note that both $C_{\mid R}$ and unprime $\left(C_{\mid R^{\prime}}\right)$ are state constraints.

A constraint describes how register values change in one step: their relative order at the beginning (when $t_{1}, t_{2} \in R$ ), at the end (when $t_{1}, t_{2} \in R^{\prime}$ ), and in between (with $t_{1} \in R$ and $t_{2} \in R^{\prime}$ ).

Example 3. For instance, the ordering $r_{1}<r_{1}^{\prime}<r_{2}^{\prime}<r_{2}$ is a constraint over $R=$ $\left\{r_{1}, r_{2}\right\}$ and can be represented by $\left\{r_{1}<r_{2}, r_{1}<r_{1}^{\prime}, r_{2}>r_{2}^{\prime}, r_{1}^{\prime}<r_{2}^{\prime}\right\}$; it is satisfied e.g. by the two successive valuations $\nu_{a}:\left\{r_{1} \mapsto 1, r_{2} \mapsto 4\right\}$ and $\nu_{b}:\left\{r_{1} \mapsto 2, r_{2} \mapsto 3\right\}$. Similarly, $r_{1}=r_{1}^{\prime}<r_{2}^{\prime}=r_{2}$ is a constraint corresponding to the set $\left\{r_{1}<r_{2}, r_{1}=\right.$ $\left.r_{1}^{\prime}, r_{2}=r_{2}^{\prime}, r_{1}^{\prime}<r_{2}^{\prime}\right\}$. Note that the set $\left\{r_{1}<r_{2}, r_{1}>r_{1}^{\prime}, r_{2}<r_{2}^{\prime}, r_{1}^{\prime}>r_{2}^{\prime}\right\}$ does not represent a constraint: it is not consistent since $r_{1}>r_{1}^{\prime}>r_{2}^{\prime}>r_{2}>r_{1}$ implies $r_{1}>r_{1}$, violating irreflexivity, and thus does not correspond to any total non-strict preorder. Another counter-example is $r \leq r^{\prime}$ for $R=\{r\}$ : it is not a constraint since it is not total.

Definition 4. A constraint sequence is then an infinite sequence of constraints $C_{0} C_{1} \ldots$ (when a sequence is finite, we explicitly state it).

It is consistent if for every $i$ : unprime $\left(C_{i \mid R^{\prime}}\right)=C_{i+1 \mid R}$, i.e. the register order at the end of step $i$ equals the register order at the beginning of step $i+1$.

A valuation $w \in \mathscr{D}^{R \cup R^{\prime}}$ satisfies a constraint $C$, written $w \models C$, if every atom holds when we replace every $r \in R \cup R^{\prime}$ by $w(r)$. A constraint sequence is satisfiable if there exists a sequence of valuations $\nu_{0} \nu_{1} \ldots \in\left(D^{R}\right)^{\omega}$ such that $\nu_{i} \cup \nu_{i+1}^{\prime} \models C_{i}$ for all $i \geq 0$. If, additionally ${ }^{4}$, $\nu_{0}=0^{R}$, then it is 0 -satisfiable. Note that satisfiability implies consistency, but not vice versa, as we show below.

[^4]Note also that the notions of constraints and constraint sequences over $(\mathbb{N}, \leq)$ and over $(\mathbb{Q}, \leq)$ syntactically coincide. This is done on purpose, to ease the comparison between the two domains. When this matters, we always make it clear on which domain a constraint sequence is meant to be interpreted.

Finally, remark that consistency also coincides for both domains, while satisfiability does not, as witnessed by the constraint sequence $\left(\left\{r>r^{\prime}\right\}\right)^{\omega}$ over $R=\{r\}$ : it is satisfiable in $\mathbb{Q}$ but not in $\mathbb{N}$.

Example 4. We give a richer example. Let $R=\left\{r_{1}, r_{2}, r_{3}, r_{4}\right\}$. Let a consistent constraint sequence $C_{0} C_{1} \ldots$ start with

$$
\left\{r_{2}^{\prime}<r_{1}=r_{1}^{\prime}<r_{2}<r_{3}=r_{4}^{\prime}<r_{4}=r_{3}^{\prime}\right\}\left\{r_{1}^{\prime}<r_{2}=r_{2}^{\prime}<r_{1}<r_{4}=r_{3}^{\prime}<r_{3}=r_{4}^{\prime}\right\}
$$

Figure 4 visualises $C_{0} C_{1}$ plus a bit more constraints. The black lines represent the evolution of the same register; ignore the colored paths for now. The constraint $C_{0}$ describes the transition from moment 0 to 1 , and $C_{1}$ the transition from moment 1 to 2. This finite constraint sequence is satisfiable in $\mathbb{Q}$ and in $\mathbb{N}$. For example, the valuations can start with $\nu_{0}=\left\{r_{4} \mapsto 6, r_{3} \mapsto 5, r_{2} \mapsto 4, r_{1} \mapsto 3\right\}$. In $\mathbb{N}$, no valuations starting with $\nu_{0}\left(r_{3}\right)<5$ can satisfy the sequence. Further, since the constraint $C_{0}$ requires all registers in $R$ to differ, the sequence is not 0 -satisfiable in $\mathbb{Q}$ nor in $\mathbb{N}$.


Figure 4: Visualisation of a constraint sequence. Individual register values are depicted by black dots, and dots are connected by black lines when they talk about the same register. Yellow/blue/green/red paths depict chains (cf infra).

## Chains

This section describes a characterisation of satisfiable constraint sequences that is amenable to being recognised by automata. The proofs are quite technical, so we defer them to Section 5 and for the time being we only give an intuition.

Definition 5 (Chains). Fix $R$ and a consistent constraint sequence $C_{0} C_{1} \ldots$ over $R$. A (decreasing) two-way chain is a finite or infinite sequence $\left(r_{0}, m_{0}\right) \triangleright_{0}\left(r_{1}, m_{1}\right) \triangleright_{1} \ldots \in$ $((R \times \mathbb{N}) \cdot\{=,>\})^{*, \omega}$ satisfying the following (note that $m_{0}$ can differ from 0 ).

- $m_{i+1}=m_{i}$, or $m_{i+1}=m_{i}+1$ (time flows forward), or $m_{i+1}=m_{i}-1$ (backwards).
- If $m_{i+1}=m_{i}$ then $\left(r_{i} \triangleright_{i} r_{i+1}\right) \in C_{m_{i}}$.
- If $m_{i+1}=m_{i}+1$ then $\left(r_{i} \triangleright_{i} r_{i+1}^{\prime}\right) \in C_{m_{i}}$.
- If $m_{i+1}=m_{i}-1$ then $\left(r_{i}^{\prime} \triangleright_{i} r_{i+1}\right) \in C_{m_{i}-1}$.

The depth of a chain is the number of $>$; when it is infinity, the chain is infinitely decreasing. Figure 4 highlights four two-way chains (there are more) with yellow, blue, green and red colors. For instance, the green-colored chain $c_{3}$, defined as $\left(r_{4}, 2\right)>$ $\left(r_{3}, 3\right)>\left(r_{2}, 2\right)>\left(r_{1}, 3\right)>\left(r_{2}, 3\right)$, has depth 4 .

Given a moment $i$ and a register $x$, a (decreasing) right two-way chain starting in $(x, i)$ ( $r 2 w$ for short) is a two-way chain $(x, i) \triangleright_{1}\left(r_{1}, m_{1}\right) \triangleright_{2}\left(r_{2}, m_{2}\right) \ldots$ such that $m_{j} \geq i, \triangleright_{j} \in\{=,>\}$, for all $j$. Thus, all elements appear to the right of the starting moment ( $x, i$ ).

We define one-way chains similarly, except that time now flows forwards or stays the same, and that they can be either increasing or decreasing:

- $m_{i+1}=m_{i}$ (time does not flow), or $m_{i+1}=m_{i}+1$ (time flows forward).
- If $m_{i+1}=m_{i}$ then $\left(r_{i} \bowtie_{i} r_{i+1}\right) \in C_{m_{i}}$.
- If $m_{i+1}=m_{i}+1$ then $\left(r_{i} \bowtie_{i} r_{i+1}^{\prime}\right) \in C_{m_{i}}$.

A one-way chain is decreasing (respectively, increasing) if for all $i \geq 0, \bowtie_{i} \in\{>,=\}$ (resp., $\bowtie_{i} \in\{<,=\}$ ).

In Figure 4 , the blue $\left(c_{2}\right)$ chain $\left(r_{4}, 0\right)>\left(r_{3}, 0\right)>\left(r_{2}, 0\right)>\left(r_{1}, 0\right)>\left(r_{2}, 1\right)>$ $\left(r_{1}, 2\right)>\left(r_{2}, 3\right)$ is one-way decreasing chain of depth 6 ; the same sequence is also a two-way chain. The red $\left(c_{4}\right)$ chain $\left(r_{2}, 3\right)<\left(r_{1}, 4\right)=\left(r_{1}, 5\right)<\left(r_{2}, 5\right)<\left(r_{4}, 5\right)<$ $\left(r_{3}, 5\right)$ is one-way increasing of depth 4 ; if we read the sequence in reverse, it represents a two-way chain (two-way chains are always decreasing). Sometimes we write "chain" omitting whether it is two- or one-way.

A stable chain is an infinite chain $\left(r_{0}, m\right)=\left(r_{1}, m+1\right)=\left(r_{2}, m+2\right)=\ldots$; it can also be written as $\left(m, r_{0} r_{1} r_{2} \ldots\right)$. In Figure 4 , the yellow $\left(c_{1}\right)$ chain $\left(0,\left(r_{4} r_{3}\right)^{\omega}\right)$ is stable. Given a stable chain $\chi_{r}=\left(m, r_{0} r_{1} \ldots\right)$ and a chain $\chi_{s}=\left(s_{0}, n_{0}\right) \bowtie_{0}\left(s_{1}, n_{1}\right) \bowtie_{1}$ ..., where $n_{i} \geq m$ for all $i$, the chain $\chi_{r}$ is above $\chi_{s}$ (equiv., $\chi_{s}$ is below $\chi_{r}$ ) if for all $i$ the constraint $C_{n_{i}}$ contains $r_{n_{i}-m}>s_{i}$ or $r_{n_{i}-m}=s_{i}$; here we used $n_{i}-m$ because the register at moment $n_{i}$ in the chain $\chi_{r}$ is $r_{n_{i}-m}$. In Figure 4, the yellow chain $\left(0,\left(r_{4} r_{3}\right)^{\omega}\right)$ is above all colored chains. A stable chain $\left(m, r_{0} r_{1} \ldots\right)$ is maximal if it is above all other stable chains starting after $m$. In Figure 4, the yellow chain $\left(0,\left(r_{4} r_{3}\right)^{\omega}\right)$ is maximal (assuming the sequence evolves in a similar fashion). Notice that if a sequence has a stable chain, then it has a maximal one. A ceiled chain is a chain that is below a maximal stable chain. A constraint sequence can have an infinite number of ceiled chains; it can also have zero, e.g. when there are no stable chains.

Note that in this section, we mostly focus on one-way chains and right two-way chains, while two-way chains are used in Section 5.1 as a technical intermediate. In the latter section, we show:

Lemma 12. A consistent constraint sequence is 0 -satisfiable in $\mathbb{N}$ iff there exists $\mathrm{B} \geq 0$ such that:

1. it has no infinitely decreasing one-way chains,
2. the ceiled one-way chains have a depth at most B
3. it starts in $C_{0}$ s.t. $C_{0 \mid R}=\{r=s \mid r, s \in R\}$, and
4. it has no decreasing one-way chains of depth $\geq 1$ from $(r, 0)$ for any $r$.

In line with Example 2, the above characterisation is not $\omega$-regular; the culprit is item 2 . We thus define quasi-feasible constraint sequences, by relaxing the condition to asking that there are no infinite increasing ceiled chains.

Definition 6. A consistent constraint sequence is quasi-feasible whenever:

- it has no infinitely decreasing one-way chains,
- it has no infinitely increasing ceiled one-way chains,
- it starts in $C_{0}$ s.t. $C_{0 \mid R}=\{r=s \mid r, s \in R\}$, and
- it has no decreasing one-way chains of depth $\geq 1$ from $(r, 0)$ for any $r$.

In Section 5.3 on page 43, we show:
Lemma 26. A lasso-shaped consistent constraint sequence is 0 -satisfiable if and only if it is quasi-feasible.

We conclude the section by formally relating action words (see Definition 2) with constraint sequences.

## Action words and constraint sequences

Every action word naturally induces a unique constraint sequence. For instance, for registers $R=\{r, s\}$, an action word starting with $(\{r<*, s<*\},\{s\})$ (test whether the current data $d$ is above the values of $r$ and $s$, store it in $s$ ) induces a constraint sequence starting with $\left\{r=s, r=r^{\prime}, s<s^{\prime}, r^{\prime}<s^{\prime}\right\}$ (the atom $r=s$ is due to all registers being equal initially). This is formalised in the next lemma, which is notation-heavy but says a simple thing: given an action word, we can construct, on the fly, a constraint sequence that is 0-satisfiable iff the action word is feasible. For technical reasons, we need a new register $r_{d}$ to remember the last Adam data. The proof is on page 36, so as not to break the flow of the argument.

Lemma 13. Let $R$ be a set of registers, $R_{d}=R \uplus\left\{r_{d}\right\}$, and $\mathscr{D}$ be $(\mathbb{N}, \leq)$ or $(\mathbb{Q}, \leq)$. There exists a mapping constr $: \Pi \times \mathrm{Tst} \times \mathrm{Asgn} \rightarrow \mathrm{C}$ from state constraints $\Pi$ over $R_{d}$ and tests-assignments over $R$ to constraints $C$ over $R_{d}$, such that for all action words $a_{0} a_{1} a_{2} \ldots \in(\mathrm{Tst} \times \mathrm{Asgn})^{\omega}, a_{0} a_{1} a_{2} \ldots$ is feasible iff $C_{0} C_{1} C_{2} \ldots$ is 0 -satisfiable, where $\forall i \geq 0: C_{i}=\operatorname{constr}\left(\pi_{i}, a_{i}\right), \pi_{i+1}=\operatorname{unprime}\left(C_{i \mid R_{d}^{\prime}}\right), \pi_{0}=\left\{r=s \mid r, s \in R_{d}\right\}$.

Then, given a set of registers $R$, we say that an action word $\bar{a}$ is quasi-feasible whenever constr $(\bar{a})$ is quasi-feasible. We correspondingly denote by QFeasible $\mathbb{N}_{\mathbb{N}}(R)$ the set of quasi-feasible action words over $R$.

### 4.3 The $\boldsymbol{\omega}$-regular game $G_{S}^{r e g}$

After this long but necessary detour through constraint sequences, we are ready to define the $\omega$-regular game associated with the automaton game. Recall that in Section 3.3, given a one-sided automaton $S$, we defined $G_{S}^{f}=\left(V_{\forall}, V_{\exists}, v_{0}, E, W_{S}^{f}\right)$. We
now let $G_{S}^{r e g}=\left(V_{\forall}, V_{\exists}, v_{0}, E, W_{S}^{\text {reg }}\right)$. Thus, it has the same vertices and edge relation: $V_{\forall}=\left\{q_{\iota}\right\} \cup\left(\Sigma \times Q_{A}\right), V_{\exists}=$ Tst $\times$ Asgn $\times Q_{E}, v_{0}=q_{\iota}, E=E_{0} \cup E_{\forall} \cup E_{\exists}$ where:

- $E_{0}=\left\{\left(v_{0},\left(\right.\right.\right.$ tst, asgn,$\left.\left.u_{0}\right)\right) \mid \delta\left(v_{0}\right.$, tst $)=\left(\right.$ asgn,$\left.\left.u_{0}\right)\right\}$,
- $E_{\forall}=\{((\sigma, v),(\mathrm{tst}$, asgn, $u)) \mid \delta(v$, tst $)=($ asgn,$u)\}$, and
- $E_{\exists}=\{((\mathrm{tst}, \operatorname{asgn}, u),(\sigma, v)) \mid \delta(u, \sigma)=v\}$.

However, the winning condition is now:

$$
W_{S}^{f}=\left\{\begin{array}{l|l}
v_{0}\left(\operatorname{tst}_{0}, \operatorname{asgn}_{0}, u_{0}\right)\left(\sigma_{0}, v_{1}\right) \ldots & \begin{array}{l}
\left(\operatorname{tst}_{0} \operatorname{asgn}_{0}\right) \ldots \in \text { QFeasible } \\
\mathbb{N}
\end{array}(R) \\
\Rightarrow v_{0} u_{0} v_{1} u_{1} \ldots \models \alpha
\end{array}\right\}
$$

i.e., we replaced Feasible $\mathbb{N}_{\mathbb{N}}(R)$ with QFeasible $\left.\mathbb{N}^{( } R\right)$.

First, by Proposition 27, we know that QFeasible $\left.\mathbb{N}^{( } R\right)$ is $\omega$-regular. Thus:
Proposition 14. Let $S$ be a one-sided automaton, and define $G_{S}^{r e g}$ as above. Then, $G_{S}^{r e g}$ is an $\omega$-regular game.

We now show that it is equivalent with the Church game $G_{S}$.
Proposition 15. Let $S$ be a one-sided automaton, $G_{S}$ the corresponding Church game, $G_{S}^{f}$ its automaton game, and $G_{S}^{\text {reg }}$ its associated $\omega$-regular game. The following are equivalent:

1. Eve has a winning strategy in $G_{S}^{\text {reg }}$
2. Eve has a finite-memory winning strategy in $G_{S}^{\text {reg }}$
3. Eve has a finite-memory winning strategy in $G_{S}^{f}$
4. Eve has a winning strategy in $G_{S}^{f}$
5. Eve has a winning strategy in $G_{S}$.

Proof. We start with the chain of implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$.
The implication (1) $\Rightarrow(2)$ holds because $G_{S}^{r e g}$ is $\omega$-regular, and we know that those games are finite-memory determined [34].

Then, $(2) \Rightarrow(3)$ follows from the fact that $G_{S}^{r e g}$ is actually harder than $G_{S}^{f}$, i.e. $W_{S}^{\text {reg }} \subseteq W_{S}^{f}$, because Feasible ${ }_{\mathbb{N}}(R) \subseteq$ QFeasible $_{\mathbb{N}}(R)$.
$(3) \Rightarrow(4)$ is immediate.
$(4) \Rightarrow(5)$ is exactly Proposition 3 .
It remains to show that $(5) \Rightarrow(1)$. We proceed by contraposition. Thus, assume that Eve does not have a winning strategy in $G_{f}^{r e g}$. By finite-memory determinacy of games with parity objectives, in $G_{f}^{r e g}$ Adam has a finite-memory winning strategy $\lambda_{\forall}^{f}: V_{\forall}\left(V_{\exists} V_{\forall}\right)^{*} \rightarrow V_{\exists}$ (equiv., $\left.\lambda_{\forall}^{f}: \Sigma^{*} \rightarrow \mathrm{Tst}\right)$. We show the following:

Proposition 16. If Adam has a winning strategy in $G_{S}^{\text {reg }}$, then he has a winning strategy in $G_{S}$.

Proof. At first, it is not clear how to instantiate it to a data strategy $\lambda_{\forall}^{\mathbb{N}}: \Sigma^{*} \rightarrow \mathbb{N}$ winning in $G_{S}$. For instance, if the strategy $\lambda_{\forall}^{f}$ in $G_{f}^{r e g}$ dictates Adam to pick the
test $*>r$, it is not clear which data should $\lambda_{\forall}^{\mathbb{N}}$ pick $(\nu(r)+1, \nu(r)+2$, more? ) because for different strategies of Eve different values may be needed. To construct $\lambda_{\forall}^{\mathbb{N}}$ from $\lambda_{\forall}^{f}$ that beats every Eve, we show that for any finite-memory strategy of Adam, there is a uniform bound on the depth of all its r 2 w chains. This is formalised by the following claim (that we prove afterwards):

Claim 17. Let $\lambda_{\forall}^{f}$ be a finite-memory strategy of Adam that is winning in $G_{f}^{r e g}$. There exists a bound $\mathrm{B} \geq 0$ such that for each play $\rho$ consistent with $\lambda_{\forall}^{f}$, for each right twoway chain $\gamma$ of the constraint sequence induced by $\rho$ (starting in some ( $r, i) \in R \times \mathbb{N}$ ), $\operatorname{depth}(\gamma) \leq$ B.

Thanks to existence of this uniform bound $B$, we can construct $\lambda_{\forall}^{\mathbb{N}}$ from $\lambda_{\forall}^{f}$ as follows. First, translate the currently played action-word prefix $\left(\mathrm{tst}_{0}, \operatorname{asgn}_{0}\right) \ldots\left(\mathrm{tst}_{m}, \operatorname{asgn}_{m}\right)$ into a constraint-sequence prefix using Lemma 13. Then apply to it the data-assignment function from Lemma 28. By construction, for each play in $G$ consistent with $\lambda_{\forall}^{\mathbb{N}}$, the corresponding run in $S$ is a play consistent with $\lambda_{\forall}^{f}$ in $G_{f}^{r e g}$. As $\lambda_{\forall}^{f}$ is winning, this run is not accepting, i.e. the play is winning for Adam in $G_{S}$.

Therefore, $\lambda_{\forall}^{\mathbb{N}}$ is a winning Adam's strategy in $G_{S}$. End of the proof of Prop. 16
As a consequence, Eve does not have a winning strategy in $G_{S}$, which means that $(5) \Rightarrow(1) . \quad$ End of the proof of Prop. 15

We are left to prove Claim 17.

## Boundedness of right two-way chains induced by Adam (Proof of Claim 17)

Proof idea. If Adam has a finite-memory strategy, then if a decreasing right two-way chain $\gamma$ is sufficiently deep, Eve can force Adam to loop in a memory state in a way such that the loop can be iterated while preserving the chain. We can additionally ensure that this chain contains a strictly decreasing or increasing segment. When iterated, this segment makes the chain unfeasible. Indeed, if the segment is decreasing, iterating the loop yields an infinite descending chain in $\mathbb{N}$, which is not feasible. The case of an increasing fragment happens when $\gamma$ is decreasing from right to left (recall that it is a two-way chain), so increasing from left to right. When iterated, this yields an infinite increasing chain, which is perfectly fine in $\mathbb{N}$. However, it can be bounded from above with the help of $\gamma$ : before decreasing from right to left, $\gamma$ has to go from left to right, since it is a right chain (i.e. it is not allowed to go to the left of its initial position). On the strictly increasing segment, this left-to-right prefix is either constant or decreasing, so when the loop is iterated it provides an upper bound for our increasing chain.

Proof. We now move to the formal proof. We could use a Ramsey argument in the spirit of Lemma 23 to extract an infinite one-way chain that is either increasing or decreasing. However, this amounts to breaking a butterfly upon the wheel, and we prefer to rely on a simpler pumping argument, which also gives a finer-grained
perception of what is happening there. In particular, it provides a bound B that does not depend on a Ramsey number.

Thus, let $\lambda_{\forall}^{f}$ be a finite-memory strategy of Adam with memory $M$ that is winning in $G_{S}$. Suppose, towards a contradiction, that there exists a play $\rho$ that is consistent with $\lambda_{\forall}^{f}$ and which contains a decreasing right two-way chain of depth $D>|M| \cdot 2^{2|R|^{2}}$. We denote it $\gamma=\left(r_{0}, m_{0}\right) \triangleright_{0}\left(r_{1}, m_{1}\right) \triangleright_{1}\left(r_{2}, m_{2}\right) \triangleright_{2} \ldots \triangleright_{n-1}\left(r_{n}, m_{n}\right)$, where for all $0 \leq i \leq n, \triangleright_{i} \in\{>,=\}, r_{i} \in R$ and $m_{i} \in \mathbb{N}$. Given a two-way chain and a position $i \geq m_{0}$, we define the crossing section at $i$ as the sequence of registers that occur at position $i$, ordered by their appearance in the chain: $2 \gamma \int_{i}$ is the maximal subword of $\gamma$ that contains letters of the form $(r, i)$ for some $r \in R$ (see Fig. 5a, where we depicted a chain that has two identical crossing sections at positions $i$ and $j$ ). This

(a) A chain with two identical crossing sections.

(b) Iterating a fragment of a play. We are able to glue the chain since the crossing sections and the order between registers are the same at positions $i$ and $j$.
construction is reminiscent of the techniques that are used to study loops in two-way automata or transducers, hence the name. At each position, there are $|M|$ distinct memory states for Adam, less than $2^{|R|^{2}}$ many distinct crossing sections and less than $2^{|R|^{2}}$ many possible orderings of the registers. As a consequence there exists two positions $m_{0} \leq i<j$ such that $2 \gamma S_{i}=2 \gamma S_{j}$, the memory state of Adam at position $i$ and $j$ is the same, the order between registers at position $i$ is the same at position $j$, and there is at least one occurrence of $>$ in the chain segment. Since $\lambda_{\forall}^{f}$ is finitememory, Eve can repeat her actions between positions $i$ and $j$ indefinitely to iterate this fragment of the play $\rho$. Since the crossing sections match and the order between registers is the same at positions $i$ and $j$, we can glue the chain fragments together
to get an infinite two-way chain (see Fig.5b), with infinitely many occurrences of $>$. There are two cases:

- There is a fragment that strictly decreases from left to right (as the chain fragment over register $r_{4}$ in Fig.5b). Then, when Eve repeats her actions indefinitely, this yields an infinite descending chain, which means that the play is not feasible (Lemma 22), so Eve wins. This contradicts the fact that $\lambda_{\forall}^{f}$ is winning.
- All decreasing fragments occur from right to left (as do the fragments over $r_{2}$ and $r_{1}$ in Fig.5b). Necessarily, the topmost fragment, i.e. the fragment of the register that appears first in $2 \gamma \int_{i}$, is left-to-right, since $\gamma$ is a right two-way chain. It is not strictly decreasing, otherwise we are back to the first case. Then, the strictly decreasing fragments are bounded from above by this constant fragment. Iterating the loop yields an infinite increasing chain that is bounded from above, which means that the play is again not feasible, so we again obtain a contradiction.

Overall, the depth of the decreasing right two-way chains induced by $\lambda_{\forall}^{f}$ is uniformly bounded by $\mathrm{B}=|M| \cdot 2^{2|R|^{2}}$, where $|M|$ is the size of Adam's memory. We finally have all the cards in hand to show:

Theorem 18. Let $S=\left(\Sigma, Q, q_{\iota}, R, \delta, \alpha\right)$ be a one-sided register automaton over ( $\mathbb{N}, \leq$ ).

1. The problem of determining if Eve wins the Church synthesis game $G=(\mathscr{D}, \mathscr{D}, S)$ is decidable in time polynomial in $|Q|$ and exponential in $c$ and $|R|$.
2. $G_{S}$ is determined, i.e. either Eve or Adam has a winning strategy in $G_{S}$.

Proof. For $(\mathbb{N}, \leq)$, item (1) follows from Proposition 15 and from the fact that $G_{f}^{\text {reg }}$ is of size polynomial in $|Q|$ and exponential in $|R|$. Item (2) on determinacy is proven as follows. Assume Eve loses $G_{S}$. By Proposition 15, Eve loses $G_{f}^{r e g}$. In the proof of Proposition 15, we have shown (Proposition 16) that in this case Adam has a strategy winning in the original Church game. As a consequence, our Church games are determined.

With the help of Proposition 15, since finite-memory winning strategies of Eve in $G_{S}^{f}$ correspond to register transducer implementations (Proposition 4), we also get:

Theorem 19. For specifications defined by deterministic input-driven output register automata over data domains $(\mathbb{N}, \leq)$, the register transducer synthesis problem is equivalent with the synthesis problem (for arbitrary implementations) and can be solved in time polynomial in $|Q|$ and exponential in $c$ and $|R|$.

## 5 Satisfiability of Constraint Sequences in ( $\mathbb{N}, \leq$ )

This section studies the problem of checking whether a given infinite sequence of constraints can be satisfied with values from domain $\mathbb{N}$. Recall that constraints and constraint sequences are respectively defined in Definitions 3 and 4 on page 22. This section's structure is:

- We start with a simple and relatively known result on satisfiability of constraint sequences in data domain $\mathbb{Q}$. We then focus completely on $\mathbb{N}$.
- Section 5.1 describes conditions on chains that characterise satisfiable constraint sequences (in $\mathbb{N}$ ).
- Section 5.2 describes "max-automata" characterisation of satisfiable constraint sequences. The max-automaton characterisation checks the conditions on chains introduced in Section 5.1.
- In the study of Church synthesis games on $\mathbb{N}$, the crucial role play lasso-shaped constraint sequences and their satisfiability. We rely on them when proving Proposition 15 . The satisfiability of such sequences is the focus of Section 5.3, which shows that the regularity of sequences allows for characterisation of the satisfiability using classical $\omega$-regular automata instead of max-automata. Thus, in the context of Church synthesis games, the max-automaton characterisation is not used.
- Section 5.4 shows that "depth-bounded" constraint sequences can be mapped to satisfying valuations on-the-fly: such a data assignment function is used when proving the decidability of Church synthesis games (Proposition 15), namely, to show that winning Adam's strategies in abstracted finite-alphabet games can be instantiated to winning data Adam's strategies in Church synthesis games.


## Satisfiability of constraint sequences in $\mathbb{Q}$

Before proceeding to our main topic of satisfiability of constraint sequences in $\mathbb{N}$, we describe, for completeness, similar results for $\mathbb{Q}$.

The following result is glimpsed in several places (e.g. in [47, Appendix C]): a constraint sequence is satisfiable in $\mathbb{Q}$ iff it is consistent. This is a consequence of the following property which holds because $\mathbb{Q}$ is dense: for every constraint $C$ and $\nu \in \mathbb{Q}^{R}$ such that $\nu \vDash C_{\mid R}$, there exists $\nu^{\prime} \in \mathbb{Q}^{R^{\prime}}$ such that $\nu \cup \nu^{\prime} \models C$. Consistency can be checked by comparing every two consecutive constraints of the sequence. Thus, it is not hard to show that consistent - hence satisfiable - constraint sequences in $\mathbb{Q}$ are recognisable by deterministic parity automata.

Theorem 20. There is a deterministic parity automaton with two colors and of size exponential in $|R|$ that accepts exactly all constraint sequences satisfiable (or 0 -satisfiable) in $\mathbb{Q}$.

To prove the result, we first show that a constraint sequence in $\mathbb{Q}$ is satisfiable iff it is consistent, then we construct an automaton checking the consistency.

Lemma 21. Let $R$ be a set of registers and $\mathscr{D}=\mathbb{Q}$. A constraint sequence $C_{0} C_{1} \ldots$ is satisfiable iff it is consistent. It is 0 -satisfiable iff it is consistent and $C_{0 \mid R}=\left\{r_{1}=\right.$ $\left.r_{2} \mid r_{1}, r_{2} \in R\right\}$.

Proof. Direction $\Rightarrow$ is simple for both claims, so we only prove direction $\Leftarrow$.
Consider the first claim, direction $\Leftarrow$. Assume the sequence is consistent. We construct $\nu_{0} \nu_{1} \cdots \in\left(\mathbb{Q}^{R}\right)^{\omega}$ such that $\nu_{i} \cup \nu_{i+1}^{\prime} \models C_{i}$ for all $i$. The construction proceeds step-by-step and relies on the following fact ( $\dagger$ ): for every constraint $C$ and $\nu \in \mathbb{Q}^{R}$ such that $\nu \models C_{\mid R}$, there exists $\nu^{\prime} \in \mathbb{Q}^{R^{\prime}}$ such that $\nu \cup \nu^{\prime} \models C$. Then define
$\nu_{0}, \nu_{1} \ldots$ as follows: start with an arbitrary $\nu_{0}$ satisfying $\nu_{0} \models C_{0 \mid R}$. Given $\nu_{i} \models C_{i \mid R}$, let $\nu_{i+1}$ be any valuation in $\mathbb{Q}^{R}$ that satisfies $\nu_{i} \cup \nu_{i+1}^{\prime} \models C_{i}$ (it exists by ( $\dagger$ )). Since $\nu_{i+1} \models C_{i \mid R^{\prime}}$, and unprime $\left(C_{i \mid R^{\prime}}\right)=C_{i+1 \mid R}$ by consistency, we have $\nu_{i+1} \models C_{i+1 \mid R}$, and we can apply the argument again.

We are left to prove the fact $(\dagger)$. The constraint $C$ completely specifies the order on $R \cup R^{\prime}$, while $\nu$ fixes the values for $R$, and $\nu \models C_{\mid R}$. Thus, we can uniquely order registers $R^{\prime}$ and the values $\{\nu(r) \mid r \in R\}$ of $R$ on the $\mathbb{Q}$-line. Since $\mathbb{Q}$ is dense, it is always possible to choose the values for $R^{\prime}$ that respect this order; we leave out the details.

Consider the second claim, direction $\Leftarrow$. Since $C_{0} C_{1} \ldots$ is consistent, then by the first claim, it is satisfiable, hence it has a witnessing valuation $\nu_{0} \nu_{1} \ldots$. The constraint $C_{0}$ requires all registers in $R$ to start with the same value, so define $d=\nu_{0}(r)$ for arbitrary $r \in R$. Let $\nu_{0}^{\prime} \nu_{1}^{\prime} \ldots$ be the valuations decreased by $d: \nu_{i}^{\prime}(r)=\nu_{i}(r)-d$ for every $r \in R$ and $i \geq 0$. The new valuations satisfy the constraint sequence because the constraints in $\mathbb{Q}$ are invariant under the shift (follows from the fact: if $r_{1}<r_{2}$ holds for some $\nu \in \mathscr{D}^{R}$, then it holds for any $\nu-\mathbb{d}$ where $\left.d \in \mathscr{D}\right)$. The equality $\nu_{0}^{\prime}=0^{R}$ means that the constraint sequence is 0 -satisfiable.

We now prove Theorem 20.
Proof of Theorem 20. The sought automaton has an alphabet consisting of all constraints. By Lemma 21, for satisfiability, it suffices to construct the automaton that checks consistency, namely that every two adjacent constraints $C_{1} C_{2}$ in the input word satisfy the condition unprime $\left(C_{1 \mid R^{\prime}}\right)=C_{2 \mid R}$. We only sketch the construction. The automaton memorises the atoms $C_{1 \mid R^{\prime}}$ of the last constraint $C_{1}$ into its state, and on reading the next constraint $C_{2}$ the automaton checks that unprime $\left(C_{1 \mid R^{\prime}}\right)=C_{2 \mid R}$. If this holds, the automaton transits into the state that remembers $C_{2 \mid R^{\prime}}$; if the check fails, the automaton goes into the rejecting sink state. And so on. The automaton for checking 0 -satisfiability additionally checks that $C_{0 \mid R}=\{r=s \mid r, s \in R\}$. The number of states is exponential in $|R|$, the number of colors is 2 , and in fact the socalled safety (aka looping) acceptance suffices.

For the rest of this section, we focus on domain $\mathbb{N}$.

### 5.1 Chains characterise satisfiability of constraint sequences

In this section we prove the characterisation of satisfiable constraint sequences that we used to $\omega$-regularly approximate the automaton game over $(\mathbb{N}, \leq)$ (Section 4.2). Recall that chains are defined in Definition 5 on page 23.

While the target characterisation relies on one-way chains, we start by presenting a characterisation using two-way chains: such chains compare register values forwards and backwards in time. This characterisation is intuitive and easy to prove but difficult to implement using one-way automata. Therefore, later we provide an alternative characterisation using one-way chains which read constraint sequences in forward direction only. The lifting from two-way to one-way chains is done using Ramsey theorem [45]. A similar proof strategy is employed in [47, Appendix C], but our notion of chains is simpler, and we describe the previously missing application
of Ramsey theorem. We start with the definitions of two-way chains, then describe the characterisations in Lemmas 22 and 23.

Lemma 22. A consistent constraint sequence is satisfiable in $\mathbb{N}$ iff
A2. it has no infinite-depth two-way chains, and
B2. every ceiled two-way chain has a bounded depth
(i.e., there exists $\mathrm{B} \in \mathbb{N}$ such that the depth of every ceiled two-way chain is $\leq \mathrm{B}$ ).

Proof. The direction $\Rightarrow$ is proven by contradiction: if $\mathrm{A}_{2}$ is not satisfied, then one needs infinitely many values below the maximal initial value of a register to satisfy the sequence, which is impossible in $\mathbb{N}$. Similarly for B2. We now state this formally. Suppose a constraint sequence $C_{0} C_{1} \ldots$ is satisfiable by some valuations $\nu_{0} \nu_{1} \ldots$. Towards a contradiction, assume that $\mathrm{A}_{2}$ does not hold, i.e. there is an infinite decreasing two-way chain $\chi=\left(r_{0}, m_{0}\right)\left(r_{1}, m_{1}\right) \ldots$ Let $\nu_{m_{0}}\left(r_{0}\right)=d^{\star}$ be the data value at the start of the chain. Each decrease $\left(r_{i}, m_{i}\right)>\left(r_{i+1}, m_{i+1}\right)$ in the chain $\chi$ requires the data to decrease as well: $\nu_{i}\left(r_{i}\right)>\nu_{i+1}\left(r_{i+1}\right)$, so there must be an infinite number of data values between $d^{\star}$ and 0 , which is impossible in $\mathbb{N}$. Hence $A_{2}$ must hold. Now consider B2. If there are no ceiled chains, we are done, so assume there is at least one ceiled chain. Then there exists a maximal stable chain, by definition. Let $d^{\star}$ be the value of the registers in the maximal stable chain. All ceiled chains lie below the maximal stable chain, therefore the values of their registers are bounded by $\ell^{\star}$. Thus the depth of each such a chain is bounded by $B=d^{\star}$, so $\mathrm{B}_{2}$ holds.

The direction $\Leftarrow$. Given a consistent constraint sequence $C_{0} C_{1} \ldots$ satisfying $\mathrm{A}_{2}$ and $\mathrm{B}_{2}$, we construct a sequence of register valuations $\nu_{0} \nu_{1} \ldots$ such that $\nu_{i} \cup \nu_{i+1}^{\prime} \models C_{i}$ for all $i \geq 0$ (recall that $\nu^{\prime}=\left\{r^{\prime} \mapsto \nu(r) \mid r \in R\right\}$ ). For a register $r$ and moment $i \in \mathbb{N}$, let $d(r, i)$ be the largest depth of two-way chains from $(r, i)$; such a number exists by assumption $\mathrm{B}_{2}$; it is not $\infty$ by assumption $\mathrm{A}_{2}$; it can be 0 . Then, for every $r \in R$ and $i \in \mathbb{N}$, set $\nu_{i}(r)=d(r, i)$.

We now prove that for all $i$, the satisfaction $\nu_{i} \cup \nu_{i+1}^{\prime} \models C_{i}$ holds, i.e. all atoms of $C_{i}$ are satisfied. Pick an arbitrary atom $t_{1} \bowtie t_{2}$ of $C_{i}$, where $t_{1}, t_{2} \in R \cup R^{\prime}$. Define $m_{t_{1}}=i+1$ if $t_{1}$ is a primed register, else $m_{t_{1}}=i$; similarly define $m_{t_{2}}$. There are two cases.

- $t_{1} \bowtie t_{2}$ is $t_{1}=t_{2}$. Then the deepest chains from $\left(t_{1}, m_{t_{1}}\right)$ and $\left(t_{2}, m_{t_{2}}\right)$ have the same depth, $d\left(t_{1}, m_{t_{1}}\right)=d\left(t_{2}, m_{t_{2}}\right)$, and hence $\nu_{i} \cup \nu_{i+1}^{\prime}$ satisfies the atom.
- $t_{1} \bowtie t_{2}$ is $t_{1}>t_{2}$. Then, any chain $\left(t_{2}, m_{t_{2}}\right) \ldots$ from $\left(t_{2}, m_{t_{2}}\right)$ can be prefixed by $\left(t_{1}, m_{t_{1}}\right)$ to create the deeper chain $\left(t_{1}, m_{t_{1}}\right)>\left(t_{2}, m_{t_{2}}\right) \ldots$. Thus, $d\left(t_{1}, m_{t_{1}}\right)>$ $d\left(t_{2}, m_{t_{2}}\right)$, therefore $\nu_{i} \cup \nu_{i+1}^{\prime}$ satisfies the atom.

This concludes the proof.
Remark. The proof describes a data-assignment function which maps a sequence of constraints to a sequence of valuations satisfying it. Such functions are widespread, see e.g. [47, Lemma C.7] or [17, Lemma 15]. Later in Section 5.4 we describe a different kind of data-assignment function, which does not see the whole constraint sequence beforehand but only the prefix read so far. This changes how much the register values get separated from each other: from B in the above proof to approx. $2^{B}$.

The previous lemma characterises satisfiability in terms of two-way chains, but our final goal is the characterisation by automata. It is hard to design a one-way automaton tracing two-way chains, so we lift the previous lemma to one-way chains.

Lemma 23. A consistent constraint sequence is satisfiable in $\mathbb{N}$ iff
Aı. it has no infinitely decreasing one-way chains, and
Bı. every ceiled one-way chain has a bounded depth
(i.e., there exists $\mathrm{B} \in \mathbb{N}$ such that the depth of every ceiled one-way chain is $\leq \mathrm{B}$ ).

We describe a proof idea then provide a full proof.
Proof idea. We start from Lemma 22 and show that hypotheses $\mathrm{A}_{2}$ and $\mathrm{B}_{2}$ can be refined to $A_{1}$ and $B_{1}$ respectively. From an infinite (decreasing) two-way chain, we can always extract an infinite decreasing one-way chain, since two-way chains are infinite to the right and not to the left. Hence, for every moment $i$, there always exists a moment $j>i$ such that one register of the chain is smaller at step $j$ than a register of the chain at step $i$. Then, given a sequence of ceiled two-way chains of unbounded depth, we are able to construct a sequence of one-way chains of unbounded depth. This construction is more difficult than in the above case. Indeed, even though there are by hypothesis deeper and deeper ceiled two-way chains, they may start at later and later moments in the constraint sequence and go to the left. Thus, one cannot simply take an arbitrarily deep two-way chain and extract an arbitrarily deep oneway chain from it. However, we demonstrate, using a Ramsey argument, that it is still possible to extract arbitrarily deep one-way chains since the two-way chains are not completely independent.

Proof. Thanks to Lemma 22, it suffices to show that $A_{1} \Leftrightarrow A_{2}$ and $B_{1} \Leftrightarrow B_{2}$. The implications $A_{2} \Rightarrow A_{1}$ and $B_{2} \Rightarrow B_{1}$ follow from the definitions of chains.

Now, let us show that $\neg \mathrm{A} 2 \Rightarrow \neg \mathrm{~A} 1$ : let $C_{0} C_{1} \ldots$ be a consistent constraint sequence, and assume that it has an infinite two-way chain $\chi=\left(r_{a}, i\right) \ldots$ We then construct an infinite descending one-way chain $\chi^{\prime}$. The construction is illustrated in Figure 6. Our one-way chain $\chi^{\prime}$ starts in $\left(r_{a}, i\right)$. The area on the left from $i$-timeline contains $i \cdot|R|$ points, but $\chi$ has an infinite depth hence at some point it must go to the right from $i$. Let $r_{b}$ be the smallest register visited at moment $i$ by $\chi$; we first assume that $r_{b}$ is different from $r_{a}$ (the other case is later). Let $\chi$ go $\left(r_{b}, i\right) \triangleright\left(r^{\prime}, i+1\right)$. We append this to $\chi^{\prime}$ and get $\chi^{\prime}=\left(r_{a}, i\right)>\left(r_{b}, i\right) \triangleright\left(r^{\prime}, i+1\right)$. If $r_{a}$ and $r_{b}$ were actually the same, so the chain $\chi$ moved $\left(r_{a}, i\right) \triangleright\left(r^{\prime}, i+1\right)$, then we would append only $\left(r_{a}, i\right) \triangleright\left(r^{\prime}, i+1\right)$. By repeating the argument from the point $\left(r^{\prime}, i+1\right)$, we construct the infinite descending one-way chain $\chi^{\prime}$. Hence $\neg A_{\boldsymbol{1}}$ holds.

Now, let us show $\neg \mathrm{B}_{2} \Rightarrow \neg \mathrm{~B} 1$. Given a sequence of ceiled two-way chains of unbounded depth, we need to create a sequence of ceiled one-way chains of unbounded depth. We extract a witnessing one-way chain of a required depth from a sufficiently deep two-way chain. To this end, we represent the two-way chain as a clique with colored edges, and whose one-colored subcliques represent all one-way chains. We then use the Ramsey theorem that says a monochromatic subclique of a required size always exists if a clique is large enough. From the monochromatic subclique we extract the sought one-way chain.


Figure 6: Proving the direction $\neg \mathrm{A}_{2} \Rightarrow \neg \mathrm{~A}_{1}$ in Lemma 23. The two-way chain is in grey, the constructed one-way chain is in blue.

The Ramsey theorem [45] is about clique graphs with colored edges. For the number $n \in \mathbb{N}$ of vertices, let $K_{n}$ denote the clique graph and let $E_{K_{n}}$ be its set of edges. Then, we let color : $E_{K_{n}} \rightarrow\{1, \ldots, \# c\}$ be an edge-coloring function, where $\# c$ is the number of edge colors in the clique. A clique is monochromatic if all its edges have the same color $(\# c=1)$. The Ramsey theorem says:

Fix the number $\# c$ of edge colors. $(\forall n)(\exists l)\left(\forall\right.$ color : $\left.E_{K_{l}} \rightarrow\{1, \ldots, \# c\}\right)$ : there exists a monochromatic subclique of $K_{l}$ with $n$ vertices. The number $l$ is called the Ramsey number for $(\# c, n)$.
I.e., for any given $n$, there is a sufficiently large size $l$ such that any colored clique of this size contains a monochromatic subclique of size $n$. Ramsey numbers depend on the number $\# c$ of colors and size $n$ of the clique and are independent of a coloring function color. We use the theorem with three colors only: $\# c=3$.

Given a sequence of two-way chains of unbounded depth, we show how to build a sequence of one-way chains of unbounded depth. Suppose we want to build a oneway chain of depth $n$, and let $l$ be the Ramsey number for $(3, n)$. Since the two-way chains from the sequence have unbounded depth, there is a two-way chain $\chi$ of depth $l$. From it we construct the following colored clique (the construction is illustrated in Figure 7).

- Remove stuttering elements from $\chi$ : whenever $\left(r_{i}, m_{i}\right)=\left(r_{i+1}, m_{i+1}\right)$ appears in $\chi$, remove $\left(r_{i+1}, m_{i+1}\right)$. We repeat this until no stuttering elements appear. Let $\chi_{>}=\left(r_{1}, m_{1}\right)>\cdots>\left(r_{l}, m_{l}\right)$ be the resulting sequence; it is strictly decreasing, and contains $l$ pairs (the same as the depth of the original $\chi$ ). Note the following property $(\dagger)$ : for every not necessarily adjacent $\left(r_{i}, m_{i}\right)>\left(r_{j}, m_{j}\right)$, there is a oneway chain $\left(r_{i}, m_{i}\right) \ldots\left(r_{j}, m_{j}\right)$; it is decreasing if $m_{i}<m_{j}$, and increasing otherwise; its depth is at least 1 . The resulting sequence may skip points in time, but this as will be explained later - does not affect the construction.
- The elements $(r, m)$ of $\chi_{>}$serve as the vertices of the colored clique. The edgecoloring function is: for every not necessarily adjacent $\left(r_{a}, m_{a}\right)>\left(r_{b}, m_{b}\right)$ in $\chi_{>}$, let $\operatorname{color}\left(\left(r_{a}, m_{a}\right),\left(r_{b}, m_{b}\right)\right)$ be $\nearrow$ if $m_{a}<m_{b}, \searrow$ if $m_{a}>m_{b}, \downarrow$ if $m_{a}=m_{b}$. Thus, we assign a color to an edge between every two vertices. Figure 7b gives an example.


Figure 7: Proving the direction $\neg \mathrm{B}_{2} \Rightarrow \neg \mathrm{~B}_{1}$ in Lemma 23

By applying the Ramsey theorem, we get a monochromatic subclique of size $n$ with vertices $V \subseteq\left\{\left(r_{1}, m_{1}\right), \ldots,\left(r_{l}, m_{l}\right)\right\}$. Its color cannot be $\downarrow$ when $n>|R|$, because a timeline has maximum $|R|$ points. Suppose the subclique's color is $\nearrow$ (the case of $\searrow$ is similar). We build the increasing sequence $\chi^{\star}=\left(r_{1}^{\star}, m_{1}^{\star}\right)<\cdots<\left(r_{n}^{\star}, m_{n}^{\star}\right)$, where $m_{i}^{\star}<m_{i+1}^{\star}$ and $\left(r_{i}^{\star}, m_{i}^{\star}\right) \in V$ for every $i$. The sequence $\chi^{\star}$ may not satisfy the definition of one-way chains, because the removal of stuttering elements that we performed at the beginning can cause time jumps i.e. $m_{i+1}>m_{i}+1$. But it is easy relying on the property $(\dagger)$ - to construct the one-way chain $\chi^{\star \star}$ of depth $n$ from $\chi^{\star}$ by inserting the necessary elements between $\left(r_{i}, m_{i}\right)$ and $\left(r_{i+1}, m_{i+1}\right)$. The case when the subclique has color $\searrow$, the resulting constructed chain is decreasing.

Thus, for every given $n$, we constructed either a decreasing or increasing ceiled one-way chain of depth $n$. In other words, a sequence of such chains of unbounded depth. Hence $\neg$ Bı holds, which concludes the proof.

The next easy lemma (first stated on page 24) refines the characterisation to 0 -satisfiability:

Lemma 12. A consistent constraint sequence is 0 -satisfiable in $\mathbb{N}$ iff there exists $\mathrm{B} \geq 0$ such that:

1. it has no infinitely decreasing one-way chains,
2. the ceiled one-way chains have a depth at most B
3. it starts in $C_{0}$ s.t. $C_{0 \mid R}=\{r=s \mid r, s \in R\}$, and
4. it has no decreasing one-way chains of depth $\geq 1$ from $(r, 0)$ for any $r$.

Proof. Direction $\Rightarrow$. The first two items follow from Lemma 23; the third one follows from the definition of satisfiability. Consider the last item: suppose there is such a chain. Then, at the moment when the chain strictly decreases and goes to some register $s$, the register $s$ would need to have a value below 0 , which is impossible in $\mathbb{N}$.

Direction $\Leftarrow$. The first two items are exactly $A_{1}$ and $B_{1}$ from Lemma 23, so the sequence is satisfiable, hence it also satisfies the conditions $A_{2}$ and $B_{2}$ from Lemma 22. In the proof of Lemma 22, we showed that in this case the following valuations $\nu_{0} \nu_{1} \ldots$ satisfy the sequence: for every $r \in R$ and moment $i \in \mathbb{N}$, set $\nu_{i}(r)$ (the value of $r$ at moment $i$ ) to the largest depth of the two-way chains starting in $(r, i)$. We construct $\nu_{0} \nu_{1} \ldots$ as above, and get a witness of satisfaction of our constraint sequence. Note that at moment $0, \nu_{0}=0^{R}$, by the last item. Hence the constraint sequence is 0 satisfiable.

## Action words and constraint sequences

In this section, we provide the proof of the following lemma, stated on page 25:
Lemma 13. Let $R$ be a set of registers, $R_{d}=R \uplus\left\{r_{d}\right\}$, and $\mathscr{D}$ be $(\mathbb{N}, \leq)$ or $(\mathbb{Q}, \leq)$. There exists a mapping constr $: \Pi \times \mathrm{Tst} \times$ Asgn $\rightarrow$ C from state constraints $\Pi$ over $R_{d}$ and tests-assignments over $R$ to constraints C over $R_{d}$, such that for all action words $a_{0} a_{1} a_{2} \ldots \in(\mathrm{Tst} \times \mathrm{Asgn})^{\omega}, a_{0} a_{1} a_{2} \ldots$ is feasible iff $C_{0} C_{1} C_{2} \ldots$ is 0 -satisfiable, where $\forall i \geq 0: C_{i}=\operatorname{constr}\left(\pi_{i}, a_{i}\right), \pi_{i+1}=\operatorname{unprime}\left(C_{i \mid R_{d}^{\prime}}\right), \pi_{0}=\left\{r=s \mid r, s \in R_{d}\right\}$.

Proof. Given $\pi$, tst, asgn, we define the mapping constr : $(\pi$, tst, asgn $) \mapsto C$ as follows. The definition is as expected, but we should be careful about handling of $r_{d}$, it is the last item.

- The constraint $C$ includes all atoms of the state constraint $\pi$ (that relates the registers at the beginning of the step).
- Recall that neither tst nor asgn talk about $r_{d}$. For readability, we shorten $\left(t_{1} \bowtie\right.$ $\left.t_{2}\right) \in C$ to simply $t_{1} \bowtie t_{2},(* \bowtie r) \in$ tst to $* \bowtie r$, and $a \leq b$ means $(a<b) \vee(a=b)$.
- We define the order at the end of the step as follows. For every two different $r, s \in R$ :
$-r^{\prime}=s^{\prime}$ iff $(r=s) \wedge r, s \notin$ asgn or $r \in \operatorname{asgn} \wedge(*=s)$ or $r, s \in$ asgn;
$-r^{\prime}<s^{\prime}$ iff $(r<s) \wedge r, s \notin \operatorname{asgn}$ or $(*<s) \wedge r \in \operatorname{asgn} \wedge s \notin$ asgn;
$-r^{\prime}=r_{d}^{\prime}$ iff $(r=*)$ or $r \in \operatorname{asgn} ;$
$-r^{\prime} \bowtie r_{d}^{\prime}$ iff $(r \bowtie *) \wedge r \notin \operatorname{asgn}$, for $\bowtie \in\{<,>\}$;
- So far we have defined the order of the registers at the beginning and the end of the step. Now we relate the values between these two moments. For every $r \in R$ :
- $r=r^{\prime}$ iff $r \notin$ asgn or $r \in \operatorname{asgn} \wedge(*=r)$;
$-r \bowtie r^{\prime}$ iff $r \in \operatorname{asgn} \wedge(r \bowtie *)$, for $\bowtie \in\{<,>\}$;
- Finally, we relate the values of $r_{d}$ between the moments. There are two cases.
- The value of $r_{d}$ crosses another register: $\exists r \in R:\left(r_{d}<r\right) \wedge(* \geq r)$. Then $\left(r_{d}^{\prime}>r_{d}\right)$. Similarly for the opposite direction: if $\exists r \in R:\left(r_{d}>r\right) \wedge(* \leq r)$ then $\left(r_{d}^{\prime}<r_{d}\right)$.
- Otherwise, the value of $r_{d}$ does not cross any register boundary. Then $r_{d}^{\prime}=r_{d}$.

Using the mapping constr, every action word $\bar{a}=\left(\operatorname{tst}_{0} \operatorname{asgn}_{0}\right)\left(\operatorname{tst}_{1} \operatorname{asgn}_{1}\right) \ldots$ can be uniquely mapped to the constraint sequence $C_{0} C_{1} \ldots$ as follows: $C_{0}=$ $\operatorname{constr}\left(\pi_{0}, \mathrm{tst}_{0}, \operatorname{asgn}_{0}\right)$, set $\pi_{1}=\operatorname{unprime}\left(C_{0 \mid R_{d}^{\prime}}\right)$, then $C_{1}=\operatorname{constr}\left(\pi_{1}, \mathrm{tst}_{1}, \operatorname{asgn}_{1}\right)$, and so on.

We now prove that an action word is feasible iff the constructed constraint sequence is 0 -satisfiable. This follows from the definitions of feasibility and 0satisfiability, and from the following simple property of feasible action words. Every feasible action word has a witness $\nu_{0} \ell_{0} \nu_{1} \ell_{1} \cdots \in\left(\mathscr{D}^{R} \cdot \mathscr{D}\right)^{\omega}$ such that: if some tst is repeated twice and no assignment is done, then the value $d$ stays the same. This property is needed due to the last item in the definition of constr where we set $r_{d}^{\prime}=r_{d}$.

### 5.2 Max-automata recognise satisfiable constraint sequences

This section presents an automaton characterisation of constraint sequences satisfiable in $\mathbb{N}$. The automaton construction verifies the conditions on one-way chains stated in Lemma 23: the absence of ( $\mathrm{A} \mathbf{1}$ ) infinite decreasing one-way chains and of (Bı) unbounded one-way ceiled chains. The boundedness requirement of the second condition cannot be checked by $\omega$-regular automata ${ }^{5}$, and for that reason in [47] the authors used nondeterministic $\omega \mathrm{B}$-automata. Since nondeterminism is usually hard to handle in synthesis, we picked deterministic max-automata [8], which are incomparable with $\omega$ B-automata, expressivity-wise. We now define max-automata and then present the characterisation.

Deterministic max-automata extend classic finite-alphabet parity automata with a finite set of counters $c_{1}, \ldots, c_{n}$ which can be incremented, reset to 0 , or updated by taking the maximal value of a set of counters, but the counters cannot be tested. On reading a word, the automaton builds a sequence of counter valuations. The acceptance condition is given as a conjunction of the parity acceptance condition and a Boolean combination of conditions "counter $c_{i}$ is bounded along the run". Such a condition on a counter is satisfied by a run if there exists a bound $B \in \mathbb{N}$ such that counter $c_{i}$ has value at most B along the run. By using negation, conditions such as " $c_{i}$ is unbounded along the run" can also be expressed. A run is accepting if it satisfies the parity condition and the Boolean formula on the counter conditions. Deterministic max-automata are strictly more expressive than $\omega$-regular automata. For instance, they can express the non- $\omega$-regular language of words of the form $a^{n_{1}} b a^{n_{2}} b \ldots$ such that $n_{i} \leq \mathrm{B}$ for all $i \geq 0$, for some $\mathrm{B} \in \mathbb{N}$ that can vary from word to word. A max-automaton recognising the language is in Figure 8.

We now prove the main result of this section.

[^5]

Figure 8: Max-automaton recognising $\left\{a^{n_{1}} b a^{n_{2}} b \ldots \mid \exists \mathrm{B} \in \mathbb{N} \forall i: n_{i} \leq \mathrm{B}\right\}$. It uses a single counter $c$, the acceptance condition is "counter $c$ is bounded", and the parity acceptance is trivial (always accept). The operation max is not used.

Theorem 24. For every $R$, there is a deterministic max-automaton accepting exactly all constraint sequences satisfiable in $\mathbb{N}$. The number of states is exponential in $|R|$, the number of counters is $O\left(|R|^{2}\right)$, and the number of priorities is polynomial in $|R|$. The same holds for 0-satisfiability in $\mathbb{N}$.

Proof idea. We design a deterministic max-automaton that checks conditions Aı and Bı of Lemma 23. Condition $A ı$, namely the absence of infinitely decreasing one-way chains, is checked as follows. We construct a nondeterministic Büchi automaton that guesses a chain and verifies that it is infinitely decreasing, i.e. that ' $>$ ' occurs infinitely often and that there is no ' $<$ ' (only ' $>$ ' and ' $=$ '). Determinising and complementing yields a deterministic parity automaton, that can be disjuncted through a synchronised product with the deterministic max-automaton checking condition Bı. The latter condition (the absence of ceiled one-way chains of unbounded depth) is more involved. We design a master automaton that tracks every chain $\chi$ that currently exhibits a stable behaviour. To every such a chain $\chi$, the master automaton assigns a tracer automaton whose task is to ensure the absence of unbounded-depth ceiled chains below $\chi$. For that, the tracers use $2|R|$ counters - one for tracking increasing and one for tracking decreasing chains - and requires them to be bounded. We use the max operation on counters to ensure that we trace the largest chains only. The overall acceptance condition ensures that if the chain $\chi$ is stable, then there are no ceiled chains below $\chi$ of unbounded depth. Finally, we take the product of all these automata, which preserves determinism.

In the next section, we provide the details of the proof.

## Proof of Theorem 24

We describe a max-automaton $A$ that accepts a constraint sequence iff it is consistent and has no infinitely decreasing one-way chains and no ceiled one-way chains of unbounded depth. By Lemma 23, such a sequence is satisfiable.

The automaton has three components $A=A_{c} \wedge A_{\neg \infty} \wedge A_{\mathrm{B}}$.
$A_{c}$ The parity automaton $A_{c}$ checks consistency, i.e. that $\forall i$ : unprime $\left(C_{i \mid R^{\prime}}\right)=$ $\left(C_{i+1}\right)_{\mid R}$. It has exponential in $|R|$ number of states and two priorities (the safety language).
$A_{\neg \infty}$ The parity automaton $A_{\neg \infty}$ ensures there are no infinitely decreasing oneway chains. First, we construct its negation, an automaton that accepts a constraint
sequence iff it has such a chain. Intuitively, the automaton guesses such a chain and then verifies that the guess is correct. It loops in the initial state $q_{\iota}$ until it nondeterministically decides that now is the starting moment of the chain and guesses the first register $r_{0}$ of the chain, and transits into the next state while memorising $r_{0}$. When the automaton is in a state with $r$ and reads a constraint $C$, it guesses the next register $r_{n}$, verifies that $\left(r_{n}^{\prime}>r\right) \in C$ or $\left(r_{n}^{\prime}=r\right) \in C$, and transits into the state that remembers $r_{n}$. The Büchi acceptance condition ensures that the automaton leaves the initial state and transits from some $r$ to some $r_{n}$ with $\left(r_{n}^{\prime}>r\right) \in C$ infinitely often. Determinising and complementing this automaton gives $A_{\neg \infty}$. The number of states is exponential and the number of priorities is polynomial in $|R|$, due to the determinisation.
$A_{\mathrm{B}}$ The max-automaton $A_{\mathrm{B}}$ ensures that all ceiled one-way chains have bounded depth. It relies on the master automaton controlling the team of $|R|$ chain tracers $\operatorname{Tr}=\left\{t r_{1}, \ldots, t r_{|R|}\right\}$. Each tracer $t r$ is equipped with a counter $i d l e_{t r}$ and a set $C n_{t r}$ of $2|R|$ of counters, thus overall there are $|R|(2|R|+1)$ counters. The construction ensures that every stable chain is tracked by a single tracer $t r$ and its counter $i d l e_{t r}$ is bounded; and vice versa, if a tracer $t r$ has its counter $i d l e_{t r}$ bounded, it tracks a stable chain. Suppose for a moment that tracer $t r$ tracks a stable chain $\chi$. Then the goal of counters $C n_{t r}$ is to track the deepest increasing and decreasing chains below $\chi$. Since there are only $|R|$ registers, it suffices to track $|R|$ decreasing chains, every chain ending in a different register (similarly for increasing chains). This is because there is no need to track two decreasing chains ending in the same register: once the two chains "meet" in a register $r$, we continue tracking only the one with the larger depth and forget about the other. We use the max operation of automata to implement this idea. Overall, the construction ensures that the counters in $C n_{t r}$ are bounded iff the increasing and decreasing chains ceiled by the stable chain tracked by the tracer $t r$ have bounded depths. The acceptance of $A_{\mathrm{B}}$ is the formula

$$
\bigwedge_{t r \in T r}\left(i d l e_{t r} \text { is bounded } \rightarrow \bigwedge_{c \in C n_{t r}} c \text { is bounded }\right)
$$

The work of tracers is controlled by the master automaton via four commands idle ("track nothing"), start ("start tracking a potentially stable chain"), move ("continue tracking"), and reset ("stop tracking"). Before we formally describe the master and the tracers, we define the concept of "levels" used in the presentation. Intuitively, the levels abstract concrete data values, and the tracers actually track the levels instead of specific registers.

Fix a constraint $C$. A level $l \subseteq R \backslash\{\emptyset\}$ is an equivalence class of registers wrt. $C_{\mid R}$ or wrt. unprime $\left(C_{\mid R^{\prime}}\right)$. Thus, in the constraint $C$ we distinguish the levels of two kinds: start levels (at the beginning of the step) and end levels (at the end of the step). A start level $l \subseteq R$ disappears when $C$ contains no atoms of the form $r=s^{\prime}$ for $r \in l$ and $s \in R$; this means that a data value abstracted by the level disappears from the registers. An end level $l \subseteq R$ is new if $C$ contains no atoms of the form $r=s^{\prime}$ where $r \in R$ and $s \in l$; intuitively, the constraint requires a new data value to appear in registers $l$. A start level $l$ morphs into an end level $l^{\prime}$ if $C$ contains an

Figure 9: Example of levels: start levels are $\left\{r_{1}, r_{2}\right\}$ and $\left\{r_{3}\right\}$, end levels are $\left\{r_{3}\right\},\left\{r_{2}\right\}$, and $\left\{r_{1}\right\}$. The start level $\left\{r_{1}, r_{2}\right\}$ morphs into end level $\left\{r_{3}\right\}$, the start level $\left\{r_{3}\right\}$ disappears, and two new end levels appear, $\left\{r_{1}\right\}$ and $\left\{r_{2}\right\}$. The constraint is
 $\left\{r_{1}=r_{2}=r_{3}^{\prime}>r_{2}^{\prime}>r_{3}>r_{1}^{\prime}\right\}$.
atom $r=s^{\prime}$ for some $r \in l$ and $s \in l^{\prime}$; i.e., the constraint requires the registers in $l^{\prime}$ to hold the data value previously held by the registers in $l$. Notice that there can be at most $|R|$ start and $|R|$ end levels, for a fixed constraint $C$. Figure 9 illustrates the definitions. We are now ready to describe the master and the tracers.
Master. States of $A_{\mathrm{B}}$ are of the form ( $\mathrm{get} \operatorname{Tr}, \vec{q}$ ), where the partial mapping get $\operatorname{Tr}$ : $l \mapsto t r$ maps a level $l \subseteq R \backslash\{\emptyset\}$ to a tracer $\operatorname{tr} \in \operatorname{Tr}$, and $\vec{q}=\left(q_{1}, \ldots, q_{|T r|}\right)$ describes the states of individual tracers. The master updates the state component get Tr while the tracers update their states. Initially, there is only one start level $R$ (assuming the registers start with the same value), so we define get $T r=\left\{R \mapsto t r_{1}\right\}$. Suppose the automaton reads a constraint $C$, let $L$ and $L^{\prime}$ be the start and end levels of $C$, and suppose the automaton is in state $(\operatorname{get} T r, \vec{q})$ and $\operatorname{get} T r: L \rightarrow \operatorname{Tr}$. We define the successor state $\left(\operatorname{getTr}{ }^{\prime}, \vec{q}^{\prime}\right)$, where $\operatorname{get} T r^{\prime}: L^{\prime} \rightarrow T r$, and operations on the counters using the following procedure.

- To every tracer $t r$ that does not currently track a level, i.e. $\operatorname{tr} \in \operatorname{Tr} \backslash \operatorname{get} \operatorname{Tr}(L)$, the master commands idle (causing the tracer to increment idle $e_{t r}$ ).
- For every start level $l \in L$ that morphs into $l^{\prime} \in L^{\prime}$ : let $\operatorname{tr}=\operatorname{get} \operatorname{Tr}(l)$, then
- the master sends move $\left(r_{\top}\right)$ to $\operatorname{tr}$ where $r_{\top} \in l$ is chosen arbitrary; this will cause the tracer $t r$ to update its counters $C n_{t r}$ and move into a successor state $q_{t r}^{\prime}$; the register $r_{\top}$ will be used as a descriptor of a stable chain tracked by $t r$.
- we set $\operatorname{get}^{\operatorname{Tr}}{ }^{\prime}\left(l^{\prime}\right)=\operatorname{get} \operatorname{Tr}(l)$, thus the tracer continues to track it.
- For every start level $l \in L$ that disappears: let $\operatorname{tr}=\operatorname{get} \operatorname{Tr}(l)$, then
- the master sends reset to $t r$, which causes the reset of the counters in $C n_{t r}$ and the increment of $i d l e_{t r}$.
- For every new end level $l^{\prime} \in L^{\prime}$ :
- we take an arbitrary $t r$ that is not yet mapped by $g e t T r^{\prime}$ and map $\operatorname{get}^{\prime} r^{\prime}\left(l^{\prime}\right)=t r$;
- the master sends start to $t r$.

Tracers. We now describe the tracer component. Its goal is to trace the depths of ceiled chains. When the counters of a tracer are bounded, the depths of the chains it tracks are also bounded. The tracer consists of two components, $B_{\downarrow}$ and $B_{\uparrow}$, which track decreasing and increasing chains. We only describe $B_{\downarrow}$, the other one is similar.

The component $B_{\downarrow}$ has a set $C n \cup\{i d l e\}$ of $|R|+1$ counters. A state of $B \downarrow$ is either the initial state $q_{\iota}$ or a partial mapping getCn : $R \rightharpoonup C n$. Intuitively, in each $\operatorname{get} C n$-state, for each register $r$ mapped by get $C n$, the value of the counter $\operatorname{get} C n(r)$ reflects the depth of the deepest ceiled decreasing one-way chain ending in $r$. When
several chains end in $r$, the counter gets the maximal value of the depths. We maintain this property of getCn during the transition of $B_{\downarrow}$ on reading a constraint $C$, using operations of max-automata on counters and register-order information from $C$. The component $B_{\downarrow}$ does the following:

- If the master's command is idle, then increment the counter idle and stay in $q_{\iota}$.
- If the master's command is reset, reset all counters in $C n$, increment the counter $i d l e$, and go into state $q_{\iota}$.
- If the master's command is start, move from state $q_{\iota}$ into the state with the empty mapping getCn.

Otherwise, the master's command is move $\left(r_{\top}\right)$, for some $r_{\top} \in R$ passed by the master and serving as a descriptor of a stable chain traced by the current tracer. The tracer performs the operations on its counters and updates the mapping getCn as follows.

- Release counters. For every $r$ such that $r<r_{\top}<r^{\prime}$, the component resets the counter $\operatorname{get} C n(r)$ and removes $r$ from the mapping getCn. I.e., we stop tracking chains ending in register $r$ since such chains are no longer below the stable chain assigned to the tracer.
- Allocate counters. For every $r$ such that $r \geq r_{\top}>r^{\prime}$ : pick a counter $c \in C n \backslash$ $\operatorname{get} C n(R)$ and map $\operatorname{get} C n(r)=c$. I.e., we start tracking chains ending in $r$.
- Update counters. For every $r$ such that $r \leq r_{\top}$ and $r^{\prime}<r_{\top}$ do the following. Let $R_{>r^{\prime}}=\left\{r_{o} \mid r^{\prime}<r_{o}<r_{\top}\right\}$ be the registers larger than the updated $r$ but below $r_{\top}$, and let $\operatorname{getCn}\left(R_{>r^{\prime}}\right)$ be the associated counters. Let $r_{=}$be a register s.t. $r_{=}=r^{\prime}$ (may not exist). We update the counter $\operatorname{get} C n(r)$ depending on the case:
- $R_{>r^{\prime}}$ is empty and $r_{=}$does not exist: the condition means that no decreasing ceiled chain can be extended into $r^{\prime}$. Then we reset the counter $\operatorname{getCn}(r)$.
- $R_{>r^{\prime}}$ is empty and $r_{=}$exists: only the chains ending in $r_{=}$can be extended into $r^{\prime}$, and since $r_{=}=r^{\prime}$, the deepest chain keeps its depth. Therefore, we $\operatorname{copy}\left(\operatorname{getCn}\left(r_{=}\right)\right)$into the counter $\operatorname{getCn}(r)$.
- $R_{>r^{\prime}}$ is not empty and $r_{=}$does not exist: the chains from registers in $R_{>r^{\prime}}$ can be extended into $r^{\prime}$, and since $r^{\prime}$ is lower than any register in $R_{>r^{\prime}}$, their depths increase. The new value of counter $\operatorname{get} \operatorname{Cn}(r)$ must reflect the deepest chain, therefore the counter gets the value $\max \left(\operatorname{get} C n\left(R_{>r^{\prime}}\right)\right)+1$.
- $R_{>r^{\prime}}$ is not empty and $r_{=}$exists: some chains from registers in $R_{>r^{\prime}}$ can be decremented into $r^{\prime}$, there is also a chain from $r=$ that can be extended into $r^{\prime}$ without its depth changed. The counter gets $\max \left(\max \left(\operatorname{get} C n\left(R_{>r^{\prime}}\right)\right)+1, \operatorname{get} C n\left(r_{=}\right)\right)$, which describes the deepest resulting chain.
The number of states in $B \downarrow$ is no more than $|R|^{|R|}+1$, and the number of counters is $|R|+1$. The construction for $B_{\boldsymbol{\nearrow}}$ is similar to this construction for $B_{\downarrow}$, except that we need to track increasing ceiled chains instead of decreasing ones. The number of counters in $B_{\downarrow}$ and $B_{\not}$ is $2|R|+1$. Since we use $|R|$ number of tracers, the total number of counters becomes $|R|(2|R|+1)$. Overall, $A_{\mathrm{B}}$ has an exponential in $|R|$ number of states, the number of counters is in $O\left(|R|^{2}\right)$, and the parity condition is trivial. This concludes the description of the tracers and of the automaton $A_{\mathrm{B}}$.

We have described all three components $A=A_{c} \wedge A_{\neg \infty} \wedge A_{\mathrm{B}}$, where $A_{c}$ expresses a safety language, $A_{\neg \infty}$ is a classic deterministic parity automaton, and $A_{\mathrm{B}}$ is a deterministic max-automaton with the trivial parity acceptance condition. All the automata has no more than an exponential in $|R|$ number of states, $A_{\neg \infty}$ has a polynomial in $|R|$ number of colors, and $A_{\mathrm{B}}$ has a polynomial in $|R|$ number of counters. It is not hard to see that the product of these automata gives the desired automaton $A$ with exponentially many states, polynomially many colors and counters, in $|R|$. The acceptance condition is the parity acceptance in conjunction with the formula of $A_{\mathrm{B}}$ described on page 39 .

Finally, for the case of 0-satisfiability, the automaton $A$ also needs to satisfy the additional conditions stated in Lemma 12, in particularly there shall be no decreasing one-way chains from moment 0 of depth $\geq 1$. This check is simple and omitted. This concludes the proof of Theorem 24.
Remark. In [47, Appendix C] it is shown that satisfiable constraint sequences in $\mathbb{N}$ are characterised by nondeterministic $\omega \mathrm{B}$-automata [6]. These automata are incomparable with deterministic max-automata.

The following two languages separate these classes: $\overline{\left(a^{B} b\right)^{\omega}}$ is recognised by det max automata but not by nondet $\omega \mathrm{B}$ automata, and $\left\{a^{n_{1}} b a^{n_{2}} b a^{n_{3}} b \ldots\right.$ | $\left.\lim \inf n_{i}<\infty\right\}$ witnesses the opposite direction. The latter language is recognisable by the nondet $\omega \mathrm{B}$ automaton which guesses a bounded subsequence of $n_{1} n_{2} \ldots$ The non-recognisability by det max automata follows from [8, Section 6].

We prove the claim about $\overline{\left(a^{B} b\right)^{\omega}}$. First, the language $\left(a^{B} b\right)^{\omega}$ is recognisable by det $\omega \mathrm{B}$ automata and hence by det max automata. Since det max automata are closed under the complement, $\overline{\left(a^{B} b\right)^{\omega}}$ is also recognisable by det max automata. Now, by contradiction, assume that $\overline{\left(a^{B} b\right)^{\omega}}$ is recognisable by nondet $\omega \mathrm{B}$ automata. The result [6, Lemma 2.5] says: if an $\omega \mathrm{B}$ language over alphabet $\{a, b\}$ contains a word with infinitely many $b$ s then it contains a word from $\left(a^{B} b\right)^{\omega}$. The language $\overline{\left(a^{B} b\right)^{\omega}}$ contains the former (e.g. take any word from $\left.\left(a^{S} b\right)^{\omega}\right)$ but not the latter. Contradiction. Hence it is not an $\omega \mathrm{B}$ language.

### 5.3 Satisfiability of lasso-shaped sequences

An infinite sequence is lasso-shaped (or regular) if it is of the form $w=u v^{\omega}$. Lasso-shaped sequences are prevalent in automata theory and in the data setting in particular. For instance, [21] studies satisfiability of logic Constraint LTL in the data domain $(\mathbb{N}, \leq)$ and shows that considering lasso-shaped witnesses of satisfiability is sufficient. Another work [26] shows that if there is an $\omega$-regular over-approximation of satisfiable constraint sequences and which is exact on lasso-shaped sequences, then a synthesis problem is decidable in $(\mathbb{N}, \leq)$. In this paper, when proving the decidability of Church synthesis problem, we do not directly rely on lasso-shaped sequences, but we use a characterisation similar to the one proven in this section.

This section shows that considering lasso-shaped constraint sequences greatly simplifies the task of characterisation of satisfiability. We first show how lasso-shaped sequences simplify the condition B1 of characterisation Lemma 23, then describe the
chain characterisation under assumption of lasso-shaped sequences, and finally state the $\omega$-regular automaton characterisation.

Lemma 25. For every lasso-shaped consistent constraint sequence, it has ceiled oneway chains of unbounded depth iff it has ceiled one-way chains of infinite depth.

Proof. Direction $\Leftarrow$ is trivial, so consider direction $\Rightarrow$. The argument uses the standard pumping technique. Fix a lasso-shaped constraint sequence $C_{0} \ldots C_{k-1}\left(C_{k} \ldots C_{k+l}\right)^{\omega}$ having ceiled chains of unbounded depth. Since these chains have unbounded depth, they pass through $C_{k}$ more and more often. At moments when the current constraint is $C_{k}$, each such a chain is in one of the finitely-many registers. Hence there is a chain, say increasing, that on two separate occasions of reading the constraint $C_{k}$ goes through the same register $r$, and the chain suffix from the first pass through $r$ until the second pass has at least one $<$. Then we create an increasing chain of infinite depth by repeating this suffix forever.

The above lemma together with Lemma 12 yields the following result.
Lemma 26. A lasso-shaped consistent constraint sequence is 0 -satisfiable iff it is quasi-feasible, i.e.:

- it has no infinite-depth decreasing one-way chains,
- it has no ceiled infinite-depth increasing one-way chains,
- it has no decreasing one-way chains of depth $\geq 1$ from moment 0 , and
- it starts with $C_{0}$ s.t. $C_{0 \mid R}=\{r=s \mid r, s \in R\}$.

The conditions of this lemma can be checked by an $\omega$-regular automaton: Its construction is similar to the components $A_{c}$ and $A_{\neg \infty}$ from the proof of Theorem 24 and is omitted. Thus, we get the theorem below.

Theorem 27. For every $R$, there is a deterministic parity automaton that accepts a lasso-shaped constraint sequence iff it is 0 -satisfiable in $\mathbb{N}$; its number of states and priorities is exponential and polynomial in $|R|$, respectively.

### 5.4 Data-assignment function

In this section, we design a data-assignment function that maps a sequence of constraints to a sequence of register valuations satisfying it, while doing it on the fly, i.e. by reading the constraint sequence from left to right. It is significant that the entire constraint sequence is not known in advance. Such a function is used in Section 3 when proving Proposition 15, namely that Adam's winning strategy in the finitealphabet game transfers to the winning strategy in the Church synthesis game. There, Adam has to produce data values given only the prefix of a play.

In the next section, we state the lemma on existence of a data-assignment function, and then devote a significant amount of space to proving it.

### 5.4.1 Lemma 28 on existence of a data-assignment function

Intuitively, a data-assignment function produces register valuations while reading a constraint sequence from left to right. We are interested in functions that produce register valuations satisfying given constraint sequences. Since data-assignment functions cannot look into the future and do not know how many values will be inserted between any two registers, knowing a certain bound on such insertions is necessary. Moreover, to simplify the presentation, we restrict how many new data values can appear during the step. In our Church synthesis games, at most one new value provided by Adam can appear. We start by defining data-assignment functions, then describe the assumptions and state the lemma.

Let C denote the set of all constraints over registers $R$, and let $\mathrm{C}_{\mid R}$ denote the set of all constraints over atoms over $R$ only. A data-assignment function has the type $\left(\mathrm{C}_{\mid R} \cup \mathrm{C}^{+}\right) \rightarrow \mathbb{N}^{R}$. A data-assignment function $f$ maps a constraint sequence $C_{0} C_{1} \ldots$ into a sequence of valuations $f\left(C_{0 \mid R}\right) f\left(C_{0}\right) f\left(C_{0} C_{1}\right) \ldots$

We now describe the two assumptions used by our data-assignment function.
Intuitively, the first assumption states that only a bounded number of insertions between any two registers can happen, and this bound is known. To formalise the assumption, we define a special kind of chains, called right two-way chains. Informally, right chains are two-way chains that operate to the right of their starting point. Knowing a bound on the depths of right chains amounts to knowing how many values in the future can be inserted between the registers. Fix a constraint sequence. Given a moment $i$ and a register $x$, a (decreasing) right two-way chain starting in $(x, i)$ (r2w for short) is a two-way chain $(x, i) \triangleright_{1}\left(r_{1}, m_{1}\right) \triangleright_{2}\left(r_{2}, m_{2}\right) \ldots$ such that $m_{j} \geq i, \triangleright_{j} \in\{=,>\}$, for all $j$. As these chains are two-way, they can start and end in the same moment $i$. Notice that in Lemma 22 on characterisation of satisfiable constraint sequences we can replace two-way chains by r2w chains. Our data-assignment function will assume the knowledge of a bound on the r 2 w chains.

We now describe the second assumption about one-new-value appearance during a step. Its formalisation uses the notion of levels introduced in Section 5.2 on page 39 (see also Figure 9). We briefly recall those notions. Recall that a constraint describes a set of totally ordered equivalence classes of registers from $R \cup R^{\prime}$. The figure on
 the right describes a constraint that can be defined by the ordered equivalence classes $\left\{r_{4}, r_{4}^{\prime}\right\}<\left\{r_{2}^{\prime}\right\}<\left\{r_{3}, r_{3}^{\prime}\right\}<\left\{r_{1}, r_{2}, r_{1}^{\prime}\right\}$. It shows two columns of levels, start levels (in the left column) and end levels (in the right column), where a level describes a set of registers that are equivalent at this point of time. The assumption $\dagger$ says:

In every constraint of a given sequence, at most one new end level appear.
The constraint depicted in the above figure satisfies this assumption, the one in Figure 9 does not. This assumption helps to simplify the proofs, and is satisfied by the constraint sequences induced in our Church synthesis games.

One final notion before stating the lemma. A constraint sequence is 0 -consistent if it is consistent, starts in $C_{0}$ with $C_{0 \mid R}=\{r=s \mid r, s \in R\}$, and has no decreasing
chains of depth $\geq 1$ starting at moment 0 . Note that a 0 -consistent constraint sequence whose r 2 w chains are bounded is 0 -satisfiable (follows from Lemma 22).

Lemma 28 (data-assignment function). For every $\mathrm{B} \geq 0$, there exists a dataassignment function $f:\left(\mathrm{C}_{\mid R} \cup \mathrm{C}^{+}\right) \rightarrow \mathbb{N}^{R}$ such that for every finite or infinite 0 -consistent constraint sequence $C_{0} C_{1} C_{2} \ldots$ satisfying assumption $\dagger$ and whose r2w chains are depth-bounded by B , the register valuations $f\left(C_{0 \mid R}\right) f\left(C_{0}\right) f\left(C_{0} C_{1}\right) \ldots$ satisfy the constraint sequence.

Proof idea. We define a special kind of $x y^{(m)}$-chains that help to estimate how many insertions between the values of registers $x$ and $y$ at moment $m$ we can expect in the future. As it turns out, without knowing the future, the distance between $x$ and $y$ has to be exponential in the maximal depth of $x y^{(m)}$-chains. We describe a data-assignment function that maintains such exponential distances. The function is surprisingly simple: if the constraint inserts a register $x$ between two registers $r$ and $s$ with already assigned values $\ell_{r}$ and $\ell_{s}$, then set $\ell_{x}=\left\lfloor\frac{d_{r}+\ell_{s}}{2}\right\rfloor$; and if the constraint puts a register $x$ above all other registers, then set $\ell_{x}=\ell_{M}+2^{\mathrm{B}}$ where $\ell_{M}$ the largest value currently held in the registers and $B$ is the given bound on the depth of r 2 w chains.

The rest of the section is devoted to the proof of this lemma.

### 5.4.2 Proof of Lemma 28

$x y^{(m)}$-connecting chains and the exponential nature of register valuations
Fix an arbitrary 0-satisfiable constraint sequence $C_{0} C_{1} \ldots$ whose 22 w chains are depthbounded by B. Consider a moment $m$ and two registers $x$ and $y$ such that $(x>y) \in$ $C_{m}$.

We would like to construct witnessing valuations $\nu_{0} \nu_{1} \ldots$ using the current history only, e.g. a register valuation $\nu_{m}$ at moment $m$ given only the prefix $C_{0} \ldots C_{m-1}$. Note that the prefix $C_{0} \ldots C_{m-1}$ defines the ordered partition of registers at moment $m$ as well, since $C_{m-1}$ is defined over $R \cup R^{\prime}$. Let us see how much space we might need between $\nu_{m}(x)$ and $\nu_{m}(y)$, relying on the fact that the depths of r 2 w chains are bounded by B. Consider decreasing two-way chains that start at moment $i \leq m$, end in $(x, m)$, and which are contained within time moments $\{i, \ldots, m\}$ (shown in blue). Further, consider decreasing two-way chains starting in $(y, m)$, ending at moment $j \in\{i, \ldots, m\}$, and contained within time moments $\{j, \ldots, m\}$ (shown in pink). Among such chains, pick two chains of depths $\alpha$ and $\beta$, respectively, that maximise the sum $\alpha+\beta$. After seeing $C_{0} C_{1} \ldots C_{m-1}$, we do not know how the constraint sequence will evolve, but by boundedness of r 2 w chains, any r 2 w chain starting in
 $(x, m)$ and ending in $(y, m)$ (contained within time moments $\geq m$ ) will have a depth $d \leq \mathrm{B}-\alpha-\beta$ (otherwise, we could add prefix $\alpha$ and postfix $\beta$ to it and construct an r 2 w chain of depth larger than B ). We conclude that $\nu_{m}(x)-\nu_{m}(y) \geq$ $\mathrm{B}-\alpha-\beta$, since the number of values in between two registers should be greater or equal than the longest two-way chain connecting them. To simplify the upcoming
arguments, we introduce $x y^{(m)}$-connecting chains which consist of $\alpha$ and $\beta$ parts and directly connect $x$ to $y$.

An $x y^{(m)}$-connecting chain is any r 2 w chain of the form $(a, i) \triangleright \ldots(x, m)>(y, m) \triangleright$ $\ldots \triangleright(b, j)$ : it starts in $(a, i)$ and ends in $(b, j)$, where $i \leq j \leq m$ and $a, b \in R$, and it directly connects $x$ to $y$ at moment $m$. Note that it is located solely within moments $\{i, \ldots, m\}$. Continuing the previous example, the $x y^{(m)}$-connecting chain starts with $\alpha$, directly connects $(x, m)>(y, m)$, and ends with $\beta$; its depth is $\alpha+\beta+1$ (we have " +1 " no matter how many registers are between $x$ and $y$, since $x$ and $y$ are connected directly).

With this new notion, the requirement $\nu_{m}(x)-\nu_{m}(y) \geq \mathrm{B}-\alpha-\beta$ becomes $\nu_{m}(x)-\nu_{m}(y) \geq \mathrm{B}-d_{x y}+1$, where $d_{x y}$ is the largest depth of $x y^{(m)}$-connecting chains.

However, since we do not know how the constraint sequence evolves after $C_{0} \ldots C_{m-1}$, we might need even more space between the registers at moment $m$. Consider an example on the right, with $R=\left\{r_{0}, r_{1}, r_{2}\right\}$ and the bound $\mathrm{B}=3$ on the depth of r 2 w chains.


- Suppose at moment 1, after seeing the constraint $C_{0}$, which is $\left\{r_{1}^{\prime}, r_{2}^{\prime}\right\}>$ $\left\{r_{0}, r_{1}, r_{2}, r_{0}^{\prime}\right\}$, the valuation is $\nu_{1}=\left\{r_{0} \mapsto 0 ; r_{1}, r_{2} \mapsto 3\right\}$. It satisfies $\nu_{1}\left(r_{2}\right)-$ $\nu_{1}\left(r_{0}\right) \geq \mathrm{B}-d_{r_{2} r_{0}}+1$ (indeed, $\mathrm{B}=3$ and $d_{r_{2} r_{0}}=1$ at this moment); similarly for $\nu\left(r_{1}\right)-\nu\left(r_{0}\right)$.
- Let the constraint $C_{1}$ be $\left\{r_{1}, r_{2}, r_{2}^{\prime}\right\}>\left\{r_{1}^{\prime}\right\}>\left\{r_{0}, r_{0}^{\prime}\right\}$. What value $\nu_{2}\left(r_{1}\right)$ should register $r_{1}$ have at moment 2? Note that the assignment should work no matter what $C_{2}$ will be in the future. Since the constraint $C_{1}$ places $r_{1}$ between $r_{0}$ and $r_{2}$ at moment 2 , we can only assign $\nu_{2}\left(r_{1}\right)=2$ or $\nu_{2}\left(r_{1}\right)=1$. If we choose 2 , then the constraint $C_{2}$ having $\left\{r_{2}, r_{2}^{\prime}\right\}>\left\{r_{1}^{\prime}\right\}>\left\{r_{1}\right\}>\left\{r_{0}, r_{0}^{\prime}\right\}$ (the red dot in the figure) shows that there is not enough space between $r_{2}$ and $r_{1}$ at moment 2 $\left(\nu_{2}\left(r_{2}\right)=3\right.$ and $\left.\nu_{2}\left(r_{1}\right)=2\right)$. Similarly for $\nu_{2}\left(r_{1}\right)=1$ : the constraint $C_{2}$ having $\left\{r_{2}, r_{2}^{\prime}\right\}>\left\{r_{1}\right\}>\left\{r_{1}^{\prime}\right\}>\left\{r_{0}, r_{0}^{\prime}\right\}$ (the blue dot in the figure) eliminates any possibility for a correct assignment.

Thus, at moment 2 , the register $r_{1}$ should be equally distanced from $r_{0}$ and $r_{2}$, i.e. $\nu_{2}\left(r_{1}\right) \approx \frac{\nu_{2}\left(r_{0}\right)+\nu_{2}\left(r_{2}\right)}{2}$, since its evolution can go either way, towards $r_{2}$ or towards $r_{0}$. This hints at the exponential nature of distances between the registers. This is formalised in the next lemma showing that any data-assignment function that places two registers $x$ and $y$ at any moment $m$ closer than $2^{\mathrm{B}-d_{x y}}$ is bound to fall. Intuitively, $\mathrm{B}-d_{x y}$ describes how many more times an insertion between the values of registers $x$ and $y$ can happen in the future. Since each newly inserted value should be equidistant from the boundaries, we get the $2^{\mathrm{B}-d_{x y}}$ lower bound.

Lemma 29 (tightness). Fix в $\geq 3$, registers $R$ of $|R| \geq 3$, a 0 -consistent constraint sequence prefix $C_{0} \ldots C_{m-1}$ where $m \geq 1$ and whose r2w chains are depth-bounded by B, two registers $x, y \in R$ s.t. $\left(x^{\prime}>y^{\prime}\right) \in C_{m-1}$, and a data-assignment function $f:\left(\mathrm{C}_{\mid R} \cup \mathrm{C}^{+}\right) \rightarrow \mathbb{N}^{R}$. Let $\nu_{m}=f\left(C_{0} \ldots C_{m-1}\right)$ and $d_{x y}$ be the maximal depth of $x y^{(m)}$-connecting chains. If $\nu_{m}(x)-\nu_{m}(y)<2^{\mathrm{B}-d_{x y}}$, then there exists a continuation
$C_{m} C_{m+1} \ldots$ such that the whole sequence $C_{0} C_{1} \ldots$ is 0 -consistent and its r2w chains are depth-bounded by в (hence 0-satisfiable), yet $f$ cannot satisfy it.

Proof. We use the idea from the previous example. The constraints $C_{m} C_{m+1} \ldots$ are:

1. If at moment $m$ there are registers different from $x$ and $y$, we add the step that makes them equal to $x$ (or to $y$ ): this does not affect the depth of $x y$-connecting chains at moments $m$ and $m+1$; also, the maximal depths of r 2 w chains defined at moments $\{0, \ldots, m\}$ and $\{0, \ldots, m+1\}$ stay the same. Therefore, below we assume that at moment $m$ every register is equal to $x$ or to $y$.
2. If B $-d_{x y}=0$, we are done: $\nu_{m}(x)-\nu_{m}(y)<2^{\mathrm{B}-d_{x y}}$ gives $\nu_{m}(x) \leq \nu_{m}(y)$ but $C_{m-1}$ requires $\nu_{m}(x)>\nu_{m}(y)$. The future constraints then simply keep the registers constant. Otherwise, when $\mathrm{B}-d_{x y}>0$, we proceed as follows.
3. To ensure consistency of constraints, $C_{m}$ contains all atoms over $R$ that are implied by atoms over $R^{\prime}$ of $C_{m-1}$.
4. $C_{m}$ contains $x=x^{\prime}$ and $y=y^{\prime}$.
5. $C_{m}$ places a register $z$ between $x$ and $y: x^{\prime}>z^{\prime}>y^{\prime}$.

This gives $d_{x z}^{\prime}=d_{z y}^{\prime}=d_{x y}+1 \leq b$, where $d_{x y}$ is the largest depth of connecting chains for $x y^{(m)}, d_{x z}^{\prime}-$ for $x z^{(m+1)}$, and $d_{z y}^{\prime}-$ for $z y^{(m+1)}$. Since $\nu_{m+1}(x)-$ $\nu_{m+1}(y)<2^{\mathrm{B}-d_{x y}}$, either $\nu_{m+1}(x)-\nu_{m+1}(z)<2^{\mathrm{B}-d_{x z}^{\prime}} \quad$ or $\quad \nu_{m+1}(z)-\nu_{m+1}(y)<$ $2^{\mathrm{B}-d_{z y}^{\prime}}$; this is the key observation. If the first case holds, we have the original setting $\nu_{m+1}(x)-\nu_{m+1}(z)<2^{\mathrm{B}-d_{x z}^{\prime}}$ but at moment $m+1$ and with registers $x$ and $z$; for the second case - with registers $z$ and $y$. Hence we repeat the entire procedure, again and again, until reaching the depth B , which gives the sought conclusion in item (2).
Finally, it is easy to prove that the whole constraint sequence $C_{0} C_{1} \ldots$ is 0 -satisfiable, e.g. by showing that it satisfies the conditions of Lemma 12. Moreover, it is 0 consistent, and all r 2 w chains of $C_{0} C_{1} \ldots$ are depth-bounded by B because: (a) in the initial moment $m$, all r 2 w chains are depth-bounded by B ; and (b) the procedure deepens only $x y$-connecting chains and only until the depth B , whereas other r 2 w chains existing at moments $\{0, \ldots, m\}$ keep their depths unchanged (or at moments $\{0, \ldots, m+1\}$, if we executed item 1 ).

## Proof of Lemma 28 under additional assumption about 0

Tightness by Lemma 29 tells us that if a data-assignment function exists, it should separate the register values by at least $2^{\mathrm{B}-d_{x y}}$. Such separation is sufficient as we show below. We first describe a data-assignment function, then prove an invariant about it, and finally conclude with the proof of Lemma 28 . For simplicity, we assume that the constraints contain a register that never changes and always holds 0 . That is not true in general, so later we will lift this assumption.
Data-assignment function. The function $f:\left(\mathrm{C}_{\mid R} \cup \mathrm{C}^{+}\right) \rightarrow \mathbb{N}^{R}$ is constructed inductively on the length of $C_{0} \ldots C_{m-1}$ as follows.

Initially, $f\left(C_{0 \mid R}\right)=\nu_{0}$ where $\nu_{0}(r)=0$ for all $r \in R$ (since $C_{0}$ has $r=s$, $\forall r, s \in R)$. Suppose at moment $m$, the register valuation is $\nu_{m}=f\left(C_{0 \mid R} C_{0} \ldots C_{m-1}\right)$. Let $C_{m}$ be the next constraint, then $\nu_{m+1}=f\left(C_{0 \mid R} C_{0} \ldots C_{m}\right)$ is as follows:
Dı. If a register $x$ at moment $m+1$ lays above all registers at moment $m$, i.e. $\left(x^{\prime}>\right.$ $r) \in C_{m}$ for every register $r$, then set $\nu_{m+1}(x)=\nu_{m}(r)+2^{\mathrm{B}}$, where $r$ is one of the largest registers at moment $m$. In Church games this case happens when the test contains the atom $*>r$.
D2. If a register $x$ at moment $m+1$ lays between two adjacent registers $a>b$ at moment $m$, then $\nu_{m+1}(x)=\left\lfloor\frac{\nu_{m}(a)+\nu_{m}(b)}{2}\right\rfloor$. In Church games this happens when the test contains $a>*>b$.
D3. If a register $x$ at moment $m+1$ equals a register $r$ at previous moment $m$, so $\left(r=x^{\prime}\right) \in C_{m}$, then $\nu_{m+1}(x)=\nu_{m}(r)$. In Church games this case corresponds to a test containing the atom $*=r$ for some register $r$.

Note that the case when a register $x$ must lay below all registers never happens, since the special register $r_{0}$ always holds 0 and a given constraint sequence is 0 -consistent and hence never requires $r_{0}>r^{\prime}$ for some register $r$. This is where $r_{0}$ comes handy.
Invariant. The data-assignment function satisfies the following invariant:

$$
\forall m \in \mathbb{N} . \forall x, y \in R \text { s.t. }(x>y) \in C_{m}: \nu_{m}(x)-\nu_{m}(y) \geq 2^{\mathrm{B}-d_{x y}}
$$

where $d_{x y}$ is the largest depth of $x y^{(m)}$-connecting chains and B is the bound on the depth of r 2 w chains.

Proof of the invariant. The invariant holds initially since $\left(r_{1}=r_{2}\right) \in C_{0}$ for all $r_{1}, r_{2} \in R$. Assuming it holds at step $m$, we show that it holds at $m+1$. Fix two arbitrary registers $x, y \in R$ such that $\left(x^{\prime}>y^{\prime}\right) \in C_{m}$; we will prove that $\nu_{m+1}(x)-\nu_{m+1}(y) \geq 2^{\mathrm{B}-d_{x y}}$, where $d_{x y}$ is the largest depth of $x y^{(m+1)}$-connecting chains. There are four cases depending on whether the levels of $x$ and $y$ at moment $m+1$ are present at moment $m$ or not, illustrated in Figure 10.
Case 1: both present. The levels of $x$ and $y$ at $m+1$ also exist at moment $m$. Let $a, b$ be registers s.t. $(a>b) \in C_{m}$ laying at moment $m$ on the same levels as $x$ and $y$ at moment $m+1$. By data-assignment function (item D3), $\nu_{m}(a)=\nu_{m+1}(x)$ and $\nu_{m}(b)=\nu_{m+1}(y)$. Note that the number of levels between $x-y$ and between $a-b$ may differ. Consider the depths of connecting chains for $a b^{(m)}$ and $x y^{(m+1)}$ : Since every $a b^{(m)}$-connecting chain can be extended to $x y^{(m+1)}$-connecting chain of the same depth as shown on the figure, we have ${ }^{6} d_{a b} \leq d_{x y}$, and hence $2^{\mathrm{B}-d_{a b}} \geq 2^{\mathrm{B}-d_{x y}}$. Using the inductive hypothesis, we conclude $\nu_{m+1}(x)-\nu_{m+1}(y)=\nu_{m}(a)-\nu_{m}(b) \geq$ $2^{\mathrm{B}-d_{a b}} \geq 2^{\mathrm{B}-d_{x y}}$.
Case 2: $x$ is new top. The register $x$ lies on the top level of both moments $m$ and $m+1$, and $y$ lies on a level that was also present at moment $m$. This corresponds to item Dı. Let $\left(b=y^{\prime}\right) \in C_{m}$ and $a$ lies on the largest level at moment $m$ ( $a$ and $b$ may coincide). Thus, $\nu_{m+1}(x)=\nu_{m}(a)+2^{\mathrm{B}}$. The invariant holds for $x, y$ because $\nu_{m+1}(x)=\nu_{m}(a)+2^{\mathrm{B}}$ and $\nu_{m}(a) \geq \nu_{m}(b)=\nu_{m+1}(y)$.

[^6]

Figure 10: Proving the invariant

Case 3: $x$ is middle new, $y$ was present. The register $x$ at moment $m+1$ lies on a new level that is between the levels of $a$ and $b$ at moment $m$, so $\nu_{m+1}(x)=\left\lfloor\frac{\nu_{m}(a)+\nu_{m}(b)}{2}\right\rfloor$ by item D 2 of data-assignment function. The register $y$ at moment $m+1$ lies on a level that was also present at moment $m$, witnessed by register $c$. Formally, $C_{m}$ contains $a>x^{\prime}>b$ for $a$ and $b$ adjacent at moment $m, c=y^{\prime}$, and $x^{\prime}>y^{\prime}$. Note that $c$ and $b$ may coincide. Then, $\nu_{m+1}(x)-\nu_{m+1}(y)=\left\lfloor\frac{\nu_{m}(a)+\nu_{m}(b)}{2}\right\rfloor-\nu_{m}(c)=\left\lfloor\frac{\nu_{m}(a)-\nu_{m}(c)}{2}+\right.$ $\left.\frac{\nu_{m}(b)-\nu_{m}(c)}{2}\right\rfloor \geq\left\lfloor\frac{\nu_{m}(a)-\nu_{m}(c)}{2}\right\rfloor+\left\lfloor\frac{\nu_{m}(b)-\nu_{m}(c)}{2}\right\rfloor \geq\left\lfloor 2^{\mathrm{B}-d_{a c}-1}\right\rfloor+\left\lfloor 2^{\mathrm{B}-d_{b c}-1}\right\rfloor \geq 2^{\mathrm{B}-d_{a c}-1}+$ $\left\lfloor 2^{\mathrm{B}-d_{b c}-1}\right\rfloor$; the latter holds because $d_{a c}<b$ while $d_{b c} \leq b$. We need to prove that the last sum is greater or equal to $2^{\mathrm{B}-d_{x y}}$. Figure 10 (case 3 ) shows how the green $x y^{(m+1)}$-connecting chain can be constructed from the pink $a c^{(m)}$-connecting chain, hence $d_{x y} \geq d_{a c}+1$, so we get $2^{\mathrm{B}-d_{a c}-1} \geq 2^{\mathrm{B}-d_{x y}}$. Hence, $\nu_{m+1}(x)-\nu_{m+1}(y) \geq$ $2^{\mathrm{B}-d_{a c}-1}+\left\lfloor 2^{\mathrm{B}-d_{b c}-1}\right\rfloor \geq 2^{\mathrm{B}-d_{x y}}$.
Case 4: $x$ was present, $y$ is middle new. The case is similar to the previous one, but we prove it for completeness. The constraint $C_{m}$ contains $a=x^{\prime}, x^{\prime}>y^{\prime}$, $b>y^{\prime}>c$, where $b$ and $c$ are adjacent ( $a$ and $b$ might be the same). Then, $\nu_{m+1}(x)-$ $\nu_{m+1}(y)=\nu_{m}(a)-\left\lfloor\frac{\nu_{m}(b)+\nu_{m}(c)}{2}\right\rfloor \geq\left\lfloor\frac{\nu_{m}(a)-\nu_{m}(b)}{2}+\frac{\nu_{m}(a)-\nu_{m}(c)}{2}\right\rfloor \geq\left\lfloor\frac{\nu_{m}(a)-\nu_{m}(b)}{2}\right\rfloor+$ $\left\lfloor\frac{\nu_{m}(a)-\nu_{m}(c)}{2}\right\rfloor \geq\left\lfloor 2^{\mathrm{B}-d_{a b}-1}\right\rfloor+\left\lfloor 2^{\mathrm{B}-d_{a c}-1}\right\rfloor \geq\left\lfloor 2^{\mathrm{B}-d_{a b}-1}\right\rfloor+2^{\mathrm{B}-d_{a c}-1}$, and since $d_{a c}+1 \leq$ $d_{x y}$, we get $\nu_{m+1}(x)-\nu_{m+1}(y) \geq\left\lfloor 2^{\mathrm{B}-d_{a b}-1}\right\rfloor+2^{\mathrm{B}-d_{a c}-1} \geq 2^{\mathrm{B}-d_{x y}}$.

Proof of Lemma 28. It is sufficient to show that for every atom ( $r \bowtie s$ ) or ( $r \bowtie s^{\prime}$ ) of $C_{m}$, where $r, s \in R$ and $\bowtie \in\{<,>,=\}$, the expressions $\nu_{m}(r) \bowtie \nu_{m}(s)$ or $\nu_{m}(r) \bowtie$ $\nu_{m+1}(s)$ hold, respectively. Depending on $r \bowtie s$, there are the following cases.

- If $C_{m}$ contains $(r=s)$ or $\left(r=s^{\prime}\right)$ for $r, s \in R$, then item D3 implies resp. $\nu_{m}(r)=\nu_{m}(s)$ or $\nu_{m}(r)=\nu_{m+1}(s)$.
- If $(r>s) \in C_{m}$, then $\nu_{m}(r)>\nu_{m}(s)$ by the invariant.
- Let $\left(r>s^{\prime}\right) \in C_{m}$ and the level of $s$ at moment $m+1$ be present at moment $m$, i.e. there is a register $t$ such that $\left(t=s^{\prime}\right) \in C_{m}$. Since $\nu_{m}(t)=\nu_{m+1}(s)$ by item D3 and since $\nu_{m}(r)>\nu_{m}(t)$ by $\left(r>t=s^{\prime}\right) \in C_{m}$, we get $\nu_{m}(r)>\nu_{m+1}(s)$. Similarly for the case $\left(r<s^{\prime}\right) \in C_{m}$ where $s$ lies on a level also present at moment $m$.
- Let $\left(r<s^{\prime}\right) \in C_{m}$ and $s$ lies on the highest level among all levels at moments $m$ and $m+1$. Then $\nu_{m}(r)<\nu_{m+1}(s)$ because $\nu_{m+1}(s) \geq \nu_{m}(r)+2^{\text {B }}$ by item Dı.
- Finally, there are two cases left: $\left(r>s^{\prime}\right) \in C_{m}$ or $\left(r<s^{\prime}\right) \in C_{m}$, where $s$ lies on a newly created level at moment $m+1$, and there are higher levels at moment $m$. This
corresponds to item D2. Let $(a>b) \in C_{m}$ be two adjacent registers at moment $m$ between which the register $s$ is inserted at moment $m+1$, so $\left(a>s^{\prime}>b\right) \in C_{m}$. Let $d_{a b}$ be the maximal depth of $a b^{(m)}$-connecting chains; fix one such chain. We change it by going through $s$ at moment $m+1$, i.e. substitute the part $(a, m)>(b, m)$ by $(a, m)>(s, m+1)>(b, m)$ : the depth of the resulting chain is $d_{a b}+1$ and it is $\leq \mathrm{B}$ by boundedness of r 2 w chains. Hence $d_{a b} \leq \mathrm{B}-1$, so $\nu_{m}(a)-\nu_{m}(b) \geq 2$, implying $\nu_{m}(a)>\left\lfloor\frac{\nu_{m}(a)+\nu_{m}(b)}{2}\right\rfloor>\nu_{m}(b)$. When $\left(r>s^{\prime}\right) \in C_{m}$ we get $\nu_{m+1}(r) \geq \nu_{m}(a)$, and when $\left(r<s^{\prime}\right) \in C_{m}$ we get $\nu_{m+1}(r) \leq \nu_{m}(b)$, therefore we are done.

Finally, the function always assigns nonnegative numbers, from $\mathbb{N}$, so we are done.

## Lifting the assumption about 0

We now lift the assumption about a register always holding 0 . This assumption was used in the definition of the data-assignment function (items $D_{1}, D_{2}, D_{3}$ ). The idea is to convert a given constraint sequence over registers $R$ into a sequence over registers $R \uplus\left\{r_{0}\right\}$ while preserving satisfiability.

Conversion function. Given a 0 -consistent constraint sequence $C_{0} C_{1} \ldots$ over $R$ without a special register holding 0 , we will construct, on-the-fly, a 0 -consistent sequence $\tilde{C}_{0} \tilde{C}_{1} \ldots$ over $R \uplus\left\{r_{0}\right\}$ that has such a register. Intuitively, we will add atoms $r=r_{0}$ only if they follow from what is already known otherwise we add atoms $r>r_{0}$.

Initially, in addition to the atoms of $C_{0}$, we require $r=r_{0}$ for every $r \in R$ (recall that the original $C_{0}$ contains $r_{1}=r_{2}$ for all $r_{1}, r_{2} \in R$ ). This gives an incomplete constraint $\tilde{C}_{0}$ over $R_{0} \cup R_{0}^{\prime}$ : it does not yet have atoms of the form $r \bowtie r_{0}^{\prime}, r_{0} \bowtie r^{\prime}$, $r_{0}^{\prime} \bowtie r^{\prime}$, where $r \in R_{0}$.

At moment $m \geq 0$, given a constraint $\tilde{C}_{m \mid R_{0}}$ over $R_{0}$ (without primed registers $\left.R_{0}^{\prime}\right)$ and a constraint $C_{m}$ over $R \cup R^{\prime}$ (without register $r_{0}$ ), we construct $\tilde{C}_{m}$ over $R_{0} \cup R_{0}^{\prime}$ as follows:

- $\tilde{C}_{m}$ contains all atoms of $C_{m}$.
- $\left(r_{0}=r_{0}^{\prime}\right) \in \tilde{C}_{m}$.
- For every $r \in R$ : if $r^{\prime}=r_{0}$ is implied by the current atoms of $\tilde{C}_{m}$, then we add it, otherwise we add $r^{\prime}>r_{0}$.
Notice that the atom $r^{\prime}<r_{0}$ is never implied by $\tilde{C}_{m}$, as we show now. Suppose the contrary. Then, since $C_{m}$ does not talk about $r_{0}$ nor $r_{0}^{\prime}$, there should be $s \in R$ such that $\left(s=r_{0}\right) \in \tilde{C}_{m \mid R_{0}}$ and $\left(r^{\prime}<s\right) \in C_{m}$. By construction, if this is the case, then there is a one-way chain $\left(r_{1}, 0\right)=\left(r_{2}, 1\right)=\ldots=(s, m)$ of zero depth. As a consequence, we can construct the one-way decreasing chain $\left(r_{1}, 0\right)=\left(r_{2}, 1\right)=$ $\ldots=(s, m)>(r, m+1)$ of depth 1 , which implies that $C_{0} C_{1} \ldots$ is not 0-consistent. We reached a contradiction, so $\left(r^{\prime}<r_{0}\right) \in \tilde{C}_{m}$ is not possible.
- Finally, to make $\tilde{C}_{m}$ maximal, we add all atoms implied by $\tilde{C}_{m}$ but not present there.

Using this construction, we can easily define conv : $C^{+} \rightarrow \tilde{C}$ and map a given $0-$ consistent constraint sequence $C_{0} C_{1} \ldots$ to $\tilde{C}_{0} \tilde{C}_{1} \ldots$ with a dedicated register holding 0 . Notice that the constructed sequence is also 0 -consistent, because we never add
inconsistent atoms and never add an atom $r^{\prime}<r_{0}$ (see the third item). Finally, in the constructed sequence the depths of r 2 w chains can increase by at most 1 , due to the register $r_{0}$ : it can increase the depth of a finite chain by one, unless the chain is already ending in a register holding 0 . Hence we get the following lemma.

Lemma 30. For every 0 -consistent constraint sequence $C_{0} C_{1} \ldots$, the sequence $\tilde{C}_{0} \tilde{C}_{1} \ldots$ constructed with c0nv is also 0-consistent. Moreover, the maximal depth of r2w chains cannot increase by more than 1 .

Final proof of Lemma 28. We lift the assumption about constraint sequences having a special register always holding zero. Using c0nv, we automatically translate a given 0 -consistent constraint sequence prefix $C_{0} \ldots C_{m}$ over $R$ into $\tilde{C}_{0} \ldots \tilde{C}_{m}$ over $R \uplus\left\{r_{0}\right\}$ that contains a register $r_{0}$ always holding 0 . Now we can apply the dataassignment function as described before. By definition of $c 0 n v$, the original constraint $C_{i} \subset \tilde{C}_{i}$ for every $i \geq 0$, so the resulting valuation satisfies the original constraints as well. This concludes the proof of Lemma 28.

## 6 Conclusion

Our main result states that one-sided Church games for specifications given as deterministic register automata over $(\mathbb{N}, \leq)$ are decidable, in ExpTimE. Moreover, we show that those games are determined, and that strategies implemented by transducers with registers suffice to win.

The decidability result involves a characterisation of satisfiable infinite constraint sequences over ( $\mathbb{N}, \leq$ ): they must not have decreasing two-way chains of infinite depth, nor ceiled (bounded from the above) chains of unbounded depth. A similar characterisation can be established for $(\mathbb{Z}, \leq)$. For instance, it should require that the two-way chains which are bounded from both above and below have bounded depth. Then, the decidability of one-sided Church synthesis for $(\mathbb{Z}, \leq)$ can be established in a similar way to $(\mathbb{N}, \leq)$. The decidability for $(\mathbb{Z}, \leq)$ can also be proven by reducing to the problem for ( $\mathbb{N}, \leq$ ) as follows. From a specification $S$, given as a set of words $d_{1} \sigma_{1} \ell_{2} \sigma_{2} \ldots$ alternating between a value $d_{i} \in \mathbb{Z}$ and a letter $\sigma_{i}$ from a finite alphabet $\Sigma$, we construct a specification $S^{\prime}$ of words of the form $\max \left(0, d_{1}\right) \# \max \left(0,-d_{1}\right) \sigma_{1} \max \left(0, d_{2}\right) \# \max \left(0,-d_{2}\right) \sigma_{2} \cdots \in(\mathbb{N}(\Sigma \cup\{\#\}))^{\omega}$, where \# acts as a waiting symbol. Non-zero values given by Adam at positions $4 n+1$ correspond to positive values, and non-zero values at positions $4 n+3$ correspond to negative values. Thus, if $S$ is given as a deterministic register automaton, one can construct a deterministic register automaton that recognises $S^{\prime}$, which preserves the existence of solutions to synthesis. An interesting future direction is to establish a general reduction between data domains such that decidability results for one-sided Church synthesis transfer from one domain to the other. A candidate notion for such a reduction was defined in the context of register-bounded transducer synthesis [26].

Another important future direction is to consider logical formalisms instead of automata to describe specifications in a more declarative and high-level manner. Data word first-order logics $[7,46]$ have been studied with respect to the satisfiability problem but when used as specification languages for synthesis, only few results are
known. The first steps in this direction were done in [30,4] for Constraint LTL on $(\mathbb{Z}, \leq)$; see also [22] for an overview of nonemptiness of constraint tree automata; and see [3] for a slightly different context of parameterised synthesis.

## References

[1] Parosh Aziz Abdulla, Mohamed Faouzi Atig, Piotr Hofman, Richard Mayr, K. Narayan Kumar, and Patrick Totzke. Infinite-state energy games. In Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), CSL-LICS '14, Vienna, Austria, July 14-18, 2014, pages 7:1-7:10, 2014.
[2] Parosh Aziz Abdulla, Ahmed Bouajjani, and Julien d'Orso. Deciding monotonic games. In International Workshop on Computer Science Logic, pages 1-14. Springer, 2003.
[3] Béatrice Bérard, Benedikt Bollig, Mathieu Lehaut, and Nathalie Sznajder. Parameterized synthesis for fragments of first-order logic over data words. In FOSSACS, volume 12077 of Lecture Notes in Computer Science, pages 97-118. Springer, 2020.
[4] Ashwin Bhaskar and M Praveen. Realizability problem for constraint ltl. arXiv preprint arXiv:2207.06708, 2022.
[5] Roderick Bloem, Krishnendu Chatterjee, and Barbara Jobstmann. Graph games and reactive synthesis. In Edmund M. Clarke, Thomas A. Henzinger, Helmut Veith, and Roderick Bloem, editors, Handbook of Model Checking, pages 921-962. Springer, 2018.
[6] M. Bojańczyk and T. Colcombet. Bounds in $\omega$-regularity. In Proc. 21st IEEE Symp. on Logic in Computer Science, pages 285-296, 2006.
[7] M. Bojanczyk, A. Muscholl, T. Schwentick, L. Segoufin, and C. David. Twovariable logic on words with data. In Proc. 21st IEEE Symp. on Logic in Computer Science, pages 7-16, 2006.
[8] Mikołaj Bojańczyk. Weak MSO with the unbounding quantifier. Theory of Computing Systems, 48(3):554-576, 2011.
[9] Mikołaj Bojańczyk. Weak MSO+U with path quantifiers over infinite trees. In Automata, Languages, and Programming - 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part II, pages 38-49, 2014.
[10] A. Bouajjani, P. Habermehl, Y. Jurski, and M. Sighireanu. Rewriting systems with data. In $F C T$, pages 1-22, 2007.
[11] A. Bouajjani, P. Habermehl, and R R. Mayr. Automatic verification of recursive procedures with one integer parameter. Theoretical Computer Science, 295:85106, 2003.
[12] A.-J. Bouquet, O. Serre, and I. Walukiewicz. Pushdown games with unboundedness and regular conditions. In Proc. 23rd Conf. on Foundations of Software Technology and Theoretical Computer Science, volume 2914 of Lecture Notes in Computer Science, pages 88-99. Springer, 2003.
[13] Véronique Bruyère. Synthesis of equilibria in infinite-duration games on graphs. ACM SIGLOG News, 8(2):4-29, 2021.
[14] J.R. Büchi and L.H. Landweber. Solving sequential conditions by finite-state strategies. Trans. AMS, 138:295-311, 1969.
[15] T. Cachat. Two-way tree automata solving pushdown games. In E. Grädel, W. Thomas, and T. Wilke, editors, Automata Logics, and Infinite Games, volume 2500 of Lecture Notes in Computer Science, chapter 17, pages 303-317. Springer, 2002.
[16] C.S. Calude, S. Jain, B. Khoussainov, W. Li, and F. Stephan. Deciding parity games in quasipolynomial time. In Proc. 49th ACM Symp. on Theory of Computing, pages 252-263, 2017.
[17] Claudia Carapelle, Alexander Kartzow, and Markus Lohrey. Satisfiability of ctl* with constraints. In Pedro R. D'Argenio and Hernán Melgratti, editors, CONCUR 2013 - Concurrency Theory, pages 455-469, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg.
[18] S. Ceri, P. Fraternali, A. Bongio, M. Brambilla, S. Comai, and M. Matera. Designing Data-Intensive Web Applications. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 2002.
[19] G. Delzanno, A. Sangnier, and R. Traverso. Parameterized verification of broadcast networks of register automata. In P. A. Abdulla and I. Potapov, editors, Reachability Problems, pages 109-121, Berlin, Heidelberg, 2013. Springer.
[20] S. Demri and R. Lazic. LTL with the freeze quantifier and register automata. ACM Trans. Comput. Log., 10(3):16:1-16:30, 2009.
[21] Stéphane Demri and Deepak D'Souza. An automata-theoretic approach to constraint LTL. Information and Computation, 205(3):380-415, 2007.
[22] Stephane Demri and Karin Quaas. Constraint automata on infinite data trees: From $\mathrm{ctl}(\mathrm{z}) / \mathrm{ctl}^{*}(\mathrm{z})$ to decision procedures. arXiv preprint arXiv:2302.05327, 2023.
[23] R. Ehlers, S. Seshia, and H. Kress-Gazit. Synthesis with identifiers. In Proc. 15th Int. Conf. on Verification, Model Checking, and Abstract Interpretation, volume 8318 of Lecture Notes in Computer Science, pages 415-433. Springer, 2014.
[24] Léo Exibard. Automatic Synthesis of Systems with Data. PhD Thesis, AixMarseille Université (AMU); Université libre de Bruxelles (ULB), September 2021.
[25] Léo Exibard, Emmanuel Filiot, and Ayrat Khalimov. Church synthesis on register automata over linearly ordered data domains. In Markus Bläser and Benjamin Monmege, editors, 38th International Symposium on Theoretical Aspects of Computer Science, STACS 2021, March 16-19, 2021, Saarbrücken, Germany (Virtual Conference), volume 187 of LIPIcs, pages 28:1-28:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.
[26] Léo Exibard, Emmanuel Filiot, and Ayrat Khalimov. A generic solution to register-bounded synthesis with an application to discrete orders. In Mikolaj Bojanczyk, Emanuela Merelli, and David P. Woodruff, editors, 49th International Colloquium on Automata, Languages, and Programming, ICALP 2022, July 4-8, 2022, Paris, France, volume 229 of LIPIcs, pages 122:1-122:19. Schloss

Dagstuhl - Leibniz-Zentrum für Informatik, 2022.
[27] Léo Exibard, Emmanuel Filiot, and Pierre-Alain Reynier. Synthesis of data word transducers. Log. Methods Comput. Sci., 17(1), 2021.
[28] Rachel Faran and Orna Kupferman. On synthesis of specifications with arithmetic. In Alexander Chatzigeorgiou, Riccardo Dondi, Herodotos Herodotou, Christos Kapoutsis, Yannis Manolopoulos, George A. Papadopoulos, and Florian Sikora, editors, SOFSEM 2020: Theory and Practice of Computer Science, pages 161-173, Cham, 2020. Springer International Publishing.
[29] Azadeh Farzan and Zachary Kincaid. Strategy synthesis for linear arithmetic games. Proceedings of the ACM on Programming Languages, 2(POPL):1-30, 2017.
[30] Diego Figueira, Anirban Majumdar, and M. Praveen. Playing with repetitions in data words using energy games. Log. Methods Comput. Sci., 16(3), 2020.
[31] B. Finkbeiner, F. Klein, R. Piskac, and M. Santolucito. Temporal stream logic: Synthesis beyond the bools. In Proc. 31st Int. Conf. on Computer Aided Verification, 2019.
[32] Stefan Göller, Richard Mayr, and Anthony Widjaja To. On the computational complexity of verifying one-counter processes. In Proceedings of the 24th Annual IEEE Symposium on Logic in Computer Science, LICS 2009, 11-14 August 2009, Los Angeles, CA, USA, pages 235-244, 2009.
[33] E. Grädel, W. Thomas, and T. Wilke. Automata, Logics, and Infinite Games: A Guide to Current Research, volume 2500 of Lecture Notes in Computer Science. Springer, 2002.
[34] Y. Gurevich and L. Harrington. Trees, automata, and games. In Proc. 14 th ACM Symp. on Theory of Computing, pages 60-65. ACM Press, 1982.
[35] R. Hojati, D.L. Dill, and R.K. Brayton. Verifying linear temporal properties of data insensitive controllers using finite instantiations. In Hardware Description Languages and their Applications, pages 60-73. Springer, 1997.
[36] M. Kaminski and N. Francez. Finite-memory automata. Theoretical Computer Science, 134(2):329-363, 1994.
[37] A. Khalimov, B. Maderbacher, and R. Bloem. Bounded synthesis of register transducers. In 16th Int. Symp. on Automated Technology for Verification and Analysis, volume 11138 of Lecture Notes in Computer Science, pages 494-510. Springer, 2018.
[38] Ayrat Khalimov and Orna Kupferman. Register-bounded synthesis. In Wan Fokkink and Rob van Glabbeek, editors, 30th International Conference on Concurrency Theory, CONCUR 2019, August 27-30, 2019, Amsterdam, the Netherlands, volume 140 of LIPIcs, pages 25:1-25:16. Schloss Dagstuhl -Leibniz-Zentrum für Informatik, 2019.
[39] Bartek Klin and Mateusz Lełyk. Scalar and Vectorial mu-calculus with Atoms. Logical Methods in Computer Science, Volume 15, Issue 4, Oct 2019.
[40] Paul Krogmeier, Umang Mathur, Adithya Murali, P. Madhusudan, and Mahesh Viswanathan. Decidable synthesis of programs with uninterpreted functions. In Shuvendu K. Lahiri and Chao Wang, editors, Computer Aided Verification, pages 634-657, Cham, 2020. Springer International Publishing.
[41] R. Lazić and D. Nowak. A unifying approach to data-independence. In Proc. 11th Int. Conf. on Concurrency Theory, pages 581-596. Springer Berlin Heidelberg, 2000.
[42] M.L. Minsky. Computation: Finite and Infinite Machines. Prentice Hall, 1 edition, 1967.
[43] A. Pnueli and R. Rosner. On the synthesis of a reactive module. In Proc. 16th ACM Symp. on Principles of Programming Languages, pages 179-190, 1989.
[44] M.O. Rabin. Automata on infinite objects and Church's problem. Amer. Mathematical Society, 1972.
[45] Frank Plumpton Ramsey. On a problem of formal logic. Proceedings of the London Mathematical Society, 30(1):264-286, 1930.
[46] Thomas Schwentick and Thomas Zeume. Two-variable logic with two order relations. Log. Methods Comput. Sci., 8(1), 2012.
[47] Luc Segoufin and Szymon Torunczyk. Automata-based verification over linearly ordered data domains. In 28th International Symposium on Theoretical Aspects of Computer Science (STACS 2011). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2011.
[48] Olivier Serre. Parity games played on transition graphs of one-counter processes. In Foundations of Software Science and Computation Structures, 9th International Conference, FOSSACS 2006, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2006, Vienna, Austria, March 25-31, 2006, Proceedings, pages 337-351, 2006.
[49] Syntcomp@CAV. The reactive synthesis competition. http://www.syntcomp.org, 2014.
[50] Wolfgang Thomas. Facets of synthesis: Revisiting church's problem. In Luca de Alfaro, editor, Foundations of Software Science and Computational Structures, 12th International Conference, FOSSACS 2009, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2009, York, UK, March 22-29, 2009. Proceedings, volume 5504 of Lecture Notes in Computer Science, pages 1-14. Springer, 2009.
[51] V. Vianu. Automatic verification of database-driven systems: a new frontier. In ICDT' '09, pages 1-13, 2009.
[52] I. Walukiewicz. Model checking CTL properties of pushdown systems. In Proc. 20th Conf. on Foundations of Software Technology and Theoretical Computer Science, volume 1974 of Lecture Notes in Computer Science, pages 127-138. Springer, 2000.
[53] P. Wolper. Expressing interesting properties of programs in propositional temporal logic. In Proc. 13th ACM Symp. on Principles of Programming Languages, pages 184-192, 1986.


[^0]:    *This article is an extended version of [25], which features full proofs and incorporates elements of [24, Chapter 7].

[^1]:    ${ }^{1}$ Lasso-shaped words are also called regular words or ultimately periodic words in the literature.

[^2]:    ${ }^{2}$ We only construct the given play, since the rest of the strategy does not matter.

[^3]:    ${ }^{3}$ What we really need is a winning Eve strategy of the form $\lambda_{\exists}^{D}: \mathscr{D}^{+} \rightarrow \Sigma$. The strategy $\lambda_{\exists}: \mathrm{Tst}^{+} \rightarrow \Sigma$ that we construct encodes $\lambda_{\exists}^{D}$ as follows: it has the same set $R$ of registers as the automaton $G_{S}$, and performs the same assignment actions as the automaton. Then, on seeing a new data value, the strategy compares it with the register values, which induces a test, and passes this test to $\lambda_{\exists}$.

[^4]:    ${ }^{4}$ Recall that over $(\mathbb{N}, \leq), 0$ denotes its minimal element. Over $(\mathbb{Q}, \leq)$, its choice is irrelevant.

[^5]:    ${ }^{5}$ For a formal statement, see [47, Theorem 4.3] saying that the class of languages of finite-alphabet projections of "constraint automata" and the class of $\omega \mathrm{B}$-languages coincide.

[^6]:    ${ }^{6}$ A stronger result holds, namely $d_{a b}=d_{x y}$, but it is not needed here.

