

An inertial forward-backward algorithm for monotone inclusions

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Abstract In this paper, we propose an inertial forward backward splitting algorithm to compute a zero of the sum of two monotone operators, with one of the two operators being co-coercive. The algorithm is inspired by the accelerated gradient method of Nesterov, but can be applied to a much larger class of problems including convex-concave saddle point problems and general monotone inclusions. We prove convergence of the algorithm in a Hilbert space setting and show that several recently proposed first-order methods can be obtained as special cases of the general algorithm. Numerical results show that the proposed algorithm converges faster than existing methods, while keeping the computational cost of each iteration basically unchanged.

Keywords convex optimization, forward-backward splitting, monotone inclusions, primal-dual algorithms, saddle-point problems, image restoration

1 Introduction

A fundamental problem is to find a zero of a maximal monotone operator T in a real Hilbert space X :

$$\text{find } x \in X : \quad 0 \in T(x). \quad (1)$$

This problem includes, as special cases, variational inequality problems, non-smooth convex optimization problems and convex-concave saddle-point problems. Therefore this problem finds many important applications in scientific fields

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such as image processing, computer vision, machine learning and signal processing.

In case, $T = \nabla f$ is the gradient of a differentiable convex function f , the most simple approach to solve (1) is to apply for each $k \geq 0$ the following recursion:

$$x^{k+1} = (\text{Id} - \lambda_k T)(x^k),$$

where the operator $(\text{Id} - \lambda_k T)$ is the so-called forward operator. Note, that the above scheme is nothing else than the classical method of steepest descend and $\lambda_k > 0$ is the step size parameter that has to be chosen according to a rule that guarantees convergence of the algorithm.

In case, T is a general monotone operator, the classical algorithm to solve (1) is the proximal point algorithm which can be traced back to the early works of Minty [31] and Martinet [30]. See also the thesis of Eckstein [20] for a detailed treatment of the subject.

The proximal point algorithm generates a sequence x^k according to the recursion

$$x^{k+1} = (\text{Id} + \lambda_k T)^{-1}(x^k), \quad (2)$$

where $\lambda_k > 0$ is a regularization parameter. The operator $(\text{Id} + \lambda_k T)^{-1}$ is the so-called resolvent operator, that has been introduced by Moreau in [32]. In the context of algorithms, the resolvent operator is often referred to as the backward operator. In the seminal paper [46], Rockafellar has shown that the sequence x^k generated by the proximal point algorithm converges weakly to a point x^* satisfying $0 \in T(x^*)$.

Unfortunately, in many interesting cases, the evaluation of the resolvent operator is as difficult as solving the original problem, which limits the practicability of the proximal point algorithm in its plain form. To partly overcome this problem, it is shown in [46], that the algorithm still converges when using inexact evaluations of the resolvent operator. In fact, the evaluation errors have to satisfy a certain

summability condition which essentially means that the resolvent operators have to be computed with increasing accuracy. This is still somewhat limiting, since in practice the errors of the resolvent operator are often hard to control.

1.1 Splitting methods

In many problems, however, the operator T can be written as the sum of two maximal monotone operators, i.e. $T = A + B$, such that the resolvent operators $(\text{Id} + \lambda A)^{-1}$ and $(\text{Id} + \lambda B)^{-1}$, are much easier to compute than the full resolvent $(\text{Id} + \lambda T)^{-1}$. Then, by combining the resolvents with respect to A , and B in a certain way, one might be able to mimic the effect of the full proximal step based on T . The two most successful instances that are based on combining forward and backward steps with respect to A and B , are the Peaceman-Rachford splitting algorithm [40],

$$x^{k+1} = (\text{Id} + \lambda B)^{-1}(\text{Id} - \lambda A)(\text{Id} + \lambda A)^{-1}(\text{Id} - \lambda B)(x^k),$$

and the Douglas-Rachford splitting algorithm [18],

$$x^{k+1} = (\text{Id} + \lambda B)^{-1}[(\text{Id} + \lambda A)^{-1}(\text{Id} - \lambda B) + \lambda B](x^k).$$

These splitting techniques have been originally proposed in the context of linear operators and therefore cannot be applied to general monotone operators. In [29], Lions and Mercier have analyzed and further developed these splitting algorithms. Their idea was to perform a change of variables $x^k = (\text{Id} + \lambda B)^{-1}(v^k)$, such that the Peaceman-Rachford and Douglas-Rachford splitting algorithms have a meaning even for A and B being multivalued operators. Regarding convergence of the algorithms, the Peaceman-Rachford algorithm still needs to assume that B is single-valued but the Douglas-Rachford algorithm converges even in the general setting, where $A + B$ is just maximal monotone.

In [21], Eckstein has pointed out that the Douglas-Rachford splitting algorithm can be re-written in the form of (2). Hence, it is basically a certain instance of the proximal point algorithm. Moreover, Eckstein has shown that the application of the Douglas-Rachford algorithm to the dual of a certain structured convex optimization problem coincides with the so-called alternating direction method of multipliers. It is remarkable, that the Douglas-Rachford splitting algorithm and its variants have seen a considerable renaissance in modern convex optimization [25, 8]. The main reason for the renewed interest lies in the fact that it is well suited for distributed convex programming. This is an important aspect for solving large scale convex optimization problems arising in recent image processing and machine learning applications.

Another important line of splitting methods is given by the so-called forward-backward splitting technique [24, 28,

9, 29]. In contrast to the more complicated splitting techniques discussed above, the forward-backward scheme is based (as the name suggests) on the recursive application of an explicit forward step with respect to B , followed by an implicit backward step with respect to A . The forward-backward algorithm is written as:

$$x^{k+1} = (\text{Id} + \lambda_k A)^{-1}(\text{Id} - \lambda_k B)(x^k) \quad (3)$$

In the most general setting, where both A and B are general monotone operators, the convergence result is rather weak [39], basically, λ_k has to fulfill the same step-size restrictions as unconstrained subgradient descend schemes. However, if in addition B is single valued and Lipschitz, e.g. B is the gradient of a smooth convex function, the situation becomes much more beneficial. In fact, if B is L -Lipschitz, and λ_k is chosen such that $\lambda_k < 2/L$, the forward backward algorithm (3) converges to a zero of $T = A + B$ [23, 47]. Similar to the Douglas-Rachford splitting algorithm, the forward-backward algorithm has seen a renewed interest. It has been proposed and further improved in the context of sparse signal recovery [17, 15], image processing [45], and machine learning [19] applications.

1.2 Inertial methods

In [44], Polyak introduced the so-called heavy ball method, a two-step iterative method for minimizing a smooth convex function f . The algorithm takes the following form:

$$\begin{cases} y^k = x^k + \alpha_k(x^k - x^{k-1}) \\ x^{k+1} = y^k - \lambda_k \nabla f(x^k), \end{cases}$$

where $\alpha_k \in [0, 1)$ is an extrapolation factor and λ_k is again a step-size parameter that has to be chosen sufficiently small. The difference compared to a standard gradient method is that in each iteration, the extrapolated point y^k is used instead of x^k . It is remarkable that this minor change greatly improves the performance of the scheme. In fact, its efficiency estimate [44] on strongly convex functions is equivalent to the known lower complexity bounds of first-order methods [35] and hence the heavy-ball method resembles an optimal method. The acceleration is explained by the fact that the new iterate is given by taking a step which is a combination of the direction $x^k - x^{k-1}$ and the current anti-gradient direction $-\nabla f(x^k)$.

The heavy ball method can also be interpreted as an explicit finite differences discretization of the time dynamical system

$$\ddot{x}(t) + \alpha_1 \dot{x}(t) + \alpha_2 \nabla f(x(t)) = 0,$$

where $\alpha_{1,2} > 0$ are free model parameters of the equation. This equation is used to describe the motion of a heavy body

in a potential field f and hence the system is coined the heavy ball with friction dynamical system.

In [2], Alvarez and Attouch translated the idea of the heavy ball method to the setting of a general maximal monotone operator using the framework of the proximal point algorithm (2). The resulting algorithm is called the inertial proximal point algorithm and it is written as

$$\begin{cases} y^k = x^k + \alpha_k(x^k - x^{k-1}) \\ x^{k+1} = (\text{Id} + \lambda_k T)^{-1}(y^k), \end{cases} \quad (4)$$

It is shown that under certain conditions on α_k and λ_k , the algorithm converges weakly to a zero of T . In fact, the algorithm converges if λ_k is non-decreasing and $\alpha_k \in [0, 1)$ is chosen such that

$$\sum_k \alpha_k \|x^k - x^{k-1}\|^2 < \infty, \quad (5)$$

which can be achieved by choosing α_k with respect to a simple on-line rule which ensures summability or in particular it is also true for $\alpha_k < 1/3$.

In subsequent work [33], Moudafi and Oliny introduced an additional single-valued and Lipschitz continuous operator B into the inertial proximal point algorithm:

$$\begin{cases} y^k = x^k + \alpha_k(x^k - x^{k-1}) \\ x^{k+1} = (\text{Id} + \lambda_k A)^{-1}(y^k - \lambda_k B(x^k)), \end{cases} \quad (6)$$

It turns out that this algorithm converges as long as $\lambda_k < 2/L$, where L is the Lipschitz constant of B and the same condition (5), which is used to ensure convergence of the inertial proximal point algorithm. Note that for $\alpha_k > 0$, the algorithm does not take the form of a forward-backward splitting algorithm, since B is still evaluated at the point x^k .

In recent work, Pesquet and Pustelnik proposed a Douglas-Rachford type parallel splitting method for finding the zero of the sum of an arbitrary number maximal monotone operators. The method also includes inertial forces [41] which numerically speeds up the convergence of the algorithm. Related algorithms also including inertial forces have been proposed and investigated in [7, 6].

1.3 Optimal methods

In a seminal paper [34], Nesterov proposed a modification of the heavy ball method in order to improve the convergence rate on smooth convex functions. While the heavy ball method evaluates the gradient in each iterate at the point x^k , the idea of Nesterov was to use the extrapolated point y^k also for evaluating the gradient. Additionally, the extrapolation parameter α_k is computed according to some special

law that allows to prove optimal convergence rates of this scheme. The scheme is given by:

$$\begin{cases} y^k = x^k + \alpha_k(x^k - x^{k-1}) \\ x^{k+1} = y^k - \lambda_k \nabla f(y^k), \end{cases} \quad (7)$$

where $\lambda_k = 1/L$. There are several choices to define an optimal sequence $\{\alpha_k\}$ [34, 35, 4, 48]. In [35], it has been shown that the efficiency estimate of the above scheme is up to some constant factor equivalent to the lower complexity bounds of first-order methods for the class of μ -strongly convex functions, $\mu \geq 0$, with L -Lipschitz gradient.

In [26], Güler has translated Nesterov's idea to the general setting of the proximal point algorithm, with the restriction that the operator T is the subdifferential of a convex function. Inexact versions of this algorithm have been proposed and studied in [50]. In [4], Beck and Teboulle have proposed the so-called fast iterative shrinkage thresholding algorithm (FISTA), that combines in a clever way the ideas of Nesterov and Güler within the forward-backward splitting framework. The algorithm features the same optimal convergence rate as Nesterov's method but it can be applied also in the presence of an additional but simple (with easy to compute proximal map) non-smooth convex function. The FISTA algorithm can be applied to a variety of practical problem arising in sparse signal recovery, image processing and machine learning and hence has become a standard algorithm. Related algorithms with similar properties have been independently proposed by Nesterov in [36, 37].

1.4 Content

In this paper we propose a modification of the forward-backward splitting algorithm (3) to solve monotone inclusions. Our method is inspired by the inertial forward-backward splitting method (6), but differs from this method in two regards. First, the operator B is evaluated at the inertial extrapolate y^k which is inspired by Nesterov's optimal gradient method (7). In addition, we consider a symmetric positive definite map M , which can be interpreted as a preconditioner or variable metric and is inspired by recently work on primal dual algorithms [10, 22, 42, 27] and forward backward splitting [14, 12, 11]. These changes allow us to define a new "meta-algorithm", that includes, as special cases for example several convex optimization algorithms that have recently attracted a lot of attention in the imaging, signal processing and machine learning communities.

In section 2 we will define the proposed algorithm and prove the general convergence in a Hilbert space setting. In section 3 we will apply the proposed algorithm to a class of convex-concave saddle-point problems and will show how several known algorithms can be recovered from the proposed "meta-algorithm". In section 4, we will apply the proposed algorithm to image processing problems including,

image restoration and image deconvolution. In the last section, we will give some concluding remarks.

2 Proposed algorithm

We consider the problem of finding a point x^* in a Hilbert space X such that

$$0 \in (A + B)(x^*), \quad (8)$$

where A, B are maximal monotone operators. We additionally assume that the operator B is single-valued and co-coercive with respect to the solution set $S := (A + B)^{-1}(0)$ and a linear, selfadjoint and positive definite map L , i.e. for all $x \in X, y \in S$

$$\langle B(x) - B(y), x - y \rangle \geq \|B(x) - B(y)\|_{L^{-1}}^2 \quad (9)$$

where, as usual, we denote $\|x\|_{L^{-1}}^2 = \langle L^{-1}x, x \rangle$. Note that in the most simple case where $L = l \text{Id}$, $l > 0$, the operator B is $1/l$ co-coercive and hence l -Lipschitz. However, we will later see that in some cases, it makes sense to consider more general L .

The algorithm we propose in this paper is a basically a modification of the forward-backward splitting algorithm (3). The scheme is as follows:

$$\begin{cases} y^k = x^k + \alpha_k(x^k - x^{k-1}) \\ x^{k+1} = (\text{Id} + \lambda_k M^{-1}A)^{-1}(\text{Id} - \lambda_k M^{-1}B)(y^k), \end{cases} \quad (10)$$

where $\alpha_k \in [0, 1)$ is an extrapolation factor, λ_k is a step-size parameter and M is a linear selfadjoint and positive definite map that can be used as a preconditioner for the algorithm (cf. Section 3.2). Note that the iteration can be equivalently expressed as

$$\begin{cases} y^k = x^k + \alpha_k(x^k - x^{k-1}) \\ x^{k+1} = (M + \lambda_k A)^{-1}(M - \lambda_k B)(y^k), \end{cases} \quad (11)$$

Observe that (10) (resp. (11)) differs from the inertial forward-backward algorithm of Moudafi and Oliny insofar that we also evaluate the operator B at the inertial extrapolate y^k . This allows us to rewrite the algorithm in the form of the standard forward-backward algorithm (3).

In the following Theorem, we analyze the basic convergence properties of the proposed algorithm.

Theorem 1 *Let X be a real Hilbert space and $A, B : X \rightrightarrows X$ be maximally monotone operators. Further assume that $M, L : X \rightarrow X$ are linear, bounded, selfadjoint and positive definite maps and that B is single valued and co-coercive w.r.t. L^{-1} (cf. (9)). Moreover, let $\lambda_k > 0, \alpha < 1, \alpha_k \in [0, \alpha], x^0 = x^{-1} \in X$ and let the sequences x^k and y^k be defined by (10) (or (11)). If*

- (i) $S_k = M - \frac{\lambda_k}{2}L$ is positive definite for all k and
- (ii) $\sum_{k=1}^{\infty} \alpha_k \|x^k - x^{k-1}\|_M^2 < \infty$

then x^k converges weakly to a solution of the inclusion $0 \in (A + B)(x)$.

Proof Denote by x^* a zero of $A + B$. From (8), it holds that $-B(x^*) \in A(x^*)$.

Furthermore, the second line in (11) can be equivalently expressed as

$$M(y^k - x^{k+1}) - \lambda_k B(y^k) \in \lambda_k A(x^{k+1}).$$

For convenience, we define for any symmetric positive definite M ,

$$\begin{aligned} \phi_M^k &= \frac{1}{2} \|x^k - x^*\|_M^2 = \frac{1}{2} \langle M(x^k - x^*), x^k - x^* \rangle, \\ \Delta_M^k &= \frac{1}{2} \|x^k - x^{k-1}\|_M^2 = \frac{1}{2} \langle M(x^k - x^{k-1}), x^k - x^{k-1} \rangle \\ \Gamma_M^k &= \frac{1}{2} \|x^{k+1} - y^k\|_M^2 = \frac{1}{2} \langle M(x^{k+1} - y^k), x^{k+1} - y^k \rangle. \end{aligned}$$

From the well-known identity

$$\langle a - b, a - c \rangle_M = \frac{1}{2} \|a - b\|_M^2 + \frac{1}{2} \|a - c\|_M^2 - \frac{1}{2} \|b - c\|_M^2 \quad (12)$$

we have by using the definition of the inertial extrapolate y_k that

$$\begin{aligned} \phi_M^k - \phi_M^{k+1} &= \Delta_M^{k+1} + \langle y^k - x^{k+1}, x^{k+1} - x^* \rangle_M \\ &\quad - \alpha_k \langle x^k - x^{k-1}, x^{k+1} - x^* \rangle_M. \end{aligned} \quad (13)$$

Then, by using the monotonicity of A we deduce that

$$\begin{aligned} \langle \lambda_k A(x^{k+1}) - \lambda_k A(x^*), x^{k+1} - x^* \rangle &\geq 0 \\ \langle M(y^k - x^{k+1}) - \lambda_k B(y^k) + \lambda_k B(x^*), x^{k+1} - x^* \rangle &\geq 0 \end{aligned}$$

and

$$\begin{aligned} \langle y^k - x^{k+1}, x^{k+1} - x^* \rangle_M \\ + \lambda_k \langle B(x^*) - B(y^k), x^{k+1} - x^* \rangle &\geq 0. \end{aligned}$$

Combining with (13), we obtain

$$\begin{aligned} \phi_M^k - \phi_M^{k+1} &\geq \Delta_M^{k+1} + \lambda_k \langle B(y^k) - B(x^*), x^{k+1} - x^* \rangle \\ &\quad - \alpha_k \langle x^k - x^{k-1}, x^{k+1} - x^* \rangle_M. \end{aligned} \quad (14)$$

From the co-coercivity property of B we have that

$$\begin{aligned} \langle B(y^k) - B(x^*), x^{k+1} - x^* \rangle \\ = \langle B(y^k) - B(x^*), x^{k+1} - y^k + y^k - x^* \rangle \\ \geq \|B(y^k) - B(x^*)\|_{L^{-1}}^2 + \langle B(y^k) - B(x^*), x^{k+1} - y^k \rangle \\ \geq \|B(y^k) - B(x^*)\|_{L^{-1}}^2 - \|B(y^k) - B(x^*)\|_{L^{-1}}^2 - \frac{1}{2} \Gamma_L^k \\ = -\frac{1}{2} \Gamma_L^k \end{aligned}$$

Substituting back into (14) we arrive at

$$\phi_M^k - \phi_M^{k+1} \geq \Delta_M^{k+1} - \frac{\lambda_k}{2} \Gamma_L^k - \alpha_k \langle x^k - x^{k-1}, x^{k+1} - x^* \rangle_M$$

Invoking again (12), it follows that

$$\begin{aligned} & \phi_M^{k+1} - \phi_M^k - \alpha_k (\phi_M^k - \phi_M^{k-1}) \\ & \leq -\Delta_M^{k+1} + \frac{\lambda_k}{2} \Gamma_L^k \\ & \quad + \alpha_k (\Delta_M^k + \langle x^k - x^{k-1}, x^{k+1} - x^k \rangle_M) \\ & = -\Gamma_M^k + \frac{\lambda_k}{2} \Gamma_L^k + (\alpha_k + \alpha_k^2) \Delta_M^k. \end{aligned} \quad (15)$$

The rest of the proof closely follows the proof of Theorem 2.1 in [2]. By the definition of S_k and using $(\alpha_k + \alpha_k^2)/2 \leq \alpha_k$, we have

$$\phi_M^{k+1} - \phi_M^k - \alpha_k (\phi_M^k - \phi_M^{k-1}) \leq -\Gamma_{S_k}^k + 2\alpha_k \Delta_M^k. \quad (16)$$

By assumption (i), the first term is non-positive and since $\alpha_k \geq 0$, the second term is non-negative.

Now, defining $\theta^k = \max(0, \phi_M^k - \phi_M^{k-1})$ and setting

$$\delta^k = 2\alpha_k \Delta_M^k = \alpha_k \|x^k - x^{k-1}\|_M^2,$$

we obtain

$$\theta^{k+1} \leq \alpha_k \theta^k + \delta^k \leq \alpha \theta^k + \delta^k$$

Applying this inequality recursively, one obtains a geometric series of the form

$$\theta^{k+1} \leq \alpha^k \theta^1 + \sum_{i=0}^{k-1} \alpha^i \delta^{k-i}$$

Summing this inequality from $k = 0, \dots, \infty$, one has

$$\sum_{k=0}^{\infty} \theta^{k+1} \leq \frac{1}{1-\alpha} \left(\theta^1 + \sum_{k=1}^{\infty} \delta^k \right)$$

Note that the series on the right hand side converges by assumption (ii).

Now we set $t^k = \phi_M^k - \sum_{i=1}^k \theta^i$ and since $\phi_M^k \geq 0$ and $\sum_{i=1}^k \theta_i$ is bounded independently of k , we see that t^k is bounded from below. On the other hand,

$$\begin{aligned} t^{k+1} &= \phi_M^{k+1} - \theta^{k+1} - \sum_{i=1}^k \theta^i \\ &\leq \phi_M^{k+1} - \phi_M^{k+1} + \phi_M^k - \sum_{i=1}^k \theta^i = t^k \end{aligned}$$

and hence, t^k is also non-decreasing, thus convergent. This implies that ϕ_M^k is convergent and especially that $\theta^k \rightarrow 0$.

From (16) we get

$$\begin{aligned} \frac{1}{2} \|x^{k+1} - y^k\|_{S_k}^2 &\leq -\theta^{k+1} - \alpha \theta^k + \delta^k \\ \frac{1}{2} \|x^{k+1} - x^k - \alpha_k (x^k - x^{k-1})\|_{S_k}^2 &\leq -\theta^{k+1} - \alpha \theta^k + \delta^k \end{aligned}$$

Since δ^k is summable it follows that $\|x^k - x^{k-1}\|_{S_k} \rightarrow 0$ and hence

$$\lim_{k \rightarrow \infty} \|x^{k+1} - x^k - \alpha_k (x^k - x^{k-1})\|_{S_k} = 0.$$

We already know that x^k is bounded hence, there is a convergent subsequence $x_{\nu} \rightarrow \bar{x}$. Then we also get that $y_{\nu} = (1 + \alpha_{\nu})x_{\nu} - \alpha_{\nu}x_{\nu-1} \rightarrow \bar{x}$. Now we get from (10) that

$$x^{\nu} = (\text{Id} + \lambda_{\nu} M^{-1} A)^{-1} (y^{\nu} - \lambda_{\nu} M^{-1} B(y^{\nu}))$$

and pass to the limit (extracting another subsequence such that $\lambda_{\nu} \rightarrow \bar{\lambda}$ if necessary) to obtain

$$\bar{x} = (\text{Id} + \bar{\lambda} M^{-1} A)^{-1} (\bar{x} - \bar{\lambda} M^{-1} B(\bar{x}))$$

which is equivalent to

$$-B(\bar{x}) \in A(\bar{x})$$

which in turn shows that \bar{x} is a solution. Opial's Theorem [38] concludes the proof. \square

Next, we address the question whether the sequence $\{\alpha_k\}$ can be chosen a-priori such that the algorithm is guaranteed to converge. Indeed, in case of the inertial proximal point algorithm (4), it has already been shown in [2] that convergence is ensured if $\{\alpha_k\}$ is a nondecreasing sequence in $[0, \alpha]$ with $\alpha < 1/3$. The next theorem presents a related result for the proposed algorithm.

Theorem 2 *In addition to the conditions to Theorem 1 assume that $\{\lambda_k\}$ and $\{\alpha_k\}$ are nondecreasing sequences and that there exists a $\varepsilon > 0$ such that for all α_k*

$$R_k = (1 - 3\alpha_k)M - (1 - \alpha_k)^2 \frac{\lambda_k}{2} L \geq \varepsilon M. \quad (17)$$

Then x^k converges weakly to a solution of the inclusion $0 \in (A + B)(x^)$.*

Proof The proof of this result is an adaption of the proof of Proposition 2.1 in [2]. From the last estimate in (15) and using the definition of y^k in (6) it follows that

$$\begin{aligned} & \phi_M^{k+1} - \phi_M^k - \alpha_k (\phi_M^k - \phi_M^{k-1}) \\ & \leq -\Gamma_{S_k}^k + \alpha_k (1 + \alpha_k) \Delta_M^k \\ & \leq -\Delta_{S_k}^{k+1} - \alpha_k^2 \Delta_{S_k}^k + \alpha_k \langle x^{k+1} - x^k, x^k - x^{k-1} \rangle_{S_k} \\ & \quad + (\alpha_k + \alpha_k^2) \Delta_M^k \\ & \leq (\alpha_k - 1) \Delta_{S_k}^{k+1} + (\alpha_k - \alpha_k^2) \Delta_{S_k}^k + (\alpha_k + \alpha_k^2) \Delta_M^k \\ & \leq (\alpha_k - 1) \Delta_{S_k}^{k+1} + \alpha_k \Delta_{T_k}^k, \end{aligned}$$

where $T_k = 2M - \frac{(1-\alpha_k)\lambda_k}{2} L$.

We define $\mu^k = \phi_M^k - \alpha_k \phi_M^{k-1} + \alpha_k \Delta_{T_k}^k$ and since $\alpha_{k+1} \geq \alpha_k$ and using the above inequality,

$$\begin{aligned} \mu^{k+1} - \mu^k &= \phi_M^{k+1} - \alpha_{k+1} \phi_M^k + \alpha_{k+1} \Delta_{T_{k+1}}^{k+1} - \phi_M^k + \alpha_k \phi_M^{k-1} - \alpha_k \Delta_{T_k}^k \\ &\leq \phi_M^{k+1} - \phi_M^k - \alpha_k (\phi_M^k - \phi_M^{k-1}) + \alpha_{k+1} \Delta_{T_{k+1}}^{k+1} - \alpha_k \Delta_{T_k}^k \\ &\leq (\alpha_k - 1) \Delta_{S_k}^{k+1} + \alpha_{k+1} \Delta_{T_{k+1}}^{k+1}. \end{aligned}$$

Then, we obtain since $\alpha_{k+1} \geq \alpha_k$

$$\begin{aligned} \mu^{k+1} - \mu^k &\leq \frac{1}{2} \langle ((\alpha_k - 1)S_k + \alpha_{k+1}T_k)(x^{k+1} - x^k), x^{k+1} - x^k \rangle \\ &\leq \frac{1}{2} \langle ((\alpha_{k+1} - 1)S_k + \alpha_{k+1}T_k)(x^{k+1} - x^k), x^{k+1} - x^k \rangle. \end{aligned}$$

Now using $\alpha_{k+1} \geq \alpha_k$ and $\lambda_{k+1} \geq \lambda_k$ we obtain

$$\begin{aligned} (\alpha_{k+1} - 1)S_k + \alpha_{k+1}T_k &\leq (3\alpha_{k+1} - 1)M + (1 - \alpha_{k+1})^2 \frac{\lambda_{k+1}}{2} L = R_k \end{aligned}$$

which finally gives

$$\mu^{k+1} - \mu^k \leq -\Delta_{R_k}^k. \quad (18)$$

Observe that by assumption (17), the sequence $\{\mu_k\}$ is non-increasing and hence

$$\phi_M^k - \alpha \phi_M^{k-1} \leq \mu^k \leq \mu^1.$$

It follows that

$$\phi_M^k \leq \alpha^k \phi^0 + \mu^1 \sum_{i=0}^{k-1} \alpha^i \leq \alpha^k \phi^0 + \frac{\mu^1}{1 - \alpha}$$

On the other hand, we have by summing up (18) from $i = 1$ to k ,

$$\mu^{k+1} - \mu^1 \leq -\sum_{i=1}^k \Delta_{R_i}^i.$$

Combining these two estimates it follows that

$$\sum_{i=1}^k \Delta_{R_i}^i \leq \mu^1 - \mu^{k+1} \leq \mu^1 + \alpha \phi_M^k \leq \alpha^{k+1} \phi^0 + \frac{\mu^1}{1 - \alpha}.$$

Since $R_k \geq \varepsilon M$, it follows that

$$\sum_{k=1}^{\infty} \Delta_M^k < \infty,$$

which especially shows (ii) in Theorem 1. The weak convergence of the x^k now follows from Theorem 1. \square

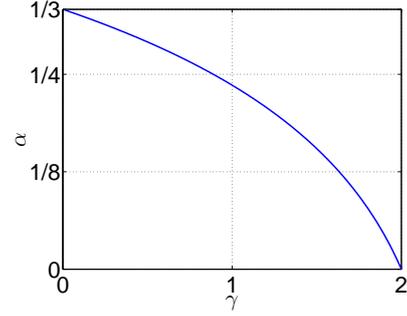


Fig. 1 Upper bound on the extrapolation factor α in dependence on γ .

Remark 3 In case, $M = m \text{ Id}$, $L = l \text{ Id}$, $\lambda_k \equiv \lambda$ and defining the normalized step size $\gamma = \frac{\lambda}{m} \in (0, 2)$, assertion (17) reduces to

$$1 - 3\alpha_k - \varepsilon - \frac{(1 - \alpha_k)^2}{2} \gamma \geq 0.$$

It easily follows that for any $\varepsilon \in (0, (9 - 4\gamma)/(2\gamma))$, the algorithm converges, if the sequence $\{\alpha_k\}$ is non-decreasing with $0 \leq \alpha_k \leq \alpha(\gamma)$, where

$$\alpha(\gamma) = 1 + \frac{\sqrt{9 - 4\gamma - 2\varepsilon\gamma} - 3}{\gamma}.$$

See Figure 1 for a plot of $\alpha(\gamma)$ using $\varepsilon = 10^{-6}$.

Remark 4 Let us consider a “fully-implicit” variant of the scheme (10), which is given by

$$\begin{cases} y^k = x^k + \alpha_k(x^k - x^{k-1}), \\ x^{k+1} = (\text{Id} + \lambda_k \bar{M}^{-1}(A + B))^{-1}(y^k), \end{cases}$$

where \bar{M} is again a linear, selfadjoint and positive definite map. In fact this algorithm, is an inertial proximal point algorithm, in the \bar{M} metric, whose convergence properties have been studied in [2]. This algorithm has less stringent convergence properties compared to the algorithm proposed in this paper, but its application to practical problems is limited since the resolvent with respect to $A + B$ can be complicated.

Interestingly, if the operator B is a linear, selfadjoint and positive semi-definite map, the above fully-implicit scheme can be significantly simplified. In fact, using $\lambda_k \equiv \lambda$ and setting $\bar{M} = M - \lambda B$, where λ is chosen such that $\bar{M} > 0$, it turns out that the fully implicit scheme in the \bar{M} metric is equivalent to our proposed inertial forward-backward splitting algorithm (10) in the M metric, which only requires to compute the resolvent with respect to A .

According to Theorem 2.1 and Proposition 2.1 in [2], condition (i) of Theorem 1 can be replaced by the simpler condition $M - \lambda B > 0$ and convergence of the algorithm is guaranteed for $\{\alpha_k\}$ non-decreasing in $[0, \alpha]$ with $\alpha < 1/3$.

3 Application to convex-concave saddle-point problems

Recently, so-called primal-dual splitting techniques have been proposed which are motivated by the need to solve large-scale non-smooth convex optimization problems in image processing [10, 22, 42, 27, 16, 13, 49]. These algorithms can be applied if the structure of the problem allows to rewrite it as certain convex-concave saddle-point problems.

Now let X and Y be two Hilbert spaces and consider the saddle point problem

$$\min_{x \in X} \max_{y \in Y} G(x) + Q(x) + \langle Kx, y \rangle - F^*(y) - P^*(y) \quad (19)$$

with convex $G, Q : X \rightarrow \mathbb{R}_\infty$, $F^*, P^* : Y \rightarrow \mathbb{R}_\infty$, $K : X \rightarrow Y$ linear and bounded and Q, P^* differentiable with Lipschitz gradient (with respective Lipschitz constants L_Q, L_P).

We define the monotone operators A, B on $X \times Y$ as

$$A = \begin{bmatrix} \partial G & K^* \\ -K & \partial F^* \end{bmatrix}, \quad B = \begin{bmatrix} \nabla Q & 0 \\ 0 & \nabla P^* \end{bmatrix}$$

and observe that the optimality system of the saddle point problem can be written as

$$0 \in (A + B) \begin{bmatrix} x \\ y \end{bmatrix}.$$

This setup fits into our general framework of section 2.

The standard splitting iterations (3) and (6) would not be applicable in general since the evaluation of the proximal mapping $(\text{Id} + \lambda A)^{-1}$ may be prohibitively expensive in this case. However, if we consider the preconditioned iteration (10) with an appropriate mapping M the iteration becomes feasible. The idea is, to choose M such that one of the off-diagonal blocks in A cancel out. However, M still has to be symmetric and positive definite and this leads to the choice

$$M = \begin{bmatrix} \frac{1}{\tau} \text{Id} & -K^* \\ -K & \frac{1}{\sigma} \text{Id} \end{bmatrix}. \quad (20)$$

From (10) we get for $\lambda_k = 1$ the following inertial primal-dual forward-backward algorithm

$$\begin{cases} \xi^k &= x^k + \alpha_k(x^k - x^{k-1}) \\ \zeta^k &= y^k + \alpha_k(y^k - y^{k-1}) \\ x^{k+1} &= (\text{Id} + \tau \partial G)^{-1}(\xi^k - \tau(\nabla Q(\xi^k) + K^* \zeta^k)) \\ \bar{\xi}^{k+1} &= 2x^{k+1} - \xi^k \\ y^{k+1} &= (\text{Id} + \sigma \partial F^*)^{-1}(\zeta^k - \sigma(\nabla P^*(\zeta^k) - K \bar{\xi}^{k+1})). \end{cases} \quad (21)$$

In the case that $Q = P^* = 0$ and $\alpha_k = 0$ we obtain the primal-dual method from [10].

The next two results characterize the conditions under which the proposed inertial primal-dual forward-backward algorithm converges.

Theorem 5 *The iterates given by method (21) converge weakly to a solution of the saddle point problem (19) if*

$$\begin{aligned} 0 < \tau < 2/L_Q, \quad 0 < \sigma < 2/L_P, \\ \|K\|^2 < \left(\frac{1}{\tau} - \frac{L_Q}{2}\right)\left(\frac{1}{\sigma} - \frac{L_P}{2}\right), \end{aligned} \quad (22)$$

and if $\alpha_k \in [0, \alpha]$ with $\alpha < 1$ and the iterates (x^k, y^k) fulfill

$$\sum_{k=1}^{\infty} \alpha_k \|(x^k, y^k) - (x^{k-1}, y^{k-1})\|_M^2 < \infty. \quad (23)$$

Furthermore, condition (23) is fulfilled, if $\{\alpha_k\}$ is nondecreasing and there exists $\varepsilon > 0$ such for all α_k

$$\begin{aligned} \frac{1-3\alpha_k-\varepsilon}{\tau} &\geq \frac{(1-\alpha_k)^2}{2} L_Q, \\ \frac{1-3\alpha_k-\varepsilon}{\sigma} &\geq \frac{(1-\alpha_k)^2}{2} L_P, \\ \left(\frac{1-3\alpha_k-\varepsilon}{\tau} - \frac{(1-\alpha_k)^2}{2} L_Q\right) &\left(\frac{1-3\alpha_k-\varepsilon}{\sigma} - \frac{(1-\alpha_k)^2}{2} L_P\right) \\ &\geq (1-3\alpha_k-\varepsilon)^2 \|K\|^2. \end{aligned} \quad (24)$$

Proof Since Q and P^* are convex with Lipschitz-continuous gradients with Lipschitz constants L_Q and L_P , respectively, it follows from the Baillon-Haddad Theorem ([3, Corollary 18.16]) that that ∇Q and ∇P^* are co-coercive w.r.t. L_Q^{-1} and L_P^{-1} , respectively. Hence, for $x, \xi \in X$ and $y, \zeta \in Y$ it holds that

$$\begin{aligned} &\langle B(x, y) - B(\xi, \zeta), (x, y) - (\xi, \zeta) \rangle_{X \times Y} \\ &= \langle \nabla Q(x) - \nabla Q(\xi), x - \xi \rangle_X + \langle \nabla P^*(y) - \nabla P^*(\zeta), y - \zeta \rangle_Y \\ &\geq L_Q^{-1} \|\nabla Q(x) - \nabla Q(\xi)\|_X^2 + L_P^{-1} \|\nabla P^*(y) - \nabla P^*(\zeta)\|_Y^2. \end{aligned}$$

Thus B is co-coercive w.r.t. the mapping

$$L^{-1} = \begin{bmatrix} L_Q^{-1} \text{Id} & 0 \\ 0 & L_P^{-1} \text{Id} \end{bmatrix}$$

It is easy to check that

$$S = M - \frac{1}{2}L = \begin{bmatrix} \left(\frac{1}{\tau} - \frac{L_Q}{2}\right) \text{Id} & -K^* \\ -K & \left(\frac{1}{\sigma} - \frac{L_P}{2}\right) \text{Id} \end{bmatrix}$$

is positive definite if (22) is fulfilled and (23) follows from Theorem 1.

Applying condition (17) to (21) we have:

$$\begin{aligned} (1-3\alpha_k) &\begin{bmatrix} \frac{1}{\tau} \text{Id} & -K^* \\ -K & \frac{1}{\sigma} \text{Id} \end{bmatrix} - \frac{(1-\alpha_k)^2}{2} \begin{bmatrix} L_Q \text{Id} & 0 \\ 0 & L_P \text{Id} \end{bmatrix} \\ &\geq \varepsilon \begin{bmatrix} \frac{1}{\tau} \text{Id} & -K^* \\ -K & \frac{1}{\sigma} \text{Id} \end{bmatrix} \end{aligned}$$

which can be checked to be true under the stated condition (24). \square

Let us present some practical rules to choose feasible parameters for the algorithm. For this we introduce the parameters $\gamma, \delta \in (0, 2)$, which can be interpreted as normalized step sizes in the primal and dual variables and the parameter $r > 0$, which controls the relative scaling between the primal and dual step sizes.

Lemma 6 Choose $\gamma, \delta \in (0, 2)$ and $r > 0$ and set

$$\tau = \frac{1}{\|K\|r + L_Q/\gamma} \quad \text{and} \quad \sigma = \frac{1}{\|K\|/r + L_P/\delta},$$

Furthermore, let $\{\alpha_k\}$ be a non-decreasing sequence satisfying $0 \leq \alpha_k \leq \alpha(\gamma, \delta)$, where

$$\alpha(\gamma, \delta) = 1 + \frac{\sqrt{9 - 4 \max(\gamma, \delta) - 2\varepsilon \max(\gamma, \delta)} - 3}{\max(\gamma, \delta)}, \quad (25)$$

and $\varepsilon \in (0, (9 - 4 \max(\gamma, \delta))/(2 \max(\gamma, \delta)))$. Then, the conditions (22) and (24) of Theorem 5 hold, i.e. algorithm (21) converges weakly to a solution of the saddle-point problem (19).

Proof It can be easily checked that the conditions (22) hold. Indeed, one has $\tau < 2/L_Q$, $\sigma < 2/L_P$ and also

$$\begin{aligned} \left(\frac{1}{\tau} - \frac{L_Q}{2}\right) \left(\frac{1}{\sigma} - \frac{L_P}{2}\right) &= \\ \|K\|^2 + \|K\| \left(\frac{rL_P(2-\delta)}{2\delta} + \frac{L_Q(2-\gamma)}{2\gamma r} \right) + L_Q L_P \frac{(2-\gamma)(2-\delta)}{4\gamma\delta} &\geq \|K\|^2. \end{aligned} \quad (26)$$

Next, we can compute the maximum value of α_k that ensures convergence of the algorithm. Observe that for any $r > 0$ assertion (24) holds in particular if

$$(1 - 3\alpha_k - \varepsilon) - \frac{(1 - \alpha_k)^2}{2} \frac{\tau L_Q}{1 - \tau\|K\|r} \geq 0,$$

and

$$(1 - 3\alpha_k - \varepsilon) - \frac{(1 - \alpha_k)^2}{2} \frac{\sigma L_P}{1 - \sigma\|K\|/r} \geq 0.$$

where $\gamma = \frac{\tau L_Q}{1 - \tau\|K\|r}$ and $\delta = \frac{\sigma L_P}{1 - \sigma\|K\|/r}$. Clearly, the two inequalities are fulfilled if

$$(1 - 3\alpha_k - \varepsilon) - \frac{(1 - \alpha_k)^2}{2} \max(\gamma, \delta) \geq 0,$$

from which the upper bound $\alpha(\gamma, \delta)$ follows. \square

Remark 7 From equation (26), we can see that in case L_P or L_Q is zero (and fixed γ respectively δ), it might be favorable to choose larger respectively smaller values of r , since it leads to a smaller product of the terms on the left hand side of (26) and hence to larger product of primal and dual step sizes.

3.1 Recovering known algorithms

The proposed algorithm includes several popular algorithms for convex optimization as special cases:

- **Forward-backward splitting:** Set $K = F^* = P^* = 0$ in (19), and set $M = \text{Id}$, $\lambda_k < 2/L_Q$ and $\alpha_k = 0$ in (21). We obtain exactly the popular forward-backward splitting algorithm (3) for minimizing the sum of a smooth and a non-smooth convex function. See [15, 17].
- **Nesterov's accelerated gradient method:** In addition to the previous setting, let $\lambda_k = 1/L_Q$ and let the sequence $\{\alpha_k\}$ be computed according to one of the laws proposed in [34, 35, 4]. We can exactly recover Nesterov's accelerated gradient method [34, 35], the accelerated proximal point algorithm [26] and FISTA [4].

These algorithms offer an optimal convergence rate of $\mathcal{O}(1/k^2)$ for the function gap $(G + Q)(x^k) - (G + Q)(x^*)$. However, it is still unclear whether the sequence of iterates $\{x^k\}$ converges. We cannot give a full answer here but we can at least modify the FISTA algorithm such that the sequence $\alpha_k \|x^k - x^{k-1}\|^2$ has finite length. Following [2], condition (ii) can be easily enforced “on-line” because it involves only past iterates. One possibility to ensure summability in (ii) is to require that $\alpha_k \|x^k - x^{k-1}\|_M^2 = \mathcal{O}(1/k^2)$, e.g.

$$\alpha_k = \min((k-1)/(k+2), c/(k^2 \|x^k - x^{k-1}\|_M^2)), \quad (27)$$

for some $c > 0$. However, since in the FISTA algorithm $\alpha_k = (k-1)/(k+2) \rightarrow 1$, Theorem 1 still does not imply convergence of the iterates. This is left for future investigation.

- **Primal-dual algorithms:** Setting in (19) $P^* = Q = 0$ and let in (21) $\alpha_k = 0$, we clearly obtain the first-order primal-dual algorithm investigated in [43, 22, 10, 27]. Furthermore, if we let Q be a convex function with Lipschitz continuous gradient ∇Q , we obtain the first-order primal-dual algorithm of Condat [16]. Moreover, in the present of smooth terms Q and P^* in the primal and dual problem, we recover Vũ's algorithm from [49]. We point out that the methods in [16, 49] involve an additional relaxation step of the form:

$$x^{k+1} = ((1 - \rho_k) \text{Id} + \rho_k (\text{Id} + \lambda_k T)^{-1})(x^k), \quad (28)$$

where ρ_k is the relaxation parameter. In case there are no smooth term Q and P^* the relaxation parameter $\rho_k \in (0, 2)$, in presence of Q and P^* the relaxation parameter is further restricted. See Section 4 for numerical comparisons. Observe, that the overrelaxation technique is quite different from the inertial technique we used in this paper, which is of the form:

$$x^{k+1} = (\text{Id} + \lambda_k T)^{-1}(x^k + \alpha_k(x^k - x^{k-1})). \quad (29)$$

Indeed, it was shown in [1] that one can even use over-relaxation and inertial forces simultaneously. However, introducing an additional overrelaxation step in the proposed framework is left for future investigation.

3.2 Preconditioning

Besides the property of the map M to make the primal-dual iterations feasible, the map M can also be interpreted as applying the algorithm (10) using $M = \text{Id}$ to the modified inclusion:

$$-M^{-1}B(x^*) \in M^{-1}A(x^*),$$

and hence, M^{-1} can be interpreted as a left preconditioner to the inclusion (8). In the context of saddle point problems, Pock and Chambolle [42] proposed a preconditioning of the form

$$M = \begin{bmatrix} T^{-1} & -K^* \\ -K & \Sigma^{-1} \end{bmatrix}.$$

where T and Σ are selfadjoint, positive definite maps. A condition for the positive definiteness of M follows from the following lemma.

Lemma 8 *Let A_1, A_2 be symmetric positive definite maps and B a bounded operator. If $\|A_2^{-\frac{1}{2}}BA_1^{\frac{1}{2}}\| < 1$, then $A = \begin{bmatrix} A_1 & B^* \\ B & A_2 \end{bmatrix}$ is positive definite.*

Proof We calculate

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} A_1 & B^* \\ B & A_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle = \langle x, A_1x \rangle + 2\langle Bx, y \rangle + \langle y, A_2y \rangle$$

and estimate the middle term from below by Cauchy-Schwarz and Young's inequality and get for every $c > 0$ that

$$\begin{aligned} \langle Bx, y \rangle &= \langle A_2^{-\frac{1}{2}}BA_1^{-\frac{1}{2}}A_1^{\frac{1}{2}}x, A_2^{\frac{1}{2}}y \rangle \\ &\geq -\frac{c}{2}\|A_2^{-\frac{1}{2}}BA_1^{-\frac{1}{2}}\|^2\|A_1^{\frac{1}{2}}x\|^2 - \frac{1}{2c}\|A_2^{\frac{1}{2}}y\|^2. \end{aligned}$$

Combining this with the assumption that $\|A_2^{-\frac{1}{2}}BA_1^{\frac{1}{2}}\| < 1$ we see that we can choose c such that

$$\begin{aligned} \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, A \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle &\geq (1 - c\|A_2^{-\frac{1}{2}}BA_1^{-\frac{1}{2}}\|^2)\|A_1^{\frac{1}{2}}x\|^2 + (1 - \frac{1}{c})\|A_2^{\frac{1}{2}}y\|^2 \\ &> 0 \end{aligned}$$

which proves the auxiliary statement. \square

We conclude that algorithm (10) converges as long as one has $\|\Sigma^{-\frac{1}{2}}KT^{-\frac{1}{2}}\| < 1$. In order to keep the proximal maps with respect to G and F^* feasible, the maps T and Σ were restricted to diagonal matrices. However, in recent work [5], it was shown that some proximal maps are still efficiently computable if T and Σ are the sum of a diagonal matrix and a rank-one matrix.

Applying the preconditioning technique to the proposed inertial primal-dual forward-backward algorithm (21), we obtain the method

$$\begin{cases} \xi^k &= x^k + \alpha_k(x^k - x^{k-1}) \\ \zeta^k &= y^k + \alpha_k(y^k - y^{k-1}) \\ x^{k+1} &= (\text{Id} + T\partial G)^{-1}(\xi^k - T(\nabla Q(\xi^k) + K^*\zeta^k)) \\ \bar{\xi}^{k+1} &= 2x^{k+1} - \xi^k \\ y^{k+1} &= (\text{Id} + \Sigma\partial F^*)^{-1}(\zeta^k - \Sigma(\nabla P^*(\zeta^k) - K\bar{\xi}^{k+1})). \end{cases} \quad (30)$$

It turns out that the resulting method converges under appropriate conditions.

Theorem 9 *In the setting of Theorem 5 let furthermore ∇Q and ∇P^* be co-coercive w.r.t. the two bound, linear, symmetric and positive linear maps D^{-1} and E^{-1} , respectively. If it holds that*

$$\Sigma^{-1} - \frac{1}{2}E > 0, \quad (31)$$

$$T^{-1} - \frac{1}{2}D > 0, \quad (32)$$

$$\|(\Sigma^{-1} - \frac{1}{2}E)^{-\frac{1}{2}}K(T^{-1} - \frac{1}{2}D)^{-\frac{1}{2}}\| < 1, \quad (33)$$

and that α_k is a nondecreasing sequence in $[0, \alpha]$ with $\alpha < 1$, and the iterates (x^k, y^k) of (30) fulfill

$$\sum_{k=1}^{\infty} \alpha_k \|(x^k, y^k) - (x^{k-1}, y^{k-1})\|_M^2 < \infty$$

then (x^k, y^k) converges weakly to a saddle point of (19). Furthermore, convergence is assured if there exists an $\varepsilon > 0$ such that for all α_k it holds that

$$\begin{aligned} (1 - 3\alpha_k - \varepsilon)\Sigma^{-1} &\geq \frac{(1-\alpha_k)^2}{2}E, \\ (1 - 3\alpha_k - \varepsilon)T^{-1} &\geq \frac{(1-\alpha_k)^2}{2}D, \\ \left\| \left((1 - 3\alpha_k - \varepsilon)\Sigma^{-1} - \frac{(1-\alpha_k)^2}{2}E \right)^{-\frac{1}{2}} K \right. \\ &\quad \left. \left((1 - 3\alpha_k - \varepsilon)T^{-1} - \frac{(1-\alpha_k)^2}{2}D \right)^{-\frac{1}{2}} \right\| \leq \frac{1}{(1-3\alpha_k-\varepsilon)}. \end{aligned} \quad (34)$$

Proof We start by setting

$$C = \begin{bmatrix} D & 0 \\ 0 & E \end{bmatrix}.$$

and by Theorem 1 we only need to check if $S = M - \frac{1}{2}C$ is positive. Obviously, the diagonal blocks of S are positive, by (31) and (32).

Now we use Lemma 8 to see that (31), (32) and (33) imply that S is positive definite. For the second claim, we employ Theorem 2 and only need to show that $R = (1 - 3\alpha_k)M - \frac{(1-\alpha_k)^2}{2}C \geq \epsilon M$ which is equivalent to showing that

$$(1 - 3\alpha_k - \epsilon) \begin{bmatrix} T^{-1} - K^* \\ -K \quad \Sigma^{-1} \end{bmatrix} - \frac{(1-\alpha_k)^2}{2} \begin{bmatrix} D & 0 \\ 0 & E \end{bmatrix} \geq 0.$$

Again using Lemma 8 we obtain that (34) ensures this. \square

3.3 Diagonal Preconditioning

In this section, we show how we can choose pointwise step sizes for both the primal and the dual variables that will ensure the convergence of the algorithm. The next result is an adaption of the preconditioner proposed in [42].

Lemma 10 *Assume that ∇Q and ∇P^* are co-coercive with respect to diagonal matrices D^{-1} and E^{-1} , where $D = \text{diag}(d_1, \dots, d_n)$ and $E = \text{diag}(e_1, \dots, e_n)$. Fix $\gamma, \delta \in (0, 2)$, $r > 0$, $s \in [0, 2]$ and let $T = \text{diag}(\tau_1, \dots, \tau_n)$ and $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_m)$ with*

$$\tau_j = \frac{1}{\frac{d_j}{\gamma} + r \sum_{i=1}^m |K_{i,j}|^{2-s}}, \quad \sigma_i = \frac{1}{\frac{e_i}{\delta} + \frac{1}{r} \sum_{j=1}^n |K_{i,j}|^s} \quad (35)$$

then it holds that

$$\Sigma^{-1} - \frac{1}{2}E > 0, \quad T^{-1} - \frac{1}{2}D > 0, \quad (36)$$

$$\|(\Sigma^{-1} - \frac{1}{2}E)^{-\frac{1}{2}} K (T^{-1} - \frac{1}{2}D)^{-\frac{1}{2}}\| \leq 1. \quad (37)$$

Furthermore, equation (34) is fulfilled if (25) is fulfilled.

Proof The first two conditions follow from the fact that for diagonal matrices, the (36) can be written pointwise. By the definition of τ_j , and σ_i it follows that for any $s \in [0, 2]$ and using the convention that $0^0 = 0$,

$$\frac{1}{\tau_i} - \frac{d_i}{2} > \frac{1}{\tau_i} - \frac{d_i}{\gamma} = r \sum_{i=1}^m |K_{i,j}|^{2-s} \geq 0,$$

and

$$\frac{1}{\sigma_i} - \frac{e_i}{2} > \frac{1}{\sigma_i} - \frac{e_i}{\delta} = \frac{1}{r} \sum_{j=1}^n |K_{i,j}|^s \geq 0.$$

For the third condition, we have that for any $s \in [0, 2]$

$$\begin{aligned} & \|(\Sigma^{-1} - \frac{1}{2}E)^{-\frac{1}{2}} K (T^{-1} - \frac{1}{2}D)^{-\frac{1}{2}} x\|^2 \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n \frac{1}{\sqrt{\frac{1}{\sigma_i} - \frac{e_i}{2}}} K_{i,j} \frac{1}{\sqrt{\frac{1}{\tau_j} - \frac{d_j}{2}}} x_j \right)^2 \\ &= \sum_{i=1}^m \frac{1}{\frac{1}{\sigma_i} - \frac{e_i}{2}} \left(\sum_{j=1}^n K_{i,j} \frac{1}{\sqrt{\frac{1}{\tau_j} - \frac{d_j}{2}}} x_j \right)^2 \\ &< \sum_{i=1}^m \frac{1}{\frac{1}{\sigma_i} - \frac{e_i}{\delta}} \left(\sum_{j=1}^n |K_{i,j}|^{\frac{s}{2}} |K_{i,j}|^{1-\frac{s}{2}} \frac{1}{\sqrt{\frac{1}{\tau_j} - \frac{d_j}{\gamma}}} x_j \right)^2 \\ &\leq \sum_{i=1}^m \frac{1}{\frac{1}{\sigma_i} - \frac{e_i}{\delta}} \left(\sum_{j=1}^n |K_{i,j}|^s \right) \left(\sum_{j=1}^n |K_{i,j}|^{2-s} \frac{1}{\frac{1}{\tau_j} - \frac{d_j}{\gamma}} x_j^2 \right), \end{aligned} \quad (38)$$

where the second line follows from $K_{i,j} \leq |K_{i,j}|$ and $\gamma, \delta < 2$ and the last line follows from the Cauchy-Schwarz inequality. By definition of σ_i and τ_j , and introducing $r > 0$, the above estimate can be simplified to

$$\begin{aligned} & \sum_{i=1}^m \frac{1/r}{\frac{1}{\sigma_i} - \frac{e_i}{\delta}} \left(\sum_{j=1}^n |K_{i,j}|^s \right) \left(\sum_{j=1}^n |K_{i,j}|^{2-s} \frac{r}{\frac{1}{\tau_j} - \frac{d_j}{\gamma}} x_j^2 \right) \\ &= \sum_{i=1}^m \sum_{j=1}^n |K_{i,j}|^{2-s} \frac{r}{\frac{1}{\tau_j} - \frac{d_j}{\gamma}} x_j^2 \\ &= \sum_{j=1}^n \left(\sum_{i=1}^m |K_{i,j}|^{2-s} \right) \frac{r}{\frac{1}{\tau_j} - \frac{d_j}{\gamma}} x_j^2 = \|x\|^2. \end{aligned} \quad (39)$$

Using the above estimate in the definition of the operator norm, we obtain the desired result

$$\begin{aligned} & \|(\Sigma^{-1} - \frac{1}{2}E)^{-\frac{1}{2}} K (T^{-1} - \frac{1}{2}D)^{-\frac{1}{2}}\|^2 \\ &= \sup_{x \neq 0} \frac{\|(\Sigma^{-1} - \frac{1}{2}E)^{-\frac{1}{2}} K (T^{-1} - \frac{1}{2}D)^{-\frac{1}{2}} x\|^2}{\|x\|^2} \leq 1. \end{aligned} \quad (40)$$

If we now assume that (25) is fulfilled, we especially obtain that

$$\frac{1 - 3\alpha_k - \epsilon}{\delta} \geq \frac{(1 - \alpha_k)^2}{2} \quad \text{and} \quad \frac{1 - 3\alpha_k - \epsilon}{\gamma} \geq \frac{(1 - \alpha_k)^2}{2}$$

and consequently, by using the definition of τ_j and σ_i from (35), that

$$\begin{aligned} & \frac{1/r}{\frac{1-3\alpha_k-\epsilon}{\sigma_i} - \frac{(1-\alpha_k)^2}{2}} \sum_j |K_{ij}|^s \leq \frac{1}{1-3\alpha_k-\epsilon} \\ \text{and} \quad & \frac{r}{\frac{1-3\alpha_k-\epsilon}{\tau_i} - \frac{(1-\alpha_k)^2}{2}} \sum_i |K_{ij}|^{2-s} \leq \frac{1}{1-3\alpha_k-\epsilon}. \end{aligned}$$

Now one can use the same arguments as in inequalities (38) and (39) to derive that (34) is fulfilled. \square

4 Numerical experiments

In this section, we provide several numerical results based on simple convex image processing problems to investigate the numerical properties of the proposed algorithm.

4.1 TV- ℓ_2 denoising

Let us investigate the well-known total variation denoising model:

$$\min_u \|\nabla u\|_{2,1} + \frac{\lambda}{2} \|u - f\|_2^2, \quad (41)$$

where $f \in \mathbb{R}^{MN}$ is a given noisy image of size $M \times N$ pixels, $u \in \mathbb{R}^{MN}$ is the restored image, $\nabla \in \mathbb{R}^{2MN \times MN}$ is a sparse matrix implementing the discretized image gradient based on simple forward differences. The operator norm of ∇ is computed as $\sqrt{8}$. The parameter $\lambda > 0$ is used to control the trade-off between smoothness and data fidelity. For more information we refer to [10]. Figure 2 shows an exemplary denoising result, where we used the noisy image on the left hand side as input image and set $\lambda = 10$.

The dual problem associated to (41) is given by the following optimization problem

$$\min_p \frac{1}{2} \|\lambda f - \nabla^T p\|_2^2 + I_P(p), \quad (42)$$

where $p \in \mathbb{R}^{2MN}$ is the dual variable and I_P is the indicator function for the set $P = \{p \in \mathbb{R}^{2MN} : \|p\|_{2,\infty} \leq 1\}$. This problem can easily be cast into the problem class (19), by setting $Q(p) = \frac{1}{2} \|\lambda f - \nabla^T p\|_2^2$, $G = I_P(p)$, and $K = F^* = P^* = 0$.

In our first experiment of this section, we study the behavior of the error $e_k = \alpha_k \|x_k - x_{k-1}\|_2^2$, which plays a central role in showing convergence of the algorithm. Figure 3 shows the convergence of the sequence $\{e_k\}$ generated by the FISTA algorithm by additionally using (27) for different choices of the constant c . The left figure depicts a case where c is not chosen large enough and hence the safeguard shrinks the extrapolation factor α_k such that the error e_k still converges with rate $1/k^2$. The right figure shows a case where c is chosen sufficiently large and hence the safeguard does not apply. In this case, the algorithm produces the same sequence of iterates as the original FISTA algorithm. From our numerical results, it seems that the asymptotic convergence of e_k is actually faster than $1/k^2$ which suggests that the iterates of FISTA are indeed convergent.

In the second experiment we consider a saddle-point formulation of (41)

$$\min_u \max_p \langle \nabla u, p \rangle + \frac{\lambda}{2} \|u - f\|_2^2 - I_P(p), \quad (43)$$

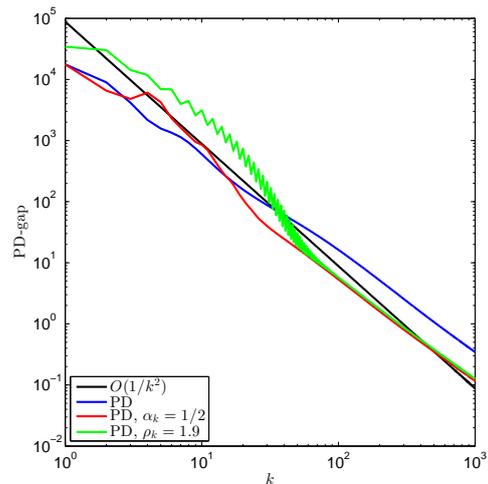


Fig. 5 Comparison between inertial forces and overrelaxation. Both techniques show similar performance improvement but overrelaxation appears numerically less stable.

Casting this problem in the general form (19), the most simple choice is $K = \nabla$, $F^*(p) = I_P(p)$, $G(u) = \|u - f\|_2^2$, $Q = P^* = 0$. Hence, algorithm 21 reduces to an inertial variant of the primal-dual algorithm of [10]. According to (22) the step sizes τ and σ need to satisfy $\tau\sigma < 1/\|K\|^2$, but the ratio τ/σ can be chosen arbitrarily.

Figure 4 shows the convergence of the primal dual gap for different choices for α_k and the ratio τ/σ . In general, one can see that the convergence becomes faster for larger values of α_k . According to (24), we can guarantee convergence for $\alpha_k < 1/3$ but we cannot guarantee convergence for larger values of α_k . In fact, it turns out that the feasible range of α_k depends on the ratio τ/σ . For $\tau/\sigma = 0.1$, fastest convergence is obtained by choosing α_k dynamically as $\alpha_k = (k-1)/(k+2) \rightarrow 1$. In this case, the primal-dual shows a very similar performance to the FISTA algorithm. For $\tau/\sigma = 0.01$, the algorithm converges for up to $\alpha_k = 1/2$, but diverges for the dynamic choice. This behavior can be explained by the fact that the ratio τ/σ directly influences the M -metric (20) which in turn leads to a divergence of the error term $\sum_{k=1}^{\infty} \alpha_k \|x^k - x^{k-1}\|_M^2$.

Next, we provide an experiment, where we compare the effect of the inertial force with the effect of overrelaxation that has already been considered in [16, 49]. Figure 5 compares the primal-dual gap of the plain primal-dual (i.e. $\alpha_k = 0$) algorithm [10] with the performance of its variants using either inertial forces using $\alpha_k = 1/2$ or overrelaxation (see (28)) using $\rho_k = 1.9$. For all methods we used $\tau/\sigma = 0.01$. Both variants improve the convergence of the plain primal-dual algorithm but we observed that overrelaxation leads to some numerical oscillations, in particular for values of ρ_k close to 2.

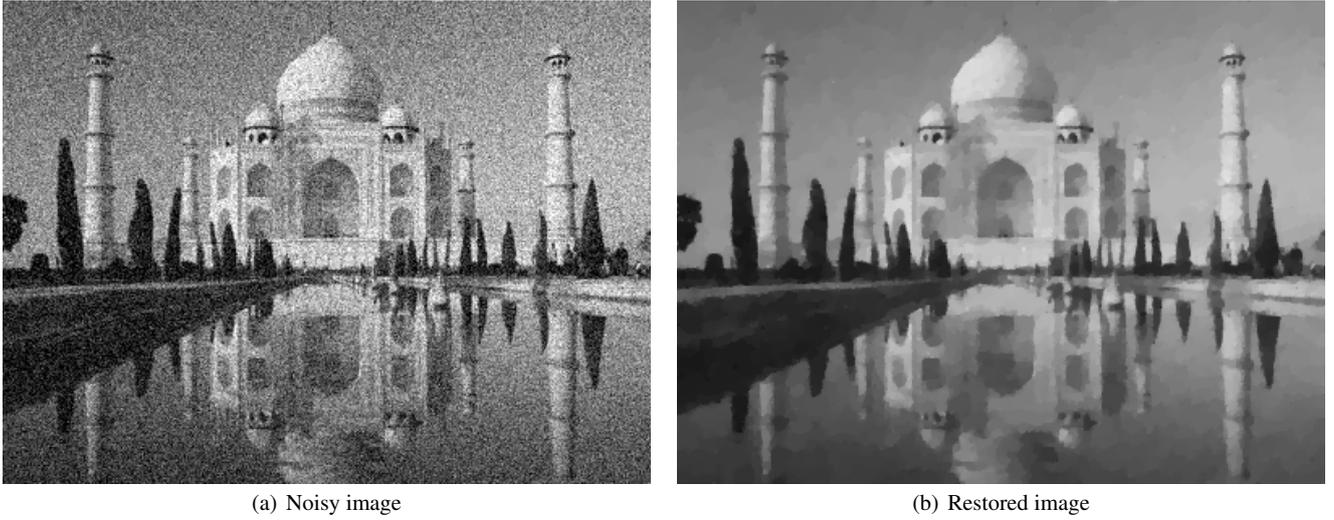


Fig. 2 Application to total variation based image denoising with ℓ_2 fitting term. (a) shows the noisy input image containing Gaussian noise with a standard deviation of $\sigma = 0.1$, (b) shows the restored image using $\lambda = 10$.

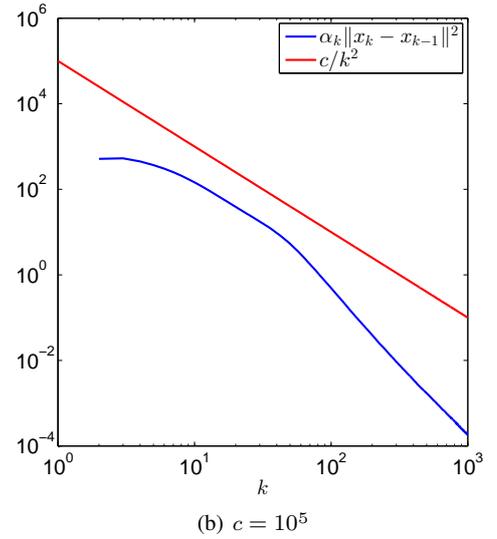
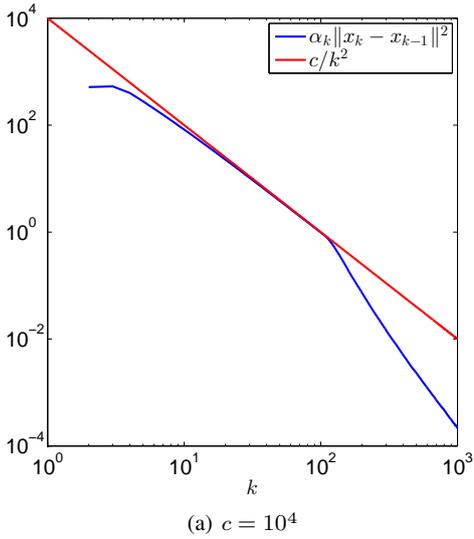


Fig. 3 Convergence of the error sequence in the FISTA algorithm.

4.2 TV- ℓ_2 deconvolution

Our next example incorporates an additional linear operator in the data fidelity. The problem is given by

$$\min_u \|\nabla u\|_{2,1} + \frac{\lambda}{2} \|Hu - f\|_2^2, \quad (44)$$

where $H \in \mathbb{R}^{MN \times MN}$ is a linear operator, for example H can be such that Hu is equivalent to the 2D convolution $h * u$, where h is a convolution kernel. We again consider a saddle-point formulation

$$\min_u \max_p \langle \nabla u, p \rangle + \frac{\lambda}{2} \|Hu - f\|_2^2 - I_P(p). \quad (45)$$

Casting this problem into the general class of problems (19), one has different possibilities. If we would choose $K = \nabla$, $F^*(p) = I_P(p)$, $G(u) = \frac{\lambda}{2} \|Hu - f\|_2^2$, $Q, P^* = 0$, we would have to compute the proximal map with respect to G in each iteration of the algorithm, which can be computationally very expensive. Instead, if we choose $G = 0$, $Q(u) = \frac{\lambda}{2} \|Hu - f\|_2^2$, we only need to compute $\nabla Q(u) = \lambda H^T (Hu - f)$ which is obviously much cheaper. We call this variant the *explicit* variant. Alternatively, we can additionally dualize the data term, which leads to the extended

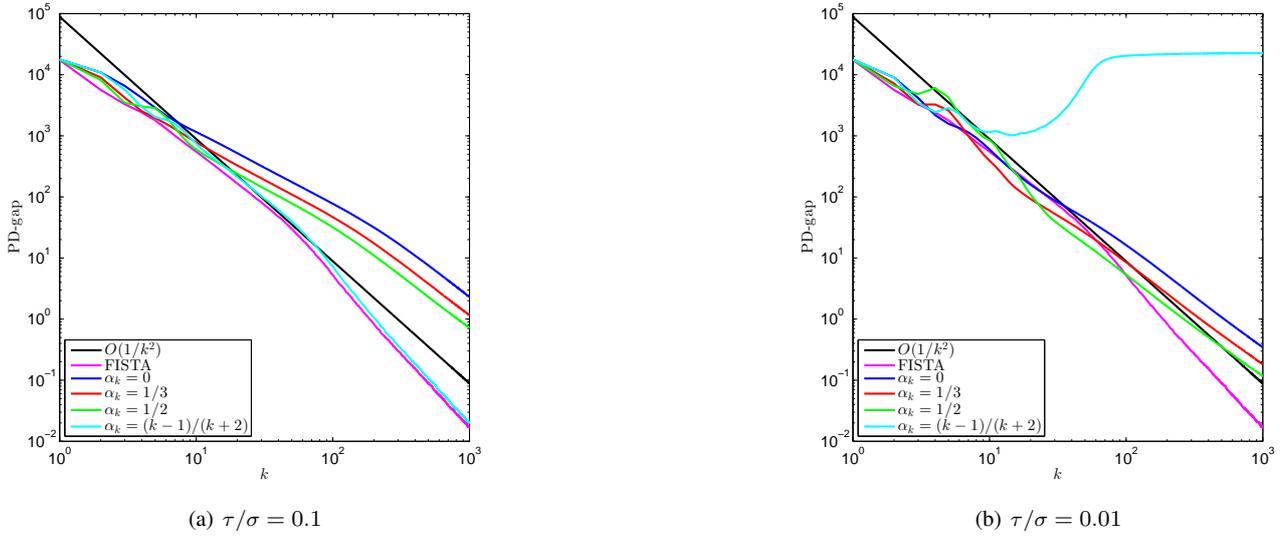


Fig. 4 Convergence of the inertial primal-dual forward-backward algorithm (21) for different choices of τ/σ and α_k .



Fig. 6 Application to total variation based image deconvolution with ℓ_2 fitting term. (a) shows the noisy ($\sigma = 0.01$) and blurry input image together with the known point spread function, and (b) shows the restored image using $\lambda = 1000$.

saddle-point problem

$$\min_u \max_{p,q} \langle \nabla u, p \rangle + \langle Hu, q \rangle + \langle f, q \rangle - \frac{1}{2\lambda} \|q\|_2^2 - I_P(p).$$

where $q \in \mathbb{R}^{MN}$ is the new dual variable vector. Casting now this problem into (19), we identify $K = \begin{pmatrix} \nabla \\ H \end{pmatrix}$, $G = 0$, $Q = 0$, $F^*(p, q) = I_P(p) + \frac{1}{2\lambda} \|q\|_2^2 - \langle f, q \rangle$, which eventually leads to proximal maps that are easy to compute. We call this variant the *split-dual* variant.

Figure 7 shows a comparison of the convergence between the explicit and the split-dual variants with and with-

out inertial forces. For the explicit variant, the maximal value of the inertial force was computed using formula (25), where we set $L_K = \sqrt{8}$, $L_Q = \lambda$, $\gamma = 1$ and $r = 100$. This results in a theoretically maximal value of $\alpha_k = 0.236$ but the algorithm also converges for $\alpha_k = 1/3$ (see Remark 4). For the split-dual variant, the formulation does not involve any explicit terms and hence the maximal feasible value for α_k is $1/3$. The primal and dual step sizes were computed according to the preconditioning rules (35) (skipping the explicit terms), where we again used $r = 100$.

The figure shows the primal energy gap, where the optimal primal energy value has been computed by running the

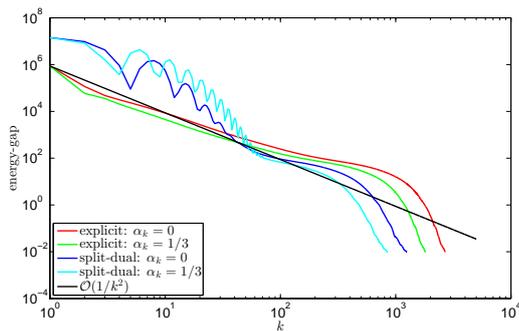


Fig. 7 Convergence of the primal dual algorithms with and without inertial forces.

explicit variant for 10000 iterations. The algorithms were stopped, after the primal energy-gap was below a threshold of 10^{-2} . From the figure, one can see that for both variants, the inertial force leads to a faster convergence. One can also see that in the early stage of the iterations, the explicit variant seems to converge faster than the split-dual variant. Finally, we point out that the asymptotic convergence of both variants is considerably faster than $\mathcal{O}(1/k^2)$.

5 Conclusion

In this paper we considered an inertial forward-backward algorithm for solving monotone inclusions given by the sum of a monotone operator with an easy-to-compute resolvent operator and another monotone operator which is co-coercive. We have proven convergence of the algorithm in a general Hilbert space setting. It turns out that the proposed algorithm generalizes several recently proposed algorithms for example the FISTA algorithm of Beck and Teboulle [4] and the primal-dual algorithm of Chambolle and Pock [10]. This gives rise to new inertial primal-dual algorithms for convex-concave programming. In several numerical experiments we demonstrated that the inertial term leads to faster convergence while keeping the complexity of each iteration basically unchanged.

Future work will mainly concentrate on trying to find worst-case convergence rates for particular problem classes.

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