Parallel Algorithms for Generating Random Networks with Given Degree Sequences

Maksudul Alam*† and Maleq Khan*
*Network Dynamics and Simulation Science Laboratory, Virginia Bioinformatics Institute

†Department of Computer Science
Virginia Tech, Blacksburg, Virginia 24061 USA
Email: {maksud, maleq}@vbi.vt.edu

Abstract—Random networks are widely used for modeling and analyzing complex processes. Many mathematical models have been proposed to capture diverse real-world networks. One of the most important aspects of these models is degree distribution. Chung-Lu (CL) model is a random network model, which can produce networks with any given arbitrary degree distribution. The complex systems we deal with nowadays are growing larger and more diverse than ever. Generating random networks with any given degree distribution consisting of billions of nodes and edges or more has become a necessity, which requires efficient and parallel algorithms. We present an MPI-based distributed memory parallel algorithm for generating massive random networks using CL model, which takes $O(\frac{m+n}{P} + P)$ time with high probability and O(n) space per processor, where n, m, and P are the number of nodes, edges and processors, respectively. The time efficiency is achieved by using a novel load-balancing algorithm. Our algorithms scale very well to a large number of processors and can generate massive powerlaw networks with one billion nodes and 250 billion edges in one minute using 1024 processors. *Index Terms*—assive networks, parallel algorithms, network generatorassive networks, parallel algorithms, network generatorm

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I. INTRODUCTION

The advancements of modern technologies are causing a rapid growth of complex systems. These systems, such as the Internet [1], biological networks [2], social networks [3, 4], and various infrastructure networks [5, 6] are sometimes modeled by random graphs for the purpose of studying their behavior. The study of these complex systems have significantly increased the interest in various random graph models such as Erdős-Rényi (ER) [7], small-world [8], Barabási-Albert (BA) [9], Chung-Lu (CL) [10], HOT [11], exponential random graph (ERGM) [12], recursive matrix (R-MAT)[13], and stochastic Kronecker graph (SKG) [14, 15] models. Among those models, the SKG model has been included in Graph500 supercomputer benchmark [16] due to its simple parallel implementation. The CL model exhibits the similar properties of the SKG model and further has the ability to generate a wider range of degree distributions [17]. To the best of our knowledge, there is no parallel algorithm for the CL model.

Analyzing a very large complex system requires generating massive random networks efficiently. As the interactions in a

larger network lead to complex collective behavior, a smaller network may not exhibit the same behavior, even if both networks are generated using the same model. In [18], by experimental analysis, it was shown that the structure of larger networks is fundamentally different from small networks and many patterns emerge only in massive datasets. Demand for large random networks necessitates efficient algorithms to generate such networks. However, even efficient sequential algorithms for generating such graphs were nonexistent until recently. Sequential algorithms are sometimes acceptable in network analysis with tens of thousands of nodes, but they are not appropriate for generating large graphs [19]. Although, recently some efficient sequential algorithms have been developed [13, 19, 14, 20], these algorithms can generate networks with only millions of nodes in a reasonable time. But, generating networks with billions of nodes can take an undesirably longer amount of time. Further, a large memory requirement may even prohibit generating such large networks using these sequential algorithms. Thus, distributedmemory parallel algorithms are desirable in dealing with these problems. Shared-memory parallel algorithms also suffer from the memory restriction as these algorithms use the memory of a single machine. Also, most shared-memory systems are limited to only a few parallel processors whereas distributedmemory parallel systems are available with hundreds or thousands of processors.

In this paper, we present a time-efficient MPI-based distributed memory parallel algorithm for generating random networks from a given sequence of expected degrees using the CL model. To the best of our knowledge, it is the first parallel algorithm for the CL model. The most challenging part of this algorithm is load-balancing. Partitioning the nodes with a balanced computational load is a non trivial problem. In a sequential setting, many algorithms for the load-balancing problem were studied [21, 22, 23]. Some of them are exact and some are approximate. These algorithms uses many different techniques such as heuristic, iterative refinement, dynamic programming, and parametric search. All of these algorithms require at least $\Omega(n + P \log n)$ time, where n, P are the number of nodes and processors respectively. To the best of our knowledge, there is no parallel algorithm for this problem. In this paper, we present a novel and efficient parallel algorithm for computing the balanced partitions in $O(\frac{n}{P} + P)$ time. The parallel algorithm for load balancing can be of independent interest and probably could be used in many other problems. Using this load balancing algorithm, the parallel algorithm for the CL model takes an overall runtime of $O(\frac{n+m}{P}+P)$ w.h.p.. The algorithm requires O(n) space per processor. Our algorithm scales very well to a large number of processors and can generate a power-law networks with a billion nodes and 250 billion edges in memory in less than a minute using 1024 processors. The rest of the paper is organized as follows. In Section III we describe the problem and the efficient sequential algorithm. In Section IV, we present the parallel algorithm along with analysis of partitioning and load balancing. Experimental results showing the performance of our parallel algorithms are presented in Section V. We conclude in Section VI.

II. PRELIMINARIES AND NOTATIONS

In the rest of the paper we use the following notations. We denote a network by G(V, E), where V and E are the sets of vertices (nodes) and edges, respectively, with m = |E| edges and n = |V| vertices labeled as $0, 1, 2, \ldots, n-1$. We use the terms *node* and *vertex* interchangeably.

We develop parallel algorithm for the message passing interface (MPI) based distributed memory system, where the processors do not have any shared memory and each processor has its own local memory. The processors can exchange data and communicate with each other by exchanging messages. The processors have a shared file system and they read-write data files from the same external memory. However, such reading and writing of the files are done independently.

We use K, M and B to denote thousands, millions and billions, respectively; e.g., 2B stands for two billion.

III. CHUNG-LU MODEL AND EFFICIENT SEQUENTIAL ALGORITHM

Chung-Lu (CL) model [10] generates random networks from a given sequence of expected degrees. We are given n nodes and a set of non-negative weights $w=(w_0,\ldots w_{n-1})$ assuming $\max_i w_i^2 < S$, where $S=\sum_k w_k$ [10]. For every pair of nodes i and j, edge (i,j) is added to the graph with probability $p_{i,j}=w_iw_j/S$. If no self loop is allowed, i.e., $i\neq j$, the expected degree of node i is given by $\sum_j w_i w_j/S = w_i - w_i^2/S$. For massive graphs, where n is very large, the average degree converges to w_i , thus w_i represents the expected degree of node i [20].

The naïve algorithm of CL model for an undirected graph with n nodes takes each of the n(n-1)/2 possible node pairs $\{i,j\}$ and creates the edge with probability $p_{i,j}$, therefore requiring $O(n^2)$ time. An O(n+m) algorithm was proposed in [20] to generate networks assuming w is sorted in non-increasing order, where m is the number of edges. It is easy to see that O(n+m) is the best possible runtime to generate m edges. The algorithm is based on the edge skipping technique introduced in [19] for Erdős–Rényi model. Adaptation of that technique leads to the efficient sequential algorithm in [20]. The pseudocode of the algorithm is given in

Algorithm III.1 Sequential Chung-Lu Algorithm

```
1: procedure SERIAL-CL(w)
          S \leftarrow \sum_k w_k
 2:
           E \leftarrow \overline{\mathbf{C}}_{\mathsf{REATE-EDGES}}(w, S, V)
 3:
 4: procedure Create-Edges(w, S, V)
          E \leftarrow \emptyset
 5:
          for all i \in V do
 6:
               j \leftarrow i+1, p \leftarrow \min(w_i w_i/S, 1)
 7:
                while j < n and p > 0 do
 8:
                     if p \neq 1 then
 9:
                          choose a random r \in (0,1)
10:
                          \delta \leftarrow \lfloor \log(r) / \log(1-p) \rfloor
11:
12:
                     else
                          \delta \leftarrow 0
13:
                     v \leftarrow j + \delta
                                                                 \triangleright skip \delta edges
14:
15:
                     if v < n then
                          q \leftarrow \min(w_i w_v / S, 1)
16:
17:
                          choose a random r \in (0, 1)
                          if r < q/p then
18:
                               E \leftarrow E \cup \{i, v\}
19:
                          p \leftarrow q, \quad j \leftarrow v + 1
20:
          return E
21:
```

Algorithm III.1, consisting of two procedures SERIAL-CL and CREATE-EDGES. Note that we restructured Algorithm III.1 by defining procedure CREATE-EDGES to use it without any changes later in our parallel algorithm. Below we provide an overview and a brief description of the algorithm (for complete explanation and correctness see [20]).

The algorithm starts at SERIAL-CL, which computes the sum S and calls procedure CREATE-EDGES(w, S, V), where V is the entire set of nodes. For each node $i \in V$, the algorithm selects some random nodes v from [i+1, n-1], and creates the edges (i, v). A naïve way to select the nodes v from [i+1, n-1]1] is: for each $j \in [i+1, n-1]$, select j independently with probability $p_{i,j} = w_i w_j / S$, leading to an algorithm with run time $O(n^2)$. Instead, the algorithm skips the nodes that are not selected by a random skip length δ as follows. For each $i \in V$ (Line 6), the algorithm starts with j = i + 1 and computes a random skip length $\delta \leftarrow \left\lfloor \frac{\log(r)}{\log(1-p)} \right\rfloor$, where r is a real number in (0,1) chosen uniformly at random and $p=p_{i,j}=w_iw_j/S$. Then node v is selected by skipping the next δ nodes (Line 14), and edge (i, v) is selected with probability q/p, where q = $p_{i,v} = w_i w_v / S$ (Line 16–19). Then from the next node j + v, this cycle of skipping and selecting edges is repeated (while loop in Line 8–20). As we always have i < j and no edge (i, j)can be selected more than once, this algorithm does not create any self-loop or parallel edges. As the set of weights w is sorted in non-increasing order, for any node i, the probability $p_{i,j} = w_i w_j / S$ decreases monotonically with the increase of j. It is shown in [20] that for any i, j, edge (i, j) is included in E with probability exactly $w_i w_j / S$, as desired, and that the algorithm runs in O(n+m) time.

 $c_u = e_u + 1 (2)$

Next we present our distributed memory parallel algorithm for the CL model. Although our algorithm generates undirected edges, for the ease of discussion we consider u as the source node and v as the destination node for any edge (u,v) generated by the procedure CREATE-EDGES. Let T_u be the task of generating the edges from source node u (Lines 6–20 in Algorithm III.1). It is easy to see that for any $u \neq u'$ tasks T_u and $T_{u'}$ are independent, i.e., tasks T_u and $T_{u'}$ can be executed independently by two different processors. Now execution of procedure CREATE-EDGES(w,S,V) is equivalent to executing the set of tasks $\{T_u:u\in V\}$. Efficient parallelization of Algorithm III.1 requires:

- Computing the sum $S = \sum_{k=0}^{n-1} w_k$ in parallel
- Dividing the task of executing CREATE-EDGES into independent subtasks
- Accurately estimating the computational cost for each task
- Balancing load among the processors

To compute the sum S efficiently, a parallel sum operation is performed on w using P processors, which takes $O(\frac{n}{P} + \log P)$ time. To divide the task of executing procedure CREATE–EDGES into independent subtasks, the set of nodes V is divided into P disjoint subsets V_1, V_2, \ldots, V_P ; that is, $V_i \subset V$, such that for any $i \neq j, \ V_i \cap V_j = \emptyset$ and $\bigcup_i V_i = V$. Then V_i is assigned to processor P_i , and P_i execute the tasks $\{T_u : u \in V_i\}$; that is, P_i executes CREATE–EDGES (w, S, V_i) .

Estimating and balancing computational loads accurately are the most challenging tasks. To achieve good speedup of the parallel algorithm, both tasks must also be done in parallel, which is a non-trivial problem. A good load balancing is achieved by properly partitioning the set of nodes V such that the computational loads are equally distributed among the processors. We use two classes of partitioning schemes named consecutive partitioning (CP) and round-robin partitioning (RRP). In CP scheme consecutive nodes are assigned to each partition, whereas in RRP scheme nodes are assigned to the partitions in a round-robin fashion. The use of various partitioning schemes is not only interesting for understanding the performance of the algorithm, but also useful in analyzing the generated networks. It is sometimes desirable to generate networks on the fly and analyze it without performing disk I/O. Different partitioning schemes can be useful for different network analysis algorithms. Many network analysis algorithms require partitioning the graph into an equal number of nodes (or edges) per processor. Some algorithms also require the consecutive nodes to be stored in the same processor. Before discussing the partitioning schemes in detail, we describe some formulations that are applicable to all of these schemes.

Let e_u be the expected number of edges produced and c_u be the computational cost in task T_u for a source node u. For the sake of simplicity, we assign one unit of time to process a node or an edge. With $S = \sum_{v=0}^{n-1} w_v$, we have:

$$e_u = \sum_{v=u+1}^{n-1} p_{u,v} = \sum_{v=u+1}^{n-1} \frac{w_u w_v}{S} = \frac{w_u}{S} \sum_{v=u+1}^{n-1} w_v$$
 (1)

For two nodes $u,v\in V$ such that u< v, we have $c_u\geq c_v$ (see Lemma 4 in Appendix VII). The expected number of edges generated by the tasks $\{T_u:u\in V_i\}$ is given by $m_i=\sum_{u\in V_i}e_u$. Note that the expected number of edges in the generated graph, i.e., the expected total number of edges generated by all processors is $m=|E|=\sum_{i=0}^{P-1}m_i=\sum_{u=0}^{n-1}e_u$. The computational cost for processor P_i is given by: $c(V_i)=\sum_{u\in V_i}c_u=\sum_{u\in V_i}(e_u+1)=m_i+|V_i|$. Therefore, the total cost for all processors is given by:

$$\sum_{i=0}^{P-1} c(V_i) = \sum_{i=0}^{P-1} (m_i + |V_i|) = m + n$$
 (3)

A. Consecutive Partitioning (CP)

Let partition V_i starts at node n_i and ends at node $n_{i+1}-1$, where $n_0=0$ and $n_P=n$, i.e., $V_i=\{n_i,n_i+1,\dots,n_{i+1}-1\}$ for all i. We say n_i is the lower boundary of partition V_i . A naïve way for partitioning V is where each partition consists of an equal number of nodes, i.e., $|V_i|=\left\lceil\frac{n}{P}\right\rceil$ for all i. To keep the discussion neat, we simply use $\frac{n}{P}$. Although the number of nodes in each partition is equal, the computational cost among the processors is very imbalanced. For two consecutive partitions V_i and V_{i+1} , $c(V_i) > c(V_{i+1})$ for all i and the difference is at least $\frac{n^2}{SP^2}\overline{W}_i\overline{W}_{i+1}$, where $\overline{W}_i=\frac{1}{|V_i|}\sum_{u\in V_i}w_u$, the average weight (degree) of the nodes in V_i (see Lemma 5 in Appendix VII). Thus $c(V_i)$ gradually decreases with i by a large amount leading to a very imbalanced distribution of the computational cost.

To demonstrate that naïve CP scheme leads to imbalanced distribution of computational cost, we generated two networks, both with one billion nodes: i) Erdős–Rényi network with an average degree of 500, and ii) Power–Law network with an average degree of 49.72. We used 512 processors, which is good enough for this experiment. Fig. 1 shows the computational cost and runtime per processor. In both cases, the cost is not balanced. For power-law network the imbalance of computational cost is more prominent. Observe that the runtime is almost directly proportional to the cost, which justifies our choice of cost function. That is balancing the cost would also balance the runtime.

We need to find the partitions V_i such that each partition has equal cost, i.e., $c(V_i) \approx \overline{Z}$, where $\overline{Z} = (m+n)/P$ is the average cost per processor. We refer such partitioning scheme as uniform cost partitioning (UCP). Although determining the partition boundaries in the naïve scheme is very easy, finding the boundaries in UCP scheme is a non trivial problem and requires: (i) computing the cost c_u for each node $u \in V$ and (ii) finding the boundaries of the partitions such that every partition has a cost of \overline{Z} . Naïvely computing costs for all nodes takes $O(n^2)$ as each node independently requires O(n) time using Equation 2 and 1. A trivial parallelization achieves $O(n^2/P)$ time. However, our goal is to parallelize the computation of the costs in $O(n/P + \log P)$ time.

Finding the partition boundaries such that the maximum cost of a partition is minimized is a well-known problem named *chains-on-chains partitioning* (CCP) problem [23]. In CCP,

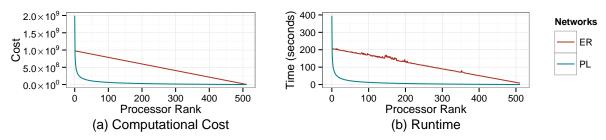


Fig. 1: Computational cost and runtime in naïve CP scheme

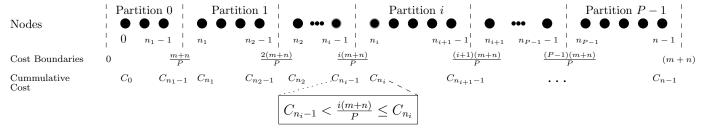


Fig. 2: Uniform cost partitioning (UCP) scheme

a sequence of P-1 separators are determined to divide a chain of n tasks with associated non-negative weights (c_u) into P partitions so that the maximum cost in the partitions is minimized. Sequential algorithms for CCP are studied quite extensively [21, 22, 23]. Since these algorithms take at least $\Omega(n+P\log n)$ time, using any of these sequential algorithms to find the partitions, along with the parallel algorithm for the CL model, does not scale well. To the best of our knowledge, there is no parallel algorithm for CCP problem. We present a novel parallel algorithm for determining the partition boundaries which takes O(n/P+P) time in the worst case.

To determine the partition boundaries, instead of using c_u directly, we use the cumulative cost $C_u = \sum_{v=0}^u c_v$. We call a partition V_i a balanced partition if the computational cost of V_i is $c(V_i) = \sum_{u=n_i}^{n_{i+1}-1} c_u = C_{n_{i+1}-1} - C_{n_i-1} \approx \overline{Z}$. Also note that for lower boundary n_i of partition V_i we have, $C_{n_i-1} < i\overline{Z} \le C_{n_i}$ for $0 < i \le P-1$. Thus, we have:

$$n_i = \arg\min_{u} \left(C_u \ge i\overline{Z} \right) \tag{4}$$

In other words, a node u with cumulative cost C_u belongs to partition V_i such that $i = \lfloor C_u/\overline{Z} \rfloor$. The partition scheme is shown visually in Fig. 2.

Computing C_u in Parallel. Computing C_u has two difficulties: i) for a node u, computing c_u by using Equation 1 and 2 directly is inefficient and ii) C_u is dependent on C_{u-1} , which is hard to parallelize. To overcome the first difficulty, we use the following form of e_u to calculate c_u . From Equation 1 we have:

$$e_{u} = \frac{w_{u}}{S} \sum_{v=u+1}^{n-1} w_{v} = \frac{w_{u}}{S} \left(\sum_{v=0}^{n-1} w_{v} - \sum_{v=0}^{u} w_{v} \right)$$

$$= \frac{w_{u}}{S} \left(\sum_{v=0}^{n-1} w_{v} - \sum_{v=0}^{u-1} w_{v} - w_{u} \right)$$

$$c_{u} = e_{u} + 1$$

$$= \frac{w_u}{S} \left(S - \sigma_u - w_u \right) + 1 \left[\text{where } \sigma_u = \sum_{v=0}^{u-1} w_v \right]$$
 (5)

Therefore, c_u can be computed by successively updating $\sigma_u = \sigma_{u-1} + w_{u-1}$.

To deal with the second difficulty, we compute C_u in several steps using procedure CALC-COST as shown in Algorithm IV.1 (see Fig. 7 in Appendix VII for a visual representation of the algorithm). In each processor, the partitioning algorithm starts with procedure UCP that calculates the cumulative costs using procedure CALC-COST. Then procedure MAKE-PARTITION is used to compute the partitioning boundaries. At the beginning of the CALC-COST procedure, the task of computing costs for the n nodes are distributed among the P processors equally, i.e., processor P_i is responsible for computing costs for the nodes from $i\frac{n}{P}$ to $(i+1)\frac{n}{P}-1$. Note that these are the nodes that processor P_i works with while executing the partitioning algorithm to find the boundaries of the partitions.

In Step 1 (Line 6), P_i computes a partial sum $s_i = \sum_{u=\frac{in}{P}}^{\frac{(i+1)n}{P}-1} w_u$ independently of other processors. In Step 2 (Line 7), exclusive prefix sum $S_i = \sum_{j=0}^{i-1} s_j$ is calculated for all s_i where $0 \le i \le P-1$ and $S_0 = 0$. This exclusive prefix sum can be computed in parallel in $O(\log P)$ time [24]. We have:

$$S_i = \sum_{j=0}^{i-1} s_j = \sum_{j=0}^{i-1} \sum_{\substack{u=\frac{jn}{P}}}^{\frac{(j+1)n}{P}-1} w_u = \sum_{u=0}^{\frac{in}{P}-1} w_u = \sigma_{\frac{in}{P}}$$

In Step 3, P_i partially computes C_u , where $\frac{in}{P} \leq u < \frac{(i+1)n}{P}$. By assigning $\sigma_{\frac{in}{P}} = S_i$, $C_{\frac{in}{P}}$ is determined partially using Equation 5 in constant time (Line 10). For each u, values of σ_u , e_u and C_u are also determined in constant time (Line 11–14), where $\frac{in}{P} + 1 \leq u \leq \frac{(i+1)n}{P} - 1$. After Step 3, we have $C_u = \sum_{v=\frac{in}{P}}^u c_v$. To get the final value of $C_u = \sum_{v=0}^u c_v$, the value $\sum_{v=0}^v \frac{e^{in}}{P} - 1$ c_v needs to be added. For a processor P_i , let

Algorithm IV.1 Uniform Consecutive Partition

```
1: procedure UCP(V, w, S)
                CALC-COST(w, V, S)
 2:
                Make-Partition(w, V, S)
 3:
  4: procedure CALC-COST(w, V, S)
               i \leftarrow \text{processor id}
 5:
               s_i \leftarrow \sum_{u=i\frac{n}{P}}^{(i+1)\frac{n}{P}-1} w_u
  6:
               In Parallel: S_i \leftarrow \sum_{j=0}^{i-1} s_j
  7:
  8:
              \sigma_u \leftarrow S_i
C_u \leftarrow e_u + 1 = \frac{w_u}{S}(S - \sigma_u - w_u) + 1
\mathbf{for} \ u = \frac{in}{P} + 1 \ \mathbf{to} \frac{(i+1)n}{P} - 1 \ \mathbf{do}
  9:
10:
11:
                      \sigma_u \leftarrow \sigma_u + w_u
e_u \leftarrow \frac{w_u}{S} (S - \sigma_u - w_u)
C_u \leftarrow C_{u-1} + e_u + 1
12:
13:
14:
               z_i \leftarrow C_{\frac{(i+1)n}{2}-1}
15:
              In Parallel: Z_i \leftarrow \sum_{j=0}^{i-1} z_j for u = \frac{in}{P} to \frac{(i+1)n}{P} - 1 do C_u = C_u + Z_i
16:
17:
18:
19: procedure Make-Partition(w, V, S)
20: In Parallel: Z \leftarrow \sum_{i=0}^{P-1} z_i
21:
               FIND-BOUNDARIES (\frac{in}{P}, \frac{(i+1)n}{P} - 1, C, \overline{Z})
22:
               for all n_k \in B_i do
23:
                       Send n_k to P_k and P_{k+1}
24:
               Receive boundaries n_i and n_{i+1}
25:
               return V_i = [n_i, n_{i+1} - 1]
26:
27: procedure FIND-BOUNDARIES(s, e, C, \overline{Z})
               if \left\lfloor \frac{C_s}{\overline{Z}} \right\rfloor = \left\lfloor \frac{C_e}{\overline{Z}} \right\rfloor then return
28:
              m \leftarrow \frac{\tilde{(e+s)}}{2}
\text{if } \left\lfloor \frac{C_m}{\overline{Z}} \right\rfloor \neq \left\lfloor \frac{C_{m+1}}{\overline{Z}} \right\rfloor \text{ then }
n_{\left\lfloor \frac{C_{m+1}}{\overline{Z}} \right\rfloor} \leftarrow m+1
29:
30:
31:
               FIND-BOUNDARIES(s, m, C, \overline{Z})
32:
               FIND-BOUNDARIES(m+1, e, C, \overline{Z})
33:
```

 $z_i=C_{\frac{(i+1)n}{P}-1}=\sum_{v=\frac{in}{P}}^{\frac{(i+1)n}{P}-1}c_v.$ In Step 4 (Line 16), another exclusive parallel prefix sum operation is performed on z_i so that

$$Z_i = \sum_{j=0}^{i-1} z_j = \sum_{j=0}^{i-1} \sum_{v=\frac{j_n}{P}}^{\frac{(j+1)n}{P}-1} c_v = \sum_{v=0}^{\frac{i_n}{P}-1} c_v.$$

Note that Z_i is exactly the value required to get the final cumulative cost C_u . In Step 5 (Lines 17–18), Z_i is added to C_u for $\frac{in}{P} \leq u \leq \frac{(i+1)n}{P}-1$.

Finding Partition Boundaries in Parallel. The partition boundaries are determined using Equation 4. The procedure Make-Partition generates the partition boundaries. In Line 20, parallel sum is performed on z_i to determine $Z = \sum_{0}^{P-1} z_i = \sum_{0}^{n-1} c_u = n+m$, the total cost and $\overline{Z} = \frac{Z}{P}$ be the average cost per processor (Line 21). FIND-

BOUNDARIES is called to determine the boundaries (Line 22). From Equation 4 it is easy to show that a partition boundary is found between two consecutive nodes u and u+1, such that $|C_u/\overline{Z}| \neq |C_{u+1}/\overline{Z}|$. Node u+1 is the lower boundary of partition V_i , where $i = |C_{u+1}/\overline{Z}|$. P_i executes FIND-BOUNDARIES from nodes in/P to (i+1)n/P-1. FIND-BOUNDARIES is a divide & conquer based algorithm to find all the boundaries in that range efficiently using the cumulative costs. All the found boundaries are stored in a local list. In Line 28, it is determined whether the range contains any boundary. If the range does not have any boundary, i.e., if $|C_s/\overline{Z}| = |C_e/\overline{Z}|$, the algorithm returns immediately. Otherwise, it determines the middle of the range m in Line 29. In Line 30, the existence of a boundary between m and m+1is evaluated. If m+1 is indeed a lower partition boundary, it is stored in local list in Line 31. In Line 32 and 33, FIND-BOUNDARIES is called with the ranges [s, m] and [m + 1, e]respectively. Note that the range [in/P, (i+1)n/P - 1] may contain none, one or more boundaries. Let B_i be the set of those boundaries. Once the set of boundaries B_i , for all i, are determined, the processors exchange these boundaries with each other as follows. Node n_k , in some B_i , is the boundary between the partitions V_k and V_{k+1} , i.e., $n_k - 1$ is the upper boundary of V_k , and n_k is the lower boundary of V_{k+1} . In Line 23, for each n_k in the range [in/P, (i+1)n/P - 1], processor P_i sends a boundary message containing n_k to processors P_k and P_{k+1} . Notice that each processor i receives exactly two boundary messages from other processors (Line 25), and these two messages determine the lower and upper boundary of the *i*-th partition V_i . That is, now each processor i has partition V_i and is ready to execute the parallel algorithm for the CL model with UCP scheme.

The runtime of parallel Algorithm IV.1 is $O(\frac{n}{P} + P)$ as shown in Theorem 1.

Theorem 1. The parallel algorithm for determining the partition boundaries of the UCP scheme runs in $O(\frac{n}{P} + P)$ time, where n and P are the number of nodes and processors, respectively.

Proof: The parallel algorithm for determining the partition boundaries is shown in Algorithm IV.1. For each processor, Line 6 takes $O(\frac{n}{P})$ time. The exclusive parallel prefix sum operation requires $O(\log P)$ time in Line 7. Lines 8–10 take constant time. The for loop at Line 11 iterates $\frac{n}{P}-1$ times. Each execution of the for loop takes constant time for Lines 12–14. Hence, the for loop at Line 11 takes $O(\frac{n}{P})$ time. The prefix sum in Line 16 takes $O(\log P)$ time. The for loop at Line 17 takes $O(\frac{n}{P})$ time.

The parallel sum operation in Line 20 takes $O(\log P)$ time using MPI_Reduce function. For each processor P_i , n_k 's are determined in FIND-BOUNDARIES on the range of [in/P, (i+1)n/P-1]. Finding a single partition boundary on these $\frac{n}{P}$ nodes require $O(\log \frac{n}{P})$ time. If the range contains x partition boundaries, then it takes $O(\min \left\{\frac{n}{P}, x \log \frac{n}{P}\right\})$ time. For each partition boundary n_k , processor i sends exactly two messages to the processors P_k and P_{k-1} . Thus each

processor receives exactly two messages. There are at most P boundaries in $\left[\frac{in}{P}, \frac{(i+1)n}{P} - 1\right]$. Thus, in the worst case, a processor may need to send at most 2P messages, which takes O(P) time. Therefore, the total time in the worst case is $O\left(\frac{n}{P} + \min\left\{\frac{n}{P}, P\log\frac{n}{P}\right\} + P\right) = O\left(\frac{n}{P} + P\right)$.

Theorem 1 shows the worst case runtime of $O(\frac{n}{D} + P)$. Notice that this bound on time is obtained considering the case that all P partition boundaries n_k can be in a single processor. However, in most real-world networks, it is an unlikely event, especially when the number of processors P is large. Thus it is safe to say that for most practical cases, this algorithm will scale to a larger number of processors than the runtime analysis suggests. Now we experimentally show the number of partition boundaries found in the first partition for some popular networks. For the ER networks, the maximum number of boundaries in a processor is 2, regardless of the number of processors. Even for the power-law networks, which has very skewed degree distribution, the maximum number of boundaries in a single processor is very small. Fig. 3 shows the maximum number of boundaries found in a single processor. Two fitted plots of $\log^2 P$ and $\log P$ is added in the figure for comparison. From the trend, it appears the maximum number of partition boundaries in a processor is somewhere between $O(\log P)$ and $O(\log^2 P)$. Since power-law has one of the most skewed degree distribution among real-world networks, we can expect the runtime to find partition boundaries to be approximately $O(\frac{n}{P} + \log^2 P)$ time.

Using the UCP scheme, our parallel algorithm for generating random networks with the CL model runs in $O(\frac{m+n}{P}+P)$ time as shown in Theorem 3. To prove Theorem 3, we need a bound on computation cost which is shown in Theorem 2.

Theorem 2. The computational cost in each processor is $O(\frac{m+n}{D})$ w.h.p.

Proof: For each $u \in V_i$ and v > u, (u,v) is a potential edge in processor P_i , and P_i creates the edge with probability $p_{u,v} = \frac{w_u w_v}{S}$ where $S = \sum_{v \in V} w_v$. Let x be the number of potential edges in P_i , and these potential edges are denoted by f_1, f_2, \ldots, f_x (in any arbitrary order). Let X_k be an indicator random variable such that $X_k = 1$ if P_i creates f_k and $X_k = 0$ otherwise. Then the number of edges created by P_i is $X = \sum_{k=1}^x X_k$.

As discussed in Section III, generating the edges efficiently by applying the edge skipping technique is stochastically equivalent to generating each edge (u,v) independently with probability $p_{u,v} = \frac{w_u w_v}{S}$. Let ξ_e be the event that edge e

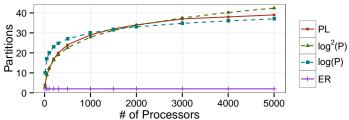


Fig. 3: Maximum number of boundaries in a single processor

is generated. Regardless of the occurrence of any event ξ_e with $e \neq (u,v)$, we always have $\Pr\{\xi_{(u,v)}\} = p_{u,v} = \frac{w_u w_v}{S}$. Thus, the events ξ_e for all edges e are mutually independent. Following the definitions and formalism given in Section IV-A, we have the expected number of edges created by P_i , denoted by μ , as

$$\mu = E[X] = \sum_{u \in V_i} e_u = m_i.$$

Now we use the following standard Chernoff bound for independent indicator random variables and for any $0 < \delta < 1$,

$$\Pr\{X \ge (1+\delta)\mu\} \le e^{-\delta^2 \mu/3}.$$

Using this Chernoff bound with $\delta = \frac{1}{2}$, we have

$$\Pr\left\{X \ge \frac{3}{2}m_i\right\} \le e^{-m_i/12} \le \frac{1}{m^3}$$

for any $m_i \geq 270$. We assume $m \gg P$ and consequently $m_i > P$ for all i. Now using the union bound,

$$\Pr\left\{X \ge \frac{3}{2}m_i\right\} \le m_i \frac{1}{m^3} = \frac{1}{m^2}$$

for all i simultaneously. Then with probability at least $1-\frac{1}{m_i^2}$, the computation cost $X+|V_i|$ is bounded by $\frac{3}{2}m_i+|V_i|=O(m_i+|V_i|)$. By construction of the partitions by our algorithm, we have $O\left(m_i+|V_i|\right)=O\left(\frac{m+n}{P}\right)$. Thus the computation cost in all processors is $O\left(\frac{m+n}{P}\right)$ w.h.p. \square

Theorem 3. Our parallel algorithm with UCP scheme for generating random networks with the CL model runs in $O(\frac{m+n}{P} + P)$ time w.h.p.

Proof: Computing the sum S in parallel takes $O\left(\frac{n}{P} + \log P\right)$ time. Using the UCP scheme, node partitioning takes $O\left(\frac{n}{P} + P\right)$ time (Theorem 1). In the UCP scheme, each partition has $O\left(\frac{m+n}{P}\right)$ computation cost w.h.p. (Theorem 2). Thus creating edges using procedure CREATE-EDGES requires $O\left(\frac{m+n}{P}\right)$ time, and the total time is $O\left(\frac{n}{P} + P + \frac{m+n}{P}\right) = O\left(\frac{m+n}{P} + P\right)$ w.h.p.

B. Round-Robin Partitioning (RRP)

In RRP scheme nodes are distributed in a round robin fashion. Partition V_i has the nodes $\langle i,i+P,i+2P,\ldots,i+kP\rangle$ such that $i+kP\leq n< i+(k+1)P;$ i.e., $V_i=\{j|j\bmod P=i\}$. In other words node i is assigned to $V_{i\bmod P}$. The number of nodes in each partition is almost equal, either $\lfloor \frac{n}{P} \rfloor$ or $\lceil \frac{n}{P} \rceil$.

In order to compare the computational cost, consider two partitions V_i and V_j with i < j. Now, for the x-th nodes in these two partitions, we have: $c_{i+(x-1)P} \geq c_{j+(x-1)P}$ as i+(x-1)P < j+(x-1)P (see Lemma 4). Therefore, $c(V_i) = \sum_{u \in V_i} c_u \geq c(V_j) = \sum_{u \in V_j} c_u$ and by the definition of RRP scheme, $|V_i| \geq |V_j|$. The difference in cost between any two partitions is at most w_0 , the maximum weight (see Lemma 6 in Appendix VII). Thus RRP scheme provides quite good load balancing. However, it is not as good as the UCP scheme. It is easy to see that in the RRP scheme, for any two partitions V_i and V_j such that i < j, we have $c(V_i) > c(V_j)$. But, by design, the UCP scheme makes the partition such that cost are equally distributed among the processors. Furthermore, although the

RRP scheme is simple to implement and provides quite good load balancing, it has another subtle problem. In this scheme, the nodes of a partition are not consecutive and are scattered in the entire range leading to some serious efficiency issues in accessing these nodes. One major issue is that the locality of reference is not maintained leading to a very high rate of cache miss during the execution of the algorithm. This contrast of performance between UCP and RRP is even more prominent when the goal is to generate massive networks as shown by experimental results in Section V.

V. EXPERIMENTAL RESULTS

In this section, we experimentally show the accuracy and performance of our algorithm. The accuracy of our parallel algorithms is demonstrated by showing that the generated degree distributions closely match the input degree distribution. The strong scaling of our algorithm shows that it scales very well to a large number of processors. We also present experimental results showing the impact of the partitioning schemes on load balancing and performance of the algorithm.

Experimental Setup. We used a 81-node HPC cluster for the experiments. Each node is powered by two octacore SandyBridge E5-2670 2.60GHz (3.3GHz Turbo) processors with 64 GB memory. The algorithm is developed with MPICH2 (v1.7), optimized for QLogic InfiniBand cards. In the experiments, degree distributions of real-world and artificial random networks were considered. The list of networks is shown in Table I. The runtime does not include the I/O time to write the graph into the disk.

Degree Distribution of Generated Networks. Fig. 4 shows the input and generated degree distributions for PL, Miami, and Twitter networks (see Appendix VII-B for other networks). As observed from the plots, the generated degree distributions closely follow the input degree distributions reassuring that our parallel algorithms generate random networks with given expected degree sequences accurately.

Effect of Partitioning Schemes. As discussed in Section IV-A, partitioning significantly affects load balancing and performance of the algorithm. We demonstrate the effects of the partitioning schemes in terms of computing time in each processor as shown in Fig. 5 using ER, Twitter, and PL networks. Computational time fo naïve scheme is skewed. For all the networks, the computational times for UCP and RRP stay almost constant in all processors, indicating good load-balancing. RRP is little slower than UCP because the locality of references is not maintained in RRP, leading to high cache miss as discussed in Section IV-B.

Strong and Weak Scaling. Strong scaling of a parallel algorithm shows it's performance with the increasing number of processors while keeping the problem size fixed. Fig. 6 shows the speedup of naïve, UCP, and RRP partitioning schemes using PL and Twitter networks. Speedups are measured as $\frac{T_s}{T_p}$, where T_s and T_p are the running time of the sequential and the parallel algorithm, respectively. The number of processors were varied from 1 to 1024. As Fig. 6 shows, UCP and RRP achieve excellent linear speedups. Naïve scheme performs

the worst as expected. The speedup of PL is greater than that of Twitter network. As Twitter is smaller than the PL network, the impact of the parallel communication overheads is higher contributing to decreased speedup. Still the algorithm to generate Twitter network has a speedup of 400 using 1024 processors.

The weak scaling measures the performance of a parallel algorithm when the input size per processor remains constant. For this experiment, we varied the number of processors from 16 to 1024. For P processors, a PL network with 10^6P nodes and 10^8P edges is generated. Note that weak scaling can only be performed on artificial networks. Fig. 6(c) shows the weak scaling for UCP and RRP schemes using PL networks. Both RRP and UCP show very good weak scaling with almost constant runtime.

Generating Large Networks. The primary objective of the parallel algorithm is to generate massive random networks. Using the algorithm with UCP scheme, we have generated power law networks with one billion nodes and 249 billion edges in one minute using 1024 processors with a speedup of about 800.

VI. CONCLUSION

We have developed an efficient parallel algorithm for generating massive networks with a given degree sequence using the Chung–Lu model. The main challenge in developing this algorithm is load balancing. To overcome this challenge, we have developed a novel parallel algorithm for balancing computational loads that results in a significant improvement in efficiency. We believe that the presented parallel algorithm for the Chung–Lu model will prove useful for modeling and analyzing emerging massive complex systems and uncovering patterns that emerges only in massive networks. As the algorithm can generate networks from any given degree sequence, its application will encompass a wide range of complex systems.

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TABLE I: Networks used in the experiments

Network	Type	Nodes	Edges
PL	Power Law Network	1 B	249 B
ER	Erdős-Rényi Network	1 M	200M
Miami [25]	Contact Network	2.1 M	51.4 B
Twitter [3]	Real-World Social Network	41.65 M	1.37 B
Friendster [4]	Real-World Social Network	65.61 M	1.81 B

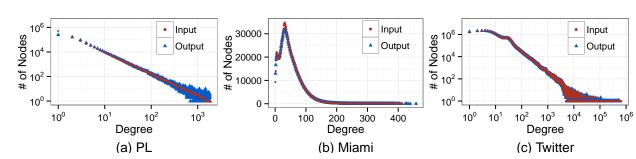
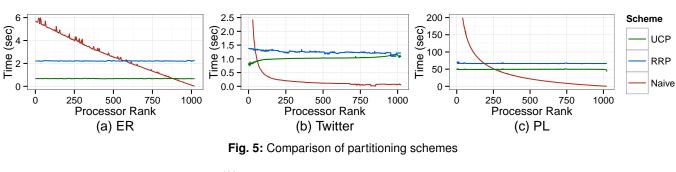


Fig. 4: Degree distributions of input and generated degree sequences



400 800 Runtime (sec.) Scheme dnpaedS 200 으300 UCP Speedd 100 **RRP** Naive 0 250 500 750 1000 250 500 750 1000 250 500 750 1000 **Number of Processors** Number of Processors Number of Processors (a) Strong Scaling (PL) (b) Strong Scaling (Twitter) (c) Weak Scaling (PL)

Fig. 6: Strong and weak scaling of the parallel algorithms

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VII. APPENDIX

Lemma 4. For any two nodes $u, v \in V$ such that u < v, $c_u \ge c_v$.

Proof: Proof omitted. The lemma follows immediately from Equation 2 and the fact that, the weights are sorted in non-increasing order. \Box

Lemma 5. Let $c(V_i)$ be the computational cost for partition V_i . In the naïve partitioning scheme, we have $c(V_i) - c(V_{i+1}) \geq \frac{n^2}{SP^2} \overline{W}_i \overline{W}_{i+1}$, where $\overline{W}_i = \frac{1}{|V_i|} \sum_{u \in V_i} w_u$, the average weight of the nodes in V_i .

Proof: In the naïve partitioning scheme, each of the partitions has $x=\frac{n}{P}$ nodes, except the last partition which can have smaller than x nodes. For the ease of discussion, assume that for $u \geq n$, $w_u = 0$ and consequently $e_u = 0$. Now, $V_i = \{ix, ix+1, \ldots, (i+1)x-1\}$. Using Equation IV, we have

$$c(V_{i}) - c(V_{i+1}) = \sum_{u \in V_{i}} (e_{u} + 1) - \sum_{u \in V_{i+1}} (e_{u} + 1)$$

$$\geq \sum_{u=ix}^{(i+1)x-1} (e_{u} + 1) - \sum_{u=(i+1)x}^{(i+2)x-1} (e_{u} + 1)$$

$$= \sum_{u=ix}^{(i+1)x-1} (e_{u} - e_{u+x})$$

$$= \sum_{u=ix}^{(i+1)x-1} \left(\frac{w_{u}}{S} \sum_{v=u+1}^{n-1} w_{v} - \frac{w_{u+x}}{S} \sum_{v=u+x+1}^{n-1} w_{v} \right)$$

$$\geq \sum_{u=ix}^{(i+1)x-1} \frac{w_{u}}{S} \sum_{v=u+1}^{u+x} w_{v} \geq \sum_{u=ix}^{(i+1)x-1} \frac{w_{u}}{S} x \overline{W}_{i+1}$$

$$= \frac{x\overline{W}_{i+1}}{S} \cdot x \overline{W}_{i} = \frac{n^{2}}{SP^{2}} \overline{W}_{i} \overline{W}_{i+1}$$

Lemma 6. In Round Robin Partitioning (RRP) scheme, for any i < j, we have $c(V_i) - c(V_j) \le w_i$.

Proof: The difference in cost between two partitions V_i and V_i is given by:

$$c(V_{i}) - c(V_{j}) = \sum_{u \in V_{i}} c_{u} - \sum_{u \in V_{j}} c_{u} = \sum_{x=0}^{k} (c_{i+xP} - c_{j+xP})$$

$$= c_{i} - \sum_{x=0}^{k-1} (c_{j+xP} - c_{i+(x+1)P}) - c_{j+kP}$$

$$\leq c_{i} - c_{j+kP} \qquad [c_{j+xP} \geq c_{i+(x+1)P}]$$

$$\leq e_{i} = \frac{w_{i}}{S} \sum_{v=i+1}^{n-1} w_{v} < \frac{w_{i}}{S} S = w_{i}$$

A. Visual Representation of Computing Cost in UCP

Fig. 7 shows the visual representation of CALC-COST procedure of Algorithm IV.1.

B. Other Networks

Fig. 8 shows input and generated degree distributions for ER and Friendster networks as shown in Table I.

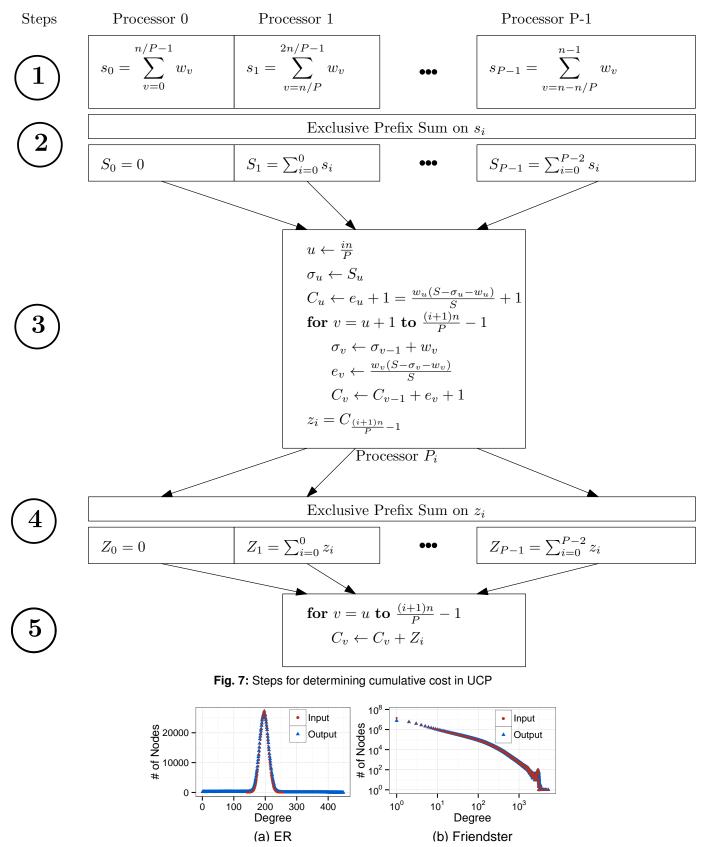


Fig. 8: Input and generated degree distributions for other networks