Church Synthesis on Register Automata over Linearly Ordered Data Domains*

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Abstract

In a Church synthesis game, two players, Adam and Eve, alternately pick some element in a *finite* alphabet, for an infinite number of rounds. The game is won by Eve if the ω -word formed by this infinite interaction belongs to a given language S, called the specification. It is well-known that for ω regular specifications, it is decidable whether Eve has a strategy to enforce the specification no matter what Adam does. We study the extension of Church synthesis games to the linearly ordered data domains $(\mathbb{Q}, <)$ and $(\mathbb{N}, <)$. In this setting, the infinite interaction between Adam and Eve results in an ω -data word, i.e., an infinite sequence of elements in the domain. We study this problem when specifications are given as register automata. Those automata consist in finite automata equipped with a finite set of registers in which they can store data values, that they can then compare with incoming data values with respect to the linear order. Church games over $(\mathbb{N}, <)$ are however undecidable, even for deterministic register automata. Thus, we introduce one-sided Church games, where Eve instead operates over a finite alphabet, while Adam still manipulates data. We show that they are determined, and that deciding the existence of a winning strategy is in ExpTime, both for \mathbb{Q} and \mathbb{N} . This follows from a study of constraint sequences, which abstract the behaviour of register automata, and allow us to reduce Church games to ω -regular games. We present an application of one-sided Church games to a transducer synthesis problem. In this application, a transducer models a reactive system (Eve) which outputs data stored in its registers, depending on its interaction with an environment (Adam) which inputs data to the system.

Keywords: Synthesis, Church Game, Register Automata, Register Transducers, Ordered Data Words

^{*}This article is an extended version of [25], which features full proofs and incorporates elements of [24, Chapter 7].

2012 ACM Subject Classification:

Theory of computation \rightarrow Logic and verification

Theory of computation \rightarrow Automata over infinite objects

Theory of computation \rightarrow Transducers

1 Introduction

Church synthesis. Reactive synthesis is the problem of automatically constructing a reactive system from a specification of correct executions, i.e. a non-terminating system which interacts with an environment, and whose executions all comply with the specification, no matter how the environment behaves. The earliest formulation of synthesis dates back to Church, who proposed to formalize it as a game problem: two players, Adam in the role of the environment and Eve in the role of the system, alternately pick the elements from two finite alphabets I and O respectively. Adam starts with $i_0 \in I$, Eve responds with $o_0 \in O$, ad infinitum. Their interaction results in the ω -word $w = i_0 o_0 i_1 o_1 \dots \in (I \cdot O)^{\omega}$. The winner is decided by a winning condition, represented as a language $S \subseteq (I \cdot O)^{\omega}$ called *specification*: if $w \in S$, the play is won by Eve, otherwise by Adam. Eve wins the game if she has a strategy $\lambda_{\exists}: I^+ \to O$ to pick elements in O, depending on what has been played so far, so that no matter the input sequence $i_0i_1...$ chosen by Adam, the resulting ω -word $i_0\lambda(i_0)i_1\lambda(i_0i_1)...$ belongs to S. Similarly, Adam wins the game if he has a strategy $\lambda_{\forall}: O^* \to I$ to win against any strategy Eve uses. In the original Church problem, specifications are ω -regular languages, i.e. languages definable in monadic second-order logic with one successor or equivalently, deterministic parity automata. The seminal papers [14, 44] have shown that Church games (for ω -regular specification) are determined: either Eve wins or otherwise Adam wins. Moreover, given a Church game, the winner of the game is computable. Finally, justifying the use of Church games as a formulation of reactive synthesis, finite-memory strategies are sufficient to win (both for Eve and Adam). This implies that if Eve wins a Church game, one can effectively construct a finite-state machine (e.g. a Mealy machine) implementing a winning strategy.

Church synthesis and games on graphs have been extensively studied for specifications given in linear-time temporal logic (LTL) [43] – recently supported by a tool competition [49] –, as well as in many other settings, for example, quantitative, distributed, non-competitive (see [5, 13] and the references therein). Yet, those works focus on control, sometimes with complex interactions between the synthesized systems, rather than on data. This is reflected already in the original formulation by Church: Adam and Eve interact via finite alphabets I and O, intended to model control actions rather than proper pieces of data. But real-life systems often operate values from a large to infinite data domain. Examples include data-independent programs [53, 35, 41], software with integer parameters [11], communication protocols with message parameters [19], and more [10, 51, 18]. The goal of this paper is to study extensions of reactive synthesis, and its formulation as Church games, to infinite data domains: (\mathbb{Q}, \leq) and (\mathbb{N}, \leq) in particular.

Church synthesis over infinite data domains. Church games naturally extend to an infinite data domain \mathcal{D} : Adam and Eve alternately pick data in \mathcal{D} , and their infinite interaction results in an ω -data word $d_0d'_0d_1d'_1\cdots\in\mathcal{D}^{\omega}$. The game is won by Eve if it belongs to a given specification $S \subseteq \mathcal{D}^{\omega}$. Accordingly, strategies for Eve have type $\mathcal{D}^+ \to \mathcal{D}$, while strategies for Adam have type $\mathcal{D}^* \to \mathcal{D}$. In this paper, we study specifications given by a standard extension of finite-state automata to infinite data domains called register automata [36]: they use a finite set of registers to store data values, and a finite set of predicates over the data domain to test those values. In each step, the automaton reads a data value from \mathcal{D} and compares it with the values held in its registers using the predicates (and possibly constants). Depending on this comparison, it decides to store the value in some of the registers, and then moves to a successor state. This way, it builds a sequence of *configurations* (pairs of state and register values) representing its run on reading a data word from \mathcal{D}^{ω} : it is accepted if the visited states satisfy a certain parity condition. In this paper, we study specifications given by deterministic register automata over \mathbb{Q} or \mathbb{N} , which can use the predicate \leq and the constant 0 to test data values.

Contributions. Our first result is an impossibility result: deciding the winner of a Church game for specifications given by deterministic register automata over $(\mathbb{N}, \leq$) is an undecidable problem (Theorem 1). We introduce the one-sided restriction on Church games: Adam still has the full power of picking data values, but Eve's behaviour is restricted to picking elements from a *finite* alphabet only. Despite being asymmetric, one-sided Church games are quite expressive. For example, they model synthesis scenarios for runtime data monitors that monitor the input data stream and raise a Boolean flag when a critical trend happens (like oscillations above a certain amplitude), and for systems that need to take control actions depending on sensor measurements (a heating controller for instance). Formally, in one-sided Church games, there is a finite set of elements Σ in which Eve picks her successive choices. Accordingly, specifications are languages $S \subseteq (\mathcal{D}\Sigma)^{\omega}$, in this paper defined by deterministic one-sided register automata (defined naturally by alternating between register automata transitions and finite-state automata transitions). Eve's strategies have type $\lambda_{\exists}: \mathcal{D}^+ \to \Sigma$ while Adam's strategies have type $\lambda_{\forall}: \Sigma^* \to \mathcal{D}$. We prove the following about one-sided Church games whose specifications are given by one-sided deterministic register automata over (\mathbb{Q}, \leq) and (\mathbb{N}, \leq) :

- 1. they are determined: every game is either won by Eve or Adam
- 2. they are decidable: the winner can be computed in time exponential in the number of registers of the specification,
- 3. if Eve wins, then she has a winning strategy which can be implemented by a transducer with registers (which can be effectively constructed).

Transducers with registers extend Mealy machines with a finite set of registers: they have finitely many states, and given any state and a test over the input data value, deterministically, they assign the current value to some registers (or none), output an element of Σ , and update their state. Therefore, the last result echoes the similar result in the ω -regular setting (finite-memory strategies can be effectively constructed for the winner), and supports the fact that one-sided Church games on register

automata are an adequate framework for effective synthesis of machines processing streams of data.

Example 1. Figure 1 illustrates a specification given by a deterministic one-sided register automaton, alternating between square and circle states, depending on whether their outgoing transitions read data values or elements in a finite alphabet $\Sigma = \{a, b\}$. It can be seen as a game arena where Adam controls the square states while Eve controls the circle states. To simplify the presentation, two parts of the automaton are not depicted and have been summarised as "Eve wins" and "Eve loses": any run going in the former part is non-accepting and any run going in the latter part is accepting (this can be modelled by a parity condition). So, Eve's objective is to force executions into "Eve wins", whatever input data values are issued by Adam. There are two registers, r_M and r_l . The test \top (true) means that the transition can be taken irrespective of the value played, the test $r_l < * < r_M$ means that the value should be between the values of registers r_l and r_M , and the test 'else' means the opposite. The writing $\downarrow r$ means that the value is stored into the register r. At first, Adam provides some data value d_M , serving as a maximal value stored in r_M . Register r_l , initially 0, holds the last data value θ_l played by Adam. Consider state C: if Adam provides a value outside of the interval d_l, d_M , he loses; if it is strictly between d_l and d_M , it is stored into register r_l and the game proceeds to state D. There, Eve can either respond with label b and move to state E, or with a to state C. In state E, Adam wins if he can provide a data value strictly between d_l and d_M , otherwise he loses. Eve wins this game in \mathbb{N} : for example, she could always respond with label a, looping in states C-D. After a finite number of steps, Adam is forced to provide a data value $\geq d_M$, losing the game. An alternative Eve winning strategy, that does depend on Adam data, is to loop in C-D until $d_M - d_l = 1$ (thus, she has to memorise the first Adam value d_M), then move to state E, where Adam loses. In the dense domain (\mathbb{Q}, \leq) , however, the game is won by Adam, because he can always provide a value within d_l, d_M for any $d_l < d_M$, so the game either loops in C-D forever or reaches "Eve loses".

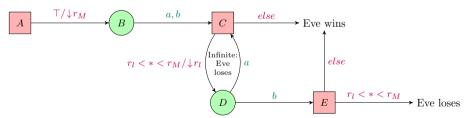


Figure 1: Eve wins this game in \mathbb{N} but loses in \mathbb{Q} .

Proof overview. We give intuitions about the main ingredients to show decidability. The key idea used to solve problems about register automata is to forget the precise values of input data and registers, and track instead the constraints (sometimes called types) describing the relations between them. In our example, all registers start in

0 so the initial constraint is $r_l^1 = r_M^1$, where r^i abstracts the value of register r at step i. Then, if Adam provides a data above the value of r_l , the constraint becomes $r_l^2 < r_M^2$ in state B. Otherwise, if Adam had provided a data equal to the value in r_l , the constraint would be $r_l^2 = r_M^2$. In this way the constraints evolve during the play, forming an infinite sequence. Looping in states C-D induces the constraint sequence $(r_l^i < r_l^{i+1} < r_M^i = r_M^{i+1})_{i>2}$. It forms an infinite chain $r_l^3 < r_l^4 < \dots$ bounded by constant $r_M^3 = r_M^4 = \dots$ from above. In $\mathbb N$, as it is a well-founded order, it is not possible to assign values to the registers at every step to satisfy all constraints, so the sequence is not satisfiable. Before elaborating on how this information can be used to solve Church games, we describe our results on satisfiability of constraint sequences. This topic was inspired by the work [47] which studies, among others, the nonemptiness problem of constraint automata, whose states and transitions are described by constraints. In particular, they show [47, Appendix C] that satisfiability of constraint sequences can be checked by nondeterministic ω B-automata [6]. Nondeterminism however poses a challenge in synthesis, and it is not known whether games with a winning objective given as a nondeterministic ω B-automaton are decidable. In contrast, we describe a deterministic max-automaton [8] characterising the satisfiable constraint sequences in N. As a consequence of [9], games over such automata are decidable. Then we study two kinds of constraint sequences inspired by Church games with register automata. First, we show that the satisfiable lasso-shaped constraint sequences, of the form uv^{ω} , are recognisable by deterministic parity automata. Second, we show how to assign values to registers on-the-fly in order to satisfy a constraint sequence induced by a play in the Church game.

To solve one-sided Church games with a specification given as a register automaton S for $(\mathbb{N}, <)$ and $(\mathbb{Q}, <)$, we reduce them to certain finite-arena zero-sum games, which we call automata games. The states and transitions of the game are those of the specification automaton S. The winning condition requires Eve to satisfy the original objective of S only on feasible plays, i.e. those that induce satisfiable constraint sequences. In our example, the play $A \cdot B \cdot (C \cdot D)^{\omega}$ does not satisfy the parity condition, yet it is won by Eve in the automaton game since it is not satisfiable in N, and therefore there is no corresponding play in the Church game. We show that if Eve wins the automaton game, then she wins the Church game, using a strategy that simulates the register automaton S and simply picks one of its transitions. It is also sufficient: if Adam wins the automaton game then he wins the Church game. To prove this, we construct, from a winning strategy of Adam in the automaton game, a winning strategy of Adam (that manipulates data) in the Church game. This step uses the previously mentioned results on satisfiability of constraint sequences. Over (\mathbb{N}, \leq) , we cannot solve the automaton game directly, as it is not ω -regular. We instead reduce it to an ω -regular approximation of it which considers quasi-feasible sequences, a notion which is more liberal than feasibility but coincides with it on lasso-shaped words.

Related works. This paper is an extended version of the conference paper [25]. It follows a line of works about synthesis from register automata specifications [23,

¹Lasso-shaped words are also called regular words or ultimately periodic words in the literature.

37, 38, 27], which focused on register automata over data domains $(\mathcal{D}, =)$ equipped with equality tests only. The synthesis of data systems has also been investigated in [31, 40]. They do not rely on register automata and are also limited to equality tests or do not study data comparison. Thus, systems that output the largest value seen so far, grant a resource to a process with the lowest ID, or raise an alert when a heart sensor reads values forming a dangerous curve, are out of reach of those synthesis methods. These systems require \leq .

In this paper, we consider specifications given by deterministic register automata. Already in the case of infinite alphabets $(\mathcal{D}, =)$, dropping the determinism requirement leads to undecidability: finding a winner of a Church game is undecidable when specifications are given as nondeterministic or universal register automata [23, 27]. To recover decidability, in the case of universal register automata, those works restrict Eve strategies to register transducers with an a priori fixed number of registers. This problem is called register-bounded synthesis. Recently in [26], register-bounded synthesis have been extended to various data domains such as (\mathbb{N}, \leq) , (\mathbb{Z}, \leq) , or (Σ^*, \preceq) where Σ is an arbitrary finite alphabet and \preceq is the prefix relation. The results of [26] are orthogonal to the results of this paper, although they rely on the study of constraint sequences we conduct here.

The paper [28] studies synthesis from variable automata with arithmetic. Those automata are incomparable with register automata: on the one hand, they allow addition on top of a dense order predicate, but on the other hand they do not allow updating the content of the registers along the run. Note that they do not consider the case of a discrete order. The paper [29] studies strategy synthesis but, again, mainly over a dense domain. A one-sided setting similar to ours was studied in [30] for Church games whose winning condition is given by formulas of the Logic of Repeating Values (a fragment of LTL with the freeze quantifier [20]), but only for $(\mathcal{D}, =)$. That work was extended to domain (\mathbb{Z}, \leq) in [4]. There, the authors show that the realisability problem in one-sided setting on (\mathbb{Z}, \leq) for Constraint LTL and its prompt variant are 2EXPTIME-complete. Deterministic register automata are more expressive than Constraint LTL, so our work subsumes their decidability result, yet the lower expressivity of Constraint LTL enables simpler arguments. We note that our proof ideas — abstracting data words by finite-alphabet words and utilising regularity of abstracted words — are somewhat similar to those in papers on Constraint LTL [21, 4]. The work on automata with atoms [39] implies our decidability result for (\mathbb{Q}, \leq) , even in the two-sided setting, but not the complexity result, and it does not apply to (\mathbb{N}, \leq) . Our setting in \mathbb{N} is loosely related to monotonic games [2]: they both forbid infinite descending behaviours, but the direct conversion is unclear. Games on infinite arenas induced by pushdown automata [52, 12, 1] or one-counter systems [48, 32] are orthogonal to our games.

Outline. In Section 2, we introduce preliminary notions. Section 3 introduces Church synthesis games along with the main tools and results (with proofs postponed). Section 4 presents the postponed proofs for Church synthesis, relying on results about satisfiability of constraint sequences over (\mathbb{N}, \leq) described in Section 5.

2 Preliminaries

In this paper, $\mathbb{N} = \{0, 1, \dots\}$ is the set of natural numbers (including 0). We assume some knowledge of ω -regular languages and ω -automata, and refer to e.g. [15] for an introduction.

 ω -data words. In this paper, an ordered data domain, or simply data domain, \mathfrak{D} is an infinite countable set of elements called data, linearly ordered by some order denoted <. We consider two data domains, \mathbb{N} and \mathbb{Q} , with their usual order. An ω -data word over \mathfrak{D} is an infinite sequence $d_0d_1\ldots$ of data in \mathfrak{D} . We denote by \mathfrak{D}^{ω} the set of ω -data words. Similarly, we denote by \mathfrak{D}^* the set of finite sequences (possibly empty) of elements in \mathfrak{D} .

Registers. Let R be a finite set of elements called *registers*, intended to contain data values, i.e. values in \mathcal{D} . A *register valuation* is a mapping $\nu: R \to \mathcal{D}$ (also written $\nu \in \mathcal{D}^R$). For any data $\ell \in \mathcal{D}$, we write ℓ^R to denote the constant valuation $\nu_{\ell}(r) = \ell$ for all $r \in R$.

A test is a maximally consistent set of atoms of the form $*\bowtie r$ for $r\in R$ and $\bowtie\in\{=,<,>\}$. We may represent tests as conjunctions of atoms instead of sets. The symbol '*' is used as a placeholder for incoming data. For example, for $R=\{r_1,r_2\}$, the expression $r_1<*$ is not a test because it is not maximal, but $(r_1<*)\wedge (*< r_2)$ is a test. We denote Tst_R the set of all tests and just Tst if R is clear from the context. A register valuation $\nu\in \mathcal{D}^R$ and data $\emptyset\in \mathcal{D}$ satisfy a test $\mathsf{tst}\in \mathsf{Tst}$, written $(\nu,\emptyset)\models \mathsf{tst}$, if all atoms of tst get satisfied when we replace the placeholder * by \emptyset and every register $r\in R$ by $\nu(r)$. An assignment is a subset $\mathsf{asgn}\subseteq R$. Given an assignment asgn , a data $\emptyset\in \mathcal{D}$, and a valuation ν , we define $update(\nu,\emptyset,\mathsf{asgn})$ to be the valuation ν' s.t. $\forall r\in \mathsf{asgn}: \nu'(r)=\emptyset$ and $\forall r\not\in \mathsf{asgn}: \nu'(r)=\nu(r)$.

Register automata. A specification deterministic register automaton, or simply deterministic register automaton is a tuple $S = (Q, q_{\iota}, R, \delta, \alpha)$ where $Q = Q_A \uplus Q_E$ is a set of states partitioned into Adam and Eve states, the state $q_{\iota} \in Q_A$ is initial, R is a set of registers, $\delta = \delta_A \uplus \delta_E$ is a (total and deterministic) transition function where, for $P \in \{A, E\}$, we have, by setting $\overline{A} = E$ and $\overline{E} = A$: $\delta_P : (Q_P \times \mathsf{Tst} \to \mathsf{Asgn} \times Q_{\overline{P}})$; and $\alpha : Q \to \{1, ..., c\}$ is a priority function where c is the priority index.

A configuration of A is a pair $(q, \nu) \in Q \times \mathcal{D}^R$, describing the state and register content; the initial configuration is $(q_\iota, 0^R)$. A run of S on a word $w = \ell_0 \ell_1 \dots \in \mathcal{D}^\omega$ is a sequence of configurations $\rho = (q_0, \nu_0)(q_1, \nu_1) \dots \in ((Q_A \times \mathcal{D}^R)(Q_E \times \mathcal{D}^R))^\omega$ starting in the initial configuration $((q_0, \nu_0) = (q_\iota, 0^R))$ and such that for every $i \geq 0$: by letting tst_i be a unique test for which $(\nu_i, \ell_i) \models \mathsf{tst}_i$, we have $\delta(q_i, \mathsf{tst}_i) = (\mathsf{asgn}_i, q_{i+1})$ for some asgn_i and $\nu_{i+1} = update(\nu_i, \ell_i, \mathsf{asgn}_i)$. Because the transition function δ is deterministic and total, every word induces a unique run in S. The run ρ is accepting if the maximal priority visited infinitely often is even. A word is accepted by S if it induces an accepting run. The language L(S) of S is the set of all words it accepts.

Interleavings. Specification register automata are meant to recognise interleavings of inputs (provided by Adam) and output (provided by Eve), hence the partitioning of states. Often, we need to combine them or conversely tell them apart. Thus, given

two words $u = u_0 u_1 \cdots \in \mathcal{D}^{\omega}$ and $v = v_0 v_1 \cdots \in \mathcal{D}^{\omega}$, we formally define their interleaving $u \otimes v = u_0 v_0 u_1 v_1 \cdots \in \mathcal{D}^{\omega}$. We note that given a word $w = w_0 w_1 \cdots \in \mathcal{D}^{\omega}$, it can be uniquely decomposed into $w = u \otimes v$, where $u = w_0 w_2 \cdots \in \mathcal{D}^{\omega}$ and $v = w_1 w_3 \cdots \in \mathcal{D}^{\omega}$.

Games. A two-player zero-sum game, or simply a game, is a tuple G = G $(V_{\forall}, V_{\exists}, v_0, E, W)$ where V_{\forall} and V_{\exists} are disjoint sets of vertices controlled by Adam and Eve, $v_0 \in V_\forall$ is initial, $E \subseteq (V_\forall \times V_\exists) \cup (V_\exists \times V_\forall)$ is a turn-based transition relation, and $W \subseteq (V_{\forall} \cup V_{\exists})^{\omega}$ is a winning objective. An Eve strategy is a mapping $\lambda_{\exists}: (V_{\forall}V_{\exists})^+ \to V_{\forall} \text{ such that } (v_{\exists}, \lambda(v_{\forall}^0v_{\exists}^0...v_{\forall}^kv_{\exists}^k)) \in E \text{ for all paths } v_{\forall}^0v_{\exists}^0...v_{\forall}^kv_{\exists}^k \text{ of } v_{\exists}^kv_{\exists}^k v_{\exists}^k v_{\exists}$ G starting in $v_{\forall}^0 = v_0$ and ending in $v_{\exists}^k \in V_{\exists}$ (where $k \geq 0$). Note that λ_{\exists} only depends on the V_{\exists} component, since the V_{\forall} part is determined by the V_{\exists} part, so we sometimes define it as $\lambda_{\exists}: V_{\exists}^+ \to V_{\forall}$. Adam strategies are defined similarly, by inverting the roles of \exists and \forall . \bar{A} strategy is *finite-memory* if it can be computed by a finite-state machine, and *positional* if it only depends on the current vertex. A play is a sequence of vertices starting in v_0 and satisfying the edge relation E. It is won by Eve if it belongs to W (otherwise it is won by Adam). An infinite play $\pi = v_0 v_1 \dots$ is compatible with an Eve strategy λ when for all $i \geq 0$ s.t. $v_i \in V_{\exists}$: $v_{i+1} = \lambda(v_0 \dots v_i)$. An Eve strategy is winning if all infinite plays compatible with it are winning. A game is determined (respectively, finite-memory determined, positionally determined) if either Adam or Eve has a winning strategy (resp., a finite-memory winning strategy, a positional winning strategy).

A finite-arena game is a game whose arena is finite, i.e. where V_{\forall} and V_{\exists} are finite. Among them, we distinguish ω -regular games, where the winning condition is an ω -regular language. In particular, a parity game is a game whose winning condition is defined through a parity function $\alpha: V_{\forall} \uplus V_{\exists} \to \{1,...,c\}$, where a play $v_0v_1...$ is winning for Eve if and only if the maximal priority seen infinitely often is even. It is well-known that ω -regular games are finite-memory determined and reduce to parity games, which are positionally determined and can be solved in n^c [33] (see also [16]), where n is the size of the game and c the priority index.

Note that in register automata, Adam is represented as A and Eve as E, while in games he is \forall and she is \exists . This is to visually distinguish automata from games.

3 Church Synthesis Games

A Church synthesis game is given as a tuple G = (I, O, S), where I is an input alphabet, O is an output alphabet, and $S \subseteq (I \cdot O)^{\omega}$ is a specification. Its semantics is provided by the game $(\{v_0\} \cup O, I, v_0, E, S)$, where $E = ((\{v_0\} \cup O) \times I) \cup (I \times O)$, but we rephrase it to provide a stronger intuition. In particular, it is at first counterintuitive that Adam owns O vertices, and Eve I vertices; this is because both players choose their move by targeting a specific vertex.

Thus, in a Church synthesis game, two players, Adam (the environment, who provides inputs) and Eve (the system, who controls outputs), interact. Their strategies are respectively represented as mappings $\lambda_{\forall}: v_0 \cdot O^* \to I$ (often simply represented as $\lambda_{\forall}: O^* \to I$ for symmetry) and $\lambda_{\exists}: I^+ \to O$. Given λ_{\forall} and λ_{\exists} , the outcome $\lambda_{\forall} \|\lambda_{\exists}$ is the infinite sequence $i_0 o_0 i_1 o_1 \dots$ such that for all $j \geq 0$: $i_j = \lambda_{\forall}(o_0 \dots o_{j-1})$

and $o_j = \lambda_\exists (i_0...i_j)$. If $\lambda_\forall \|\lambda_\exists \in S$, the outcome is won by Eve, otherwise by Adam. Eve wins the game if she has a strategy λ_\exists such that for every Adam strategy λ_\forall , the outcome $\lambda_\forall \|\lambda_\exists$ is won by Eve. Solving a synthesis game amounts to finding whether Eve has a winning strategy. Synthesis games are parameterised by classes of alphabets and specifications. A game class is *determined* if every game in the class is either won by Eve or by Adam.

The class of synthesis games where I and O are finite and where S is an ω -regular language is known as *Church games*; they are decidable and determined. They also enjoy the finite-memoriness property: if Eve wins a game then she can win it with a strategy that is represented as a finite-state machine [14] (see also [50] for a game-theoretic presentation of those results).

We study synthesis games where $I = O = \mathcal{D}$ is an ordered data domain and the specifications are described by deterministic register automata. In the following, we let $G_S^{\mathcal{D}} = (\mathcal{D}, \mathcal{D}, S)$ be the Church synthesis game with input and output alphabet \mathcal{D} and specification S, and simply write G_S when \mathcal{D} is clear from the context.

3.1 Church games on register automata

We start our study with a negative result, that highlights the difficulty of the problem: over the data domain (\mathbb{N}, \leq) , Church games are undecidable. Indeed, if the two players pick data values, one can simulate a two-counter machine as follows: one player provides the values of the counters, while the other checks that no cheating happens on the increments and decrements. This can be done using the fact that c' = c + 1 whenever there does not exist any ℓ such that $c < \ell < c'$.

Theorem 1. Deciding the existence of a winning strategy for Eve in a Church game whose specification is a deterministic register automaton over (\mathbb{N}, \leq) is undecidable.

Proof idea. We reduce from the halting problem of 2-counter machines, which is undecidable [42]. We define a specification with 4 registers r_1, r_2, z and t. Registers r_1 and r_2 each store the value of one counter; z stores 0 to conduct zero tests and t is used as a buffer. We now describe how to increment c_1 (see Figure 2a); the cases of c_2 and of decrementing are similar. Eve suggests a value $d > r_1$, which is stored in t. Then, Adam checks that the increment was done correctly: Eve cheated if and only if Adam can provide a data d' such that $r_1 < d' < d$. If he cannot, d is stored in r_1 , thus updating the value of the counter. The acceptance condition is then a reachability one, asking that a halting instruction is eventually met. Now, if M halts, then its run is finite and the values of the counters are bounded by some B. As a consequence, there exists a strategy of Eve which simulates the run by providing the values of the counters along the run. Conversely, if M does not halt, then no halting instruction is reachable by simulating M correctly, and Adam is able to check that Eve does not cheat during its simulation.

Proof. We reduce from the halting problem of deterministic 2-counter machines, which is undecidable [42]. Among multiple formalisations of counter machines, we pick the following one: a 2-counter machine has two counters which contain integers,

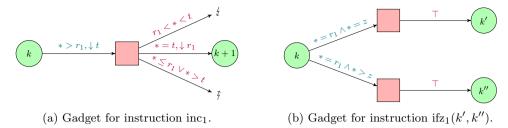


Figure 2: Gadgets for 2CM instructions. The instruction number k is stored in the state of the automaton. The state $\frac{1}{2}$ (resp. $\frac{2}{3}$) is a rejecting sink (resp. accepting sink). Non-depicted transitions go to the sink state that is losing for the player that takes them.

initially valued 0. It is composed of a finite set of instructions $M = (I_1, \ldots, I_m)$, each instruction being of the form $\operatorname{inc}_j, \operatorname{dec}_j, \operatorname{ifz}_j(k', k'')$ for j = 1, 2 and $k', k'' \in \{1, \ldots, m\}$, or halt. The semantics are defined as follows: a configuration of M is a triple (k, c_1, c_2) , where $1 \leq k \leq m$ and $c_1, c_2 \in \mathbb{N}$. The transition relation (which is actually a function, as M is deterministic) is then, from a configuration (k, c_1, c_2) :

- If $I_k = \text{inc}_1$, then the machine increments c_1 and jumps to the next instruction I_{k+1} : $(k, c_1, c_2) \to (k+1, c_1+1, c_2)$. Similarly for inc₂.
- If $I_k = \text{dec}_1$ and $c_1 > 0$, then $(k, c_1, c_2) \to (k + 1, c_1 1, c_2)$. If $c_1 = 0$, then the computation fails and there is no successor configuration. Similarly for dec_2 .
- If $I_k = \text{ifz}_1(k', k'')$, then M jumps to k' or k'' according to a zero-test on c_1 : if $c_1 = 0$, then $(k, c_1, c_2) \to (k', c_1, c_2)$, otherwise $(k, c_1, c_2) \to (k'', c_1, c_2)$. Similarly for ifz₂.

A run of the machine is then a finite or infinite sequence of successive configurations, starting at (1,0,0). We say that M halts whenever it admits a finite run which ends in a configuration (k, c_1, c_2) such that $I_k = \text{halt}$.

Let $M=(I_1,\ldots,I_m)$ be a 2-counter machine. We associate to it the following specification deterministic register automaton: S has states $Q=Q_A \uplus Q_E$, where, for $P \in \{A,E\}, Q_P=\left(\{0,\ldots,m+1\} \cup (\{0,\ldots,m+1\} \times \{y,n\}) \cup \{\not z,\not z\}\right) \times \{P\}$. The letters y and n are used to remember whether an ifz test evaluated to true or false; they are only used by A, but we included them in Q_E for symmetry. The initial state of S is (0,A). The automaton has four registers r_1,r_2,t,z . The acceptance is defined by the reachability condition $F=\{(\not z,A)\}$, while $\not z$ signals rejecting sink states. The transitions of S are defined by the following procedure:

- Initially, there is a transition $(0, A) \xrightarrow{\top} (1, E)$ so that the implementation can start the simulation.
- Then, for each $k \in \{1, \ldots, m\}$:

- If $I_k = \operatorname{inc}_j$ for j = 1, 2, then we add to the transitions of S the gadget from Figure 2a, i.e. output transition $(k, E) \xrightarrow{* > r_1, \downarrow t} (k, A)$ and input transitions $(k, A) \xrightarrow{r_1 < * < t} (\not \downarrow, E), (k, A) \xrightarrow{* = t, \downarrow r_1} (k + 1, E)$ and $(k, A) \xrightarrow{* \le r_1} (\not \uparrow, E), (k, A) \xrightarrow{* > t} (\not \uparrow, E).$
- The case $I_k = \operatorname{dec}_j$ for j = 1, 2 is similar: we add output transition $(k, E) \xrightarrow{*< r_1, \downarrow t} (k, A)$ and input transitions $(k, A) \xrightarrow{t<*< r_1} (\not{t}, E)$, $(k, A) \xrightarrow{*=t, \downarrow r_1} (k+1, E)$ and $(k, A) \xrightarrow{*\geq r_1} (\not{t}, E)$, $(k, A) \xrightarrow{*< t} (\not{t}, E)$. Note that in our definition, if $c_j = 0$, then the instruction dec_j should be blocking, i.e. the computation should fail, which is consistent with the fact that in that case, the implementation cannot provide $d < r_1$.
- If $I_k = \operatorname{ifz}_j(k', k'')$, then we add the gadget of Figure 2b, i.e. output transitions $(k, E) \xrightarrow{*=r_1 \wedge *=z} (k, y, A)$, $(k, E) \xrightarrow{*=r_1 \wedge *>z} (k, n, A)$ and input transitions $(k, y, A) \xrightarrow{\top} (k', E)$ and $(k, n, A) \xrightarrow{\top} (k'', E)$.
- If $I_k = \text{halt}$, we add a transition $(k, E) \xrightarrow{\top} (?, A)$.
- Finally, $(?, P) \xrightarrow{\top} (?, \overline{P})$ and $(?, P) \xrightarrow{\top} (?, \overline{P})$ for $P \in \{A, E\}$, so that both ? and ? are sink states alternating between the players. In the following, we sometimes write ? for (?, P) and ? for (?, P), since the owner of the state does not matter.

Now, assume that M admits an accepting run $\rho = (k_1, c_1^1, c_2^1) \to \cdots \to (k_n, c_1^n, c_2^n)$, where $n \in \mathbb{N}$, $k_1 = 1$, $c_1^1 = c_2^1 = 0$ and $I_{k_n} = \text{halt}$. The values of the counters are bounded by some $B \leq n$. Then, let λ^ρ be the strategy of Eve which ignores the input provided by Adam and plays the output $w_\rho = c_0^{j_0} \dots c_{n-1}^{j_{n-1}} 0^\omega$, where for $1 \leq l < n$, j_l is the index of the counter modified or tested at step l (i.e. $j_l = 1, 2$ is such that $I_{k_l} = \text{inc}_{j_l}$, dec_{j_l} of $\text{ifz}_{j_l}(k', k'')$). Formally, for all $u \in \mathbb{N}^+$ of length $l \geq 0$, we let $\lambda^\rho(u) = c_l^{j_l}$ if $l \leq n-1$ and $\lambda^\rho(u) = 0$ otherwise.

Let us show that λ^{ρ} is a winning strategy for Eve. Let $u \in \mathbb{N}^{\omega}$ be an input word provided by Adam. We show by induction on l that in S the partial run over $(u \otimes w)[:2l+1]$ is either in state $\hat{\tau}$ or S is in configuration $((k_l, E), \tau_l)$, where $\tau_l(r_1) = c_l^1$ and $\tau_l(r_2) = c_l^2$.

Initially, S is in configuration $((0, A), \tau_R^0)$. Then, whatever Adam plays, it transitions to $((1, E), \tau_R^0)$, so the invariant holds. Now, assume it holds up to step l. If S is in (\ref{r}, E) , the only available transition is $(\ref{r}, E) \xrightarrow{\top} (\ref{r}, A)$, and then $(\ref{r}, A) \xrightarrow{\top} (\ref{r}, E)$, so the invariant holds at step l + 2 (\ref{r}) is a sink state). Otherwise, necessarily l < n, S is in configuration $((k_l, E), \tau_l)$ and there are four cases:

• $I_{k_l} = \text{inc}_j$. By definition, $j = j_l$. We treat the case j = 1, the other case is similar. Then, Eve plays $c_l^1 = c_{l-1}^1 + 1$, which is such that $c_l^1 > \tau_l(r_1)$. Then, there does not exist d such that $\tau_l(r_1) < d < \tau_l(t)$ since $\tau_l(r_1) = c_{l-1}^1$ and $\tau_l(t) = c_{l-1}^1 + 1$, so the play cannot transition to $(\not z, E)$. Now, either Adam plays $u_{l+1} = \tau_l(t) = c_{l-1}^1 + 1$, in which case S evolves to configuration $((k_{l+1}, E), c_{l+1}^1, c_{l+1}^2)$, and the invariant holds. Otherwise, $u_{l+1} \neq \tau_l(t)$ and S goes to $(\not z, E)$ and the invariant holds as well.

- The case of $I_{k_l} = \text{dec}_j$ is similar. Let us just mention that the computation does not block at this step, otherwise ρ is not a run of M, so the transition $d < r_j$ can indeed be taken by Eve.
- $I_{k_l} = \text{ifz}_j(k', k'')$. Again, $j = j_l$, and we treat the case j = 1. Eve plays c_l^1 ; there are two cases. If $c_l^1 = 0$, the transition $* = r_1 \land * = z$ is taken in S, since at every step, $\tau_l(z) = 0$ (this register is never modified). If $c_l^1 \neq 0$, then the transition $* = r_1 \land * > z$ is taken. In both cases, whatever Adam plays, S then evolves to $((k_{l+1}, E), \tau_{l+1})$ (where $\tau_{l+1} = \tau_l$) and the invariant holds.
- Finally, if $I_{k_l} = \text{halt}$, then whatever Eve plays, S transitions to (?, A), and whatever Adam plays, the automaton transitions to (?, E).

As a consequence, \mathring{f} is eventually reached whatever the input, which means that for all $u \in \mathbb{N}^{\omega}$, $u \otimes I(u) \in S$, i.e. I is indeed an implementation of S.

Conversely, assume that Eve has a winning strategy λ_{\exists} in G_S . Let ρ be the maximal run of M (i.e. either ρ ends in a configuration with no successor, or it is infinite). It is unique since M is deterministic. Let $n = \|\rho\|$, with the convention that $n = \infty$ if ρ is infinite. Let us build by induction a play of a strategy² of Adam λ_{\forall} such that for all $l < n, (\lambda_{\forall} \| \lambda_{\exists})[:2l] = c_l^{j_l}$. and the configuration reached by S over $(\lambda_{\forall} \otimes \lambda_{\exists})[:2l]$ is $((k_l, E), \tau_l)$. Initially, let $u_0 = 0$. As the initial test is \top , S anyway evolves to state (1, E), with $\tau(r_1) = \tau(r_2) = 0$.

Now, assume we built such input u up to l. There are again four cases:

- $I_{k_l} = \text{inc}_j$. Then, Eve provides some output data $d_E > \tau_l(r_j)$. Assume by contradiction that $d_E > \tau_l(r_j) + 1$. Then, λ_\exists is not winning because if Adam plays $d_A = \tau_l(r_j) + 1$, S goes to state $(\not z, E)$, which is a sink rejecting state, so the play is losing irrelevant of what both players play after this move. So, necessarily, $d_E = \tau_l(r_j) + 1 = c_l^{j_l}$, and S evolves to configuration (k_{l+1}, τ_{l+1}) .
- The case $I_{k_l} = \text{dec}_j$ is similar. Necessarily, $c_j^l > 0$, otherwise Eve cannot provide any output data and the play is losing for Eve, which contradicts the fact that λ_{\exists} is winning. Thus, the computation does not block here.
- $I_{k_l} = \text{ifz}_j(k', k'')$. The output transitions of the gadget constrain Eve to output $d_E = \tau_l(r_j) = c_l^{j_l}$, and irrelevant of what Adam plays S then evolves to configuration $((k_{l+1}, E), \tau_{l+1})$.
- I_{k_l} = halt. Then, it means that $n < \infty$ and l = n, so the invariant vacuously holds.

Now, ρ cannot be infinite, otherwise $\lambda_{\forall} \| \lambda_{\exists}$ is not accepted by S because \dagger is never reached and Eve would not win. It moreover cannot block on some dec_j instruction, as demonstrated in the induction. Thus, a halt instruction is eventually reached, which means that ρ is a halting run of M: M halts.

3.2 Church games on one-sided register automata

In light of this undecidability result, we consider one-sided synthesis games, where Adam provides data but Eve reacts with labels from a *finite* alphabet (a similar restriction was studied in [30] for domain $(\mathcal{D}, =)$). Specifications are now given as a language $S \subseteq (\mathcal{D} \cdot \Sigma)^{\omega}$, recognised by a one-sided deterministic register automaton.

²We only construct the given play, since the rest of the strategy does not matter.

Definition 1. A one-sided deterministic register automaton, or simply one-sided register automaton $S = (\Sigma, Q, q_{\iota}, R, \delta, \alpha)$ is a deterministic register automaton that additionally has a finite alphabet Σ of Eve labels. Its states are again partitioned into Adam and Eve states $Q = Q_A \uplus Q_E$, and it has an initial state $q_{\iota} \in Q_A$. Its transition function $\delta = \delta_A \uplus \delta_E$ is again total, but now has $\delta_E : Q_E \times \Sigma \to Q_A$. The rest is defined as for deterministic register automata: $\delta_A : Q_A \times \mathsf{Tst} \to \mathsf{Asgn} \times Q_E$; R is a set of registers, and finally $\alpha : Q \to \{1, ..., c\}$ is a priority function where c is the priority index.

The notions of configurations and runs are defined analogously, except for the asymmetry between input and output: a configuration of A is a pair $(q, \nu) \in Q \times \mathcal{D}^R$, describing the state and register content; the initial configuration is $(q_\iota, 0^R)$. A run of S on a word $w = \ell_0 a_0 \ell_1 a_1 \dots \in (\mathcal{D}\Sigma)^\omega$ (note the interleaving of \mathcal{D} and Σ) is a sequence of configurations $\rho = (q_0, \nu_0)(p_0, \nu_1)(q_1, \nu_1)(p_0, \nu_2) \dots \in ((Q_A \times \mathcal{D}^R)(Q_E \times \mathcal{D}^R))^\omega$ starting in the initial configuration (i.e. $(q_0, v_0) = (q_\iota, 0^R)$) and such that for every $i \geq 0$:

- (reading an input data value) by letting tst_i be a unique test for which $(\nu_i, d_i) \models \mathsf{tst}_i$, we have $\delta(q_i, \mathsf{tst}_i) = (\mathsf{asgn}_i, p_i)$ for some asgn_i and $\nu_{i+1} = update(\nu_i, d_i, \mathsf{asgn}_i)$, as for deterministic register automata;
- (reading an output letter from Σ) $\delta(p_i, a_i) = q_{i+1}$, as for finite-state automata.

Again, because the transition function δ is deterministic and total, every word induces a unique run in S. The run ρ is accepting if the maximal priority visited infinitely often is even. A word is accepted by S if it induces an accepting run. The language L(S) of S is the set of all words it accepts.

Figure 1 shows an example of a one-sided automaton. For instance, it rejects the words $3a1b2(\Sigma \mathcal{D})^{\omega}$ and accepts the words $3a1a2b(\mathcal{D}\Sigma)^{\omega}$.

The rest of this paper is dedicated to showing that Church games whose specification are defined by one-sided register automata over (\mathbb{Q}, \leq) or (\mathbb{N}, \leq) are decidable in exponential time, and that those games are determined. Formally,

Theorem 2. Let $S = (\Sigma, Q, q_{\iota}, R, \delta, \alpha)$ be a one-sided register automaton over (\mathbb{N}, \leq) or (\mathbb{Q}, \leq) .

- 1. The problem of determining the winner of the Church synthesis game $G = (\mathcal{D}, \mathcal{D}, S)$ is decidable in time polynomial in |Q| and exponential in c and |R|.
- 2. G_S is determined, i.e. either Eve or Adam has a winning strategy in G_S .

The above is a wrapper theorem, that aggregates Theorems 9 for (\mathbb{Q}, \leq) and 18 for (\mathbb{N}, \leq) . We defer the proof to Section 4. The result for (\mathbb{Q}, \leq) can be derived from [21] or [39, Section 7], but we include it for pedagogical reasons, as it allows us to introduce the main tools in a simple setting and to highlight the difficulties that creep up when we shift to (\mathbb{N}, \leq) .

In the case of a finite alphabet, the game-theoretic approach to solving Church games whose specification is given by a deterministic finite-state automaton consists in playing on the automaton, in the following sense: the arena consists of the automaton, and Adam and Eve alternately choose an input (respectively, output) letter, or

equivalently (since the automaton is deterministic) an input (resp., output) transition of the automaton. Then, Eve wins whenever the word they jointly produced is accepted by the automaton.

Here, we follow the same approach, with the additional difficulty that the players manipulate data values from an infinite alphabet. Thus, it is not immediate to relate the data values they choose with the corresponding transitions of the automaton. To that end, we study the link between the automaton game (where players pick transitions in the automaton) and the corresponding Church game. This is done through the key notion of feasible action words: a sequence of transition labels is feasible whenever it labels a run over some data word. Adam is then asked to provide feasible action words, otherwise he loses. To show that the automaton game is equivalent with the Church game, it remains to show that a strategy of Adam in the automaton game can be translated to a strategy in the Church game. The key ingredient is to be able to instantiate a given action by a data value on-the-fly, while the play unfolds.

Over (\mathbb{Q}, \leq) , as we demonstrate, the set of feasible action words is ω -regular, so the automaton game is ω -regular as well. Moreover, from a given configuration, one can locally determine whether an action can be instantiated with a data value, and pick it accordingly, which yields the sought strategy translation. Thus, both games are equivalent, and we get decidability since ω -regular games are decidable. The case of (\mathbb{N}, \leq) is much more involved and requires further developments, so we start the presentation with (\mathbb{Q}, \leq) to sharpen our tools.

3.3 The automaton game

For the rest of this section, fix a one-sided register automaton $S = (\Sigma, Q, q_{\iota}, R, \delta, \alpha)$ over an ordered data domain \mathcal{D} (it can be either (\mathbb{Q}, \leq) or (\mathbb{N}, \leq)).

Before introducing the game itself, we define the main technical notion, which relates the syntax and semantics of register automata.

Definition 2. An action word is a sequence $(tst_0, asgn_0)(tst_1, asgn_1)...$ from $(Tst \times asgn_0)(tst_1, asgn_1)$... Asgn)*, ω . It is \mathcal{D} -feasible (or simply feasible when \mathcal{D} is clear from the context) if there exists a sequence $\nu_0 \ell_0 \nu_1 \ell_1 \dots$ of register valuations ν_i and data ℓ_i over \mathcal{D} such that $\nu_0 = 0^R$ and for all $i: \nu_{i+1} = update(\nu_i, \ell_i, \mathsf{asgn}_i)$ and $(\nu_i, \ell_i) \models \mathsf{tst}_i$.

We denote by $\mathsf{Feasible}_{\mathcal{D}}(R)$ the set of action words over R feasible in \mathcal{D} .

With the Church game $(\mathcal{D}, \mathcal{D}, S)$, we associate the following automaton game, which is a finite-arena game $G_S^f = (V_{\forall}, V_{\exists}, v_0, E, W_S^f)$. Essentially, it memorises the transitions taken by the automaton S during the play of Adam and Eve. It has $V_{\forall} = \{q_{\iota}\} \cup (\Sigma \times Q_A), V_{\exists} = \mathsf{Tst} \times \mathsf{Asgn} \times Q_E, v_0 = q_{\iota}, E = E_0 \cup E_{\forall} \cup E_{\exists} \text{ where:}$

- $E_0 = \{(v_0, (\mathsf{tst}, \mathsf{asgn}, u_0)) \mid \delta(v_0, \mathsf{tst}) = (\mathsf{asgn}, u_0)\},\$
- $E_{\forall} = \left\{ \left((\sigma, v), (\mathsf{tst}, \mathsf{asgn}, u) \right) \mid \delta(v, \mathsf{tst}) = (\mathsf{asgn}, u) \right\}, \text{ and}$ $E_{\exists} = \left\{ \left((\mathsf{tst}, \mathsf{asgn}, u), (\sigma, v) \right) \mid \delta(u, \sigma) = v \right\}.$

We let:

$$W_S^f = \left\{ v_0(\mathsf{tst}_0, \mathsf{asgn}_0, u_0)(\sigma_0, v_1) \dots \left| \begin{array}{l} (\mathsf{tst}_0 \mathsf{asgn}_0) \dots \in \mathsf{Feasible}_{\mathcal{D}}(R) \\ \Rightarrow v_0 u_0 v_1 u_1 \dots \models \alpha \end{array} \right\} \right.$$

The strategies of Adam and Eve in the automaton game are of the form $\lambda_{\forall}^f: V_{\forall}(V_{\exists}V_{\forall})^* \to V_{\exists}$ and $\lambda_{\exists}^f: (V_{\forall}V_{\exists})^+ \to V_{\forall}$. Since the automaton S is deterministic, they can equivalently be expressed as $\lambda_{\forall}^f: \Sigma^* \to \mathsf{Tst}$ and $\lambda_{\exists}^f: \mathsf{Tst}^+ \to \Sigma$.

Let us show that G_S^f is a sound abstraction of G_S , in the sense that a winning strategy of Eve in G_S^f can be translated to a winning strategy of Eve in G_S , for both (\mathbb{Q}, \leq) and (\mathbb{N}, \leq) :

Proposition 3. Let S be a deterministic register automaton. If Eve has a winning strategy in G_S^f , then she has a winning strategy in the Church game G_S .

Proof. The main idea of the proof is that is G_S , Eve has more information than in G_S^f , since she knows what data values Adam played, while in G_S^f she can only access the corresponding tests.

Formally, let $\lambda_{\exists}^f : (V_{\forall}V_{\exists})^+ \to V_{\forall}$ be a winning Eve strategy in G_S^f . We construct a winning Eve strategy $\lambda_{\exists} : \mathsf{Tst}^+ \to \Sigma$ in G_S as follows³. Fix an arbitrary sequence $\mathsf{tst}_0...\mathsf{tst}_k$; we define $\lambda_{\exists}(\mathsf{tst}_0...\mathsf{tst}_k)$. First, for all $0 \le i \le k-1$, we inductively define $v_0, u_0, v_1, u_1, \ldots, v_k \in (Q_A \cup Q_E)$, $\mathsf{asgn}_0, \ldots, \mathsf{asgn}_k$, and $\sigma_1, \ldots, \sigma_k \in \Sigma$:

- The state $v_0 = q_\iota$ is the initial state of S.
- For all $0 \le i \le k$, define $u_i \in Q_E$ and asgn_i to be such that $(\mathsf{asgn}_i, u_i) = \delta(v_i, \mathsf{tst}_i)$, $\sigma_{i+1} = \lambda_{\exists}^f \big(v_0(\mathsf{tst}_0, \mathsf{asgn}_0, u_0)(\sigma_1, v_1) \dots \big(\mathsf{tst}_i, \mathsf{asgn}_i, u_i) \big)$, and $v_{i+1} = \delta(u_i, \sigma_i)$.

We then set $\lambda_{\exists}(\mathsf{tst}_0...\mathsf{tst}_k) = \sigma_{k+1}$. We now show that the constructed Eve strategy λ_{\exists} is winning in G_S . Consider an arbitrary Adam data strategy $\lambda_{\forall}^{\mathcal{D}}$, and let $(v_0, \nu_0)(u_0, \nu_1)(v_1, \nu_1)(u_1, \nu_2)...$ be an infinite run in G_S on reading the outcome $\lambda_{\forall}^{\mathcal{D}} \| \lambda_{\exists}$; it is enough to show that $v_0 u_0 v_1 u_1...$ satisfies the parity condition. Let $\ell_0 \ell_1...$ be the sequence of data produced by Adam during the play, let $\sigma_0 \sigma_1...$ be the labels produced by Eve strategy λ_{\exists} , and let $\overline{a} = (\mathsf{tst}_0, \mathsf{asgn}_0)(\mathsf{tst}_1, \mathsf{asgn}_1)...$ be the tests and assignments performed by the automaton during the run. Then, the sequence $v_0(\mathsf{tst}_0, \mathsf{asgn}_0, u_0)(\sigma_0, v_1)(\mathsf{tst}_1, \mathsf{asgn}_1, u_1)...$ constitutes a play in G_S^f , which is compatible with λ_{\exists}^f . Moreover, as witnessed by $\nu_0 \ell_0 \nu_1 \ell_1...$, the action word \overline{a} is feasible. Therefore, since λ_{\exists}^f is winning, the sequence $v_0 u_0 v_1 u_1...$ satisfies the parity condition.

The converse direction of the above proposition is in general harder, as it amounts to showing that the information provided by tests is enough. For the case of (\mathbb{Q}, \leq) , the density of the domain allows to instantiate tests on-the-fly, in a way that does

³What we really need is a winning Eve strategy of the form $\lambda_{\exists}^{\mathcal{D}}:\mathcal{D}^+\to\Sigma$. The strategy $\lambda_{\exists}:\mathsf{Tst}^+\to\Sigma$ that we construct encodes $\lambda_{\exists}^{\mathcal{D}}$ as follows: it has the same set R of registers as the automaton G_S , and performs the same assignment actions as the automaton. Then, on seeing a new data value, the strategy compares it with the register values, which induces a test, and passes this test to λ_{\exists} .

not jeopardise the feasibility of the overall sequence (Section 4.1). The case of (\mathbb{N}, \leq) is much harder, and is the subject of most of Section 4.

3.4 Application to transducer synthesis

The Church synthesis game models the reactive synthesis problem: S is a specification, and a winning strategy in G corresponds to a reactive program which implements S, i.e. whose set of behaviours abides by S.

In the finite alphabet case, Church synthesis games are ω -regular. Since those games are finite-memory determined, it means that if a specification admits an implementation, then it admits a finite-state one [14], that can be modelled as a finite-state transducer (i.e., a Mealy machine). In this section, we study at which conditions we can get an analogue of this result for specifications defined by input-driven register automata [27]. Those specifications consist in two-sided automata where the output data values are restricted to be the content of some register (in other words, the implementation is not allowed to generate data). Input-driven automata can be simulated by one-sided automata, in that output registers can be seen as finite labels. Correspondingly, we target register transducers, which generalise finite-state transducers to data domains in the same way as register automata generalise finite-state automata. We then show that finite-memory strategies in the automaton game induce register transducer implementations. Indeed, a finite-memory strategy corresponds to a sub-automaton of S, which picks output transitions in S with the help of its memory. This sub-automaton can then be interpreted as a register transducer with R registers. Note that this result is reminiscent of Proposition 5 in [27].

We now define input-driven register automata, register transducers, and then define the synthesis problem and show that it is decidable.

Input-driven register automata. An input-driven deterministic register automaton is a two-sided register automaton whose output data are required to be the content of some register. Formally, it is a tuple $S = (Q, q_{\iota}, R, \delta, \alpha)$ where $Q = Q_A \uplus Q_E$, $q_{\iota} \in Q_A$ and the transition function is

$$\delta: (Q_A \times \mathsf{Tst} \to \mathsf{Asgn} \times Q_E) \cup (Q_E \times \mathsf{Tst}_= \to \mathsf{Asgn}_\emptyset \times Q_A),$$

where $\mathsf{Tst}_{=}$ consists of tests which contain at least one atom of the form *=r for some $r \in R$, i.e. the output data value must be equal to some specification register, and $\mathsf{Asgn}_{\emptyset} = \{\emptyset\}$ meaning that output data values are never assigned to any register. This is without loss of generality, given that the output value has to be equal to the content of some register.

Correspondence with one-sided register automata. To an input-driven register automaton specification, we associate a one-sided register automaton by treating output registers as finite labels. Formally, let $S = (Q, q_\iota, R, \delta, \alpha)$ be an input-driven register automaton. Its associated one-sided automaton is $S' = (\mathsf{Tst}_=, Q, q_\iota, R, \delta', \alpha)$ (note that the finite output alphabet is $\mathsf{Tst}_=$). Up to remembering equality relations between registers, we can assume that from an output state, all outgoing transitions can be taken, independently of the registers' configuration, i.e. that from a reachable

output configuration (q_E, τ) , for all transitions $t = q_E \xrightarrow{\mathsf{tst}_=,\varnothing} q'_A$, there exists d such that $q_E \xrightarrow{d} q'_A$. This however induces a blowup of Q exponential in |R|.

The transition function is $\delta'_A = \delta_A$, and $\delta'_E(q_E, \mathsf{tst}) = q'_A$ if and only if $\delta_E(q_E, \mathsf{tst}) = (\emptyset, q'_A)$. Overall, the size of S' is exponential in |R| (because of the assumption we made on output transitions) and polynomial in |Q|.

Register transducers. A register transducer (RT) is a tuple $T = (Q, q_{\iota}, R, \delta)$, where Q is a set of states and $q_{\iota} \in Q$ is initial, R is a finite set of registers. The transition function δ is a (total) function $\delta : Q \times \mathsf{Tst} \to \mathsf{Asgn} \times R \times Q$.

The semantics of T are provided by the associated register automaton S_T . It has states $Q' = (Q_A \cup \{ \not \downarrow_A \}) \uplus (Q_E \cup \{ \not \downarrow_E \})$, where Q_A and Q_E are two disjoint copies of Q and $\not \downarrow_A$, $\not \downarrow_E$ jointly form a rejecting sink. It has initial state q_ι and set of registers R. Its transition function is defined as $q_A \xrightarrow{\mathsf{tst},\mathsf{asgn}} q_E \xrightarrow{r^=,\varnothing} q'_A$ and $q_E \xrightarrow{r^\neq,\varnothing} \not \downarrow_A$ whenever $q \xrightarrow{\mathsf{tst}|\mathsf{asgn},r} q'$, where $q \xrightarrow{\mathsf{tst}|\mathsf{asgn},r} q'$ stands for $\delta(q,\mathsf{tst}) = (\mathsf{asgn},r,q')$ (similarly for A_T). Additionally, we let $\not \downarrow_A \xrightarrow{T,\varnothing} \not \downarrow_E \xrightarrow{T,\varnothing} \not \downarrow_A$. The priority function is defined as $\alpha:q\in Q'\mapsto 2$ and $\not \downarrow_A, \not \downarrow_E\mapsto 1$, i.e. all states but $\not \downarrow_A, \not \downarrow_E$ are accepting. Then, T recognises the (total) function $f_T: \partial_0^A \partial_1^A \cdots \mapsto \partial_0^E \partial_1^E \ldots$ such that $\partial_0^A \partial_0^E \partial_1^A \partial_1^E \cdots \in L(A_T)$. For each input ω -data word, the associated output ω -data word exists since all states but $\not \downarrow_A, \not \downarrow_E$ are accepting. It is moreover unique since the output transitions that avoid the sink state are determined by the input ones, and they only contain equality tests so the corresponding output data values are unique.

Synthesis for input-driven output specifications

Given a specification S, we say that a function f realises S if they have the same domain and its graph is included in S, i.e. dom(f) = dom(S) and for all input $x \in dom(S)$, $(x, f(x)) \in S$. We then say that a register transducer T realises the register automaton specification S if f_T does, i.e. $L(T) \subseteq L(S)$.

The register transducer synthesis problem then asks to produce a T that realises S when such T exists, otherwise output "unrealisable". Note that T and S can have different sets of registers.

Proposition 4. Let $S = (Q, q_\iota, R, \delta, \alpha)$ be an input-driven register automaton, and S' its associated one-sided register automaton. If S admits a register transducer implementation, then Eve has a winning strategy in the Church game $G_{S'}$ associated with S'.

Proof. Assume that there exists a register transducer T which realises S. From T, we define a strategy λ^T in G, which simulates T and S in parallel. Given a history $\ell_0^i \dots \ell_n^i$, let ℓ_n^0 be the data output by T. As S is deterministic, there exists a unique run over the history $\ell_0^i \ell_0^0 \dots \ell_n^i \ell_n^0$; let $t = q_E \xrightarrow{\mathsf{tst}_=,\emptyset} q'_A$ be the transition taken by S on reading ℓ_n^0 . Then, define $\lambda^T(\ell_0^i \dots \ell_n^i) = \mathsf{tst}_=$. Now, for a play in G consistent with ℓ_n^T , consider the associated run in ℓ_n^T . As ℓ_n^T is an implementation and the sequence of

transitions is feasible (as witnessed by the data given as input), this run is necessarily accepting, so λ^T is indeed a winning strategy in G.

Proposition 5. Let $S = (Q, q_i, R, \delta, \alpha)$ be an input-driven register automaton, and S' its associated one-sided register automaton. If Eve wins $G_{S'}^f$ with a finite-memory strategy, then S admits a register transducer implementation.

Proof. Let $S=(Q,q_\iota,R,\delta,\alpha)$ be an input-driven register automaton, and S' its associated one-sided register automaton. Assume that Eve has a finite-memory winning strategy in G_S^f that is computed by a finite-state automaton M with states P, initial memory p_0 , transition function $\mu:P\times V_\exists\to P$ and move selection $s:P\to V_\forall$. Thus, given a history $h=v_0\ldots v_n\in V_\exists^+$, $\lambda_\exists(h):V_\exists^+\to T\mathsf{st}_\equiv$ is defined as s(p), where $p_0\xrightarrow{h}p$. Then, consider $T=(Q\times P,(q_\iota,p_0),R,\delta')$. We define δ' as follows: assume the transducer is in state (q,p). Then, the transducer receives input satisfying some test tst. In S, it corresponds to some input transition $\delta(q,\mathsf{tst})=(\mathsf{asgn},q')$. The memory is updated to $\mu(p,(\mathsf{tst},\mathsf{asgn}))=p',$ and $s(p')=\mathsf{tst}_\equiv$. Let r be such that $\mathsf{tst}_\equiv\Rightarrow r^\equiv$ (such r necessarily exists by definition of Tst_\equiv). Then, we let $\delta((q,p),\mathsf{tst})=(\mathsf{asgn},r,(q',p'))$. Now, let $w=\ell_0^A\ell_1^A\ldots$ be an input data word, and $T(w)=\ell_0^E\ell_1^E\ldots$ By construction, the run of S over $w\otimes T(w)=\ell_0^A\ell_0^E\ell_1^A\ell_1^A\ldots$ corresponds to a play consistent with λ_\exists , so it is accepting (since it is feasible, as witnessed by $w\otimes T(w)$). As a consequence, $w\otimes T(w)\in L(S)$, which means that T is indeed a register transducer implementation of S.

In the proof of Theorem 1, Eve's strategy consists in outputting a finite data word with $B \geq 0$ distinct data values, and then only zeroes. Thus, it can be implemented with a register transducer with B registers, provided that its registers can be initialised with non-zero data values (in our setting, we assume all registers are initialised to 0). As a consequence, we get:

Theorem 6. For specifications defined by two-sided deterministic register automata over data domains (\mathbb{Q}, \leq) , the register transducer synthesis problem is undecidable, provided that registers can be initialised to an arbitrary valuation.

Remark 1. The decidability status of the synthesis problem for register transducers with a fixed initial valuation 0^R is open.

4 Solving Church Synthesis Games on (\mathbb{N}, \leq)

We now have the main tools in hand to solve Church synthesis games over ordered data domains. As an introduction, before the case of (\mathbb{N}, \leq) , we apply those tools to (\mathbb{Q}, \leq) .

4.1 Warm-up: the case of (\mathbb{Q}, \leq)

First, let us observe that in that case, the automaton game is ω -regular:

Proposition 7. Let S be a one-sided register automaton over (\mathbb{Q}, \leq) . Then G_S^f is an ω -regular game.

Proof. Let $S = (\Sigma, Q, q_\iota, R, \delta, \alpha)$ be a one-sided register automaton over (\mathbb{Q}, \leq) , and let $G_S^f = (V_\forall, V_\exists, v_0, E, W_S^f)$ be its associated automaton game. G_S^f is a finite-arena game; it remains to show that it is ω-regular, i.e. that W_S^f is ω-regular. Recall that $W_S^f = \{v_0(\mathsf{tst}_0, \mathsf{asgn}_0, u_0)(\sigma_0, v_1) \dots \mid (\mathsf{tst}_0 \mathsf{asgn}_0) \dots \in \mathsf{Feasible}_{\mathbb{Q}}(R) \Rightarrow v_0 u_0 v_1 u_1 \dots \models \alpha\}$. By Theorem 20 (on page 30), we know that $\mathsf{Feasible}_{\mathbb{Q}}(R)$ is ω-regular; since α is a parity condition, one can then build an ω-regular automaton recognising W_S^f using standard automata constructions.

From Proposition 3, we already know that for all one-sided register automata S (over (\mathbb{Q}, \leq) or (\mathbb{N}, \leq)), G_S^f soundly abstracts G_S . We now show the converse for (\mathbb{Q}, \leq) :

Proposition 8. Let S be a one-sided register automaton over (\mathbb{Q}, \leq) . If Eve has a winning strategy in G_S , then she has a winning strategy in the Church game G_S^f .

Proof. We show the result by contraposition. Assume that Eve does not win G_S^f . As G_S^f is ω -regular (Proposition 7), it is determined, so Adam has a winning strategy $\lambda_{\forall}^f: V_{\forall}(V_{\forall}V_{\exists})^* \to V_{\exists}$ in G_S^f . We construct the winning Adam data strategy $\lambda_{\forall}^{\mathbb{Q}}$ in G_S step-by-step, by instantiating the tests on-the-fly. When the test is an equality, pick the corresponding data, and when it is of the form r < * < r', take some rational number strictly in the interval.

Formally, suppose we are in the middle of a play: $\ell_0...\ell_{k-1}$ has been played by Adam $\lambda_{\forall}^{\mathbb{Q}}$ and $\sigma_0...\sigma_{k-1}$ has been played by Eve; both sequences are empty initially. We want to know the value ℓ_k for $\lambda_{\forall}^{\mathbb{Q}}(\sigma_0...\sigma_{k-1})$. Let $(v_0, \nu_0)(u_0, \nu_1)(v_1, \nu_1)(u_1, \nu_2)...(v_k, \nu_k)$ be the current run prefix of the register automaton G_S (initially (v_0, ν_0)). We construct the corresponding play prefix $v_0(\mathsf{tst}_0, \mathsf{asgn}_0, u_0)(\sigma_0, v_1)(\mathsf{tst}_1, \mathsf{asgn}_1, u_1)(\sigma_1, v_2)...(\sigma_{k-1}, v_k)$ of G_f (initially v_0). We assume that this play prefix adheres to λ_{\forall}^f (this holds initially). We now consult λ_{\forall}^f : let $(\mathsf{tst}_k, \mathsf{asgn}_k, u_k) = \lambda_{\forall}^f(\sigma_{k-1}, v_k)$. Using tst_k and ν_k , we construct ℓ_k as follows.

- If tst_k contains * = r for some $r \in R$, we set $d_k = \nu_k(r)$.
- If tst_k is of the form r < * for all $r \in R$, then set $d_k = \max(\nu_k) + 1$, i.e. take the largest value held in the registers plus 1.
- Similarly, if tst_k is of the form * < r for all $r \in R$, then set $d_k = \min(\nu_k) 1$.
- Otherwise, for every $r \in R$, the test tst_k has either r < * or * < r. We now pick two registers r, s such that the test contains r < * and * < s and no register holds a value between $\nu_k(r)$ and $\nu_k(s)$. Then we set $d_k = \frac{\nu_k(r) + \nu_k(s)}{2}$.

It is easy to see that d_k satisfies tst_k , i.e. $(\nu_k, d_k) \models \mathsf{tst}_k$. Finally, define $\nu_{k+1} = update(\nu_k, d_k, \mathsf{asgn}_k)$. Thus, the next configuration of the run in the register automaton is (u_k, ν_{k+1}) . In G_f , the play is extended by $(\mathsf{tst}_k, \mathsf{asgn}_k, u_k)$; notice that the resulting extended play again adheres to the winning Adam strategy λ_\forall^f . Therefore, starting from the empty sequences of Adam data choices and Eve label choices, step-by-step we construct the values for $\lambda_\forall^\mathbb{Q}$.

Then, each play consistent with this strategy in G_S corresponds to a unique run in S, which is also a play in G_f . As λ_{\forall}^f is winning, such a run is accepting, so λ_{\forall} is winning: Eve does not win G_S .

We are now ready to show:

Theorem 9. Let $S = (\Sigma, Q, q_{\iota}, R, \delta, \alpha)$ be a one-sided register automaton over (\mathbb{Q}, \leq) .

- 1. The problem of determining if Eve wins the Church synthesis game $G = (\mathcal{D}, \mathcal{D}, S)$ is decidable in time polynomial in |Q| and exponential in c and |R|.
- 2. G_S is determined, i.e. either Eve or Adam has a winning strategy in G_S .

Proof of Theorem 9. First, by Propositions 3 and 8, we know that Eve G_S iff she wins G_S^f .

By analysing the constructions of Propositions 7 and Theorem 20, we get that the automaton game G_S^f is of size polynomial in |Q| and exponential in |R|, and has a number of priorities linear in c, so it can be solved in $O((poly(|Q|)2^{poly(|R|)})^c)$, which yields item 1 of the theorem.

Then, determinacy (item 2) follows from the determinacy of G_S^f , since it is equivalent with G_S .

As a consequence of Propositions 4 and 5, we also get:

Proposition 10. Let S be an input-driven register automaton, and S' its associated one-sided register automaton. The following are equivalent:

- Eve has a winning strategy in $G_{S'}$
- Eve has a winning strategy in $G_{S'}^f$
- Eve has a finite-memory winning strategy in $G_{S'}^f$
- $\bullet \ S \ admits \ a \ register \ transducer \ implementation$
- ullet S admits an implementation

Thus, we have:

Theorem 11. For specifications defined by deterministic input-driven output register automata over data domains (\mathbb{Q}, \leq) , the register transducer synthesis problem is equivalent with the synthesis problem (for arbitrary implementations) and can be solved in time polynomial in |Q| and exponential in C and |R|.

Remark 2. For data domain (\mathbb{Q}, \leq) , the synthesis problem for specifications defined by two-sided register automata is also decidable, if the target implementation is any program, as the Church game again reduces to a parity game: checking feasibility is still doable using a parity automaton. However, in general, register transducers might not suffice; e.g. the environment can ask the system to produce an infinite sequence of data values in increasing order. Yet, it can be shown that implementations can be restricted to simple programs, which can be modelled by register transducers which have the additional ability to pick a data between two others, e.g. by computing $\frac{d_1+d_2}{2}$: such ability suffices to translate a finite-memory strategy in the automaton game to an implementation.

We now shift to the main result of the paper, namely that Church synthesis games are decidable over (\mathbb{N}, \leq) . We start by providing some results on actions sequences over (\mathbb{N}, \leq) that highlight the difficulties and hint at how to overcome them (Section 4.2). We then use those results to define an ω -regular approximation of the automaton game that we show to be sound and complete (Section 4.3).

4.2 Action sequences over (\mathbb{N}, \leq)

Action sequences over (\mathbb{N}, \leq) are not ω -regular

First, contrary to (\mathbb{Q}, \leq) , one needs a global condition on action sequences to check whether they are feasible. To get an intuition, consider the action sequence $(\top\{r\})((r>*)r)^{\omega}$, that asks for an initial data value (stored in r), and then repeatedly asks to provide smaller and smaller data values. While feasible in (\mathbb{Q}, \leq) , such a sequence is not feasible in (\mathbb{N}, \leq) , as it would yield an infinite descending chain in \mathbb{N} . And, actually, the discreteness of (\mathbb{N}, \leq) implies that the set of feasible action sequences is not ω -regular in (\mathbb{N}, \leq) (see, e.g., [21, Corollary 6.5] or [47, Appendix C]). We provide an example, for self-containedness.

Example 2. consider the automaton of Figure 3, which essentially consists in that of Figure 1 (on page 4) where we allow Adam to repeatedly try his luck by taking the transition from C to B. Note that the priorities (written above the states) ensure that if he does so, he loses. Then, consider sequences of states in $A(BC(DC)^*)^{\omega}$,

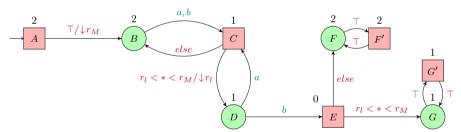


Figure 3: Eve wins this game in \mathbb{N} (but loses in \mathbb{Q}).

where Adam initially picks a value, the game transitions to B then C, then Adam and Eve loop between B and C for some time, until at some point Adam transitions back to B, and so on. To check whether such a sequence actually corresponds to a play, one needs to check that there exists a uniform bound (the content of r_M) over the iterations of DC. Formally, plays in $A(BC(DC)^*)^{\omega}$ are of the form $A(BC(DC)^{n_0})(BC(DC)^{n_1})...$ where there exists $B \geq 0$ such that for all $i \geq 0$, $n_i \leq B$. By an elementary pumping argument, one can show that this language is not ω -regular [6].

This implies that $\mathsf{Feasible}_{\mathbb{N}}(R)$ is not ω -regular whenever $|R| \geq 2$, and neither is the automaton game. We thus consider an ω -regular over-approximation of the automaton game, and show that both games are actually equivalent.

Constraint sequences, consistency and satisfiability

To introduce the said approximation, we first require a further study of $\mathsf{Feasible}_{\mathbb{N}}(R)$, that we conduct through the notion of constraint sequences. To ease the comparison between (\mathbb{Q}, \leq) and (\mathbb{N}, \leq) , we define them for both domains. Thus, in this section, fix an ordered domain \mathcal{D} .

Given a set of registers R (which can also be thought of as variables), we let $R' = \{r' \mid r \in R\}$ be the set of their *primed* versions. Given a valuation $\nu \in \mathcal{D}^R$, define $\nu' \in \mathcal{D}^{R'}$ to be the valuation that maps $\nu'(r') = \nu(r)$ for every $r \in R$.

Definition 3. A constraint over R is a total non-strict preorder over $R \cup R'$, i.e. a total order with ties allowed. It can be represented as a maximally consistent set of atoms of the form $t_1 \bowtie t_2$ where $t_1, t_2 \in R \cup R'$, where the symbol \bowtie denotes one of >, <, or =.

Given a constraint C, the writing $C_{|R}$ denotes the subset of its atoms $r \bowtie s$ for $r, s \in R$, and $C_{|R'|}$ denotes the subset of atoms over primed registers. Given a set S of atoms $r' \bowtie s'$ over $r', s' \in R'$, let unprime(S) be the set of atoms derived by replacing every $r' \in R'$ by r.

A state constraint relates registers in the current moment only: it contains atoms over non-primed registers, so it has no atoms over primed registers. Note that both $C_{|R|}$ and $unprime(C_{|R'})$ are state constraints.

A constraint describes how register values change in one step: their relative order at the beginning (when $t_1, t_2 \in R$), at the end (when $t_1, t_2 \in R'$), and in between (with $t_1 \in R$ and $t_2 \in R'$).

Example 3. For instance, the ordering $r_1 < r'_1 < r'_2 < r_2$ is a constraint over $R = \{r_1, r_2\}$ and can be represented by $\{r_1 < r_2, r_1 < r'_1, r_2 > r'_2, r'_1 < r'_2\}$; it is satisfied e.g. by the two successive valuations $\nu_a : \{r_1 \mapsto 1, r_2 \mapsto 4\}$ and $\nu_b : \{r_1 \mapsto 2, r_2 \mapsto 3\}$. Similarly, $r_1 = r'_1 < r'_2 = r_2$ is a constraint corresponding to the set $\{r_1 < r_2, r_1 = r'_1, r_2 = r'_2, r'_1 < r'_2\}$. Note that the set $\{r_1 < r_2, r_1 > r'_1, r_2 < r'_2, r'_1 > r'_2\}$ does not represent a constraint: it is not consistent since $r_1 > r'_1 > r'_2 > r_2 > r_1$ implies $r_1 > r_1$, violating irreflexivity, and thus does not correspond to any total non-strict preorder. Another counter-example is $r \le r'$ for $R = \{r\}$: it is not a constraint since it is not total.

Definition 4. A constraint sequence is then an infinite sequence of constraints $C_0C_1...$ (when a sequence is finite, we explicitly state it).

It is *consistent* if for every i: $unprime(C_{i|R'}) = C_{i+1|R}$, i.e. the register order at the end of step i equals the register order at the beginning of step i + 1.

A valuation $\omega \in \mathcal{D}^{R \cup R'}$ satisfies a constraint C, written $\omega \models C$, if every atom holds when we replace every $r \in R \cup R'$ by $\omega(r)$. A constraint sequence is satisfiable if there exists a sequence of valuations $\nu_0 \nu_1 \dots \in (\mathcal{D}^R)^\omega$ such that $\nu_i \cup \nu'_{i+1} \models C_i$ for all $i \geq 0$. If, additionally⁴, $\nu_0 = 0^R$, then it is 0-satisfiable. Note that satisfiability implies consistency, but not vice versa, as we show below.

⁴Recall that over (\mathbb{N}, \leq) , 0 denotes its minimal element. Over (\mathbb{Q}, \leq) , its choice is irrelevant.

Note also that the notions of constraints and constraint sequences over (\mathbb{N}, \leq) and over (\mathbb{Q}, \leq) syntactically coincide. This is done on purpose, to ease the comparison between the two domains. When this matters, we always make it clear on which domain a constraint sequence is meant to be interpreted.

Finally, remark that consistency also coincides for both domains, while satisfiability does not, as witnessed by the constraint sequence $(\{r > r'\})^{\omega}$ over $R = \{r\}$: it is satisfiable in \mathbb{Q} but not in \mathbb{N} .

Example 4. We give a richer example. Let $R = \{r_1, r_2, r_3, r_4\}$. Let a consistent constraint sequence $C_0C_1...$ start with

$$\{r_2' < r_1 = r_1' < r_2 < r_3 = r_4' < r_4 = r_3'\} \{r_1' < r_2 = r_2' < r_1 < r_4 = r_3' < r_3 = r_4'\}$$

Figure 4 visualises C_0C_1 plus a bit more constraints. The black lines represent the evolution of the same register; ignore the colored paths for now. The constraint C_0 describes the transition from moment 0 to 1, and C_1 the transition from moment 1 to 2. This finite constraint sequence is satisfiable in \mathbb{Q} and in \mathbb{N} . For example, the valuations can start with $\nu_0 = \{r_4 \mapsto 6, r_3 \mapsto 5, r_2 \mapsto 4, r_1 \mapsto 3\}$. In \mathbb{N} , no valuations starting with $\nu_0(r_3) < 5$ can satisfy the sequence. Further, since the constraint C_0 requires all registers in R to differ, the sequence is not 0-satisfiable in \mathbb{Q} nor in \mathbb{N} .

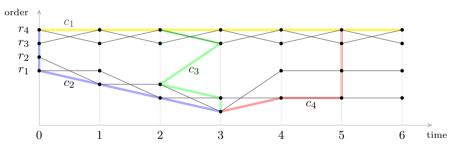


Figure 4: Visualisation of a constraint sequence. Individual register values are depicted by black dots, and dots are connected by black lines when they talk about the same register. Yellow/blue/green/red paths depict chains (cf infra).

Chains

This section describes a characterisation of satisfiable constraint sequences that is amenable to being recognised by automata. The proofs are quite technical, so we defer them to Section 5 and for the time being we only give an intuition.

Definition 5 (Chains). Fix R and a consistent constraint sequence $C_0C_1...$ over R. A (decreasing) two-way chain is a finite or infinite sequence $(r_0, m_0) \triangleright_0 (r_1, m_1) \triangleright_1... \in ((R \times \mathbb{N}) \cdot \{=, >\})^{*,\omega}$ satisfying the following (note that m_0 can differ from 0).

• $m_{i+1} = m_i$, or $m_{i+1} = m_i + 1$ (time flows forward), or $m_{i+1} = m_i - 1$ (backwards).

- If $m_{i+1} = m_i$ then $(r_i \triangleright_i r_{i+1}) \in C_{m_i}$.
- If $m_{i+1} = m_i + 1$ then $(r_i \triangleright_i r'_{i+1}) \in C_{m_i}$.
- If $m_{i+1} = m_i 1$ then $(r'_i \triangleright_i r_{i+1}) \in C_{m_i-1}$.

The depth of a chain is the number of >; when it is infinity, the chain is infinitely decreasing. Figure 4 highlights four two-way chains (there are more) with yellow, blue, green and red colors. For instance, the green-colored chain c_3 , defined as $(r_4, 2) > (r_3, 3) > (r_2, 2) > (r_1, 3) > (r_2, 3)$, has depth 4.

Given a moment i and a register x, a (decreasing) right two-way chain starting in(x,i) (r2w for short) is a two-way chain $(x,i) \triangleright_1 (r_1,m_1) \triangleright_2 (r_2,m_2) \dots$ such that $m_j \ge i, \triangleright_j \in \{=, >\}$, for all j. Thus, all elements appear to the right of the starting moment (x,i).

We define *one-way* chains similarly, except that time now flows forwards or stays the same, and that they can be either increasing or decreasing:

- $m_{i+1} = m_i$ (time does not flow), or $m_{i+1} = m_i + 1$ (time flows forward).
- If $m_{i+1} = m_i$ then $(r_i \bowtie_i r_{i+1}) \in C_{m_i}$.
- If $m_{i+1} = m_i + 1$ then $(r_i \bowtie_i r'_{i+1}) \in C_{m_i}$.

A one-way chain is decreasing (respectively, increasing) if for all $i \geq 0$, $\bowtie_i \in \{>, =\}$ (resp., $\bowtie_i \in \{<, =\}$).

In Figure 4, the blue (c_2) chain $(r_4,0) > (r_3,0) > (r_2,0) > (r_1,0) > (r_2,1) > (r_1,2) > (r_2,3)$ is one-way decreasing chain of depth 6; the same sequence is also a two-way chain. The red (c_4) chain $(r_2,3) < (r_1,4) = (r_1,5) < (r_2,5) < (r_4,5) < (r_3,5)$ is one-way increasing of depth 4; if we read the sequence in reverse, it represents a two-way chain (two-way chains are always decreasing). Sometimes we write "chain" omitting whether it is two- or one-way.

A stable chain is an infinite chain $(r_0, m) = (r_1, m+1) = (r_2, m+2) = ...;$ it can also be written as $(m, r_0r_1r_2...)$. In Figure 4, the yellow (c_1) chain $(0, (r_4r_3)^{\omega})$ is stable. Given a stable chain $\chi_r = (m, r_0r_1...)$ and a chain $\chi_s = (s_0, n_0) \bowtie_0 (s_1, n_1) \bowtie_1$..., where $n_i \geq m$ for all i, the chain χ_r is above χ_s (equiv., χ_s is below χ_r) if for all i the constraint C_{n_i} contains $r_{n_i-m} > s_i$ or $r_{n_i-m} = s_i$; here we used $n_i - m$ because the register at moment n_i in the chain χ_r is r_{n_i-m} . In Figure 4, the yellow chain $(0, (r_4r_3)^{\omega})$ is above all colored chains. A stable chain $(m, r_0r_1...)$ is maximal if it is above all other stable chains starting after m. In Figure 4, the yellow chain $(0, (r_4r_3)^{\omega})$ is maximal (assuming the sequence evolves in a similar fashion). Notice that if a sequence has a stable chain, then it has a maximal one. A ceiled chain is a chain that is below a maximal stable chain. A constraint sequence can have an infinite number of ceiled chains; it can also have zero, e.g. when there are no stable chains.

Note that in this section, we mostly focus on one-way chains and right two-way chains, while two-way chains are used in Section 5.1 as a technical intermediate. In the latter section, we show:

Lemma 12. A consistent constraint sequence is 0-satisfiable in \mathbb{N} iff there exists $B \geq 0$ such that:

1. it has no infinitely decreasing one-way chains,

- 2. the ceiled one-way chains have a depth at most B
- 3. it starts in C_0 s.t. $C_{0|R} = \{r = s \mid r, s \in R\}$, and
- 4. it has no decreasing one-way chains of depth ≥ 1 from (r,0) for any r.

In line with Example 2, the above characterisation is not ω -regular; the culprit is item 2. We thus define quasi-feasible constraint sequences, by relaxing the condition to asking that there are no infinite increasing ceiled chains.

Definition 6. A consistent constraint sequence is *quasi-feasible* whenever:

- it has no infinitely decreasing one-way chains,
- it has no infinitely increasing ceiled one-way chains,
- it starts in C_0 s.t. $C_{0|R} = \{r = s \mid r, s \in R\}$, and
- it has no decreasing one-way chains of depth ≥ 1 from (r,0) for any r.

In Section 5.3 on page 43, we show:

Lemma 26. A lasso-shaped consistent constraint sequence is 0-satisfiable if and only if it is quasi-feasible.

We conclude the section by formally relating action words (see Definition 2) with constraint sequences.

Action words and constraint sequences

Every action word naturally induces a unique constraint sequence. For instance, for registers $R = \{r, s\}$, an action word starting with $(\{r < *, s < *\}, \{s\})$ (test whether the current data ℓ is above the values of r and s, store it in s) induces a constraint sequence starting with $\{r = s, r = r', s < s', r' < s'\}$ (the atom r = s is due to all registers being equal initially). This is formalised in the next lemma, which is notation-heavy but says a simple thing: given an action word, we can construct, on the fly, a constraint sequence that is 0-satisfiable iff the action word is feasible. For technical reasons, we need a new register r_d to remember the last Adam data. The proof is on page 36, so as not to break the flow of the argument.

Lemma 13. Let R be a set of registers, $R_d = R \uplus \{r_d\}$, and $\mathfrak D$ be $(\mathbb N, \leq)$ or $(\mathbb Q, \leq)$. There exists a mapping constr : $\Pi \times \mathsf{Tst} \times \mathsf{Asgn} \to \mathsf C$ from state constraints Π over R_d and tests-assignments over R to constraints $\mathsf C$ over R_d , such that for all action words $a_0a_1a_2... \in (\mathsf{Tst} \times \mathsf{Asgn})^\omega$, $a_0a_1a_2...$ is feasible iff $C_0C_1C_2...$ is 0-satisfiable, where $\forall i \geq 0$: $C_i = constr(\pi_i, a_i)$, $\pi_{i+1} = unprime(C_{i|R'_2})$, $\pi_0 = \{r = s \mid r, s \in R_d\}$.

Then, given a set of registers R, we say that an action word \overline{a} is quasi-feasible whenever $constr(\overline{a})$ is quasi-feasible. We correspondingly denote by $\mathsf{QFeasible}_{\mathbb{N}}(R)$ the set of quasi-feasible action words over R.

4.3 The ω -regular game G_S^{reg}

After this long but necessary detour through constraint sequences, we are ready to define the ω -regular game associated with the automaton game. Recall that in Section 3.3, given a one-sided automaton S, we defined $G_S^f = (V_{\forall}, V_{\exists}, v_0, E, W_S^f)$. We

now let $G_S^{reg} = (V_\forall, V_\exists, v_0, E, W_S^{reg})$. Thus, it has the same vertices and edge relation: $V_\forall = \{q_\iota\} \cup (\Sigma \times Q_A), \ V_\exists = \mathsf{Tst} \times \mathsf{Asgn} \times Q_E, \ v_0 = q_\iota, \ E = E_0 \cup E_\forall \cup E_\exists \ \text{where:}$

- $$\begin{split} \bullet \ E_0 &= \big\{ \big(v_0, (\mathsf{tst}, \mathsf{asgn}, u_0)\big) \mid \delta(v_0, \mathsf{tst}) = (\mathsf{asgn}, u_0) \big\}, \\ \bullet \ E_\forall &= \big\{ \big((\sigma, v), (\mathsf{tst}, \mathsf{asgn}, u)\big) \mid \delta(v, \mathsf{tst}) = (\mathsf{asgn}, u) \big\}, \text{ and } \\ \bullet \ E_\exists &= \big\{ \big((\mathsf{tst}, \mathsf{asgn}, u), (\sigma, v)\big) \mid \delta(u, \sigma) = v \big\}. \end{split}$$

However, the winning condition is now:

$$W_S^f = \left\{ v_0(\mathsf{tst}_0, \mathsf{asgn}_0, u_0)(\sigma_0, v_1) \dots \middle| \begin{array}{l} (\mathsf{tst}_0 \mathsf{asgn}_0) \dots \in \mathsf{QFeasible}_{\mathbb{N}}(R) \\ \Rightarrow v_0 u_0 v_1 u_1 \dots \models \alpha \end{array} \right\}$$

i.e., we replaced $\mathsf{Feasible}_{\mathbb{N}}(R)$ with $\mathsf{QFeasible}_{\mathbb{N}}(R)$.

First, by Proposition 27, we know that $\mathsf{QFeasible}_{\mathbb{N}}(R)$ is ω -regular. Thus:

Proposition 14. Let S be a one-sided automaton, and define G_S^{reg} as above. Then, G_{S}^{reg} is an ω -regular game.

We now show that it is equivalent with the Church game G_S .

Proposition 15. Let S be a one-sided automaton, G_S the corresponding Church game, G_S^f its automaton game, and G_S^{reg} its associated ω -regular game. The following are equivalent:

- 1. Eve has a winning strategy in G_S^{reg}
- 2. Eve has a finite-memory winning strategy in G_S^{reg}
- 3. Eve has a finite-memory winning strategy in G_s^f
- 4. Eve has a winning strategy in G_S^f
- 5. Eve has a winning strategy in G_S .

Proof. We start with the chain of implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$.

The implication (1) \Rightarrow (2) holds because G_S^{reg} is ω -regular, and we know that those games are finite-memory determined [34].

Then, (2) \Rightarrow (3) follows from the fact that G_S^{reg} is actually harder than G_S^f , i.e. $W_S^{reg} \subseteq W_S^f$, because $\mathsf{Feasible}_{\mathbb{N}}(R) \subseteq \mathsf{QFeasible}_{\mathbb{N}}(R)$.

- $(3) \Rightarrow (4)$ is immediate.
- $(4) \Rightarrow (5)$ is exactly Proposition 3.

It remains to show that $(5) \Rightarrow (1)$. We proceed by contraposition. Thus, assume that Eve does not have a winning strategy in G_f^{reg} . By finite-memory determinacy of games with parity objectives, in G_f^{reg} Adam has a finite-memory winning strategy $\lambda_{\forall}^f: V_{\forall}(V_{\exists}V_{\forall})^* \to V_{\exists}$ (equiv., $\lambda_{\forall}^f: \Sigma^* \to \mathsf{Tst}$). We show the following:

Proposition 16. If Adam has a winning strategy in G_S^{reg} , then he has a winning strategy in G_S .

Proof. At first, it is not clear how to instantiate it to a data strategy $\lambda_{\forall}^{\mathbb{N}}: \Sigma^* \to \mathbb{N}$ winning in G_S . For instance, if the strategy λ_{\forall}^f in G_f^{reg} dictates Adam to pick the test * > r, it is not clear which data should $\lambda_{\forall}^{\mathbb{N}}$ pick $(\nu(r) + 1, \nu(r) + 2, \text{more?})$ because for different strategies of Eve different values may be needed. To construct $\lambda_{\forall}^{\mathbb{N}}$ from λ_{\forall}^{f} that beats every Eve, we show that for any finite-memory strategy of Adam, there is a uniform bound on the depth of all its r2w chains. This is formalised by the following claim (that we prove afterwards):

Claim 17. Let λ_{\forall}^f be a finite-memory strategy of Adam that is winning in G_f^{reg} . There exists a bound $B \geq 0$ such that for each play ρ consistent with λ_{\forall}^f , for each right two-way chain γ of the constraint sequence induced by ρ (starting in some $(r, i) \in R \times \mathbb{N}$), $depth(\gamma) \leq B$.

Thanks to existence of this uniform bound B, we can construct $\lambda_{\forall}^{\mathbb{N}}$ from λ_{\forall}^f as follows. First, translate the currently played action-word prefix $(\mathsf{tst}_0, \mathsf{asgn}_0)...(\mathsf{tst}_m, \mathsf{asgn}_m)$ into a constraint-sequence prefix using Lemma 13. Then apply to it the data-assignment function from Lemma 28. By construction, for each play in G consistent with $\lambda_{\forall}^{\mathbb{N}}$, the corresponding run in S is a play consistent with λ_{\forall}^f in G_f^{reg} . As λ_{\forall}^f is winning, this run is not accepting, i.e. the play is winning for Adam in G_S .

Therefore, $\lambda_{\forall}^{\mathbb{N}}$ is a winning Adam's strategy in G_S . End of the proof of Prop. 16 \square As a consequence, Eve does not have a winning strategy in G_S , which means that $(5) \Rightarrow (1)$. End of the proof of Prop. 15 \square

We are left to prove Claim 17.

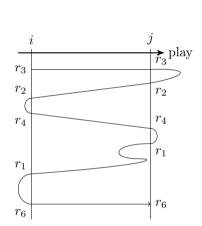
Boundedness of right two-way chains induced by Adam (Proof of Claim 17)

Proof idea. If Adam has a finite-memory strategy, then if a decreasing right two-way chain γ is sufficiently deep, Eve can force Adam to loop in a memory state in a way such that the loop can be iterated while preserving the chain. We can additionally ensure that this chain contains a strictly decreasing or increasing segment. When iterated, this segment makes the chain unfeasible. Indeed, if the segment is decreasing, iterating the loop yields an infinite descending chain in \mathbb{N} , which is not feasible. The case of an increasing fragment happens when γ is decreasing from right to left (recall that it is a two-way chain), so increasing from left to right. When iterated, this yields an infinite increasing chain, which is perfectly fine in \mathbb{N} . However, it can be bounded from above with the help of γ : before decreasing from right to left, γ has to go from left to right, since it is a right chain (i.e. it is not allowed to go to the left of its initial position). On the strictly increasing segment, this left-to-right prefix is either constant or decreasing, so when the loop is iterated it provides an upper bound for our increasing chain.

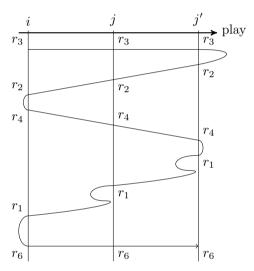
Proof. We now move to the formal proof. We could use a Ramsey argument in the spirit of Lemma 23 to extract an infinite *one-way* chain that is either increasing or decreasing. However, this amounts to breaking a butterfly upon the wheel, and we prefer to rely on a simpler pumping argument, which also gives a finer-grained

perception of what is happening there. In particular, it provides a bound B that does not depend on a Ramsey number.

Thus, let λ_{\forall}^f be a finite-memory strategy of Adam with memory M that is winning in G_S . Suppose, towards a contradiction, that there exists a play ρ that is consistent with λ_{\forall}^f and which contains a decreasing right two-way chain of depth $D > |M| \cdot 2^{2|R|^2}$. We denote it $\gamma = (r_0, m_0) \triangleright_0 (r_1, m_1) \triangleright_1 (r_2, m_2) \triangleright_2 \dots \triangleright_{n-1} (r_n, m_n)$, where for all $0 \le i \le n$, $\triangleright_i \in \{>,=\}$, $r_i \in R$ and $m_i \in \mathbb{N}$. Given a two-way chain and a position $i \ge m_0$, we define the crossing section at i as the sequence of registers that occur at position i, ordered by their appearance in the chain: $\langle \gamma \rangle_i$ is the maximal subword of γ that contains letters of the form (r,i) for some $r \in R$ (see Fig. 5a, where we depicted a chain that has two identical crossing sections at positions i and j). This



(a) A chain with two identical crossing sections.



(b) Iterating a fragment of a play. We are able to glue the chain since the crossing sections and the order between registers are the same at positions i and j.

construction is reminiscent of the techniques that are used to study loops in two-way automata or transducers, hence the name. At each position, there are |M| distinct memory states for Adam, less than $2^{|R|^2}$ many distinct crossing sections and less than $2^{|R|^2}$ many possible orderings of the registers. As a consequence there exists two positions $m_0 \leq i < j$ such that $\langle \gamma \gamma_i \rangle = \langle \gamma \gamma_j \rangle$, the memory state of Adam at position i and j is the same, the order between registers at position i is the same at position i, and there is at least one occurrence of i in the chain segment. Since i is finite-memory, Eve can repeat her actions between positions i and i indefinitely to iterate this fragment of the play i. Since the crossing sections match and the order between registers is the same at positions i and i, we can glue the chain fragments together

to get an infinite two-way chain (see Fig.5b), with infinitely many occurrences of >. There are two cases:

- There is a fragment that strictly decreases from left to right (as the chain fragment over register r_4 in Fig.5b). Then, when Eve repeats her actions indefinitely, this yields an infinite descending chain, which means that the play is not feasible (Lemma 22), so Eve wins. This contradicts the fact that λ_{\forall}^f is winning.
- All decreasing fragments occur from right to left (as do the fragments over r_2 and r_1 in Fig.5b). Necessarily, the topmost fragment, i.e. the fragment of the register that appears first in $(\gamma)_i$, is left-to-right, since γ is a right two-way chain. It is not strictly decreasing, otherwise we are back to the first case. Then, the strictly decreasing fragments are bounded from above by this constant fragment. Iterating the loop yields an infinite increasing chain that is bounded from above, which means that the play is again not feasible, so we again obtain a contradiction.

Overall, the depth of the decreasing right two-way chains induced by λ_{\forall}^f is uniformly bounded by $\mathbf{B} = |M| \cdot 2^{2|R|^2}$, where |M| is the size of Adam's memory.

We finally have all the cards in hand to show:

Theorem 18. Let $S = (\Sigma, Q, q_{\iota}, R, \delta, \alpha)$ be a one-sided register automaton over (\mathbb{N}, \leq) .

- 1. The problem of determining if Eve wins the Church synthesis game $G = (\mathcal{D}, \mathcal{D}, S)$ is decidable in time polynomial in |Q| and exponential in c and |R|.
- 2. G_S is determined, i.e. either Eve or Adam has a winning strategy in G_S .

Proof. For (\mathbb{N}, \leq) , item (1) follows from Proposition 15 and from the fact that G_f^{reg} is of size polynomial in |Q| and exponential in |R|. Item (2) on determinacy is proven as follows. Assume Eve loses G_S . By Proposition 15, Eve loses G_f^{reg} . In the proof of Proposition 15, we have shown (Proposition 16) that in this case Adam has a strategy winning in the original Church game. As a consequence, our Church games are determined.

With the help of Proposition 15, since finite-memory winning strategies of Eve in G_S^f correspond to register transducer implementations (Proposition 4), we also get:

Theorem 19. For specifications defined by deterministic input-driven output register automata over data domains (\mathbb{N}, \leq) , the register transducer synthesis problem is equivalent with the synthesis problem (for arbitrary implementations) and can be solved in time polynomial in |Q| and exponential in c and |R|.

5 Satisfiability of Constraint Sequences in (\mathbb{N}, \leq)

This section studies the problem of checking whether a given infinite sequence of constraints can be satisfied with values from domain N. Recall that constraints and constraint sequences are respectively defined in Definitions 3 and 4 on page 22. This section's structure is:

- We start with a simple and relatively known result on satisfiability of constraint sequences in data domain \mathbb{Q} . We then focus completely on \mathbb{N} .
- Section 5.1 describes conditions on chains that characterise satisfiable constraint sequences (in N).
- Section 5.2 describes "max-automata" characterisation of satisfiable constraint sequences. The max-automaton characterisation checks the conditions on chains introduced in Section 5.1.
- In the study of Church synthesis games on \mathbb{N} , the crucial role play lasso-shaped constraint sequences and their satisfiability. We rely on them when proving Proposition 15. The satisfiability of such sequences is the focus of Section 5.3, which shows that the regularity of sequences allows for characterisation of the satisfiability using classical ω -regular automata instead of max-automata. Thus, in the context of Church synthesis games, the max-automaton characterisation is not used.
- Section 5.4 shows that "depth-bounded" constraint sequences can be mapped to satisfying valuations on-the-fly: such a data assignment function is used when proving the decidability of Church synthesis games (Proposition 15), namely, to show that winning Adam's strategies in abstracted finite-alphabet games can be instantiated to winning data Adam's strategies in Church synthesis games.

Satisfiability of constraint sequences in \mathbb{Q}

Before proceeding to our main topic of satisfiability of constraint sequences in \mathbb{N} , we describe, for completeness, similar results for \mathbb{Q} .

The following result is glimpsed in several places (e.g. in [47, Appendix C]): a constraint sequence is satisfiable in $\mathbb Q$ iff it is consistent. This is a consequence of the following property which holds because $\mathbb Q$ is dense: for every constraint C and $\nu \in \mathbb Q^R$ such that $\nu \models C_{|R}$, there exists $\nu' \in \mathbb Q^{R'}$ such that $\nu \cup \nu' \models C$. Consistency can be checked by comparing every two consecutive constraints of the sequence. Thus, it is not hard to show that consistent – hence satisfiable – constraint sequences in $\mathbb Q$ are recognisable by deterministic parity automata.

Theorem 20. There is a deterministic parity automaton with two colors and of size exponential in |R| that accepts exactly all constraint sequences satisfiable (or 0-satisfiable) in \mathbb{O} .

To prove the result, we first show that a constraint sequence in \mathbb{Q} is satisfiable iff it is consistent, then we construct an automaton checking the consistency.

Lemma 21. Let R be a set of registers and $\mathfrak{D} = \mathbb{Q}$. A constraint sequence $C_0C_1...$ is satisfiable iff it is consistent. It is 0-satisfiable iff it is consistent and $C_{0|R} = \{r_1 = r_2 \mid r_1, r_2 \in R\}$.

Proof. Direction \Rightarrow is simple for both claims, so we only prove direction \Leftarrow .

Consider the first claim, direction \Leftarrow . Assume the sequence is consistent. We construct $\nu_0\nu_1\cdots\in(\mathbb{Q}^R)^\omega$ such that $\nu_i\cup\nu'_{i+1}\models C_i$ for all i. The construction proceeds step-by-step and relies on the following fact (†): for every constraint C and $\nu\in\mathbb{Q}^R$ such that $\nu\models C_{|R}$, there exists $\nu'\in\mathbb{Q}^{R'}$ such that $\nu\cup\nu'\models C$. Then define

 $\nu_0, \nu_1 \dots$ as follows: start with an arbitrary ν_0 satisfying $\nu_0 \models C_{0|R}$. Given $\nu_i \models C_{i|R}$, let ν_{i+1} be any valuation in \mathbb{Q}^R that satisfies $\nu_i \cup \nu'_{i+1} \models C_i$ (it exists by (†)). Since $\nu_{i+1} \models C_{i|R'}$, and $unprime(C_{i|R'}) = C_{i+1|R}$ by consistency, we have $\nu_{i+1} \models C_{i+1|R}$, and we can apply the argument again.

We are left to prove the fact (†). The constraint C completely specifies the order on $R \cup R'$, while ν fixes the values for R, and $\nu \models C_{\mid R}$. Thus, we can uniquely order registers R' and the values $\{\nu(r) \mid r \in R\}$ of R on the \mathbb{Q} -line. Since \mathbb{Q} is dense, it is always possible to choose the values for R' that respect this order; we leave out the details.

Consider the second claim, direction \Leftarrow . Since $C_0C_1\ldots$ is consistent, then by the first claim, it is satisfiable, hence it has a witnessing valuation $\nu_0\nu_1\ldots$. The constraint C_0 requires all registers in R to start with the same value, so define $d = \nu_0(r)$ for arbitrary $r \in R$. Let $\nu'_0\nu'_1\ldots$ be the valuations decreased by $d:\nu'_i(r) = \nu_i(r) - d$ for every $r \in R$ and $i \geq 0$. The new valuations satisfy the constraint sequence because the constraints in $\mathbb Q$ are invariant under the shift (follows from the fact: if $r_1 < r_2$ holds for some $\nu \in \mathcal D^R$, then it holds for any $\nu - d$ where $d \in \mathcal D$). The equality $\nu'_0 = 0^R$ means that the constraint sequence is 0-satisfiable.

We now prove Theorem 20.

Proof of Theorem 20. The sought automaton has an alphabet consisting of all constraints. By Lemma 21, for satisfiability, it suffices to construct the automaton that checks consistency, namely that every two adjacent constraints C_1C_2 in the input word satisfy the condition $unprime(C_{1|R'}) = C_{2|R}$. We only sketch the construction. The automaton memorises the atoms $C_{1|R'}$ of the last constraint C_1 into its state, and on reading the next constraint C_2 the automaton checks that $unprime(C_{1|R'}) = C_{2|R}$. If this holds, the automaton transits into the state that remembers $C_{2|R'}$; if the check fails, the automaton goes into the rejecting sink state. And so on. The automaton for checking 0-satisfiability additionally checks that $C_{0|R} = \{r = s \mid r, s \in R\}$. The number of states is exponential in |R|, the number of colors is 2, and in fact the so-called safety (aka looping) acceptance suffices.

For the rest of this section, we focus on domain \mathbb{N} .

5.1 Chains characterise satisfiability of constraint sequences

In this section we prove the characterisation of satisfiable constraint sequences that we used to ω -regularly approximate the automaton game over (\mathbb{N}, \leq) (Section 4.2). Recall that chains are defined in Definition 5 on page 23.

While the target characterisation relies on one-way chains, we start by presenting a characterisation using two-way chains: such chains compare register values forwards and backwards in time. This characterisation is intuitive and easy to prove but difficult to implement using *one*-way automata. Therefore, later we provide an alternative characterisation using *one*-way chains which read constraint sequences in forward direction only. The lifting from two-way to one-way chains is done using Ramsey theorem [45]. A similar proof strategy is employed in [47, Appendix C], but our notion of chains is simpler, and we describe the previously missing application

of Ramsey theorem. We start with the definitions of two-way chains, then describe the characterisations in Lemmas 22 and 23.

Lemma 22. A consistent constraint sequence is satisfiable in \mathbb{N} iff

A2. it has no infinite-depth two-way chains, and

B2. every ceiled two-way chain has a bounded depth

(i.e., there exists $B \in \mathbb{N}$ such that the depth of every ceiled two-way chain is $\leq B$).

Proof. The direction \Rightarrow is proven by contradiction: if A2 is not satisfied, then one needs infinitely many values below the maximal initial value of a register to satisfy the sequence, which is impossible in \mathbb{N} . Similarly for B2. We now state this formally. Suppose a constraint sequence $C_0C_1...$ is satisfiable by some valuations $\nu_0\nu_1...$ Towards a contradiction, assume that A2 does not hold, i.e. there is an infinite decreasing two-way chain $\chi=(r_0,m_0)(r_1,m_1)...$ Let $\nu_{m_0}(r_0)=\emptyset^*$ be the data value at the start of the chain. Each decrease $(r_i,m_i)>(r_{i+1},m_{i+1})$ in the chain χ requires the data to decrease as well: $\nu_i(r_i)>\nu_{i+1}(r_{i+1})$, so there must be an infinite number of data values between \emptyset^* and 0, which is impossible in \mathbb{N} . Hence A2 must hold. Now consider B2. If there are no ceiled chains, we are done, so assume there is at least one ceiled chain. Then there exists a maximal stable chain, by definition. Let \emptyset^* be the value of the registers in the maximal stable chain. All ceiled chains lie below the maximal stable chain, therefore the values of their registers are bounded by \emptyset^* . Thus the depth of each such a chain is bounded by $\mathbb{B}=\emptyset^*$, so B2 holds.

The direction \Leftarrow . Given a consistent constraint sequence C_0C_1 ... satisfying A2 and B2, we construct a sequence of register valuations $\nu_0\nu_1$... such that $\nu_i \cup \nu'_{i+1} \models C_i$ for all $i \geq 0$ (recall that $\nu' = \{r' \mapsto \nu(r) \mid r \in R\}$). For a register r and moment $i \in \mathbb{N}$, let d(r,i) be the largest depth of two-way chains from (r,i); such a number exists by assumption B2; it is not ∞ by assumption A2; it can be 0. Then, for every $r \in R$ and $i \in \mathbb{N}$, set $\nu_i(r) = d(r,i)$.

We now prove that for all i, the satisfaction $\nu_i \cup \nu'_{i+1} \models C_i$ holds, i.e. all atoms of C_i are satisfied. Pick an arbitrary atom $t_1 \bowtie t_2$ of C_i , where $t_1, t_2 \in R \cup R'$. Define $m_{t_1} = i + 1$ if t_1 is a primed register, else $m_{t_1} = i$; similarly define m_{t_2} . There are two cases

- $t_1 \bowtie t_2$ is $t_1 = t_2$. Then the deepest chains from (t_1, m_{t_1}) and (t_2, m_{t_2}) have the same depth, $d(t_1, m_{t_1}) = d(t_2, m_{t_2})$, and hence $\nu_i \cup \nu'_{i+1}$ satisfies the atom.
- $t_1 \bowtie t_2$ is $t_1 > t_2$. Then, any chain (t_2, m_{t_2}) ... from (t_2, m_{t_2}) can be prefixed by (t_1, m_{t_1}) to create the deeper chain $(t_1, m_{t_1}) > (t_2, m_{t_2})$ Thus, $d(t_1, m_{t_1}) > d(t_2, m_{t_2})$, therefore $\nu_i \cup \nu'_{i+1}$ satisfies the atom.

This concludes the proof.

Remark. The proof describes a data-assignment function which maps a sequence of constraints to a sequence of valuations satisfying it. Such functions are widespread, see e.g. [47, Lemma C.7] or [17, Lemma 15]. Later in Section 5.4 we describe a different kind of data-assignment function, which does not see the whole constraint sequence beforehand but only the prefix read so far. This changes how much the register values get separated from each other: from B in the above proof to approx. 2^B .

The previous lemma characterises satisfiability in terms of two-way chains, but our final goal is the characterisation by automata. It is hard to design a *one*-way automaton tracing *two*-way chains, so we lift the previous lemma to one-way chains.

Lemma 23. A consistent constraint sequence is satisfiable in \mathbb{N} iff

A1. it has no infinitely decreasing one-way chains, and

B1. every ceiled one-way chain has a bounded depth

(i.e., there exists $B \in \mathbb{N}$ such that the depth of every ceiled one-way chain is $\leq B$).

We describe a proof idea then provide a full proof.

Proof idea. We start from Lemma 22 and show that hypotheses A2 and B2 can be refined to A1 and B1 respectively. From an infinite (decreasing) two-way chain, we can always extract an infinite decreasing one-way chain, since two-way chains are infinite to the right and not to the left. Hence, for every moment i, there always exists a moment j > i such that one register of the chain is smaller at step j than a register of the chain at step i. Then, given a sequence of ceiled two-way chains of unbounded depth, we are able to construct a sequence of one-way chains of unbounded depth. This construction is more difficult than in the above case. Indeed, even though there are by hypothesis deeper and deeper ceiled two-way chains, they may start at later and later moments in the constraint sequence and go to the left. Thus, one cannot simply take an arbitrarily deep two-way chain and extract an arbitrarily deep one-way chain from it. However, we demonstrate, using a Ramsey argument, that it is still possible to extract arbitrarily deep one-way chains since the two-way chains are not completely independent. \Box

Proof. Thanks to Lemma 22, it suffices to show that $A1 \Leftrightarrow A2$ and $B1 \Leftrightarrow B2$. The implications $A2 \Rightarrow A1$ and $B2 \Rightarrow B1$ follow from the definitions of chains.

Now, let us show that $\neg A2 \Rightarrow \neg A1$: let $C_0C_1...$ be a consistent constraint sequence, and assume that it has an infinite two-way chain $\chi = (r_a, i)...$ We then construct an infinite descending one-way chain χ' . The construction is illustrated in Figure 6. Our one-way chain χ' starts in (r_a, i) . The area on the left from i-timeline contains $i \cdot |R|$ points, but χ has an infinite depth hence at some point it must go to the right from i. Let r_b be the smallest register visited at moment i by χ ; we first assume that r_b is different from r_a (the other case is later). Let χ go $(r_b, i) \triangleright (r', i+1)$. We append this to χ' and get $\chi' = (r_a, i) > (r_b, i) \triangleright (r', i+1)$. If r_a and r_b were actually the same, so the chain χ moved $(r_a, i) \triangleright (r', i+1)$, then we would append only $(r_a, i) \triangleright (r', i+1)$. By repeating the argument from the point (r', i+1), we construct the infinite descending one-way chain χ' . Hence $\neg A_1$ holds.

Now, let us show $\neg B2 \Rightarrow \neg B1$. Given a sequence of ceiled two-way chains of unbounded depth, we need to create a sequence of ceiled one-way chains of unbounded depth. We extract a witnessing one-way chain of a required depth from a sufficiently deep two-way chain. To this end, we represent the two-way chain as a clique with colored edges, and whose one-colored subcliques represent all one-way chains. We then use the Ramsey theorem that says a monochromatic subclique of a required size always exists if a clique is large enough. From the monochromatic subclique we extract the sought one-way chain.

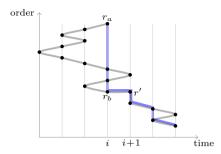


Figure 6: Proving the direction $\neg A2 \Rightarrow \neg A1$ in Lemma 23. The two-way chain is in grey, the constructed one-way chain is in blue.

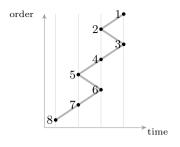
The Ramsey theorem [45] is about clique graphs with colored edges. For the number $n \in \mathbb{N}$ of vertices, let K_n denote the clique graph and let E_{K_n} be its set of edges. Then, we let $color: E_{K_n} \to \{1, \dots, \#c\}$ be an edge-coloring function, where #c is the number of edge colors in the clique. A clique is monochromatic if all its edges have the same color (#c = 1). The Ramsey theorem says:

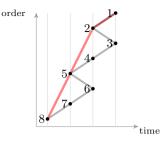
Fix the number #c of edge colors. $(\forall n)(\exists l)(\forall color: E_{K_l} \to \{1, \dots, \#c\})$: there exists a monochromatic subclique of K_l with n vertices. The number l is called the Ramsey number for (#c, n).

I.e., for any given n, there is a sufficiently large size l such that any colored clique of this size contains a monochromatic subclique of size n. Ramsey numbers depend on the number #c of colors and size n of the clique and are independent of a coloring function color. We use the theorem with three colors only: #c = 3.

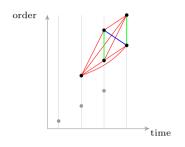
Given a sequence of two-way chains of unbounded depth, we show how to build a sequence of one-way chains of unbounded depth. Suppose we want to build a one-way chain of depth n, and let l be the Ramsey number for (3, n). Since the two-way chains from the sequence have unbounded depth, there is a two-way chain χ of depth l. From it we construct the following colored clique (the construction is illustrated in Figure 7).

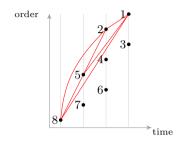
- Remove stuttering elements from χ : whenever $(r_i, m_i) = (r_{i+1}, m_{i+1})$ appears in χ , remove (r_{i+1}, m_{i+1}) . We repeat this until no stuttering elements appear. Let $\chi_{>} = (r_1, m_1) > \cdots > (r_l, m_l)$ be the resulting sequence; it is strictly decreasing, and contains l pairs (the same as the depth of the original χ). Note the following property (\dagger): for every not necessarily adjacent $(r_i, m_i) > (r_j, m_j)$, there is a oneway chain $(r_i, m_i) \ldots (r_j, m_j)$; it is decreasing if $m_i < m_j$, and increasing otherwise; its depth is at least 1. The resulting sequence may skip points in time, but this as will be explained later does not affect the construction.
- The elements (r, m) of $\chi_{>}$ serve as the vertices of the colored clique. The edgecoloring function is: for every not necessarily adjacent $(r_a, m_a) > (r_b, m_b)$ in $\chi_{>}$, let $color((r_a, m_a), (r_b, m_b))$ be \nearrow if $m_a < m_b$, \searrow if $m_a > m_b$, \downarrow if $m_a = m_b$. Thus, we assign a color to an edge between every two vertices. Figure 7b gives an example.





(a) A given two-way chain (wo stuttering) (d) Constructed increasing one-way chain





- (b) Clique: shown the edges for the top 5 points only. Try completing the rest.
- (c) Monochromatic subclique with elements 1, 2, 5, 8

Figure 7: Proving the direction $\neg B2 \Rightarrow \neg B1$ in Lemma 23

By applying the Ramsey theorem, we get a monochromatic subclique of size n with vertices $V\subseteq\{(r_1,m_1),\ldots,(r_l,m_l)\}$. Its color cannot be \downarrow when n>|R|, because a timeline has maximum |R| points. Suppose the subclique's color is \nearrow (the case of \searrow is similar). We build the increasing sequence $\chi^*=(r_1^*,m_1^*)<\cdots<(r_n^*,m_n^*)$, where $m_i^*< m_{i+1}^*$ and $(r_i^*,m_i^*)\in V$ for every i. The sequence χ^* may not satisfy the definition of one-way chains, because the removal of stuttering elements that we performed at the beginning can cause time jumps i.e. $m_{i+1}>m_i+1$. But it is easy—relying on the property (\dagger) —to construct the one-way chain χ^{**} of depth n from χ^* by inserting the necessary elements between (r_i,m_i) and (r_{i+1},m_{i+1}) . The case when the subclique has color \searrow , the resulting constructed chain is decreasing.

Thus, for every given n, we constructed either a decreasing or increasing ceiled one-way chain of depth n. In other words, a sequence of such chains of unbounded depth. Hence $\neg B1$ holds, which concludes the proof.

The next easy lemma (first stated on page 24) refines the characterisation to 0-satisfiability:

Lemma 12. A consistent constraint sequence is 0-satisfiable in \mathbb{N} iff there exists B > 0 such that:

- 1. it has no infinitely decreasing one-way chains,
- 2. the ceiled one-way chains have a depth at most B

```
3. it starts in C_0 s.t. C_{0|R} = \{r = s \mid r, s \in R\}, and 4. it has no decreasing one-way chains of depth \geq 1 from (r, 0) for any r.
```

Proof. Direction \Rightarrow . The first two items follow from Lemma 23; the third one follows from the definition of satisfiability. Consider the last item: suppose there is such a chain. Then, at the moment when the chain strictly decreases and goes to some register s, the register s would need to have a value below 0, which is impossible in \mathbb{N} .

Direction \Leftarrow . The first two items are exactly A1 and B1 from Lemma 23, so the sequence is satisfiable, hence it also satisfies the conditions A2 and B2 from Lemma 22. In the proof of Lemma 22, we showed that in this case the following valuations $\nu_0\nu_1$... satisfy the sequence: for every $r \in R$ and moment $i \in \mathbb{N}$, set $\nu_i(r)$ (the value of r at moment i) to the largest depth of the two-way chains starting in (r, i). We construct $\nu_0\nu_1$... as above, and get a witness of satisfaction of our constraint sequence. Note that at moment 0, $\nu_0 = 0^R$, by the last item. Hence the constraint sequence is 0-satisfiable.

Action words and constraint sequences

In this section, we provide the proof of the following lemma, stated on page 25:

Lemma 13. Let R be a set of registers, $R_d = R \uplus \{r_d\}$, and \mathfrak{D} be (\mathbb{N}, \leq) or (\mathbb{Q}, \leq) . There exists a mapping constr : $\Pi \times \mathsf{Tst} \times \mathsf{Asgn} \to \mathsf{C}$ from state constraints Π over R_d and tests-assignments over R to constraints C over R_d , such that for all action words $a_0a_1a_2... \in (\mathsf{Tst} \times \mathsf{Asgn})^\omega$, $a_0a_1a_2...$ is feasible iff $C_0C_1C_2...$ is 0-satisfiable, where $\forall i \geq 0$: $C_i = constr(\pi_i, a_i)$, $\pi_{i+1} = unprime(C_{i|R'_j})$, $\pi_0 = \{r = s \mid r, s \in R_d\}$.

Proof. Given π , tst, asgn, we define the mapping $constr: (\pi, tst, asgn) \mapsto C$ as follows. The definition is as expected, but we should be careful about handling of r_d , it is the last item.

- The constraint C includes all atoms of the state constraint π (that relates the registers at the beginning of the step).
- Recall that neither tst nor asgn talk about r_d . For readability, we shorten $(t_1 \bowtie t_2) \in C$ to simply $t_1 \bowtie t_2$, $(* \bowtie r) \in$ tst to $* \bowtie r$, and $a \leq b$ means $(a < b) \lor (a = b)$.
- We define the order at the end of the step as follows. For every two different $r, s \in R$:

```
\begin{array}{l} -\ r'=s' \ \text{iff} \ (r=s) \land r, s \not\in \operatorname{asgn} \ \operatorname{or} \ r \in \operatorname{asgn} \land (*=s) \ \operatorname{or} \ r, s \in \operatorname{asgn}; \\ -\ r' < s' \ \text{iff} \ (r < s) \land r, s \not\in \operatorname{asgn} \ \operatorname{or} \ (* < s) \land r \in \operatorname{asgn} \land s \not\in \operatorname{asgn}; \\ -\ r' = r'_d \ \text{iff} \ (r = *) \ \operatorname{or} \ r \in \operatorname{asgn}; \\ -\ r' \bowtie r'_d \ \text{iff} \ (r \bowtie *) \land r \not\in \operatorname{asgn}, \ \operatorname{for} \bowtie \in \{<, >\}; \end{array}
```

• So far we have defined the order of the registers at the beginning and the end of the step. Now we relate the values between these two moments. For every $r \in R$:

```
\begin{array}{ll} - \ r = r' \ \text{iff} \ r \not\in \operatorname{asgn} \ \text{or} \ r \in \operatorname{asgn} \wedge (* = r); \\ - \ r \bowtie r' \ \text{iff} \ r \in \operatorname{asgn} \wedge (r \bowtie *), \ \text{for} \bowtie \in \{<, >\}; \end{array}
```

• Finally, we relate the values of r_d between the moments. There are two cases.

- The value of r_d crosses another register: $\exists r \in R : (r_d < r) \land (* \ge r)$. Then $(r'_d > r_d)$. Similarly for the opposite direction: if $\exists r \in R : (r_d > r) \land (* \le r)$ then $(r'_d < r_d)$.
- Otherwise, the value of r_d does not cross any register boundary. Then $r'_d = r_d$.

Using the mapping constr, every action word $\overline{a} = (tst_0 asgn_0)(tst_1 asgn_1) \dots$ can be uniquely mapped to the constraint sequence $C_0C_1 \dots$ as follows: $C_0 = constr(\pi_0, tst_0, asgn_0)$, set $\pi_1 = unprime(C_0|_{R'_d})$, then $C_1 = constr(\pi_1, tst_1, asgn_1)$, and so on.

We now prove that an action word is feasible iff the constructed constraint sequence is 0-satisfiable. This follows from the definitions of feasibility and 0-satisfiability, and from the following simple property of feasible action words. Every feasible action word has a witness $\nu_0 \ell_0 \nu_1 \ell_1 \cdots \in (\mathfrak{D}^R \cdot \mathfrak{D})^{\omega}$ such that: if some tst is repeated twice and no assignment is done, then the value ℓ stays the same. This property is needed due to the last item in the definition of constr where we set $r'_d = r_d$.

5.2 Max-automata recognise satisfiable constraint sequences

This section presents an automaton characterisation of constraint sequences satisfiable in \mathbb{N} . The automaton construction verifies the conditions on one-way chains stated in Lemma 23: the absence of (A1) infinite decreasing one-way chains and of (B1) unbounded one-way ceiled chains. The boundedness requirement of the second condition cannot be checked by ω -regular automata⁵, and for that reason in [47] the authors used nondeterministic ω B-automata. Since nondeterminism is usually hard to handle in synthesis, we picked deterministic max-automata [8], which are incomparable with ω B-automata, expressivity-wise. We now define max-automata and then present the characterisation.

Deterministic max-automata extend classic finite-alphabet parity automata with a finite set of counters c_1, \ldots, c_n which can be incremented, reset to 0, or updated by taking the maximal value of a set of counters, but the counters cannot be tested. On reading a word, the automaton builds a sequence of counter valuations. The acceptance condition is given as a conjunction of the parity acceptance condition and a Boolean combination of conditions "counter c_i is bounded along the run". Such a condition on a counter is satisfied by a run if there exists a bound $B \in \mathbb{N}$ such that counter c_i has value at most B along the run. By using negation, conditions such as " c_i is unbounded along the run" can also be expressed. A run is accepting if it satisfies the parity condition and the Boolean formula on the counter conditions. Deterministic max-automata are strictly more expressive than ω -regular automata. For instance, they can express the non- ω -regular language of words of the form $a^{n_1}ba^{n_2}b\dots$ such that $n_i \leq B$ for all $i \geq 0$, for some $B \in \mathbb{N}$ that can vary from word to word. A max-automaton recognising the language is in Figure 8.

We now prove the main result of this section.

 $^{^5}$ For a formal statement, see [47, Theorem 4.3] saying that the class of languages of finite-alphabet projections of "constraint automata" and the class of ω B-languages coincide.



Figure 8: Max-automaton recognising $\{a^{n_1}ba^{n_2}b\dots \mid \exists B \in \mathbb{N} \ \forall i : n_i \leq B\}$. It uses a single counter c, the acceptance condition is "counter c is bounded", and the parity acceptance is trivial (always accept). The operation max is not used.

Theorem 24. For every R, there is a deterministic max-automaton accepting exactly all constraint sequences satisfiable in \mathbb{N} . The number of states is exponential in |R|, the number of counters is $O(|R|^2)$, and the number of priorities is polynomial in |R|. The same holds for 0-satisfiability in \mathbb{N} .

Proof idea. We design a deterministic max-automaton that checks conditions A₁ and B1 of Lemma 23. Condition A1, namely the absence of infinitely decreasing one-way chains, is checked as follows. We construct a nondeterministic Büchi automaton that guesses a chain and verifies that it is infinitely decreasing, i.e. that '>' occurs infinitely often and that there is no '<' (only '>' and '='). Determinising and complementing yields a deterministic parity automaton, that can be disjuncted through a synchronised product with the deterministic max-automaton checking condition B1. The latter condition (the absence of ceiled one-way chains of unbounded depth) is more involved. We design a master automaton that tracks every chain χ that currently exhibits a stable behaviour. To every such a chain χ , the master automaton assigns a tracer automaton whose task is to ensure the absence of unbounded-depth ceiled chains below χ . For that, the tracers use 2|R| counters – one for tracking increasing and one for tracking decreasing chains – and requires them to be bounded. We use the max operation on counters to ensure that we trace the largest chains only. The overall acceptance condition ensures that if the chain χ is stable, then there are no ceiled chains below χ of unbounded depth. Finally, we take the product of all these automata, which preserves determinism.

In the next section, we provide the details of the proof.

Proof of Theorem 24

We describe a max-automaton A that accepts a constraint sequence iff it is consistent and has no infinitely decreasing one-way chains and no ceiled one-way chains of unbounded depth. By Lemma 23, such a sequence is satisfiable.

The automaton has three components $A = A_c \wedge A_{\neg \infty} \wedge A_B$.

 A_c The parity automaton A_c checks consistency, i.e. that $\forall i : unprime(C_{i|R'}) = (C_{i+1})_{|R}$. It has exponential in |R| number of states and two priorities (the safety language).

 $A_{\neg \infty}$ The parity automaton $A_{\neg \infty}$ ensures there are no infinitely decreasing one-way chains. First, we construct its negation, an automaton that accepts a constraint

sequence iff it has such a chain. Intuitively, the automaton guesses such a chain and then verifies that the guess is correct. It loops in the initial state q_{ι} until it nondeterministically decides that now is the starting moment of the chain and guesses the first register r_0 of the chain, and transits into the next state while memorising r_0 . When the automaton is in a state with r and reads a constraint C, it guesses the next register r_n , verifies that $(r'_n > r) \in C$ or $(r'_n = r) \in C$, and transits into the state that remembers r_n . The Büchi acceptance condition ensures that the automaton leaves the initial state and transits from some r to some r_n with $(r'_n > r) \in C$ infinitely often. Determinising and complementing this automaton gives $A_{\neg \infty}$. The number of states is exponential and the number of priorities is polynomial in |R|, due to the determinisation.

 $A_{\rm B}$ The max-automaton $A_{\rm B}$ ensures that all ceiled one-way chains have bounded depth. It relies on the master automaton controlling the team of |R| chain tracers $Tr = \{tr_1, ..., tr_{|R|}\}$. Each tracer tr is equipped with a counter $idle_{tr}$ and a set Cn_{tr} of 2|R| of counters, thus overall there are |R|(2|R|+1) counters. The construction ensures that every stable chain is tracked by a single tracer tr and its counter $idle_{tr}$ is bounded; and vice versa, if a tracer tr has its counter $idle_{tr}$ bounded, it tracks a stable chain. Suppose for a moment that tracer tr tracks a stable chain χ . Then the goal of counters Cn_{tr} is to track the deepest increasing and decreasing chains below χ . Since there are only |R| registers, it suffices to track |R| decreasing chains, every chain ending in a different register (similarly for increasing chains). This is because there is no need to track two decreasing chains ending in the same register: once the two chains "meet" in a register r, we continue tracking only the one with the larger depth and forget about the other. We use the max operation of automata to implement this idea. Overall, the construction ensures that the counters in Cn_{tr} are bounded iff the increasing and decreasing chains ceiled by the stable chain tracked by the tracer tr have bounded depths. The acceptance of $A_{\rm B}$ is the formula

$$\bigwedge_{tr \in Tr} (idle_{tr} \text{ is bounded} \to \bigwedge_{c \in Cn_{tr}} c \text{ is bounded}).$$

The work of tracers is controlled by the master automaton via four commands idle ("track nothing"), start ("start tracking a potentially stable chain"), move ("continue tracking"), and reset ("stop tracking"). Before we formally describe the master and the tracers, we define the concept of "levels" used in the presentation. Intuitively, the levels abstract concrete data values, and the tracers actually track the levels instead of specific registers.

Fix a constraint C. A level $l \subseteq R \setminus \{\emptyset\}$ is an equivalence class of registers wrt. $C_{|R|}$ or wrt. $unprime(C_{|R'})$. Thus, in the constraint C we distinguish the levels of two kinds: $start\ levels$ (at the beginning of the step) and $end\ levels$ (at the end of the step). A start level $l \subseteq R$ disappears when C contains no atoms of the form r = s' for $r \in l$ and $s \in R$; this means that a data value abstracted by the level disappears from the registers. An end level $l \subseteq R$ is new if C contains no atoms of the form r = s' where $r \in R$ and $s \in l$; intuitively, the constraint requires a new data value to appear in registers l. A start level l morphs into an end level l' if C contains an

Figure 9: Example of levels: start levels are $\{r_1, r_2\}$ and $\{r_3\}$, end levels are $\{r_3\}$, $\{r_2\}$, and $\{r_1\}$. The start level $\{r_1, r_2\}$ morphs into end level $\{r_3\}$, the start level $\{r_3\}$ disappears, and two new end levels appear, $\{r_1\}$ and $\{r_2\}$. The constraint is $\{r_1 = r_2 = r'_3 > r'_2 > r_3 > r'_1\}$.



atom r = s' for some $r \in l$ and $s \in l'$; i.e., the constraint requires the registers in l' to hold the data value previously held by the registers in l. Notice that there can be at most |R| start and |R| end levels, for a fixed constraint C. Figure 9 illustrates the definitions. We are now ready to describe the master and the tracers.

Master. States of $A_{\rm B}$ are of the form $(getTr,\vec{q})$, where the partial mapping $getTr: l \mapsto tr$ maps a level $l \subseteq R \setminus \{\emptyset\}$ to a tracer $tr \in Tr$, and $\vec{q} = (q_1, ..., q_{|Tr|})$ describes the states of individual tracers. The master updates the state component getTr while the tracers update their states. Initially, there is only one start level R (assuming the registers start with the same value), so we define $getTr = \{R \mapsto tr_1\}$. Suppose the automaton reads a constraint C, let L and L' be the start and end levels of C, and suppose the automaton is in state $(getTr, \vec{q})$ and $getTr: L \to Tr$. We define the successor state $(getTr', \vec{q}')$, where $getTr': L' \to Tr$, and operations on the counters using the following procedure.

- To every tracer tr that does not currently track a level, i.e. $tr \in Tr \setminus getTr(L)$, the master commands idle (causing the tracer to increment $idle_{tr}$).
- For every start level $l \in L$ that morphs into $l' \in L'$: let tr = getTr(l), then
 - the master sends $move(r_{\top})$ to tr where $r_{\top} \in l$ is chosen arbitrary; this will cause the tracer tr to update its counters Cn_{tr} and move into a successor state q'_{tr} ; the register r_{\top} will be used as a descriptor of a stable chain tracked by tr.
 - we set getTr'(l') = getTr(l), thus the tracer continues to track it.
- For every start level $l \in L$ that disappears: let tr = getTr(l), then
 - the master sends reset to tr, which causes the reset of the counters in Cn_{tr} and the increment of $idle_{tr}$.
- For every new end level $l' \in L'$:
 - we take an arbitrary tr that is not yet mapped by getTr' and map getTr'(l') = tr;
 - the master sends start to tr.

Tracers. We now describe the tracer component. Its goal is to trace the depths of ceiled chains. When the counters of a tracer are bounded, the depths of the chains it tracks are also bounded. The tracer consists of two components, B_{\downarrow} and B_{\uparrow} , which track decreasing and increasing chains. We only describe B_{\downarrow} , the other one is similar.

The component B_{\downarrow} has a set $Cn \cup \{idle\}$ of |R| + 1 counters. A state of B_{\downarrow} is either the initial state q_t or a partial mapping $getCn : R \rightharpoonup Cn$. Intuitively, in each getCn-state, for each register r mapped by getCn, the value of the counter getCn(r) reflects the depth of the deepest ceiled decreasing one-way chain ending in r. When

several chains end in r, the counter gets the maximal value of the depths. We maintain this property of getCn during the transition of B_{\downarrow} on reading a constraint C, using operations of max-automata on counters and register-order information from C. The component B_{\downarrow} does the following:

- If the master's command is idle, then increment the counter idle and stay in q_{l} .
- If the master's command is reset, reset all counters in Cn, increment the counter idle, and go into state q_{ι} .
- If the master's command is start, move from state q_t into the state with the empty mapping getCn.

Otherwise, the master's command is $move(r_{\top})$, for some $r_{\top} \in R$ passed by the master and serving as a descriptor of a stable chain traced by the current tracer. The tracer performs the operations on its counters and updates the mapping getCn as follows.

- Release counters. For every r such that $r < r_{\top} < r'$, the component resets the counter getCn(r) and removes r from the mapping getCn. I.e., we stop tracking chains ending in register r since such chains are no longer below the stable chain assigned to the tracer.
- Allocate counters. For every r such that $r \geq r_{\top} > r'$: pick a counter $c \in Cn \setminus getCn(R)$ and map getCn(r) = c. I.e., we start tracking chains ending in r.
- Update counters. For every r such that $r \leq r_{\top}$ and $r' < r_{\top}$ do the following. Let $R_{>r'} = \{r_o \mid r' < r_o < r_{\top}\}$ be the registers larger than the updated r but below r_{\top} , and let $getCn(R_{>r'})$ be the associated counters. Let r_{\equiv} be a register s.t. $r_{\equiv} = r'$ (may not exist). We update the counter getCn(r) depending on the case:
 - $-R_{>r'}$ is empty and $r_{=}$ does not exist: the condition means that no decreasing ceiled chain can be extended into r'. Then we reset the counter getCn(r).
 - $-R_{>r'}$ is empty and $r_{=}$ exists: only the chains ending in $r_{=}$ can be extended into r', and since $r_{=} = r'$, the deepest chain keeps its depth. Therefore, we $copy(getCn(r_{=}))$ into the counter getCn(r).
 - $-R_{>r'}$ is not empty and $r_{=}$ does not exist: the chains from registers in $R_{>r'}$ can be extended into r', and since r' is lower than any register in $R_{>r'}$, their depths increase. The new value of counter getCn(r) must reflect the deepest chain, therefore the counter gets the value $max(getCn(R_{>r'})) + 1$.
 - $-R_{>r'}$ is not empty and $r_{=}$ exists: some chains from registers in $R_{>r'}$ can be decremented into r', there is also a chain from $r_{=}$ that can be extended into r' without its depth changed. The counter gets $max(max(getCn(R_{>r'})) + 1, getCn(r_{=}))$, which describes the deepest resulting chain.

The number of states in B_{\downarrow} is no more than $|R|^{|R|}+1$, and the number of counters is |R|+1. The construction for B_{\uparrow} is similar to this construction for B_{\downarrow} , except that we need to track *increasing* ceiled chains instead of decreasing ones. The number of counters in B_{\downarrow} and B_{\uparrow} is 2|R|+1. Since we use |R| number of tracers, the total number of counters becomes |R|(2|R|+1). Overall, $A_{\rm B}$ has an exponential in |R| number of states, the number of counters is in $O(|R|^2)$, and the parity condition is trivial. This concludes the description of the tracers and of the automaton $A_{\rm B}$.

We have described all three components $A = A_c \wedge A_{\neg \infty} \wedge A_{\rm B}$, where A_c expresses a safety language, $A_{\neg \infty}$ is a classic deterministic parity automaton, and $A_{\rm B}$ is a deterministic max-automaton with the trivial parity acceptance condition. All the automata has no more than an exponential in |R| number of states, $A_{\neg \infty}$ has a polynomial in |R| number of colors, and $A_{\rm B}$ has a polynomial in |R| number of counters. It is not hard to see that the product of these automata gives the desired automaton A with exponentially many states, polynomially many colors and counters, in |R|. The acceptance condition is the parity acceptance in conjunction with the formula of $A_{\rm B}$ described on page 39.

Finally, for the case of 0-satisfiability, the automaton A also needs to satisfy the additional conditions stated in Lemma 12, in particularly there shall be no decreasing one-way chains from moment 0 of depth ≥ 1 . This check is simple and omitted. This concludes the proof of Theorem 24.

Remark. In [47, Appendix C] it is shown that satisfiable constraint sequences in \mathbb{N} are characterised by nondeterministic ω B-automata [6]. These automata are incomparable with deterministic max-automata.

The following two languages separate these classes: $(a^B b)^{\omega}$ is recognised by det max automata but not by nondet ωB automata, and $\{a^{n_1}b\,a^{n_2}b\,a^{n_3}b\dots$ | $\liminf n_i < \infty\}$ witnesses the opposite direction. The latter language is recognisable by the nondet ωB automaton which guesses a bounded subsequence of $n_1 n_2 \dots$ The non-recognisability by det max automata follows from [8, Section 6].

We prove the claim about $\overline{(a^Bb)^\omega}$. First, the language $(a^Bb)^\omega$ is recognisable by det ωB automata and hence by det max automata. Since det max automata are closed under the complement, $\overline{(a^Bb)^\omega}$ is also recognisable by det max automata. Now, by contradiction, assume that $\overline{(a^Bb)^\omega}$ is recognisable by nondet ωB automata. The result [6, Lemma 2.5] says: if an ωB language over alphabet $\{a,b\}$ contains a word with infinitely many bs then it contains a word from $(a^Bb)^\omega$. The language $\overline{(a^Bb)^\omega}$ contains the former (e.g. take any word from $(a^Sb)^\omega$) but not the latter. Contradiction. Hence it is not an ωB language.

5.3 Satisfiability of lasso-shaped sequences

An infinite sequence is lasso-shaped (or regular) if it is of the form $w = uv^{\omega}$. Lasso-shaped sequences are prevalent in automata theory and in the data setting in particular. For instance, [21] studies satisfiability of logic Constraint LTL in the data domain (\mathbb{N}, \leq) and shows that considering lasso-shaped witnesses of satisfiability is sufficient. Another work [26] shows that if there is an ω -regular over-approximation of satisfiable constraint sequences and which is exact on lasso-shaped sequences, then a synthesis problem is decidable in (\mathbb{N}, \leq) . In this paper, when proving the decidability of Church synthesis problem, we do not directly rely on lasso-shaped sequences, but we use a characterisation similar to the one proven in this section.

This section shows that considering lasso-shaped constraint sequences greatly simplifies the task of characterisation of satisfiability. We first show how lasso-shaped sequences simplify the condition B₁ of characterisation Lemma 23, then describe the

chain characterisation under assumption of lasso-shaped sequences, and finally state the ω -regular automaton characterisation.

Lemma 25. For every lasso-shaped consistent constraint sequence, it has ceiled one-way chains of unbounded depth iff it has ceiled one-way chains of infinite depth.

Proof. Direction \Leftarrow is trivial, so consider direction \Rightarrow . The argument uses the standard pumping technique. Fix a lasso-shaped constraint sequence $C_0 \dots C_{k-1}(C_k \dots C_{k+l})^{\omega}$ having ceiled chains of unbounded depth. Since these chains have unbounded depth, they pass through C_k more and more often. At moments when the current constraint is C_k , each such a chain is in one of the finitely-many registers. Hence there is a chain, say increasing, that on two separate occasions of reading the constraint C_k goes through the same register r, and the chain suffix from the first pass through r until the second pass has at least one <. Then we create an increasing chain of infinite depth by repeating this suffix forever.

The above lemma together with Lemma 12 yields the following result.

Lemma 26. A lasso-shaped consistent constraint sequence is 0-satisfiable iff it is quasi-feasible, i.e.:

- it has no infinite-depth decreasing one-way chains,
- it has no ceiled infinite-depth increasing one-way chains,
- it has no decreasing one-way chains of depth > 1 from moment 0, and
- it starts with C_0 s.t. $C_{0|R} = \{r = s \mid r, s \in R\}$.

The conditions of this lemma can be checked by an ω -regular automaton: Its construction is similar to the components A_c and $A_{\neg\infty}$ from the proof of Theorem 24 and is omitted. Thus, we get the theorem below.

Theorem 27. For every R, there is a deterministic parity automaton that accepts a lasso-shaped constraint sequence iff it is 0-satisfiable in \mathbb{N} ; its number of states and priorities is exponential and polynomial in |R|, respectively.

5.4 Data-assignment function

In this section, we design a data-assignment function that maps a sequence of constraints to a sequence of register valuations satisfying it, while doing it on the fly, i.e. by reading the constraint sequence from left to right. It is significant that the entire constraint sequence is not known in advance. Such a function is used in Section 3 when proving Proposition 15, namely that Adam's winning strategy in the finite-alphabet game transfers to the winning strategy in the Church synthesis game. There, Adam has to produce data values given only the prefix of a play.

In the next section, we state the lemma on existence of a data-assignment function, and then devote a significant amount of space to proving it.

5.4.1 Lemma 28 on existence of a data-assignment function

Intuitively, a data-assignment function produces register valuations while reading a constraint sequence from left to right. We are interested in functions that produce register valuations satisfying given constraint sequences. Since data-assignment functions cannot look into the future and do not know how many values will be inserted between any two registers, knowing a certain bound on such insertions is necessary. Moreover, to simplify the presentation, we restrict how many new data values can appear during the step. In our Church synthesis games, at most one new value provided by Adam can appear. We start by defining data-assignment functions, then describe the assumptions and state the lemma.

Let C denote the set of all constraints over registers R, and let $C_{|R}$ denote the set of all constraints over atoms over R only. A data-assignment function has the type $(C_{|R} \cup C^+) \to \mathbb{N}^R$. A data-assignment function f maps a constraint sequence $C_0C_1...$ into a sequence of valuations $f(C_{0|R})f(C_0)f(C_0C_1)...$

We now describe the two assumptions used by our data-assignment function.

Intuitively, the first assumption states that only a bounded number of insertions between any two registers can happen, and this bound is known. To formalise the assumption, we define a special kind of chains, called right two-way chains. Informally, right chains are two-way chains that operate to the right of their starting point. Knowing a bound on the depths of right chains amounts to knowing how many values in the future can be inserted between the registers. Fix a constraint sequence. Given a moment i and a register x, a (decreasing) right two-way chain starting in (x,i) (r2w for short) is a two-way chain $(x,i) \triangleright_1 (r_1, m_1) \triangleright_2 (r_2, m_2) \dots$ such that $m_j \geq i$, $\triangleright_j \in \{=, >\}$, for all j. As these chains are two-way, they can start and end in the same moment i. Notice that in Lemma 22 on characterisation of satisfiable constraint sequences we can replace two-way chains by r2w chains. Our data-assignment function will assume the knowledge of a bound on the r2w chains.

We now describe the second assumption about one-new-value appearance during a step. Its formalisation uses the notion of levels introduced in Section 5.2 on page 39 (see also Figure 9). We briefly recall those notions. Recall that a constraint describes a set of totally ordered equivalence classes of registers from $R \cup R'$. The figure on



the right describes a constraint that can be defined by the ordered equivalence classes $\{r_4, r_4'\} < \{r_2'\} < \{r_3, r_3'\} < \{r_1, r_2, r_1'\}$. It shows two columns of levels, start levels (in the left column) and end levels (in the right column), where a level describes a set of registers that are equivalent at this point of time. The assumption † says:

In every constraint of a given sequence, at most one new end level appear. (†)

The constraint depicted in the above figure satisfies this assumption, the one in Figure 9 does not. This assumption helps to simplify the proofs, and is satisfied by the constraint sequences induced in our Church synthesis games.

One final notion before stating the lemma. A constraint sequence is 0-consistent if it is consistent, starts in C_0 with $C_{0|R} = \{r = s \mid r, s \in R\}$, and has no decreasing

chains of depth > 1 starting at moment 0. Note that a 0-consistent constraint sequence whose r2w chains are bounded is 0-satisfiable (follows from Lemma 22).

Lemma 28 (data-assignment function). For every B > 0, there exists a dataassignment function $f:(C_{|R}\cup C^+)\to \mathbb{N}^R$ such that for every finite or infinite 0-consistent constraint sequence $C_0C_1C_2...$ satisfying assumption \dagger and whose r2wchains are depth-bounded by B, the register valuations $f(C_{0|B})f(C_0)f(C_0C_1)...$ satisfy the constraint sequence.

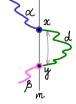
Proof idea. We define a special kind of $xy^{(m)}$ -chains that help to estimate how many insertions between the values of registers x and y at moment m we can expect in the future. As it turns out, without knowing the future, the distance between xand y has to be exponential in the maximal depth of $xy^{(m)}$ -chains. We describe a data-assignment function that maintains such exponential distances. The function is surprisingly simple: if the constraint inserts a register x between two registers r and s with already assigned values d_r and d_s , then set $d_x = \lfloor \frac{d_r + d_s}{2} \rfloor$; and if the constraint puts a register x above all other registers, then set $d_x = d_M + 2^B$ where d_M the largest value currently held in the registers and B is the given bound on the depth of r2w chains.

The rest of the section is devoted to the proof of this lemma.

5.4.2 Proof of Lemma 28

 $xy^{(m)}$ -connecting chains and the exponential nature of register valuations Fix an arbitrary 0-satisfiable constraint sequence C_0C_1 ... whose r2w chains are depthbounded by B. Consider a moment m and two registers x and y such that $(x > y) \in$ C_m .

We would like to construct witnessing valuations $\nu_0\nu_1$... using the current history only, e.g. a register valuation ν_m at moment m given only the prefix $C_0...C_{m-1}$. Note that the prefix $C_0...C_{m-1}$ defines the ordered partition of registers at moment m as well, since C_{m-1} is defined over $R \cup R'$. Let us see how much space we might need between $\nu_m(x)$ and $\nu_m(y)$, relying on the fact that the depths of r2w chains are bounded by B. Consider decreasing two-way chains that start at moment $i \leq m$, end in (x, m), and which are contained within time moments $\{i, ..., m\}$ (shown in blue). Further, consider decreasing two-way chains starting in (y, m), ending at moment $j \in \{i, ..., m\}$, and contained within time moments $\{j,...,m\}$ (shown in pink). Among such chains, pick two chains of depths α and β , respectively, that maximise the sum $\alpha + \beta$. After seeing $C_0C_1...C_{m-1}$, we do not know how the constraint sequence will



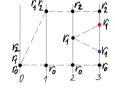
will have a depth $d \leq B - \alpha - \beta$ (otherwise, we could add prefix α and postfix β to it and construct an r2w chain of depth larger than B). We conclude that $\nu_m(x) - \nu_m(y) \geq$ $B - \alpha - \beta$, since the number of values in between two registers should be greater or equal than the longest two-way chain connecting them. To simplify the upcoming

evolve, but by boundedness of r2w chains, any r2w chain starting in (x,m) and ending in (y,m) (contained within time moments $\geq m$) arguments, we introduce $xy^{(m)}$ -connecting chains which consist of α and β parts and directly connect x to y.

An $xy^{(m)}$ -connecting chain is any r2w chain of the form $(a,i) \triangleright \ldots (x,m) > (y,m) \triangleright \ldots \triangleright (b,j)$: it starts in (a,i) and ends in (b,j), where $i \leq j \leq m$ and $a,b \in R$, and it directly connects x to y at moment m. Note that it is located solely within moments $\{i,\ldots,m\}$. Continuing the previous example, the $xy^{(m)}$ -connecting chain starts with α , directly connects (x,m) > (y,m), and ends with β ; its depth is $\alpha + \beta + 1$ (we have "+1" no matter how many registers are between x and y, since x and y are connected directly).

With this new notion, the requirement $\nu_m(x) - \nu_m(y) \ge B - \alpha - \beta$ becomes $\nu_m(x) - \nu_m(y) \ge B - d_{xy} + 1$, where d_{xy} is the largest depth of $xy^{(m)}$ -connecting chains.

However, since we do not know how the constraint sequence evolves after $C_0...C_{m-1}$, we might need even more space between the registers at moment m. Consider an example on the right, with $R = \{r_0, r_1, r_2\}$ and the bound B = 3 on the depth of r2w chains.



- Suppose at moment 1, after seeing the constraint C_0 , which is $\{r'_1, r'_2\} > \{r_0, r_1, r_2, r'_0\}$, the valuation is $\nu_1 = \{r_0 \mapsto 0; r_1, r_2 \mapsto 3\}$. It satisfies $\nu_1(r_2) \nu_1(r_0) \geq \mathbb{B} d_{r_2r_0} + 1$ (indeed, $\mathbb{B} = 3$ and $d_{r_2r_0} = 1$ at this moment); similarly for $\nu(r_1) \nu(r_0)$.
- Let the constraint C_1 be $\{r_1, r_2, r_2'\} > \{r_1'\} > \{r_0, r_0'\}$. What value $\nu_2(r_1)$ should register r_1 have at moment 2? Note that the assignment should work no matter what C_2 will be in the future. Since the constraint C_1 places r_1 between r_0 and r_2 at moment 2, we can only assign $\nu_2(r_1) = 2$ or $\nu_2(r_1) = 1$. If we choose 2, then the constraint C_2 having $\{r_2, r_2'\} > \{r_1'\} > \{r_1\} > \{r_0, r_0'\}$ (the red dot in the figure) shows that there is not enough space between r_2 and r_1 at moment 2 $(\nu_2(r_2) = 3$ and $\nu_2(r_1) = 2)$. Similarly for $\nu_2(r_1) = 1$: the constraint C_2 having $\{r_2, r_2'\} > \{r_1\} > \{r_1'\} > \{r_0, r_0'\}$ (the blue dot in the figure) eliminates any possibility for a correct assignment.

Thus, at moment 2, the register r_1 should be equally distanced from r_0 and r_2 , i.e. $\nu_2(r_1) \approx \frac{\nu_2(r_0) + \nu_2(r_2)}{2}$, since its evolution can go either way, towards r_2 or towards r_0 . This hints at the exponential nature of distances between the registers. This is formalised in the next lemma showing that any data-assignment function that places two registers x and y at any moment m closer than $2^{\mathbf{B}-d_{xy}}$ is bound to fall. Intuitively, $\mathbf{B}-d_{xy}$ describes how many more times an insertion between the values of registers x and y can happen in the future. Since each newly inserted value should be equidistant from the boundaries, we get the $2^{\mathbf{B}-d_{xy}}$ lower bound.

Lemma 29 (tightness). Fix $B \ge 3$, registers R of $|R| \ge 3$, a 0-consistent constraint sequence prefix $C_0...C_{m-1}$ where $m \ge 1$ and whose r2w chains are depth-bounded by B, two registers $x, y \in R$ s.t. $(x' > y') \in C_{m-1}$, and a data-assignment function $f: (C_{|R} \cup C^+) \to \mathbb{N}^R$. Let $\nu_m = f(C_0...C_{m-1})$ and d_{xy} be the maximal depth of $xy^{(m)}$ -connecting chains. If $\nu_m(x) - \nu_m(y) < 2^{B-d_{xy}}$, then there exists a continuation

 $C_mC_{m+1}...$ such that the whole sequence $C_0C_1...$ is 0-consistent and its r2w chains are depth-bounded by B (hence 0-satisfiable), yet f cannot satisfy it.

Proof. We use the idea from the previous example. The constraints $C_m C_{m+1}$... are:

- 1. If at moment m there are registers different from x and y, we add the step that makes them equal to x (or to y): this does not affect the depth of xy-connecting chains at moments m and m+1; also, the maximal depths of r2w chains defined at moments $\{0,...,m\}$ and $\{0,...,m+1\}$ stay the same. Therefore, below we assume that at moment m every register is equal to x or to y.
- 2. If $B-d_{xy}=0$, we are done: $\nu_m(x)-\nu_m(y)<2^{B-d_{xy}}$ gives $\nu_m(x)\leq\nu_m(y)$ but C_{m-1} requires $\nu_m(x)>\nu_m(y)$. The future constraints then simply keep the registers constant. Otherwise, when $B-d_{xy}>0$, we proceed as follows.
- 3. To ensure consistency of constraints, C_m contains all atoms over R that are implied by atoms over R' of C_{m-1} .
- 4. C_m contains x = x' and y = y'.
- 5. C_m places a register z between x and y: x' > z' > y'. This gives $d'_{xz} = d'_{zy} = d_{xy} + 1 \le b$, where d_{xy} is the largest depth of connecting chains for $xy^{(m)}$, d'_{xz} for $xz^{(m+1)}$, and d'_{zy} for $zy^{(m+1)}$. Since $\nu_{m+1}(x) \nu_{m+1}(y) < 2^{\mathbf{B}-d_{xy}}$, either $\nu_{m+1}(x) \nu_{m+1}(z) < 2^{\mathbf{B}-d'_{xz}}$ or $\nu_{m+1}(z) \nu_{m+1}(y) < 2^{\mathbf{B}-d'_{zy}}$; this is the key observation. If the first case holds, we have the original setting $\nu_{m+1}(x) \nu_{m+1}(z) < 2^{\mathbf{B}-d'_{xz}}$ but at moment m+1 and with registers x and x; for the second case with registers x and y. Hence we repeat the entire procedure, again and again, until reaching the depth x, which gives the sought conclusion in item (2).

Finally, it is easy to prove that the whole constraint sequence C_0C_1 ... is 0-satisfiable, e.g. by showing that it satisfies the conditions of Lemma 12. Moreover, it is 0-consistent, and all r2w chains of C_0C_1 ... are depth-bounded by B because: (a) in the initial moment m, all r2w chains are depth-bounded by B; and (b) the procedure deepens only xy-connecting chains and only until the depth B, whereas other r2w chains existing at moments $\{0, ..., m\}$ keep their depths unchanged (or at moments $\{0, ..., m+1\}$, if we executed item 1).

Proof of Lemma 28 under additional assumption about 0

Tightness by Lemma 29 tells us that if a data-assignment function exists, it should separate the register values by at least $2^{\mathbf{B}-d_{xy}}$. Such separation is sufficient as we show below. We first describe a data-assignment function, then prove an invariant about it, and finally conclude with the proof of Lemma 28. For simplicity, we assume that the constraints contain a register that never changes and always holds 0. That is not true in general, so later we will lift this assumption.

Data-assignment function. The function $f: (C_{|R} \cup C^+) \to \mathbb{N}^R$ is constructed inductively on the length of $C_0...C_{m-1}$ as follows.

Initially, $f(C_{0|R}) = \nu_0$ where $\nu_0(r) = 0$ for all $r \in R$ (since C_0 has r = s, $\forall r, s \in R$). Suppose at moment m, the register valuation is $\nu_m = f(C_{0|R}C_0...C_{m-1})$. Let C_m be the next constraint, then $\nu_{m+1} = f(C_{0|R}C_0...C_m)$ is as follows:

- D1. If a register x at moment m+1 lays above all registers at moment m, i.e. $(x' > r) \in C_m$ for every register r, then set $\nu_{m+1}(x) = \nu_m(r) + 2^{\mathbb{B}}$, where r is one of the largest registers at moment m. In Church games this case happens when the test contains the atom * > r.
- D2. If a register x at moment m+1 lays between two adjacent registers a>b at moment m, then $\nu_{m+1}(x)=\lfloor\frac{\nu_m(a)+\nu_m(b)}{2}\rfloor$. In Church games this happens when the test contains a>*>b.
- D3. If a register x at moment m+1 equals a register r at previous moment m, so $(r=x') \in C_m$, then $\nu_{m+1}(x) = \nu_m(r)$. In Church games this case corresponds to a test containing the atom *=r for some register r.

Note that the case when a register x must lay below all registers never happens, since the special register r_0 always holds 0 and a given constraint sequence is 0-consistent and hence never requires $r_0 > r'$ for some register r. This is where r_0 comes handy.

Invariant. The data-assignment function satisfies the following invariant:

$$\forall m \in \mathbb{N}. \ \forall x, y \in R \ s.t. \ (x > y) \in C_m: \ \nu_m(x) - \nu_m(y) \ge 2^{\mathbf{B} - d_{xy}},$$

where d_{xy} is the largest depth of $xy^{(m)}$ -connecting chains and B is the bound on the depth of r2w chains.

Proof of the invariant. The invariant holds initially since $(r_1 = r_2) \in C_0$ for all $r_1, r_2 \in R$. Assuming it holds at step m, we show that it holds at m+1. Fix two arbitrary registers $x, y \in R$ such that $(x' > y') \in C_m$; we will prove that $\nu_{m+1}(x) - \nu_{m+1}(y) \ge 2^{\mathbb{B}-d_{xy}}$, where d_{xy} is the largest depth of $xy^{(m+1)}$ -connecting chains. There are four cases depending on whether the levels of x and y at moment m+1 are present at moment m or not, illustrated in Figure 10.

Case 2: x is new top. The register x lies on the top level of both moments m and m+1, and y lies on a level that was also present at moment m. This corresponds to item D1. Let $(b=y') \in C_m$ and a lies on the largest level at moment m (a and b may coincide). Thus, $\nu_{m+1}(x) = \nu_m(a) + 2^{\mathrm{B}}$. The invariant holds for x,y because $\nu_{m+1}(x) = \nu_m(a) + 2^{\mathrm{B}}$ and $\nu_m(a) \geq \nu_m(b) = \nu_{m+1}(y)$.

⁶A stronger result holds, namely $d_{ab} = d_{xy}$, but it is not needed here.

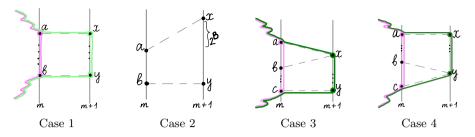


Figure 10: Proving the invariant

Case 3: x is middle new, y was present. The register x at moment m+1 lies on a new level that is between the levels of a and b at moment m, so $\nu_{m+1}(x) = \lfloor \frac{\nu_m(a) + \nu_m(b)}{2} \rfloor$ by item D2 of data-assignment function. The register y at moment m+1 lies on a level that was also present at moment m, witnessed by register c. Formally, C_m contains a > x' > b for a and b adjacent at moment m, c = y', and x' > y'. Note that c and b may coincide. Then, $\nu_{m+1}(x) - \nu_{m+1}(y) = \lfloor \frac{\nu_m(a) + \nu_m(b)}{2} \rfloor - \nu_m(c) = \lfloor \frac{\nu_m(a) - \nu_m(c)}{2} \rfloor + \lfloor \frac{\nu_m(b) - \nu_m(c)}{2} \rfloor \geq \lfloor 2^{B - d_{ac} - 1} \rfloor + \lfloor 2^{B - d_{bc} - 1} \rfloor \geq 2^{B - d_{ac} - 1} + \lfloor 2^{B - d_{bc} - 1} \rfloor$; the latter holds because $d_{ac} < b$ while $d_{bc} \le b$. We need to prove that the last sum is greater or equal to $2^{B - d_{xy}}$. Figure 10 (case 3) shows how the green $xy^{(m+1)}$ -connecting chain can be constructed from the pink $ac^{(m)}$ -connecting chain, hence $d_{xy} \ge d_{ac} + 1$, so we get $2^{B - d_{ac} - 1} \ge 2^{B - d_{xy}}$. Hence, $\nu_{m+1}(x) - \nu_{m+1}(y) \ge 2^{B - d_{ac} - 1} + \lfloor 2^{B - d_{bc} - 1} \rfloor \ge 2^{B - d_{xy}}$.

Case 4: x was present, y is middle new. The case is similar to the previous one, but we prove it for completeness. The constraint C_m contains $a=x', \ x'>y', b>y'>c$, where b and c are adjacent (a and b might be the same). Then, $\nu_{m+1}(x)-\nu_{m+1}(y)=\nu_m(a)-\lfloor\frac{\nu_m(b)+\nu_m(c)}{2}\rfloor\geq \lfloor\frac{\nu_m(a)-\nu_m(b)}{2}+\frac{\nu_m(a)-\nu_m(c)}{2}\rfloor\geq \lfloor\frac{\nu_m(a)-\nu_m(b)}{2}\rfloor+\lfloor\frac{\nu_m(a)-\nu_m(c)}{2}\rfloor\geq \lfloor2^{\mathbf{B}-d_{ab}-1}\rfloor+\lfloor2^{\mathbf{B}-d_{ac}-1}\rfloor\geq \lfloor2^{\mathbf{B}-d_{ab}-1}\rfloor+2^{\mathbf{B}-d_{ac}-1},$ and since $d_{ac}+1\leq d_{xy}$, we get $\nu_{m+1}(x)-\nu_{m+1}(y)\geq \lfloor2^{\mathbf{B}-d_{ab}-1}\rfloor+2^{\mathbf{B}-d_{ac}-1}\geq 2^{\mathbf{B}-d_{xy}}$.

Proof of Lemma 28. It is sufficient to show that for every atom $(r \bowtie s)$ or $(r \bowtie s')$ of C_m , where $r, s \in R$ and $\bowtie \in \{<,>,=\}$, the expressions $\nu_m(r) \bowtie \nu_m(s)$ or $\nu_m(r) \bowtie \nu_{m+1}(s)$ hold, respectively. Depending on $r \bowtie s$, there are the following cases.

- If C_m contains (r = s) or (r = s') for $r, s \in R$, then item D₃ implies resp. $\nu_m(r) = \nu_m(s)$ or $\nu_m(r) = \nu_{m+1}(s)$.
- If $(r > s) \in C_m$, then $\nu_m(r) > \nu_m(s)$ by the invariant.
- Let $(r > s') \in C_m$ and the level of s at moment m+1 be present at moment m, i.e. there is a register t such that $(t = s') \in C_m$. Since $\nu_m(t) = \nu_{m+1}(s)$ by item D3 and since $\nu_m(r) > \nu_m(t)$ by $(r > t = s') \in C_m$, we get $\nu_m(r) > \nu_{m+1}(s)$. Similarly for the case $(r < s') \in C_m$ where s lies on a level also present at moment m.
- Let $(r < s') \in C_m$ and s lies on the highest level among all levels at moments m and m+1. Then $\nu_m(r) < \nu_{m+1}(s)$ because $\nu_{m+1}(s) \ge \nu_m(r) + 2^{\mathrm{B}}$ by item D1.
- Finally, there are two cases left: $(r > s') \in C_m$ or $(r < s') \in C_m$, where s lies on a newly created level at moment m+1, and there are higher levels at moment m. This

corresponds to item D2. Let $(a>b)\in C_m$ be two adjacent registers at moment m between which the register s is inserted at moment m+1, so $(a>s'>b)\in C_m$. Let d_{ab} be the maximal depth of $ab^{(m)}$ -connecting chains; fix one such chain. We change it by going through s at moment m+1, i.e. substitute the part (a,m)>(b,m) by (a,m)>(s,m+1)>(b,m): the depth of the resulting chain is $d_{ab}+1$ and it is $\leq B$ by boundedness of r2w chains. Hence $d_{ab}\leq B-1$, so $\nu_m(a)-\nu_m(b)\geq 2$, implying $\nu_m(a)>\lfloor\frac{\nu_m(a)+\nu_m(b)}{2}\rfloor>\nu_m(b)$. When $(r>s')\in C_m$ we get $\nu_{m+1}(r)\geq\nu_m(a)$, and when $(r<s')\in C_m$ we get $\nu_{m+1}(r)\leq\nu_m(b)$, therefore we are done.

Finally, the function always assigns nonnegative numbers, from \mathbb{N} , so we are done.

Lifting the assumption about 0

We now lift the assumption about a register always holding 0. This assumption was used in the definition of the data-assignment function (items D1, D2, D3). The idea is to convert a given constraint sequence over registers R into a sequence over registers $R \uplus \{r_0\}$ while preserving satisfiability.

Conversion function. Given a 0-consistent constraint sequence $C_0C_1...$ over R without a special register holding 0, we will construct, on-the-fly, a 0-consistent sequence $\tilde{C}_0\tilde{C}_1...$ over $R \uplus \{r_0\}$ that has such a register. Intuitively, we will add atoms $r = r_0$ only if they follow from what is already known otherwise we add atoms $r > r_0$.

Initially, in addition to the atoms of C_0 , we require $r = r_0$ for every $r \in R$ (recall that the original C_0 contains $r_1 = r_2$ for all $r_1, r_2 \in R$). This gives an incomplete constraint \tilde{C}_0 over $R_0 \cup R'_0$: it does not yet have atoms of the form $r \bowtie r'_0$, $r_0 \bowtie r'$, $r'_0 \bowtie r'$, where $r \in R_0$.

At moment $m \geq 0$, given a constraint $\tilde{C}_{m|R_0}$ over R_0 (without primed registers R'_0) and a constraint C_m over $R \cup R'$ (without register r_0), we construct \tilde{C}_m over $R_0 \cup R'_0$ as follows:

- \tilde{C}_m contains all atoms of C_m .
- $(r_0 = r_0') \in \tilde{C}_m$.
- For every $r \in R$: if $r' = r_0$ is implied by the current atoms of \tilde{C}_m , then we add it, otherwise we add $r' > r_0$.

Notice that the atom $r' < r_0$ is never implied by \tilde{C}_m , as we show now. Suppose the contrary. Then, since C_m does not talk about r_0 nor r'_0 , there should be $s \in R$ such that $(s = r_0) \in \tilde{C}_{m|R_0}$ and $(r' < s) \in C_m$. By construction, if this is the case, then there is a one-way chain $(r_1,0) = (r_2,1) = \dots = (s,m)$ of zero depth. As a consequence, we can construct the one-way decreasing chain $(r_1,0) = (r_2,1) = \dots = (s,m) > (r,m+1)$ of depth 1, which implies that C_0C_1 ... is not 0-consistent. We reached a contradiction, so $(r' < r_0) \in \tilde{C}_m$ is not possible.

• Finally, to make \tilde{C}_m maximal, we add all atoms implied by \tilde{C}_m but not present

Using this construction, we can easily define $c\theta nv: C^+ \to \tilde{C}$ and map a given 0-consistent constraint sequence $C_0C_1...$ to $\tilde{C}_0\tilde{C}_1...$ with a dedicated register holding 0. Notice that the constructed sequence is also 0-consistent, because we never add

inconsistent atoms and never add an atom $r' < r_0$ (see the third item). Finally, in the constructed sequence the depths of r2w chains can increase by at most 1, due to the register r_0 : it can increase the depth of a finite chain by one, unless the chain is already ending in a register holding 0. Hence we get the following lemma.

Lemma 30. For every 0-consistent constraint sequence C_0C_1 ..., the sequence $\tilde{C}_0\tilde{C}_1$... constructed with c0nv is also 0-consistent. Moreover, the maximal depth of r2w chains cannot increase by more than 1.

Final proof of Lemma 28. We lift the assumption about constraint sequences having a special register always holding zero. Using $c\partial nv$, we automatically translate a given 0-consistent constraint sequence prefix $C_0...C_m$ over R into $\tilde{C}_0...\tilde{C}_m$ over $R \uplus \{r_0\}$ that contains a register r_0 always holding 0. Now we can apply the data-assignment function as described before. By definition of $c\partial nv$, the original constraint $C_i \subset \tilde{C}_i$ for every $i \geq 0$, so the resulting valuation satisfies the original constraints as well. This concludes the proof of Lemma 28.

6 Conclusion

Our main result states that one-sided Church games for specifications given as deterministic register automata over (\mathbb{N}, \leq) are decidable, in ExpTime. Moreover, we show that those games are determined, and that strategies implemented by transducers with registers suffice to win.

The decidability result involves a characterisation of satisfiable infinite constraint sequences over (\mathbb{N}, \leq) : they must not have decreasing two-way chains of infinite depth, nor ceiled (bounded from the above) chains of unbounded depth. A similar characterisation can be established for (\mathbb{Z}, \leq) . For instance, it should require that the two-way chains which are bounded from both above and below have bounded depth. Then, the decidability of one-sided Church synthesis for (\mathbb{Z}, \leq) can be established in a similar way to (\mathbb{N}, \leq) . The decidability for (\mathbb{Z}, \leq) can also be proven by reducing to the problem for (\mathbb{N}, \leq) as follows. From a specification S, given as a set of words $d_1\sigma_1d_2\sigma_2...$ alternating between a value $d_i \in \mathbb{Z}$ and a letter σ_i from a finite alphabet Σ , we construct a specification S' of words of the form $\max(0, d_1) \# \max(0, -d_1) \sigma_1 \max(0, d_2) \# \max(0, -d_2) \sigma_2 \cdots \in (\mathbb{N}(\Sigma \cup \{\#\}))^{\omega}, \text{ where } \#$ acts as a waiting symbol. Non-zero values given by Adam at positions 4n + 1 correspond to positive values, and non-zero values at positions 4n+3 correspond to negative values. Thus, if S is given as a deterministic register automaton, one can construct a deterministic register automaton that recognises S', which preserves the existence of solutions to synthesis. An interesting future direction is to establish a general reduction between data domains such that decidability results for one-sided Church synthesis transfer from one domain to the other. A candidate notion for such a reduction was defined in the context of register-bounded transducer synthesis [26].

Another important future direction is to consider logical formalisms instead of automata to describe specifications in a more declarative and high-level manner. Data word first-order logics [7, 46] have been studied with respect to the satisfiability problem but when used as specification languages for synthesis, only few results are

known. The first steps in this direction were done in [30, 4] for Constraint LTL on (\mathbb{Z}, \leq) ; see also [22] for an overview of nonemptiness of constraint tree automata; and see [3] for a slightly different context of parameterised synthesis.

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