# Labelling Algorithms for Paired-domination Problems in Block and Interval Graphs * 

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#### Abstract

Let $G=(V, E)$ be a graph without isolated vertices. A set $S \subseteq V$ is a paired-domination set if every vertex in $V-S$ is adjacent to a vertex in $S$ and the subgraph induced by $S$ contains a perfect matching. The paired-domination problem is to determine the paired-domination number, which is the minimum cardinality of a paired-dominating set. Motivated by a mistaken algorithm given by Chen, Kang and Ng [ Paired domination on interval and circular-arc graphs, Disc. Appl. Math. 155(2007),2077-2086], we present two linear time algorithms to find a minimum cardinality paired-dominating set in block and interval graphs. In addition, we prove that paired-domination problem is $N P$-complete for bipartite graphs, chordal graphs, even split graphs.


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## 1 Introduction

Domination and its variations in graphs have been extensively studied, cf. [1, 2]. A set of vertices $S$ is a dominating set for a graph $G=(V, E)$ if every vertex in $V-S$ is adjacent to a vertex in $S$. The domination problem is to determine the domination number $\gamma(G)$, which is the minimum cardinality of a dominating set for $G$.

Let $G=(V, E)$ be a simple graph without isolated vertices. For a vertex $v \in V$, the open neighborhood of $v$ is defined as $N(v)=\{u \in V \mid u v \in E\}$ and the closed neighborhood of $v$ is defined as $N[v]=N(v) \cup\{v\}$. The distance between $u$ and $v$, denoted by $d_{G}(u, v)$, is the minimum length of a path between $u$ and $v$. For a subset $S$ of $V$, the subgraph of $G$ induced by the vertices in $S$ is denoted by $G[S]$. A matching in a graph $G$ is a set of pairwise nonadjacent edges in $G$. A perfect matching $M$ in $G$ is a matching such that every vertex of $G$ is incident to an edge of $M$. Some other notations and terminology not introduced in here can be found in [9].

A set $S \subseteq V$ is a paired-dominating set of $G$ if $S$ is a dominating set of $G$ and the induced subgraph $G[S]$ has a perfect matching. If $e=u v \in M$, where $M$ is a perfect matching of $G[S]$, we say that $u$ and $v$ are paired in $S$. The paired-domination problem is to determine the paireddomination number $\gamma_{p}(G)$, which is the minimum cardinality of a paired-dominating set for a graph $G$. The paired-domination problem was introduced by Haynes and Slater [3]. If we think of each $s \in S \subseteq V$ as the location of a guard capable of protecting each vertex in $N[S]$, then "domination" requires every vertex to be protected, and for paired-domination, we will require the guards' location to be selected as adjacent pairs of vertices so that each guard is assigned one other and they are designated as backups for each other.

Linear time algorithm for paired domination problem is available for trees [6] Polynomial time algorithm for paired domination problem is available for circular-arc graphs [5]. Other results on this subject can be found in [4, 8, Although a linear time algorithm for paireddomination problem on interval graphs was given in [5], it is incorrect. In this paper, we employ the labelling technique to give efficient algorithms for finding a minimum paired-dominating set in block graphs (which contains trees) and interval graphs. In section 2, we begin with presenting a linear time algorithm for paired-domination problem in block graphs and then prove the correctness of the algorithm. Our algorithm can deduce a quite simple algorithm for paired-domination problem in trees. In section 3, we first show that the algorithm in 5 for finding a minimum paired-dominating set in interval graphs is false. Then we give an intuitive algorithm for the paired-domination problem of interval graphs. In [3], authors proved that the paired domination problem is $N P$-complete for undirected graphs. In section 4 , we show that it is still $N P$-complete for bipartite graphs, chordal graphs, even split graphs.

## 2 Algorithm for paired-domination problem in Block graphs

In a graph $G=(V, E)$ with $|V|=n$ and $|E|=m$, a vertex $x$ is a cut-vertex if deleting $x$ and all edges incident to it increases the number of connected components. A block of $G$ is a maximal connected subgraph of $G$ without a cut-vertex. If $G$ itself is connected and has no cut-vertex, then $G$ is a block. The intersection of two blocks contains at most one vertex, and a vertex is a cut-vertex if and only if it is the intersection of two or more blocks. A block graph is a connected graph whose blocks are complete graphs. If every block is $K_{2}$, then it is a tree.

As we know, every block graph not isomorphic to complete graph has at least two end blocks, which are blocks with only one cut-vertex. Beginning with an end block and working recursively inward, we can find a vertex ordering $v_{1}, v_{2}, \cdots, v_{n}$ in $O(n+m)$ time such that $v_{i} v_{j} \in E$ and $v_{i} v_{k} \in E$ implies that $v_{j} v_{k} \in E$ for $i<j<k \leq n$. Note that if $B$ is an end block with cut-vertex $x$ of block graph $G$, then the vertices in $B$ is following continually and $x$ is the last vertex of $B$ in the vertex ordering $v_{1}, v_{2}, \cdots, v_{n}$.

Let $v_{1}, v_{2}, \cdots, v_{n}$ be the vertex ordering of block graph $G$ such that $v_{i} v_{j} \in E$ and $v_{i} v_{k} \in E$ implies that $v_{j} v_{k} \in E$ for $i<j<k \leq n$. We define the following notations:

1. $F\left(v_{i}\right)=v_{j}, j=\max \left\{k \mid v_{i} v_{k} \in E, i<k\right\} . v_{j}$ is called the father of $v_{i}$ and $v_{i}$ is a child of $v_{j}$. For technical reasons, we say complete graph has an end block and $v_{n}$ is a cutvertex. Obviously, $v_{j}$ must be a cut-vertex in block graphs.
2. $C\left(v_{i}\right)=\left\{v_{j} \mid F\left(v_{j}\right)=v_{i}\right\}$.
3. $P D(G)$ is a minimum paired-dominating set of $G$.
4. For any block graph $G$, we define a rooted tree $T(G)$ about $G$, whose vertex set is $V(G)$, and $u v$ is an edge of $T(G)$ if and only if $F(u)=v$. The root of $T(G)$ is $v_{n}$. Moreover let $T_{v_{i}}$ is a subtree of $T(G)$ rooted at $v_{i}$ and every vertex in $T_{v_{i}}$ except $v_{i}$ is a descendant of $v_{i}$. For a vertex $v_{i} \in V(G), D_{G}\left(v_{i}\right)$ denotes the vertex set consisting of the descendants of $v_{i}$ in $T(G)$ and $D_{G}\left[v_{i}\right]=D_{G}\left(v_{i}\right) \cup\left\{v_{i}\right\}$. That is, $D_{G}\left[v_{i}\right]=V\left(T_{v_{i}}\right)$.

In our algorithm, we will use two labels on each vertex $u$, denoted by $(D(u), L(u))$ :

$$
\begin{aligned}
& D(u)=\left\{\begin{array}{lll}
0 & \text { if } & u \text { is not dominated } ; \\
1 & \text { if } & u \text { is dominated. }
\end{array}\right. \\
& L(u)=\left\{\begin{array}{lll}
0 & \text { if } & \text { u is not put into } P D(G) ; \\
1 & \text { if } & u \text { is put into } P D(G), \text { but it has no paired vertex in } P D(G) ; \\
2 & \text { if } & u \text { is put into } P D(G), \text { and it has a paired vertex in } P D(G) .
\end{array}\right.
\end{aligned}
$$

Now, we give an algorithm to determine a minimum paired-dominating set in block graphs.
Algorithm MPDB. Find a minimum paired-dominating set of a block graph.
Input. A block graph $G=(V, E)$ with a vertex ordering $v_{1}, v_{2}, \cdots, v_{n}(n \geq 2)$ such that $v_{i} v_{j} \in E$ and $v_{i} v_{k} \in E$ implies that $v_{j} v_{k} \in E$ for $i<j<k \leq n$. Each vertex $v_{i}$ has a label $\left(D\left(v_{i}\right), L\left(v_{i}\right)\right)=(0,0) . F\left(v_{i}\right)=v_{j}$ with $j=\max \left\{k \mid v_{k} v_{i} \in E, k>i\right\}, C\left(v_{i}\right)=\left\{v_{j} \mid F\left(v_{j}\right)=v_{i}\right\}$.

Output. A minimum paired-dominating set $P D$ of $G$.

## Method.

For $i=1$ to $n$ do
If $\left(D\left(v_{i}\right)=0\right.$ and $\left.i \neq n\right)$ then
$L\left(F\left(v_{i}\right)\right)=1 ;$
$D(u)=1$ for every vertex $u \in N\left[F\left(v_{i}\right)\right] ;$
endif
If $\left(D\left(v_{i}\right)=1\right)$ then
Let $C^{\prime}\left(v_{i}\right)=\left\{w \mid w \in C\left(v_{i}\right)\right.$ and $\left.L(w)=1\right\}$;
$L(w)=2$ for every vertex $w \in C^{\prime}\left(v_{i}\right)$;
Let $C^{\prime \prime}\left(v_{i}\right)=\left\{w \mid w \in C^{\prime}\left(v_{i}\right)\right.$ and $w \in V(M), M$ is a maximum matching in $\left.G\left[C^{\prime}\left(v_{i}\right)\right]\right\}$.
If $\left(C^{\prime}\left(v_{i}\right)-C^{\prime \prime}\left(v_{i}\right) \neq \emptyset\right)$, then
$L\left(v_{i}\right)=2 ;$
$D(u)=1$ for every vertex $u \in N\left[v_{i}\right]$;
Take a vertex $w \in C^{\prime}\left(v_{i}\right)-C^{\prime \prime}\left(v_{i}\right)$, for every vertex $v \in C^{\prime}\left(v_{i}\right)-C^{\prime \prime}\left(v_{i}\right)-\{w\}$
$L\left(v^{\prime}\right)=2$ for some vertex $v^{\prime} \in C(v)$ such that $L\left(v^{\prime}\right)=0$;
endif
endif
endfor
If $\left(D\left(v_{n}\right)=0\right.$ or $\left.L\left(v_{n}\right)=1\right)$ then
$L\left(v_{n}\right)=2 ;$
$L(w)=2$ for some vertex $w \in C\left(v_{n}\right)$ such that $L(w)=0 ;$
$D\left(v_{n}\right)=1 ;$
endif
Output $P D=\{v \mid L(v)=2\}$
end

Next we will verify the validity of the algorithm MPDB. For a block graph $G$ of order $n \geq 2$. when the algorithm MPDB terminates, any vertex $u \in V(G)$ has a label $D(u)=1$ and any vertex $v \in P D$ has a label $L(v)=2$. Hence, the output $P D$ is a paired-dominating set of $G$. It suffices to prove that $P D$ is a minimum paired-dominating set of $G$.
$S_{i}$ is the set of vertices defined by $v \in S_{i}$ if and only if $v$ has the label $(1,2)$ when $v_{i}$ is the considering vertex in the loop of the algorithm MPDB for $i=1,2, \cdots, n$. In particular, we define $S_{n+1}$ as the set of vertices with label $(1,2)$ after $v_{n}$ is considered in the loop of the algorithm MPDB. In order to prove that the output $P D$ is a minimum paired-dominating set of $G$, we proceed by induction on $i$ and show that, when $v_{i}(1 \leq i \leq n)$ is the considering vertex in the loop of the algorithm MPDB, there is a minimum paired-dominating set $S$ in $G$ such that $S_{i} \subseteq S$. Obviously, $S_{1}=\emptyset$. This is certainly true for $i=1$. Assume that there is a minimum
paired-dominating set $S$ in $G$ such that $S_{i} \subseteq S(1 \leq i \leq n)$. We show that $S_{i+1}$ holds for a given $i+1$ by the following lemmas.

Lemma 1 Let $B$ be an end block of a block graph $G$. If there is a vertex $u \in V(B)$ such that $u$ is not a cut-vertex with $D(u)=0$ and $F(u)=v(v \neq u)$, then $L(v)=1$ and $D(w)=1$ for every vertex $w \in N[v]$. Moreover $P D(G)=P D\left(G^{\prime}\right)$, where $G^{\prime}=G-\left(D_{G}(v)-\{u\}\right)$.

Proof $G^{\prime}$ has at least two vertices, hence $P D\left(G^{\prime}\right)$ is also a paired-dominating set of $G$. Then $|P D(G)| \leq\left|P D\left(G^{\prime}\right)\right|$.

Now we show that $\left|P D\left(G^{\prime}\right)\right| \leq|P D(G)|$. If $v \in P D(G)$ and $v$ is paired with $x \in B$, then $(P D(G)-\{x\}) \cup\{u\}$ is also a paired-domination of $G^{\prime}$. If $v \in P D(G)$ and $v$ is paired with $x \notin B$, it is clear that $P D(G)$ is a paired-domination of $G^{\prime}$. Assume that $v \notin P D(G)$, then there are two vertices $x, y \in B$ such that $x, y \in P D(G)$. Hence, $(P D(G)-\{x, y\}) \cup\{u, v\}$ is also a paired-domination of $G^{\prime}$.
$v$ must be in $P D\left(G^{\prime}\right)$ implies that $v$ is also in $P D(G)$. Note that the paired vertex with $v$ may be in $B$ or not. So we set $L(v)=1$ and $D(w)=1$ for every vertex $w \in N[v]$.

From Lemman, when we consider a vertex $v_{i}$ in an end block whose label is $(0,0)$ and $i \neq n$, then we will put its father $F\left(v_{i}\right)$ into $P D$. Since its paired vertex can not be determined at this time, we temporarily label $L\left(F\left(v_{i}\right)\right)=1$. Lemma 1 implies that we can consider a possible smaller block graph $G^{\prime}$ since $P D(G)=P D\left(G^{\prime}\right)$, where $G^{\prime}=G-\left(D_{G}(v)-\{u\}\right)$.

When we consider a vertex $v_{i}$ such that $D\left(v_{i}\right)=1$, we will take its child set to see if there is a child must be paired with $v_{i}$. After that we will also consider a possible smaller block graph.

Lemma 2 Let $G$ be a block graph and $v_{i}$ be a considering vertex with $D\left(v_{i}\right)=1$ in some step of the loop. Set $C^{\prime}\left(v_{i}\right)=\left\{v_{j} \mid v_{j} \in C\left(v_{i}\right)\right.$ and $\left.L\left(v_{j}\right)=1\right\}, C^{\prime \prime}\left(v_{i}\right)=\left\{w \mid w \in C^{\prime}\left(v_{i}\right)\right.$ and $w \in V(M)$, $M$ is a maximum matching in $\left.G\left[C^{\prime}\left(v_{i}\right)\right]\right\}$ and $S_{i}\left(v_{j}\right)=S_{i} \cap D_{G}\left(v_{j}\right)$ for $j \leq i$. Then, $L(w)=2$ for every vertex $w \in C^{\prime}\left(v_{i}\right)$.
(1) If $L\left(v_{i}\right)=0$ and $C^{\prime}\left(v_{i}\right)-C^{\prime \prime}\left(v_{i}\right)=\emptyset$, then $P D(G)=P D\left(G^{\prime}\right) \cup S_{i+1}\left(v_{i}\right)$, where $G^{\prime}=$ $G-D_{G}\left[v_{i}\right] \neq K_{1}$ or $G^{\prime}=G-D_{G}\left(v_{i}\right)=K_{2}$.
(2) If $L\left(v_{i}\right)=1$ and $C^{\prime}\left(v_{i}\right)-C^{\prime \prime}\left(v_{i}\right)=\emptyset$, then $P D(G)=P D\left(G^{\prime}\right) \cup S_{i+1}\left(v_{i}\right)$, where $G^{\prime}=$ $G-\left(D_{G}\left(v_{i}\right)-\{x\}\right)$ and $x$ is a vertex whose father is $v_{i}$ and $L(x)=0$.
(3) If $C^{\prime}\left(v_{i}\right)-C^{\prime \prime}\left(v_{i}\right) \neq \emptyset$, then $L\left(v_{i}\right)=2$ and $D(u)=1$ for every $u \in N\left[v_{i}\right]$, and for every vertex $v \in C^{\prime}\left(v_{i}\right)-C^{\prime \prime}\left(v_{i}\right)$ except one vertex $w, L\left(v^{\prime}\right)=2$ for some vertex $v^{\prime} \in C(v)$ such that $L\left(v^{\prime}\right)=0$. Moreover, $P D(G)=P D\left(G^{\prime}\right) \cup\left(S_{i+1}\left(v_{i}\right)-S_{i+1}(w)-\{w\}\right)$, where $G^{\prime}=$ $G-\left(D_{G}\left(v_{i}\right)-D_{G}[w]\right)$.

Proof $L(w)=1$ for $w \in C^{\prime}\left(v_{i}\right)$ implies that $w$ must be put into $P D$, so $L(w)=2$ for every vertex $w \in C^{\prime}\left(v_{i}\right)$ and we will determine their paired vertices in this step.
(1) If $v_{i}$ is not a cut-vertex in graph $G$, then $S_{i+1}=S_{i}$ and we do nothing. Suppose that $v_{i}$ is a cut-vertex. Obviously, $S_{i+1}\left(v_{i}\right)=S_{i}\left(v_{i}\right) \cup C^{\prime}\left(v_{i}\right)$ and $P D\left(G^{\prime}\right) \cup S_{i+1}\left(v_{i}\right)$ is a paireddominating set of $G$, where $G^{\prime}=G-D_{G}\left[v_{i}\right] \neq K_{2}$ or $G^{\prime}=G-D_{G}\left(v_{i}\right)=K_{2}$. Now we will show that $P D\left(G^{\prime}\right) \cup S_{i+1}\left(v_{i}\right)$ is a minimum paired-dominating set of $G$.

By the hypothesis on induction, there exists a minimum paired-dominating set $S$ of $G$ with $S_{i}\left(v_{i}\right) \subseteq S$. Since the label of every vertex in $C^{\prime}\left(v_{i}\right)$ is $(1,1)$, it must be in a minimum paireddominating set of $G$. When $v_{i}$ is the considering vertex, these vertices in $S_{i}$ have no influence on the label of remained vertices in possible smaller graph. We can assume that $S$ is a minimum paired-dominating set with $S_{i}\left(v_{i}\right) \cup C^{\prime}\left(v_{i}\right) \subseteq S$. We first claim that every vertex in $C^{\prime}\left(v_{i}\right)$ is paired with another vertex in $C^{\prime}\left(v_{i}\right)$. Let $C_{p}^{\prime}=\left\{x \mid x \notin C^{\prime}\left(v_{i}\right)\right.$ and $x$ is paired with a vertex $y$ in $S$ such that $\left.y \in C^{\prime}\left(v_{i}\right)\right\}$. Since vertices in $S_{i}$ have been paired each other, $C_{p}^{\prime} \cap S_{i}=\emptyset$. If $v_{i} \notin S$, since vertices in $C^{\prime}\left(v_{i}\right)$ can be paired each other, $S-C_{p}^{\prime}$ is a smaller paired-dominating set of $G$, a contradiction. Then, $v_{i} \in C_{p}^{\prime}$ and we assume that $v_{i}$ is paired with $x \in C^{\prime}\left(v_{i}\right)$. Note that $\left|C_{p}^{\prime}\right| \geq 2$. Let $z \in C_{p}^{\prime}$ be paired with $y \in C^{\prime}\left(v_{i}\right)$ and $x y \in E$. Obviously, $z \in D_{G}\left(v_{i}\right)$. If $t \in S$ for every $t \in N\left(v_{i}\right)$, then $x$ can be paired with $y$ and $S-\left\{v_{i}, z\right\}$ is a smaller paired-dominating set of $G$, a contradiction. If there exists some vertex $t \in N\left(v_{i}\right)$ and $t \notin S$, then $S \cup\{t\}-\{z\}$ is also a paired-dominating set of $G$. Anyway, we get that these vertices in $C^{\prime}\left(v_{i}\right)$ are paired each other in $S$.

If $G^{\prime}=G-D_{G}\left(v_{i}\right)=K_{2}$, it is easy to know that $S-S_{i+1}\left(v_{i}\right)$ is a paired-dominating set of $G^{\prime}=K_{2}$. Hence, we know that $P D\left(G^{\prime}\right) \cup S_{i+1}\left(v_{i}\right)=P D(G)$, where $G^{\prime}=G-D_{G}\left(v_{i}\right)=K_{2}$.

Suppose that $G^{\prime}=G-D_{G}\left[v_{i}\right] \neq K_{1}$. We will show that $S-S_{i+1}\left(v_{i}\right)$ is a paired-dominating set of $G^{\prime}$. Hence, $P D\left(G^{\prime}\right) \cup S_{i+1}\left(v_{i}\right)=P D(G)$, where $G^{\prime}=G-D_{G}\left[v_{i}\right] \neq K_{1}$.

If $v_{i} \notin S$, then it is obvious that $S-S_{i+1}\left(v_{i}\right)$ is a paired-dominating set of $G^{\prime}$. Assume that $v_{i} \in S$ and its paired vertex in $S$ is $x$. If the father of $v_{i}$, denoted by $F\left(v_{i}\right)$, is in $S$, and its paired vertex is not $v_{i}$ (that is $\left.x \neq F\left(v_{i}\right)\right), S-\left\{x, v_{i}\right\}$ is a smaller paired domination set of $G$, a contradiction. If $F\left(v_{i}\right)$ is not in $S, S-\{x\} \cup\left\{F\left(v_{i}\right)\right\}$ is also a paired-dominating set of $G$. Without loss of generality, we assume that $v_{i}$ is paired with $F\left(v_{i}\right)$ in $S$. If $N\left(F\left(v_{i}\right)\right) \subseteq S$, then $S-\left\{v_{i}, F\left(v_{i}\right)\right\}$ is a smaller paired-dominating set of $G$, a contradiction. Assume that $z \in N\left(F\left(v_{i}\right)\right)$ is not in $S . S-\left\{v_{i}\right\} \cup\{z\}$ is also a paired-dominating set of $G$. Hence, we can find a minimum paired domination set $S$ of $G$ such that $S_{i+1}\left(v_{i}\right) \subseteq S$ and $v_{i} \notin S$. Note that $D\left(v_{i}\right)=1$. i.e., $v_{i}$ has been dominated by some vertex in $S_{i+1}\left(v_{i}\right)$. Hence, $S-S_{i+1}\left(v_{i}\right)$ is a paired-dominating set of $G^{\prime}$, where $G^{\prime}=G-D_{G}\left[v_{i}\right] \neq K_{1}$.
(2) When $v_{i}$ is the considering vertex in the loop of the algorithm MPDB, $v_{i}$ has been labelled by $(1,1)$. There exists a vertex $x \in C\left(v_{i}\right)$ with label $(1,0)$ dominated by $v_{i}$. In this situation, $v_{i}$ must be put into $P D$ after $F\left(v_{i}\right)$ is considered in the algorithm MPDB, and the paired vertex of $v_{i}$ may be a vertex in $C\left(v_{i}\right)$. Similarly, we can prove that those vertices in $C^{\prime}\left(v_{i}\right)$ are paired each other, hence, the paired vertex of $v_{i}$ may be a vertex in $C\left(v_{i}\right)$ and its label is $(1,0)$. Thus, let $G^{\prime}=G-\left(D_{G}\left(v_{i}\right)-\{x\}\right)$, where $x$ is a vertex whose father is $v_{i}$ and $L(x)=0$. Using the
same argument, we get that $P D(G)=P D\left(G^{\prime}\right) \cup S_{i+1}\left(v_{i}\right)$. The detail is left to readers.
(3) Since $G\left[C^{\prime \prime}\left(v_{i}\right)\right]$ has a perfect matching and it is also a maximum matching in $G\left[C^{\prime}\left(v_{i}\right)\right]$, the set $C^{\prime}\left(v_{i}\right)-C^{\prime \prime}\left(v_{i}\right)$ is an independent set. For every vertex $v \in C^{\prime}\left(v_{i}\right)-C^{\prime \prime}\left(v_{i}\right)$ except one $w$, let $v^{\prime} \in C(v)$ with $L\left(v^{\prime}\right)=0$. (It always can be found since the label of $v$ is $(1,1)$ ) and the set of these vertices is denoted by $C C$, then $|C C|=\left|C^{\prime}\left(v_{i}\right)\right|-\left|C^{\prime \prime}\left(v_{i}\right)\right|-1$. After $v_{i}$ was considered in the loop of the algorithm, every vertex in $C^{\prime}\left(v_{i}\right) \cup C C \cup\left\{v_{i}\right\}$ has the label (1,2). Hence, $S_{i+1}=S_{i} \cup C^{\prime}\left(v_{i}\right) \cup C C \cup\left\{v_{i}\right\}$.

Let $G^{\prime}=G-\left(D_{G}\left(v_{i}\right)-D_{G}\left[w^{\prime}\right]\right)$. It is easy to check that $P D\left(G^{\prime}\right) \cup\left(S_{i+1}\left(v_{i}\right)-\{w\}\right)$ is a paired-dominating set of $G$. Next we will show that $P D\left(G^{\prime}\right) \cup\left(S_{i+1}\left(v_{i}\right)-\{w\}\right)$ is a minimum paired-dominating set of $G$. We can similarly find a minimum paired-dominating set $S$ of $G$ such that $S_{i} \cup C^{\prime}\left(v_{i}\right) \subseteq S$.

Let $C_{p}^{\prime}=\left\{x \mid x \in S, x \notin C^{\prime}\left(v_{i}\right)\right.$ and its paired vertex is in $\left.C^{\prime}\left(v_{i}\right)\right\}$. Then $\left|C_{p}^{\prime}\right| \geq\left|C^{\prime}\left(v_{i}\right)\right|-$ $\left|C^{\prime \prime}\left(v_{i}\right)\right|$. If $v_{i} \notin S$, then the set $S-C_{p}^{\prime} \cup C C \cup\left\{v_{i}\right\}$ is a minimum paired-dominating set of $G$. If $v_{i} \in S$ and $v_{i} \in C_{p}^{\prime}$, then $S-\left(C_{p}^{\prime}-\left\{v_{i}\right\}\right) \cup C C$ is also a minimum paired-dominating set of $G$. If $v_{i} \in S$ and $v_{i} \notin C_{p}^{\prime}$, we also get that $S-C_{p}^{\prime} \cup C C \cup\left\{w^{\prime}\right\}$ is a minimum paired-dominating set of $G$, where $w^{\prime} \in C(w)$ has the label $(1,0)$. Anyway, we can assume that these vertices in $C^{\prime \prime}\left(v_{i}\right)$ are paired in $S$ each other and $\left\{v_{i}\right\} \cup C C \subseteq S$. And for every $v \in C^{\prime}\left(v_{i}\right)-C^{\prime \prime}\left(v_{i}\right)-\{w\}$, $v$ is paired in $S$ with $v^{\prime} \in C C$.

Next we will say that $v_{i}$ and $w$ can be paired in $S$. Let $w^{\prime}\left(v^{\prime}\right.$, respectively) is the paired vertex of $w\left(v_{i}\right.$, respectively) in $S$. We assert that there is a vertex $v^{\prime \prime} \in N_{G}\left(v^{\prime}\right)$ such that $v^{\prime \prime} \notin S$, for, otherwise, $S-\left\{w^{\prime}, v^{\prime}\right\}$ is a smaller paired-dominating set of $G$. Hence, $S-\left\{w^{\prime}\right\} \cup\left\{v^{\prime \prime}\right\}$ is also a minimum paired-dominating set of $G$. Thus we can assume $w$ is paired with $v_{i}$ in $S$.

Let $S^{\prime}=S-\left(S_{i+1}\left(v_{i}\right)-S_{i+1}(w)-\{w\}\right)$. Then $S^{\prime}$ is a paired-dominating set of $G^{\prime}$, where $G^{\prime}=G-\left(D_{G}\left(v_{i}\right)-D_{G}[w]\right)$. Hence, that is $P D\left(G^{\prime}\right) \cup\left(S_{i+1}\left(v_{i}\right)-S_{i+1}(w)-\{w\}\right)$ is a minimum paired-dominating set of $G$. In this case, $P D(G)=P D\left(G^{\prime}\right) \cup\left(S_{i+1}\left(v_{i}\right)-S_{i+1}(w)-\{w\}\right)$, where $G^{\prime}=G-\left(D_{G}\left(v_{i}\right)-D_{G}\left[w^{\prime}\right]\right)$.

The discussion above implies that $w, v_{i}$ are paired in $S$, so let $L\left(v_{i}\right)=2$ and $D(u)=1$ for every vertex in $N\left[v_{i}\right]$. Since every vertex $v \in C^{\prime}\left(v_{i}\right)-C^{\prime \prime}\left(v_{i}\right)$ except one vertex $w$ is paired with $v^{\prime} \in C(v)$, which is labelled by $(1,0)$, we label $v^{\prime}$ by $(1,2)$ in this step.

Using Lemmas 1 and 2 recursively, we have that $P D(G)=P D\left(G^{\prime}\right) \cup S_{n+1}$ when $v_{n}$ has been considered in the loop of the algorithm, where $G^{\prime}$ is empty or $G^{\prime}$ is $K_{2}$. If $G^{\prime}$ is $K_{2}$, it implies that $D\left(v_{n}\right)=0$ or $L\left(v_{n}\right)=1$ after the loop of the algorithm. For the former, $v_{n}$ need to be dominated and no vertex in $N\left(v_{n}\right)$ is put in $P D$. For the latter, $v_{n}$ must be put in to $P D$ and it need a paired vertex. Hence, in the end of the algorithm MPDB, we change the labels of $v_{n}$ and $w$ to $(1,2)$, where $w \in C\left(v_{n}\right)$ with the label $(1,0)$.

Theorem 3 Given a vertex ordering, described in the beginning of this section, of a block graph $G$, the algorithm MPDB can produce a minimum paired-dominating set of $G$ in $O(m+n)$ time,
where $m=|E(G)|$ and $n=|V(G)|$.

Proof From discussion above, the algorithm MPDB can produce a minimum paired-dominating set of a block graph $G$. Furthermore, every vertex in $V(G)$ and every edge in $E(G)$ are scanned in a constant number. Although we must find a maximum matching in $G\left[C^{\prime}\left(v_{i}\right)\right]$, it can be done in linear time, because $G\left[C^{\prime}\left(v_{i}\right)\right]$ is disjoint union of some clique. If every clique has even number of vertices, then $G\left[C^{\prime}\left(v_{i}\right)\right]$ has a perfect matching, otherwise it has not a perfect matching.

In [6], a linear time algorithm was given to determine a minimum paired-dominating set in trees. Since block graphs contain trees, we can also use the algorithm MPDB to produce a paired-dominating set in trees. The only difference is that $G\left[C^{\prime}\left(v_{i}\right)\right]$ can not have a perfect matching in tree. Here, we give a very simple algorithm for tree which can be deduced from MPDB at once.

Algorithm MPDT. Find a minimum paired-dominating set of a tree.
Input. A tree $T=(V, E)$ with a vertex ordering $v_{1}, v_{2}, \cdots, v_{n}$ such that $d_{T}\left(v_{i}, v_{n}\right)<d_{T}\left(v_{j}, v_{n}\right)$ implies that $i<j$. Each vertex $v_{i}$ has a label $\left(D\left(v_{i}\right), L\left(v_{i}\right)\right)=(0,0) . F\left(v_{i}\right)=v_{j}$ with $v_{i} v_{j} \in E$ and $j>i ; C\left(v_{i}\right)=\left\{v_{j} \mid F\left(v_{j}\right)=v_{i}\right\}$.
Output. A minimum paired-dominating set $P D$ of $T$.

## Method.

For $i=1$ to $n$ do
If $\left(D\left(v_{i}\right)=0\right.$ and $\left.i \neq n\right)$ then
$L\left(F\left(v_{i}\right)\right)=1 ;$
$D(u)=1$ for every vertex $u \in N\left[F\left(v_{i}\right)\right] ;$
endif
If $\left(D\left(v_{i}\right)=1\right)$ then
Let $C^{\prime}\left(v_{i}\right)=\left\{w \mid w \in C\left(v_{i}\right)\right.$ and $\left.L(w)=1\right\}$;
If $C^{\prime}\left(v_{i}\right) \neq \emptyset$ then
$L\left(v_{i}\right)=2 ;$
$D(u)=1$ for every vertex $u \in N\left[v_{i}\right] ;$
$L(w)=2$ for every vertex $w \in C^{\prime}\left(v_{i}\right)$;
Take a vertex $w \in C^{\prime}\left(v_{i}\right)$, for every vertex $v \in C^{\prime}\left(v_{i}\right)-\{w\}$
$L\left(v^{\prime}\right)=2$ for some vertex $v^{\prime} \in C(v)$ such that $L\left(v^{\prime}\right)=0$;
endif
endif
endfor
If $\left(D\left(v_{n}\right)=0\right.$ or $\left.L\left(v_{n}\right)=1\right)$ then
$L\left(v_{n}\right)=2 ;$
$L(w)=2$ for some vertex $w \in C\left(v_{n}\right)$ such that $L(w)=0 ;$

$$
\begin{aligned}
& \qquad D\left(v_{n}\right)=1 \\
& \text { endif } \\
& \text { Output } P D=\{v \mid L(v)=2\} \\
& \text { end }
\end{aligned}
$$

Corollary 4 Algorithm MPDT can produce a minimum paired-dominating set of a tree $T$ in $O(m+n)$, where $m=|E(T)|$ and $n=|V(T)|$.

## 3 Algorithm for paired-domination problem in Interval graph

An interval representation of a graph is a family of intervals assigned to the vertices so that vertices are adjacent if and only if the corresponding intervals intersect. A graph having such a representation is an interval graph. Booth and Lueker [7] gave an $O(|V(G)|+|E(G)|)$-time algorithm for recognizing an interval graph and constructing an interval representation using $P Q$-tree. In [5], a linear algorithm was given to produce a minimum paired-dominating set of an interval graph. But this algorithm is incorrect. Next, we will introduce this algorithm and given a counterexample.

In [5], it is assumed that the input graph is given by an interval representation $I$ that is a set of $n$ sorted intervals labelled by $1,2, \cdots, n$ in increasing order of their right endpoints. The left endpoint of interval $i$ is denoted by $a_{i}$ and the right endpoint by $b_{i}\left(1<a_{i} \leq b_{i} \leq 2 n\right)$. Define the following notations.
(1) For a set $S$ of intervals, the largest left(right) endpoint of the intervals in $S$ is denoted by $\max a(S)(\max b(S))$; the interval in $S$ with the largest right endpoint is denoted by last $(S)$. Let $\max a(S)=0(\max b(S)=0)$ if $S$ is empty. For endpoint $e$, use $I F B(e)$ to denote the set of all intervals whose right endpoint are less than $e$. For any interval $j$, let $l_{j}$ be the interval such that intervals $l_{j}$ and $j$ have nonempty intersection and $a\left(l_{j}\right)$ is minimum.
(2) For $j \in\{1,2, \cdots, n\}$, define $V_{j}=\left\{i: i \in\{1,2, \cdots, n\}\right.$ and $\left.a_{i} \leq b_{j}\right\}$. Let $P D(j)=$ $\left\{S: S \subseteq V_{j}, S\right.$ is a paired-dominating set of $\left.G\left[V_{j}\right]\right\}$. Let $P D(i, j)=\left\{S: S \subseteq V_{j}, S\right.$ is a paired-dominating set of $G\left[V_{j}\right], i, j \in S$ and $i, j$ are paired in $\left.S\right\}$. Let $\operatorname{MPD}(j)=\operatorname{Min}(P D(j))$, $\operatorname{MPD}(i, j)=\operatorname{Min}(P D(i, j))$. The left endpoint sets $A_{i}=\left\{a_{j}: b_{i-1}<a_{j}<b_{j}\right\}$ for $i \in I$, where $b_{0}=0$.

Introduce two intervals $n+1$ and $n+2$ with $a_{n+1}=2 n+1, a_{n+2}=2 n+2, b_{n+1}=2 n+3$, and $b_{n+2}=2 n+4$. Let $I_{p}$ be the set of intervals obtained by augmenting $I$ with the two intervals $n+1$ and $n+2$.

## Algorithm MPD

Input: A set $I_{p}$ of sorted intervals.
Output: A minimum cardinality paired-dominating set of $G$ with interval representation $I_{p}$.

1. Find $\max a\left(\operatorname{IFB}\left(a_{j}\right)\right)$ for all $j \in I_{p}$.
2. Find $l_{j}$ for all $j \in I_{p}$.
3. Scan the endpoints of $I_{p}$ to find the left endpoint sets $A_{i}=\left\{a_{j}: b_{i-1}<a_{j}<b_{j}\right\}$ for $i \in I$, where $b_{0}=0$.
4. $M P D(0)=\emptyset$.
5. for $j=1$ to $n+2$ do
6. Find the left endpoint set $A_{k}$ containing $\max a\left(\operatorname{IFB}\left(\min \left(a_{j}, a_{l_{j}}\right)\right)\right)$.
7. Let $b_{k}$ be the right endpoint of the interval $k$ associated with the left endpoint set $A_{k}$.
8. $M P D(j)=\left\{l_{j}, j\right\} \cup M P D(k)$.
9. end for

Output MPD $(n+2)$.
It is obvious that $\operatorname{MPD}(n+2)-\{n+1, n+2\}$ got from the algorithm MPD is a paireddominating set of $G$. The following counterexample implies that $\operatorname{MPD}(n+2)-\{n+1, n+2\}$ may be not a minimum paired-dominating set of $G$.


The figure above is a counterexample. The left figure is an interval representation of the graph in the right figure. The number is ordered by the right endpoint of intervals. The parameters used in MPD are as follows:

| $i$ | $a_{i}$ | $b_{i}$ | $\max a\left(\operatorname{IFB}\left(a_{i}\right)\right)$ | $l_{i}$ | $A_{i}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0 | 3 | 0 | 2 | $\{1,2\}$ |
| 2 | 1 | 4 | 0 | 1 | $\emptyset$ |
| 3 | 2 | 6 | 0 | 1 | $\{5\}$ |
| 4 | 7 | 9 | 2 | 5 | $\{7,8\}$ |
| 5 | 5 | 10 | 1 | 3 | $\emptyset$ |
| 6 | 8 | 11 | 2 | 5 | $\emptyset$ |
| 7 | 13 | 15 | 8 | 8 | $\emptyset$ |
| 8 | 14 | 16 | 8 | 7 | $\emptyset$ |

Execute algorithm MPD as follows:
$j=1, \max a\left(\operatorname{IFB}\left(\min \left(a_{1}, a_{2}\right)\right)\right)=0, k=0, M P D(1)=\{1,2\} \cup M P D(0)=\{1,2\} ;$
$j=2, \max a\left(\operatorname{IFB}\left(\min \left(a_{1}, a_{2}\right)\right)\right)=0, k=0, M P D(2)=\{1,2\} \cup M P D(0)=\{1,2\} ;$
$j=3, \max a\left(\operatorname{IFB}\left(\min \left(a_{3}, a_{1}\right)\right)\right)=0, k=0, M P D(3)=\{1,3\} \cup M P D(0)=\{1,3\} ;$
$j=4, \max a\left(\operatorname{IFB}\left(\min \left(a_{4}, a_{5}\right)\right)\right)=1, k=1, M P D(4)=\{4,5\} \cup M P D(1)=\{4,5,1,2\} ;$
$j=5, \max a\left(\operatorname{IFB}\left(\min \left(a_{5}, a_{3}\right)\right)\right)=0, k=0, M P D(5)=\{3,5\} \cup M P D(0)=\{3,5\} ;$
$j=6, \max a\left(\operatorname{IFB}\left(\min \left(a_{6}, a_{5}\right)\right)\right)=1, k=1, M P D(6)=\{5,6\} \cup M P D(1)=\{5,6,1,2\} ;$
$j=7, \max a\left(\operatorname{IFB}\left(\min \left(a_{7}, a_{8}\right)\right)\right)=8, k=4, M P D(7)=\{1,2\} \cup M P D(4)=\{7,8,4,5,1,2\} ;$
$j=8, \max a\left(\operatorname{IFB}\left(\min \left(a_{7}, a_{8}\right)\right)\right)=8, k=4, M P D(8)=\{1,2\} \cup M P D(4)=\{7,8,4,5,1,2\} ;$

Hence the result set of the algorithm MPD is $\{4,5,1,2\}$. But it is easy to see that $\{3,5\}$ is a minimum paired-dominating set of this graph. Note that Lemma 2.3 and Lemma 2.5 in 5 are not right. The detail is left to readers.

Next, we employ the labelling technique to give a linear algorithms for finding a minimum paired-dominating set of an interval graph. Let $G=(V, E)$ be an interval graph and its interval representation is $I$. For every vertex $u_{i} \in V, I_{i}$ is the corresponding interval, and let $a_{i}\left(b_{i}\right.$, respectively) denote the left endpoint (right endpoint, respectively) of interval $I_{i}$. We order the vertices of $G$ by $u_{1}, u_{2}, \cdots, u_{n}$ in increasing order of their left endpoints. Then we have following two observations.

Observation $5 u_{1}, u_{2}, \cdots, u_{n}$ is a ordering of an interval graph $G$ by the increasing order of their left endpoints. If $u_{i} u_{j} \in E$ with $j<i$, then $u_{j} u_{k} \in E$ for every $j+1 \leq k \leq i$.

Let $V_{i}=\left\{u_{j} \mid j \leq i\right\}$ and $G\left[V_{i}\right]$ be an induced subgraph of $G$. It is obvious that $G\left[V_{n}\right]=G$. Let $F\left(u_{i}\right)=u_{j}$, where $j=\min \left\{k \mid u_{k} u_{i} \in E\right.$ and $\left.k<i\right\}$. In particular, $F\left(u_{1}\right)=u_{1}$. Let $w\left(u_{i}\right)=u_{j}$, where $j=\max \left\{k \mid u_{k} u_{i} \notin E\right.$ and $\left.k<i\right\}$. In particular, If $w\left(u_{i}\right)$ does not exist, we assume that $w\left(u_{i}\right)=u_{0}\left(u_{0} \notin V\right) . P D_{i}$ denotes a minimum paired-dominating set of $G\left[V_{i}\right]$. In this paper, we only consider connected interval graph.

Observation 6 If $G$ is a connected interval graph, then $G\left[V_{i}\right]$ is also connected.
Lemma 7 If $F\left(u_{i}\right) \neq u_{i}$ and $F\left(F\left(u_{i}\right)\right)=F\left(u_{i}\right)$, then $P D_{i}=\left\{u_{i}, F\left(u_{i}\right)\right\}$.
Proof Since $F\left(F\left(u_{i}\right)\right)=F\left(u_{i}\right)$, hence $F\left(u_{i}\right)=u_{1}$. By Observation for every $1<j \leq i$, $u_{1} u_{j} \in E$. So $\left\{u_{i}, u_{1}\right\}=\left\{u_{i}, F\left(u_{i}\right)\right\}$ is a minimum paired-dominating set of $G\left[V_{i}\right]$. Hence, $P D_{i}=\left\{u_{i}, F\left(u_{i}\right)\right\}$.

Lemma $8\left|P D_{i+1}\right| \geq\left|P D_{i}\right|$ for $2 \leq i \leq n-1$.
Proof If $u_{i+1} \notin P D_{i+1}$, then $P D_{i+1}$ is also a paired-dominating set of $G\left[V_{i}\right]$. So $\left|P D_{i+1}\right| \geq$ $\left|P D_{i}\right|$. If $u_{i+1} \in P D_{i+1}$ and $u_{k}, u_{i+1}$ are paired in $P D_{i+1}$. We consider two cases.
Case 1: $k=i$.
That is $u_{i}, u_{i+1}$ are paired in $P D_{i+1}$. If $N_{G\left[V_{i}\right]}\left(u_{i}\right) \subseteq P D_{i+1}$, then $P D_{i+1}-\left\{u_{i}, u_{i+1}\right\}$ is a paireddominating set of $G\left[V_{i}\right]$. So $\left|P D_{i+1}\right|>\left|P D_{i+1}\right|-2 \geq\left|P D_{i}\right|$. If there is a vertex $w \in N_{G\left[V_{i}\right]}\left(u_{i}\right)$ with $w \notin P D_{i+1}$, then $P D_{i+1}-\left\{u_{i+1}\right\} \cup\{w\}$ is also a paired-dominating set of $G\left[V_{i}\right]$. We also get $\left|P D_{i+1}\right| \geq\left|P D_{i}\right|$.
Case 2: $k<i$.
If $u_{i} \notin P D_{i+1}$, then $P D_{i+1}-\left\{u_{i+1}\right\} \cup\left\{u_{i}\right\}$ is a paired-dominating set of $G\left[V_{i}\right]$, so $\left|P D_{i+1}\right| \geq$ $\left|P D_{i}\right|$. If $u_{i} \in P D_{i+1}$ and $u_{i}$ is paired with $u_{l}$, then $u_{l} u_{k} \in E$ and $P D_{i+1}-\left\{u_{i}, u_{i+1}\right\}$ is a paired-dominating set of $G\left[V_{i}\right]$, so $\left|P D_{i+1}\right|>\left|P D_{i+1}\right|-2 \geq\left|P D_{i}\right|$.

Lemma 9 Let $F\left(u_{i}\right)=u_{k}$ and $F\left(u_{k}\right)=u_{j}$ with $j<k<i$.
(1) $P D_{i}=P D_{l} \cup\left\{u_{j}, u_{k}\right\}$ if $w\left(u_{j}\right)=u_{l}$ with $l \geq 2$.
(2) $P D_{i}=\left\{u_{1}, u_{2}, u_{j}, u_{k}\right\}$ if $w\left(u_{j}\right)=u_{1}$.
(3) $P D_{i}=\left\{u_{j}, u_{k}\right\}$ if $w\left(u_{j}\right)=u_{0}$.

Proof (1) It is obvious that $P D_{l} \cup\left\{u_{j}, u_{k}\right\}$ is a paired-dominating set of $G\left[V_{i}\right]$. It is sufficient to prove $\left|P D_{i}\right| \geq\left|P D_{l}\right|+2$. Since $F\left(u_{i}\right)=u_{k}$, there must exist a vertex $u_{i_{1}} \in P D_{i}$ with $k \leq i_{1} \leq i$, which dominates $u_{i}$. We may assume that $u_{i_{1}}$ is the last vertex in $P D_{i}$ which dominates $u_{i}$ and $u_{i_{1}}$ is paired with $u_{k_{1}}$. It is obvious that $k_{1} \geq j$. Let $l^{\prime}=\min \{a, b\}$, where $w\left(u_{i_{1}}\right)=u_{a}$ and $\left.w\left(u_{k_{1}}\right)=u_{b}\right\}$. Since $u_{l} u_{k_{1}} \notin E$ and $u_{l} u_{i_{1}} \notin E$ (otherwise $u_{l} u_{j} \in E$, a contradiction), so $l^{\prime} \geq l \geq 2$. Let $u_{c}$ is the last vertex in $P D_{i}-\left\{u_{i_{1}}, u_{k_{1}}\right\}$. If $c \geq l^{\prime}$, then $P D_{i}-\left\{u_{i_{1}}, u_{k_{1}}\right\}$ is a paired-dominating set of $G\left[V_{c}\right]$. So $\left|P D_{i}\right|-2 \geq\left|P D_{c}\right|$. On the other hand, since $c \geq l^{\prime} \geq l$, by Lemma 8, $\left|P D_{c}\right| \geq\left|P D_{l}\right|$. Then $\left|P D_{i}\right| \geq\left|P D_{l}\right|+2$. If $c<l^{\prime}$, then $P D_{i}-\left\{u_{i_{1}}, u_{k_{1}}\right\}$ is a paired-domination set of $G\left[V_{l^{\prime}}\right]$. Since $l^{\prime} \geq l,\left|P D_{i}\right|-2 \geq\left|P D_{l^{\prime}}\right| \geq\left|P D_{l}\right|$. So $\left|P D_{i}\right| \geq\left|P D_{l}\right|+2$. Thus $P D_{i}=P D_{l} \cup\left\{u_{j}, u_{k}\right\}$.
(2) Note that $u_{1} u_{2} \in E$ and $j \geq 3$ in this situation. So it is easy to know that $P D_{i}=$ $\left\{u_{1}, u_{2}, u_{j}, u_{k}\right\}$ if $w\left(u_{j}\right)=u_{1}$.
(3) It is obvious.

Now we give an intuitive algorithm for determining a minimum paired-dominating set in interval graphs.

Algorithm MPDI. Find a minimum paired-dominating set of an interval graph.
Input. An interval graph $G=(V, E)$ with a vertex ordering $u_{1}, u_{2}, \cdots, u_{n}$ ordered by the increasing order of left endpoints, in which each vertex $u_{i}$ has a label $D\left(u_{i}\right)=0$. Let $F\left(u_{i}\right)=$ $u_{j}\left(F\left(u_{1}\right)=u_{1}\right)$ such that $j=\min \left\{k \mid u_{k} u_{i} \in E\right.$ and $\left.k<i\right\}$.
Output. A minimum paired-dominating set $P D$ of $G$.

## Method.

$P D=\emptyset ;$
For $i=n$ to 1 do
If $\left(D\left(u_{i}\right)=0\right)$ then
If $\left(F\left(u_{i}\right) \neq u_{i}\right.$ and $\left.F\left(F\left(u_{i}\right)\right) \neq F\left(u_{i}\right)\right)$ then
$P D=P D \cup\left\{F\left(u_{i}\right), F\left(F\left(u_{i}\right)\right)\right\} ;$
$D(u)=1$ for every vertex $u \in N\left[F\left(u_{i}\right)\right]$;
$D(w)=1$ for every vertex $w \in N\left[F\left(F\left(u_{i}\right)\right)\right] ;$
else if $\left(F\left(u_{i}\right) \neq u_{i}\right)$ then
$P D=P D \cup\left\{u_{i}, F\left(u_{i}\right)\right\} ;$
$D(u)=1$ for every vertex $u \in N\left[F\left(u_{i}\right)\right] ;$
else
$P D=P D \cup\left\{u_{i}, u_{2}\right\} ;$

$$
\begin{aligned}
& \qquad D\left(u_{i}\right)=1 ; \\
& \text { endif } \\
& \text { endif } \\
& \text { endfor }
\end{aligned}
$$

Theorem 10 Given a vertex ordering ordered by the increasing order of left endpoints, the algorithm MPDI can produce a minimum paired-dominating set of $G$ in $O(m+n)$, where $m=$ $|E(G)|$ and $n=|V(G)|$.

Proof By Lemmas 7 and 9 , we know that the algorithm MPDI can produce a minimum paireddominating set of an interval graph $G$. Since each vertex and edge are scanned in a constant number, hence the algorithm MPDI can finish in $O(m+n)$, where $m=|E(G)|$ and $n=|V(G)|$.

## 4 NP-completeness of paired-domination problem

A graph is chordal if every cycle of length greater than three has a chord, i.e. an edge jointing two nonconsecutive vertices in the cycle. A graph is split if its vertex set can be partitioned into a stable set and a clique. Obviously, block graphs, interval graphs and split graphs are three subclasses of chordal graphs. This section establishes $N P$-complete results for the paireddomination problem in bipartite graphs and chordal graphs. The transformation is from the vertex cover problem, which is known to be NP-complete. The vertex cover problem is for a given nontrivial graph and a positive integer $k$ to answer if there is a vertex set of size at most $k$ such that each edge of the graph has at least one end vertex in this set.

Theorem 11 Paired-domination problem is NP-complete for bipartite graphs.
Proof For a bipartite graph $G=(V, E)$, a positive even integer $k$, and an arbitrary subset $S \subseteq V$ with $|S| \leq k$, it is easy to verify in polynomial time whether $S$ is a paired-dominating set of $G$. Hence, paired-domination problem is in $N P$.

We construct a reduction from the vertex cover problem. Given a nontrivial graph $G=$ $(V, E)$, where $V=\left\{v_{1}, v_{2}, \cdots, v_{n}\right\}$ and $E=\left\{e_{1}, e_{2}, \cdots, e_{m}\right\}$, Let $V_{i}=\left\{v_{1}^{i}, v_{2}^{i}, \cdots, v_{n}^{i}\right\}$ and $E_{i}=\left\{e_{1}^{i}, e_{2}^{i}, \cdots, e_{m}^{i}\right\}(i=1,2)$. Construct the graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ with vertex set $V^{\prime}=$ $V_{1} \cup V_{2} \cup E_{1} \cup E_{2}$, and edge set $E^{\prime}=\left\{u v \mid u \in V_{1}\right.$ and $\left.v \in V_{2}\right\} \cup\left\{v_{j}^{i} e_{k}^{i} \mid i=1,2\right.$ and $v_{j}$ is incident to $e_{k}$ in $\left.G\right\}$. Note that $G^{\prime}$ is a bipartite graph.

Next, we will show that $G$ has a vertex cover of size at most $k$ if and only if $G^{\prime}$ has a paireddominating set of size at most $2 k$. Let $V C=\left\{v_{i_{1}}, v_{i_{2}}, \cdots, v_{i_{k}}\right\}$ be a vertex cover of $G$. Then it is obvious that $\left\{v_{i_{1}}^{1}, v_{i_{2}}^{1}, \cdots, v_{i_{k}}^{1}\right\} \cup\left\{v_{i_{1}}^{2}, v_{i_{2}}^{2}, \cdots, v_{i_{k}}^{2}\right\}$ is a paired-dominating set of $G^{\prime}$ and its size is $2 k$. For the converse, let $P D$ be a paired-dominating set of $G^{\prime}$ with $|P D| \leq 2 k$. Obviously,
we can assume that $k \leq n-1$ since $V=\left\{v_{1}, v_{2}, \cdots, v_{n}\right\}$ is a vertex cover of $G$. If $P D \cap E_{1} \neq \emptyset$, without loss of generality, we assume that $e_{1}^{1} \in P D$ and its paired vertex in $P D$ is $v_{i}^{1}$. Since $k \leq n-1$, there exists a vertex $v_{j}^{2} \notin P D$. Hence, $P D \cup\left\{v_{j}^{2}\right\}-\left\{e_{1}^{1}\right\}$ is also a paired-dominating set of $G^{\prime}$. Then we may assume that $P D \cap\left(E_{1} \cup E_{2}\right)=\emptyset$. Suppose that $V C_{1}=P D \cap V_{1}$, and note that $\left|V C_{1}\right| \leq k$. Let $V C=\left\{v_{i} \mid v_{i}^{1} \in V C_{1}\right\}$ and $V C$ is a vertex cover of $G$ such that $|V C| \leq k$.

Finally, one can construct $G^{\prime}$ from $G$ in polynomial time. This implies that paired-domination problem is $N P$-complete for bipartite graphs.

Theorem 12 Paired-domination problem is NP-complete for chordal graphs.
Proof We still construct a reduction from the vertex cover problem. Given a nontrivial graph $G=(V, E)$, where $V=\left\{v_{1}, v_{2}, \cdots, v_{n}\right\}$ and $E=\left\{e_{1}, e_{2}, \cdots, e_{m}\right\}$, Let $V_{i}=\left\{v_{1}^{i}, v_{2}^{i}, \cdots, v_{n}^{i}\right\}(i=$ $1,2)$ and $E_{i}=\left\{e_{1}^{i}, e_{2}^{i}, \cdots, e_{m}^{i}\right\}(i=1,2)$. Construct the graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ with vertex set $V^{\prime}=V_{1} \cup V_{2} \cup E_{1} \cup E_{2}$, and edge set $E^{\prime}=\left\{u v \mid u \in V_{1} \cup V_{2}\right.$ and $\left.u \neq v\right\} \cup\left\{v_{j}^{i} e_{k}^{i} \mid i=1,2\right.$ and $v_{j}$ is incident to $e_{k}$ in $\left.G\right\}$. Note that $G^{\prime}$ is a chordal graph.

It is straightforward to show that $G$ has a vertex cover of size at most $k$ if and only if $G^{\prime}$ has a paired-dominating set of size at most $2 k$. The proof is almost similar with that of Theorem 11 . In here, we can also assume that $k \leq n-1$ and $P D \cap\left(E_{1} \cup E_{2}\right)=\emptyset$. So, either $V C_{1}=P D \cap V_{1}$ or $V C_{2}=P D \cap V_{2}$ has size at most $k$. The detail is left to readers.

Note that $G^{\prime}$ in Theorem $[12$ is also a split graph. Hence we get a stronger result as follows.
Corollary 13 Paired-domination problem is NP-complete for split graphs.

## References

[1] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (eds), Fundamentals of Domination in Graphs, New York, Marcel Dekker 1998.
[2] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (eds), Domination in Graphs: Advanced Topics, New York, Marcel Dekker 1998.
[3] T. W. Haynes and P. J. Slater, Paired-domination in graphs, Networks 32(1998), 199-206.
[4] M. A. Henning, Graphs with large paired-domination number, J. Comb. Optim. 13(2007), 61-78.
[5] T. C. E. Chen, L. Y. Kang and C. T. Ng, Paired domination on interval and circular-arc graphs, Discrete Appl. Math. 155(2007), 2077-2086.
[6] H. Qiao, L. Y. Kang, M. Caedei and D. Z. Du, Paired-domination of trees, J. Global Optim. 25(2003), 43-54.
[7] K. S. Booth and G. S. Lueker, Testing for consecutive ones property, interval graphs and graph planarity using $P Q$-tree algorithms, J.Comput.Syst.Sci. 13(1976), 335-379.
[8] P. D. Sylvain Gravier and M. A. Henning, Paired-domination in generalized claw-free graphs, J. Comb. Optim. 14(2007), 1-7.
[9] D. B. West, Introduction to Graph Theory, 2nd ed., Prentice Hall, Inc., NJ, 2001.


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