Computing the output distribution and selection probabilities of a stack filter from the DNF of its positive Boolean function

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ABSTRACT: Many nonlinear filters used in practise are *stack filters*. An algorithm is presented which calculates the output distribution of an arbitrary stack filter S from the disjunctive normal form (DNF) of its underlying positive Boolean function (PBF). Our algorithm avoids to enumerate the models of the PBF one by one, and thus is considerably more efficient than previous methods. The so called rank selection probabilities can be computed along the way.

1 Introduction

Stack filters were invented in 1986 and have been a key topic of research in nonlinear signal processing ever since. Simply put, all aspects of a stack filter are reflected in its underlying positive Boolean function, and a basic familiarity of the latter concept is all that is required to understand this article. Using Google Scholar one can easily track the literature on various other aspects of stack filters, e.g. their output distribution. In this article we present a new algorithm to calculate the output distribution. The new method, called stack filter *n*-algorithm, is an extension of the noncover *n*-algorithm [13] which generates, in compact form, all noncovers X of given sets A_1^*, \ldots, A_h^* (i.e. $X \not\supseteq A_i^*$ for all $1 \le i \le h$).

The stack filter *n*-algorithm is introduced by means of a medium-size example in Section 2. Section 3 is dedicated to its theoretic assessment. Section 4 touches upon five related matters, among which a numeric evaluation, and extensions of the stack filter *n*-algorithm that deliver the telling selection probabilities of [8], respectively handle the balanced stack filters of [12].

2 The stack filter *n*-algorithm

Fix $m \ge 1$ and put w := 2m + 1. Let $b : \{0, 1\}^w \to \{0, 1\}$ be a positive Boolean function (PBF), i.e. one without negated variables. Referring to e.g. [2], an operator S from $\mathbb{R}^{\mathbb{Z}}$ in itself defined by the k-th component of Sz being

$$[Sz]_k := b(z_{k-m}, \dots, z_k, \dots, z_{k+m}) \quad (k \in \mathbb{Z})$$

$$\tag{1}$$

is called a *stack filter* of window size w based on b. Notice that the PBF b in (1) has been extended from $\{0,1\}^w \to \{0,1\}$ to $\mathbb{R}^w \to \mathbb{R}$ in the usual way, i.e. by replacing the logical connectives \wedge and \vee by the minimum respectively maximum operation for pairs of real numbers (while keeping the symbols). So, if

$$b(x_{-1}, x_0, x_1) \quad := \quad ((x_0 \lor x_1) \land x_{-1}) \lor x_0 \qquad (x_i \in \{0, 1\}),$$

then

$$b(3,2,4) = ((2 \lor 4) \land 3) \lor 2 = (4 \land 3) \lor 2 = 3 \lor 2 = 3.$$

By construction each stack filter S is *translation invariant* in the sense that pushing the series x ten units to the right and then applying S yields the same as first applying S and then pushing ten units to the right. So S is completely determined by formula (1) for k = 0.

Let $Z = (..., Z_{-1}, Z_0, Z_1, ...)$ be a doubly infinite sequence of independent indentically distributed (i.i.d.) random variables. Let $F_Z(t)$ be their common (cumulative) distribution function, i.e. $F_Z(t) := Prob(Z_i \leq t)$ is the probability that Z_i is at most t. By translation invariance the *output distribution* $F_{SZ}(t) := Prob((SZ)_i \leq t)$ is independent of i. It is known that there is a well defined function $\phi_S(p)$, called the *distribution transfer* of S, such that

$$F_{SZ}(t) = \phi_S(F_Z(t)) \quad (t \in \mathbb{R}).$$

What's more, $\phi_S(p)$ is a *polynomial* which can be calculated [15], [2, p.223] as

$$\phi_S(p) = \sum_{b(x)=0} p^{|\operatorname{Zero}(x)|} \cdot q^{|\operatorname{One}(x)|}$$
(2)

where q := 1 - p and b is as in (1). The summation is over all bitstrings $x \in \{0, 1\}^w$ with b(x) = 0, where by definition

Zero
$$(x)$$
 := {1 $\leq i \leq w | x_i = 0$ },
One (x) := {1 $\leq i \leq w | x_i = 1$ }.

For instance, consider this positive Boolean function b_1 which is already in disjunctive normal form (DNF). It is of type $\{0,1\}^9 \rightarrow \{0,1\}$ but we like to scale as $\{0,1\}^W \rightarrow \{0,1\}$ with $W := \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$:

$$b_{1}(x_{-4}, \dots, x_{4}) = (x_{-2} \wedge x_{-1} \wedge x_{0}) \vee (x_{-1} \wedge x_{0} \wedge x_{1}) \vee (x_{0} \wedge x_{1} \wedge x_{2})$$
(3)
$$\vee (x_{-4} \wedge x_{-3} \wedge x_{-2} \wedge x_{1} \wedge x_{2} \wedge x_{3}) \vee (x_{-3} \wedge x_{-2} \wedge x_{-1} \wedge x_{1} \wedge x_{2} \wedge x_{3}) \\\vee (x_{-3} \wedge x_{-2} \wedge x_{-1} \wedge x_{2} \wedge x_{3} \wedge x_{4}).$$

In view of (2) we wish to encode the family Mod of all $x = (x_{-4}, x_{-3}, \ldots, x_4)$ in $\{0, 1\}^W$ with b(x) = 0 in a compact way^{*}. First note that

$$Mod = Mod_1 \cap Mod_2 \cap Mod_3 \cap Mod_4 \cap Mod_5 \cap Mod_6,$$

^{*}If we were to start with the *conjunctive* normal form (CNF) of b, we would end up with a compact representation of the set Mod' of all $x \in \{0, 1\}^W$ with b(x) = 1. Hence, instead of (2), a dual kind of formula would yield $\phi_S(p)$.

where the family Mod_i corresponds to the *i*-th conjunction in (3). For instance we write

$$Mod_1 := \{x \in \{0,1\}^W | x_{-2} \land x_{-1} \land x_0 = 0\} = \{2,2,n,n,n,2,2,2,2\}$$

because $x_{-2} \wedge x_{-1} \wedge x_0 = 0$ (nul) if and only if at least one of x_{-2}, x_{-1}, x_0 is nul, and the other variables $x_{-4}, x_{-3}, x_1, x_2, x_3, x_4$ can independently assume the **2** values 0 and 1. Thus $(1, 1, 0, 1, 0, 1, 0, 1, 1) \in \text{Mod}_1$ but $(0, 0, 1, 1, 1, 0, 0, 1, 0) \notin \text{Mod}_1$. If we identify a 0, 1-string x with the subset $X = \{i \in W : x_i = 1\}$ of W then Mod_1 consists of all noncovers X of $A_1^* := \{-2, -1, 0\}$ in the sense that $X \not\supseteq A_1^*$. The noncover n-algorithm from [13] (more on that in Section 3) generates all simultaneous noncovers of the given sets (here deriving from the terms of a PBF) $A_1^*, A_2^*, \ldots, A_6^*$ as follows:

—4	1 -	-3	-2	-	-1	0	1	2	3	4	
2		2	n	1	n	n	2	2	2	2	PC = 2
2	2	2	n	n	2	2	2	2		PC	r = 3
2	2	0	1	1	0	2	2	2		PC	r = 3
2	2	2	2	0	2	2	2	2		PC	r = 4
2	2	2	0	1	n	n	2	2		PC	r = 4
2	2	0	1	1	0	2	2	2		PC	r = 3
n	n	n	2	0	n	n	n	2		P0	C = 5
2	2	2	0	1	n	n	2	2		P0	C = 4
2	2	0	1	1	0	2	2	2		P0	C = 3
2	n	n	2	0	n	n	n	2		P0	C = 6
0	1	1	0	0	1	1	1	2		P	C = 6
2	2	2	0	1	n	n	2	2		P0	C = 4
2	2	0	1	1	0	2	2	2		P0	C = 3
2	n	n	2	0	2	n	n	2		fir	nal
2	1	1	n	0	0	1	1	n	L	fir	nal
0	1	1	0	0	1	1	1	2		\overline{P}	C = 6
2	2	2	0	1	n	n	2	2		P	C = 4
2	2	0	1	1	0	2	2	2		\overline{P}	C = 3

 Table 1: The workings of the noncover n-algorithm

By PC = 2 we mean that at this stage the *pending conjunction* is the second one, i.e. the one that defines Mod₂. In other words, we need to sieve out those $x \in \text{Mod}_1$ that happen to be in $\text{Mod}_2 = (2, 2, 2, n, n, n, 2, 2, 2)$. In order to do so we determine the intersection $\{-2, -1, 0\} \cap \{-1, 0, 1\} = \{-1, 0\}$ of the "n-pools" of Mod₁ and Mod₂ and then split the $\{0, 1, 2, n\}$ -valued row $r := \text{Mod}_1$ accordingly into a disjoint union $r = r' \cup r''$ where

$$\begin{array}{rcl} r' & := & \{x \in r \mid x_{-1} = 0 \text{ or } x_0 = 0\} & = & (2, 2, 2, \mathbf{n}, \mathbf{n}, 2, 2, 2, 2) \\ r'' & := & \{x \in r \mid x_{-1} = x_0 = 1\} & = & (2, 2, 0, \mathbf{1}, \mathbf{1}, 2, 2, 2, 2). \end{array}$$

While all $x \in r'$ trivially satisfy $x_{-1} \wedge x_0 \wedge x_1 = 0$, i.e. belong to Mod₂, this is not the case for all $x \in r''$. However, turning at the 6-th position the 2 to 0 does the job. This yields the current working stack with the two rows labelled PC = 3; see top of Table 1. (Of course this "stack" has nothing to do with its namesake in "stack filter".) As a general rule, the topmost row in the stack is always treated first ("last in, first out"). This may entail "local changes", or a splitting of the top row into several sons. In this way we proceed up to the second last stack in Table 1. Let us pick its top row r = (2, n, n, 2, 0, n, n, n, 2) and illustrate once more the splitting process. The intersection of the *n*-pool of *r* with (the index set of) the pending 6th conjunction is $\{-3, -2, 1, 2, 3\} \cap \{-3, -2, -1, 2, 3, 4\} = \{-3, -2, 2, 3\}$. Accordingly split *r* into the disjoint union of *r*' and *r*":

$$\begin{array}{rcl} r & = & (2,n,n,2,0,n,n,n,2) \\ r' & = & (2,\mathbf{n},\mathbf{n},2,0,2,\mathbf{n},\mathbf{n},2) \\ r'' & = & (2,\mathbf{1},\mathbf{1},2,0,0,\mathbf{1},\mathbf{1},2). \end{array}$$

Since $r' \subseteq Mod_6$, r' is the first son of r. We have $r'' \not\subseteq Mod_6$, but $r'' \cap Mod_6 = (2, 1, 1, n, 0, 0, 1, 1, n)$ becomes the second son. Both rows are *final*, i.e. are subsets of Mod and thus collected in a steadily increasing *final stack*. The working stack now contains three rows with pending conjunctions 6, 4, 3 respectively. In our case it just so happens that they are in fact already final (so e.g. *all* x in the row labelled PC = 4 happen to satisfy the 4th, 5th and 6th conjunction). The final stack comprises thus the five rows in Table 2 (for the moment ignore p^2q^2 and so forth):

-4	-3	-2	-1	0	1	2	3	4	
2	2	0	1	1	0	2	2	2	p^2q^2
2	2	2	0	1	n	n	2	2	$pq(1-q^2) = pq - pq^3$
0	1	1	0	0	1	1	1	2	p^3q^5
2	1	1	n	0	0	1	1	n	$p^2q^4(1-q^2) = p^2q^4 - p^2q^6$
2	n	n	2	0	2	n	n	2	$p(1-q^4) = p - pq^4$

Table 2: The probability contributions of the final rows

For instance, the second row in Table 2 contains $2^5 \cdot (2^2 - 1)$ noncovers, where $(2^2 - 1)$ comes from nn. The total number N of noncovers evaluates to

$$N = 32 + 32 \cdot 3 + 2 + 2 \cdot 3 + 16 \cdot 15 = 376.$$

which is much higher than the number R = 5 of final multivalued rows. As we shall see in Section 3, in general the *n*-pool of rows is a bit more subtle.

Let us now calculate the output distribution. The first row in Table 2 contains $2^5 = 32$ bitstrings x with $b_1(x) = 0$. Each contributes some probability $\alpha_1 \alpha_2 p q q p \alpha_3 \alpha_4 \alpha_5$ to the sum in (2). Since each α_i can independently be chosen to be p or q, the sum of these 32 terms is

$$p^2 q^2 (ppppp + \dots + pqqpq + \dots + qqqqq) = p^2 q^2 (p+q)^5 = p^2 q^2.$$
 (4)

The fact that e.g. $nn = \{00, 01, 10\}$ yields $pp + pq + qp = 1 - q^2$, explains the contribution $pq(1-q^2)$ of the second row. Similarly for the three other rows. Summing up the terms in Table 2 yields

$$\phi_S(p) = p^2 q^2 + pq - pq^3 + p^3 q^5 + p^2 q^4 - p^2 q^6 + p - pq^4$$

= $7p^2 - 8p^3 - 8p^4 + 25p^5 - 24p^6 + 11p^7 - 2p^8.$ (5)

3 Theoretic assessment

Suppose the constraint $A^* = \{3, 4\}$ is to be imposed on a row r = (1, 2, 1, 1) in the process of the stack filter *n*-algorithm. Then *r* needs to be cancelled since *no* member $X \in r$ satisfies $X \not\supseteq A^*$. Fortunately, with some precautions the cancellation of rows can be avoided, which is essential in the Theorem below. Another remark about the proof is in order. Apart from the probabilities coupled to the final $\{0, 1, 2, n\}$ -valued rows, the stack filter *n*-algorithm coincides with the noncover *n*-algorithm of [13], which is a special "homogeneous" case of the Horn *n*algorithm, which in turn is an instance of some *principle of exclusion*. Since our special case is somewhat buried by this and the technical machinery of [13], yet admits a comparatively smooth proof from scratch, we give that proof below.

Theorem: Suppose the stack filter S has window size w and its positive Boolean function b(x) is given as a disjunction of h conjunctions (DNF). Then the stack filter n-algorithm computes the output distribution of S in time $O(Nw^2h^2)$. Here N is the number of bitstrings x with b(x) = 0.

Proof. As in the introductory example, the terms in the DNF of b(x) yield subsets A_1^*, \ldots, A_h^* of W := [w] whose models (= simultaneous noncovers) $Y \subseteq W$ we wish to pack in disjoint $\{0, 1, 2, n\}$ -valued rows. Any (not necessarily final) row \overline{r} is called *feasible* if $Y \in \overline{r}$ for at least one model Y. As opposed to other applications of the principle of exclusion, here feasibility is easily tested. Namely, \overline{r} is feasible if and only if

$$(\forall 1 \le i \le h) \quad A_i^* \not\subseteq \text{ ones}(\overline{r}). \tag{6}$$

Initially our "working stack" solely comprises the row $r_0 = (2, 2, ..., 2)$ of length w which we identify with the powerset of W. Note that r_0 is feasible since $\emptyset \in r$. Row r_0 carries the pointer $PC(r_0) = 1$, where PC stands for "pending constraint". Generally, the top row r of the working stack is treated as follows. If PC(r) = j (for some $j \in [h]$) then the set A_j^* is "imposed" upon r, that is, the set U of all $X \in r$ with $X \not\supseteq A_j^*$ is represented as a disjoint union of rows r_1, \ldots, r_s where $s \leq w$. That this is always possible (the "core" claim), and costs $O(w^2)$, will be shown in a moment.

Because r was feasible by induction, at least one of its "candidate" sons r_1, \ldots, r_s will be as well. Since the feasibility of $\overline{r} = r_j$ amounts to the truth of (6), it costs $O(shw) = O(hw^2)$ to sieve the sons of r, i.e. the feasible rows among r_1, \ldots, r_s . Altogether the cost of one imposition of a constraint upon a row is $O(w^2) + O(hw^2) = O(hw^2)$.

The *R* final rows can be viewed as the leaves of a tree with root (2, 2, ..., 2) that has height *h*; each imposition triggers all sons of some node. Therefore the number of impositions is at most *Rh* (distinct final rows, possibly having some of their forefathers in common). It follows that producing the *R* final rows costs $O(Rh \cdot hw^2) = O(Nh^2w^2)$ in view of $R \leq N$, by the disjointness of final rows. Calculating (as in (4)) the contributions to $\phi_S(p)$ of all final rows, and adding them, costs O(Nw), which is swallowed by $O(Nh^2w^2)$.

It remains to verify the core claim, i.e. that $U := \{X \in r : X \not\supseteq A^*\}$ $(A^* := A_j^*)$ can be represented as promised.

Case (a): $A^* \cap \operatorname{zeros}(r) \neq \emptyset$ or A^* wholly contains an *n*-bubble of *r*. Then U = r, and so put $r_s = r_1 = r$.

Since r is feasible, $A^* \subseteq \text{ones}(r)$ is impossible, and so the only remaining possibility is

Case (b): $A^* \cap \operatorname{zeros}(r) = \emptyset$ and A^* does not wholly contain an *n*-bubble of *r* and $A^* \not\subseteq \operatorname{ones}(r)$. This is exactly Case 7 in Section 5 of [13], whose essense we repeat here.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	
2	2	n_1	n_1	n_2	n_3	n_3	n_4	n_1	n_2	n_3	n_3	n_4	n_4	r
2	2	n	n	n_2	n_3	n_3	n_4	2	n_2	n_3	n_3	n_4	n_4	r_1
2	2	1	1	0	n_3	n_3	n_4	0	2	n_3	n_3	n_4	n_4	r_2
2	2	1	1	1	n	n	n_4	0	0	2	2	n_4	n_4	r_3
2	2	1	1	1	1	1	0	0	0	n_3	n_3	2	2	r_4
n	n	1	1	1	1	1	1	0	0	n_3	n_3	n_4	n_4	r_5

Table 3: Five candidate sons of some $\{0, 1, 2, n\}$ -valued row

Suppose W = [14], $A^* = [8]$, and r is as in Table 3. Note that the *n*-pool of r is the disjoint union $\{3, 4, 9\} \cup \{5, 10\} \cup \{6, 7, 11, 12\} \cup \{8, 13, 14\}$ of four mutually independent *n*-bubbles, each one defined by "at least one nul there", as in Section 2. Putting

 $r_{1} := \{X \in r : X \not\supseteq \{3, 4\}\}$ $r_{2} := \{X \in r : X \supseteq \{3, 4\} \text{ and } X \not\supseteq \{5\}\}$ $r_{3} := \{X \in r : X \supseteq \{3, 4, 5\} \text{ and } X \not\supseteq \{6, 7\}\}$ $r_{4} := \{X \in r : X \supseteq \{3, 4, 5, 6, 7\} \text{ and } X \not\supseteq \{8\}\}$ $r_{5} := \{X \in r : X \supseteq \{3, 4, 5, 6, 7, 8\} \text{ and } X \not\supseteq \{1, 2\}\}$

it is clear that U is the disjoint union of r_1, \ldots, r_5 . A minute's reflection shows that, crucially, these sets can again be written as $\{0, 1, 2, n\}$ -valued rows as shown in Table 3, and that generally (full details in [13]) splitting a row in $s \leq w$ candidate sons like this costs $O(sw) = O(w^2)$. \Box

It follows from the proof that $O(Nw^2h^2)$ could be substituted by $O(Rw^2h^2)$ where $R \leq N$ is the number of final $\{0, 1, 2, n\}$ -valued rows. Unfortunately R is unpredictable. Theoretically R = N is possible[†] but in practise R is usually orders of magnitudes smaller than N (see Subsection 4.1).

It has been pointed out that b(x) may not initially be given in disjunctive normal form. However, if not, there are efficient methods to compute the DNF from any reasonable kind of presentation of b(x); this e.g. applies to the erosion - dilation cascades in subsection 4.1. In any case, the bigger problem arguably is to find the bitstrings $x \in \{0, 1\}^w$ with b(x) = 0.

[†]Even then, provided $N \approx \frac{1}{2} \cdot 2^W$ as is to be expected for random PBF's, the stack filter algorithm would beat by a factor 2 a brute force search of all of $\{0, 1\}^W$.

4 Related matters

Subsection 4.1 glimpses at the practical performance of a Mathematica implementation of the stack filter *n*-algorithm, and 4.2 shows that the so called rank selection probabilities p_i can be gleaned from the $\{0, 1, 2, n\}$ -valued rows along the way. Subsections 4.3 and 4.4 are about the joint distribution of stack filters, respectively about a certain generalization of "ordinary" stack filters to "balanced" stack filters. The required adaptions of our algorithm stay within the realm of $\{0, 1, 2, n\}$ -valued rows. Finally, as powerful as binary decision diagrams often are, it is argued in 4.5 that they are not appropriate in our situation.

4.1 Numerics exemplified on the LULU filter C_5

Certain stack filters L_n , their duals U_n , and compositions thereof (called *LULU* filters) have been proposed in [9] and earlier[‡], as alternatives to the popular median filters. Actually, the function $b_1(x_{-4}, \ldots, x_4)$ from Section 2 is the PBF underlying U_2L_2 .

The natural definition of each LULU filter is as a *cascade* of so called *erosions* and *dilations* (CED), two dual concepts from Mathematical Morphology [9, III.C]. Computing the DNF of any CED essentially amounts[§] to calculating CNF's and DNF's of successively bigger (details in [3]) positive Boolean functions. For instance,

$$C_n := L_n U_n L_{n-1} U_{n-1} \dots L_1 U_1$$

is a CED stack filter with window size $w = 2n^2 + 2n + 1$. Using Berge's algorithm to compute the DNF of C_5 from its CED-representation took about 46 hours. Calculating the output distribution $\phi_{C_5}(p) = p^5 + 7p^6 - \cdots + 114680p^{43} + \cdots + p^{53}$ with the stack filter *n*-algorithm took another 12 hours. At least as illuminating as $\phi_{C_5}(p)$ are the so called rank selection probabilities that can be calculated along the way as discussed in the next subsection. The underlying PBF of C_5 had N = 639'173'390'187'370'752 models, which were packed in a mere R = 179'244final rows. More extensive numerical evaluations of similar implementations of the principle of exclusion (and how they compare to say BDD's) are provided in upcoming publications.

Due to the specific regularities of U_nL_n its DNF has in fact been discovered by other means [9, p.112] and its distribution transfer was computed independent of its DNF in [3]; it equals

$$\phi_{U_n L_n} = 1 - q^{n+1} - npq^{n+1} - pq^{2n+2} - \frac{1}{2}(n-1)(n+2)p^2q^{2n+2}.$$
 (7)

One verifies that (7) coincides with (5) for n = 2. Even the distribution transfer of C_n can be determined [3], albeit only by an efficient recursive formula as opposed to the closed form in (7). For all $n \leq 5$ the results agreed with the ones obtained with the stack filter *n*-algorithm, which is a strong indication that both methods are correct.

[‡]Using terminology of Mathematical Morphology, L_n (dually for U_n) is an opening induced by a line segment of length n + 1, whence the underlying PBF has window size 2n + 1. As opposed to the median filters, all *LULU* filters S are idempotent ($S \circ S = S$) and even co-idempotent ($(id - S) \circ (id - S) = id - S$).

[§]Computing the DNF of a PBF from its CNF is a well researched topic [5], which also amounts to get all minimal transversals of a set system. The author used a refinement of the classic "Berge-algorithm" for the task, does not claim that it competes with the cutting edge algorithms for DNF \leftrightarrow CNF, but feels that the stack filter *n*-algorithm is the right approach once the DNF is *given*.

4.2 Rank selection probabilities

Let S be a stack filter. Given a sequence Z of i.i.d. random variables, the so called *rank selection* probability p_i is defined as the probability that a fixed component of the output series SZ is the *i*-th smallest in the sliding window of length w. It is known, [8], [2, p.236] that

$$p_i = \frac{A_{w-i}}{\binom{w}{w-i}} - \frac{A_{w-i+1}}{\binom{w}{w-i+1}},$$

where A_i is the number of bitstrings x with i ones and w - i zeros that have b(x) = 0. The A_i 's can be conveniently calculated in tandem with the evaluation of (2). For instance, as the reader can easily verify, the contribution of the last row in Table 2 to A_0 up to A_7 is:

A_0	:	$\begin{pmatrix} 8\\0 \end{pmatrix}$	=	1
A_1	:	$\binom{8}{1}$	=	8
A_2	:	$\binom{8}{2}$	=	28
A_3	:	$\binom{8}{3}$	=	56
A_4	:	$\binom{4}{0}\binom{4}{4} + \binom{4}{1}\binom{4}{3} + \binom{4}{2}\binom{4}{2} + \binom{4}{3}\binom{4}{1}$	=	69
A_5	:	$\binom{4}{1}\binom{4}{4} + \binom{4}{2}\binom{4}{3} + \binom{4}{3}\binom{4}{2}$	=	52
A_6	:	$\binom{4}{2}\binom{4}{4} + \binom{4}{3}\binom{4}{3}$	=	22
A_7	:	$\binom{4}{3}\binom{4}{4}$	=	4.

We mention that in [6] the optimization of stack filters with respect to certain constraints leads to specific desirable values of A_1, \ldots, A_w . Finding a stack filter S that features these values (at least approximately) is however hard. One may hence be led to compile a catalogue of CED's (see 4.1) with corresponding vectors (A_1, \ldots, A_w) from which a suitable candidate S can be picked.

4.3 The joint output distribution of two stack filters

Let Z be a doubly infinite sequence of i.i.d. random variables. For two stack filters S and T with corresponding positive Boolean functions $b_1(x)$ and $b_2(y)$ their joint output distribution $F_{SZ,TZ}(s,t)$, or simply JD(s,t), is defined as

$$JD(s,t) := \operatorname{Prob}((SZ)_0 \le s \text{ and } (TZ)_0 \le t).$$

If we set $p := \operatorname{Prob}(Z_0 \leq s)$, $\pi := \operatorname{Prob}(Z_0 \leq t)$ and assume $p \leq \pi$ (the case $p > \pi$ is similar) then it is shown in [2, p.230] that

$$JD(s,t) = \sum_{i=0}^{w} \sum_{j=0}^{w} A_{i,j} p^{i} (\pi - p)^{w-i-j} (1 - \pi)^{j}, \qquad (8)$$

where A_{ij} is the number of $(x, y) \in \{0, 1\}^w \times \{0, 1\}^w$ such that

$$x \ge y$$
, $b_1(x) = b_2(y) = 0$, $v_{-,-}(x,y) = i$, $v_{+,+}(x,y) = j$,

and where \P

 $v_{-,-}(x_1,\ldots,x_w,y_1,\ldots,y_w) := |\{1 \le k \le w : x_k = y_k = 0\}|$

$$v_{+,+}(x_1,\ldots,x_w,y_1,\ldots,y_w) := |\{1 \le k \le w : x_k = y_k = 1\}|$$

The calculation of the coefficients A_{ij} works row-wise. So suppose r in Table 4 is one of the final rows obtained after applying the noncover n-algorithm to b_1 . Obviously the set

$$\mathcal{F} \quad := \quad \{y : (\exists x \in r) \ x \ge y\}$$

is represented by row r_0 . If say $b_2(y) = y_3 \wedge y_9 \wedge y_{10}$ then the set

$$\mathcal{F}(b_2) \quad := \quad \{y \in \mathcal{F} : b_2(y) = 0\}$$

is the disjoint union $\rho_1 \cup \rho_2 \cup \rho_3$:

	1	2	3	4	5	6	7	8	9	10	11
r =	n_1	n_1	n_1	2	2	0	n_2	n_2	n_2	1	1
$r_0 =$	n_1	n_1	n_1	2	2	0	n_2	n_2	n_2	2	2
$\rho_1 =$	2	2	0	2	2	0	n_2	n_2	n_2	2	2
$\rho_2 =$	n_1	n_1	1	2	2	0	2	2	0	2	2
$\rho_3 =$	n_1	n_1	1	2	2	0	n_2	n_2	1	0	2
x =	0	1	1	1	0	0	1	1	0	1	1
$\sigma =$	0	2	2	2	0	0	0	2	0	0	2
$\tau =$	0	2	2	2	0	0	1	0	0	0	2

Table 4: Adapting the algorithm to joint output distributions

For each $x \in r$ and $k \in \{1, 2, 3\}$ one now records $v_{-,-}(x, y)$ and $v_{+,+}(x, y)$ for all $y \in \rho_k$ with $y \leq x$. For instance, taking the x indicated in Table 4 one verifies that

$$\{y \in \rho_3 : y \le x\} = \sigma \cup \tau$$

where the later union is disjoint (see n_2n_2 in ρ_3 and the corresponding boldface entries in σ, τ). It is easy to see that σ contributes an amount of $\binom{5}{j}$ to the value of $A_{4,j}$ for all $0 \leq j \leq 5$. Similarly τ contributes an amount of $\binom{4}{j}$ to the value of $A_{4,j+1}$ ($0 \leq j \leq 4$). Calculations can be sped up by clumping together suitable x's rather than processing them one by one. We discuss a similar phenomenon in more detail in the next subsection.

4.4 Balanced stack filters

In [11], [12] the concept of a *balanced*^{\parallel} stack filter S is introduced. Citing from [11]: "They are much more versatile, being empowered not only with lowpass filtering characteristics, but with

[¶]This notation is not used in [2] but ties in well with the notation used in subsection 4.4, which in turn is akin to the notation of [11]. For instance, our $v_{-,+}(x,y)$ in 4.4 corresponds to $w(\overline{x} \wedge s)$ in equation (17) of [11].

^{||}Actually, Arce, Paredes and Shmulevich propose to reserve the term "stack filter" to their new concept, and to relabel the "old" stack filters as stack smoothers. As suggested by one referee, we stick to the old, well established terminology.

bandpass or highpass filtering characteristics as well." They are based on "mirrored thresholding" which entails t and -t to play symmetric roles. Most important for us, S is based again upon a PBF albeit in a manner more sophisticated than (1). For instance, the PBF is of the kind $b(x, y) = b(x_1, \ldots, x_w, y_1, \ldots, y_w)$, and in this set up a stack filter turns out to be a balanced stack filter where b does not depend on y_1, \ldots, y_w (i.e., these variables are fictitious). As usual let Z be a doubly infinite sequence of i.i.d. random variables with common cumulative distribution function $F_Z(t) = Prob(Z_i \leq t)$ ($i \in \mathbb{Z}$). Put $F(t) = F_Z(t)$ and

$$p_{+,+} := \begin{cases} F(-t) - F(t) & \text{if } t \leq 0\\ 0 & \text{if } t > 0 \end{cases}$$

$$p_{-,-} := \begin{cases} 0 & \text{if } t \leq 0\\ F(t) - F(-t) & \text{if } t > 0 \end{cases}$$

$$p_{-,+} := \begin{cases} F(t) & \text{if } t \leq 0\\ F(-t) & \text{if } t > 0 \end{cases}$$

$$p_{+,-} := \begin{cases} 1 - F(-t) & \text{if } t \leq 0\\ 1 - F(t) & \text{if } t > 0. \end{cases}$$

Besides $v_{+,+}(x,y)$ and $v_{-,-}(x,y)$ from 4.3 we also put

$$v_{-,+}(x,y) := |\{1 \le k \le w : x_k = 0 \text{ and } y_k = 1\}|$$
$$v_{+,-}(x,y) := |\{1 \le k \le w : x_k = 1 \text{ and } y_k = 0\}|.$$

Modulo some obvious typos, it is shown in [11, (17)] that the output distribution, i.e. $F_{SZ}(t) = Prob((SZ)_0 \leq t)$, can be calculated as

$$F_{SZ}(t) = \sum_{b(x,y)=0} p_{+,+}^{v_{+,+}(x,y)} \cdot p_{+,-}^{v_{+,-}(x,y)} \cdot p_{-,+}^{v_{-,+}(x,y)} \cdot p_{-,-}^{v_{-,-}(x,y)}$$

As opposed to JD(s,t) in (8), which is a polynomial of $Prob(Z_0 \leq s)$ and $Prob(Z_0 \leq t)$, here $F_{SZ}(t)$ is not quite a polynomial in terms of $Prob((SZ)_0 \leq t)$ and $Prob((SZ)_0 \leq -t)$.

Nevertheless the noncover *n*-algorithm is of good use. Suppose it has (among others) returned the final row *r* in Table 5. Take any bitstring $x^* = (x_1, \ldots, x_9)$ "contained" in the left hand side $(n_1, n_2, n_3, 1, n_4, n_4, 0, 2, n_3)$ of *r*. More precisely, any bitstring x^* which is *extendible*^{**} to a bitstring $(X^*, y) \in r$. Say $x^* = (1, 1, 1, 1, 1, 0, 0, 0, 0)$. For each fixed $k \in \{0, 1, \ldots, 5\}$ and $k' \in \{0, 1, \cdots, 4\}$ we now show how the number f(k, k') of bitstrings $y = (y_1, \ldots, y_9)$ with

$$v_{+,+}(x^*, y) = k$$
 and $v_{-,+}(x^*, y) = k'$

(whence
$$v_{+,-}(x^*, y) = 5 - k$$
 and $v_{-,-}(x^*, y) = 4 - k'$)

can be calculated fast. First, notice that the subset

$$r(x^*) \quad := \quad \{(x,y) \in r : \ x = x^*\}$$

of r can be written as multi-valued row as shown in Table 5.

^{**}It is easily seen that the extendible bitstrings are exactly the members of $(2, 2, 2, 1, n_4, n_4, 0, 2, 2)$.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
r =	n_1	n_2	n_3	1	n_4	n_4	0	2	n_3	n_1	n_1	n_3	n_2	n_2	n_1	n_1	n_2	n_2
$r(x^{*}) =$	1	1	1	1	1	0	0	0	0	n_1	n_1	2	n_2	n_2	n_1	n_1	n_2	n_2
$r_1 =$	1	1	1	1	1	0	0	0	0	$\mathbf{n_1}$	$\mathbf{n_1}$	2	n_2	n_2	2	2	2	2
$r_2 =$	1	1	1	1	1	0	0	0	0	$\mathbf{n_1}$	n_1	2	1	1	2	2	n_2	n_2
$r_3 =$	1	1	1	1	1	0	0	0	0	1	1	2	n_2	n_2	n_1	n_1	2	2
$r_4 =$	1	1	1	1	1	0	0	0	0	1	1	2	1	1	n_1	n_1	n_2	n_2

Table 5: Adapting the algorithm to balanced stack filters

Problem is we cannot freely choose k 1's among $\{y_1, \ldots, y_5\}$ and k' 1's among $\{y_6, \ldots, y_9\}$ because e.g. the choice (1, 1, 0, 0, 0, 1, 1, 0, 0) clashes with $n_1n_1n_1n_1$. But when one partitions $r(x^*)$ as $r_1 \cup r_2 \cup r_3 \cup r_4$ as indicated, then for each r_i the choices within $\{y_1, \ldots, y_5\}$ respectively $\{y_6, \ldots, y_9\}$ can be made independently. To fix ideas, say k = 2 and k' = 3. Then the contribution of $r(x^*) = r_1 \cup r_2 \cup r_3 \cup r_4$ to the coefficient of the monom

$$p^k_{+,+} \; p^{5-k}_{+,-} \; p^{k'}_{-,+} \; p^{4-k'}_{-,-}$$

occuring in $F_{SX}(t)$ is

 $f(k,k') = 8 \cdot 4 + 1 \cdot 2 + 1 \cdot 2 + 0 \cdot 0 = 36.$

Generally, the number of bitstrings with a fixed number k of 1's that are contained in a $\{0, 1, 2, n\}$ -valued row can be determined fast. Similar to 4.3, but more obvious, time can be saved by clumping together suitable bitstrings (x_1, \ldots, x_9) . For instance, (1, 1, 0, 1, 0, 0, 0, 1, 1) causes the same right hand side $(n_1, n_1, 2, n_2, n_2, n_1, n_1, n_2, n_2)$ as did x^* . As another example, (0, 0, 1, 1, 1, 0, 0, 0, 0) is one among ten left hand sides of weight 3 that cause the right hand side (2, 2, 2, 2, 2, 2, 2, 2, 2).

4.5 On binary decision diagrams

Shmulevich et al. [10] proposed to evaluate (2) by setting up a binary decision diagram (BDD) for the Boolean function b(x) that underlies the stack filter S whose distribution transfer needs to be calculated. Suppose one has indeed spent time to get a BDD that represents b(x). While the *number* of models $x \in \{0, 1\}^w$ with b(x) = 0 can be determined fast from a BDD, it is more cumbersome to generate all models, as is forced by (2). True, from the BDD one can get the set of models as a disjoint union of $\{0, 1, 2\}$ -valued rows in recursive fashion. (See [1, p.22] or the long chapter on BDDs in Donald Knuth's forthcoming book.) However, these rows are far more numerous than the ones produced by the stack filter *n*-algorithm; not surprisingly since our algorithm uses one *additional* symbol and hence more flexibility in its $\{0, 1, 2, n\}$ -valued rows. Finally, the enhancements discussed in subsections 4.2, 4.3, 4.4 are cumbersome to be handled by BDD's.

Conclusion

The present article can be viewed as the realization of a fifth benefit of DNF's that was announced in [14], i.e. the calculation of a stack filter's output distribution and (even more useful) its selection probabilities. The so doing stack filter *n*-algorithm is accessible from the author's home page. It has the form of a Mathematica Notebook. The indicated enhancements in 4.3 and 4.4 have not been programmed by the author; anybody is welcome to do so.

Last not least we draw attention to [7], a comprehensive framework in which stack filters, alias *lattice polynomial functions* (LPF), constitute but one type of aggregation function. However, there are no references to nonlinear signal theory or Mathematical Morphology in [7]. For instance, other than might appear from [7, p.361], cumulative distribution functions of "nice" LPF's (i.e. their underlying PBF's are more regular than ours) have a long history - in the case of Order Statistics dating back to 1932 [4].

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