

Vertices in all minimum paired-dominating sets of block graphs *

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Abstract Let $G = (V, E)$ be a simple graph without isolated vertices. A set $S \subseteq V$ is a paired-dominating set if every vertex in $V - S$ has at least one neighbor in S and the subgraph induced by S contains a perfect matching. In this paper, we present a linear-time algorithm to determine whether a given vertex in a block graph is contained in all its minimum paired-dominating sets.

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1 Introduction

Let $G = (V, E)$ be a simple graph without isolated vertices. The distance between u and v in G , denoted by $d_G(u, v)$, is the minimum length of a path between u and v in G . For a vertex $v \in V$, the *neighborhood* of v in G is defined as $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is defined as $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v , denoted by $d_G(v)$, is defined as $|N_G(v)|$. We use $d(u, v)$ for $d_G(u, v)$, $N(v)$ for $N_G(v)$, $N[v]$ for $N_G[v]$ and $d(v)$ for $d_G(v)$ if there is no ambiguity. For a subset S of V , the subgraph of G induced by the vertices in S is denoted by $G[S]$ and $G - S$ denote the subgraph induced by $V - S$. A *matching* in a graph G is a set of pairwise nonadjacent edges in G . A *perfect matching* M in G is a matching such that every vertex of G is incident to an edge of M . Some other notations and terminology not introduced in here can be found in [1].

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Domination and its variations in graphs have been extensively studied [2, 3]. A set $S \subseteq V$ is a *paired-dominating set* of G , denoted PDS, if every vertex in $V - S$ has at least one neighbor in S and the induced subgraph $G[S]$ has a perfect matching M . Two vertices joined by an edge of M are said to be paired in S . The *paired-domination number*, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a PDS. A paired-dominating set of cardinality $\gamma_{pr}(G)$ is called a $\gamma_{pr}(G)$ -set. The paired-domination was introduced by Haynes and Slater [4, 5]. There are many results on this problem [6, 7, 8, 9, 10, 11].

The study of characterizing vertices contained in all various kinds of minimum dominating set, such as dominating set, total dominating set and paired-dominating set, has received considerable attention (see [12],[13], [14]). Those results are all restricted in trees. In this paper, we will extend the result in [14] to block graphs, which contain trees as its subclass. In fact, we give a linear-time algorithm to determine whether a given vertex in a block graph is contained in all its minimum paired-dominating sets. If changing the pruning rules and judgement rules in our algorithm, our method is also available to determine whether a given vertex is contained in all minimum (total) dominating sets of a block graph .

2 Pruning block graphs

Let $G = (V, E)$ be a simple graph. A vertex v is a *cut-vertex* if deleting v and all edges incident to it increases the number of connected components. A *block* of G is a maximal connected subgraph of G without cut-vertices. A *block graph* is a connected graph whose blocks are complete graphs. If every block is K_2 , then it is a tree.

Let $G = (V, E)$ be a block graph. As we know, every block graph not isomorphic to complete graph has at least two *end blocks*, which are blocks with only one cut-vertex. A vertex in G is a *leaf* if its degree is one. If a vertex is adjacent to a leaf, then we call it a *support vertex*.

Lemma 1 [14] *Let T be a tree of order at least three. If u is a leaf in T , then there exists a $\gamma_{pr}(T)$ -set not containing u .*

For block graphs, we have the following generalized result. The proof is almost same as that of Lemma 1, so it is omitted.

Lemma 2 *Let G be a block graph of order at least three. If u is not a cut-vertex of G , then there exists a $\gamma_{pr}(G)$ -set not containing u .*

If G is a block graph with order two, then every vertex is contained in the only minimum paired-dominating set. If G is a complete graph with order at least three, no vertex of G is contained in all minimum paired-dominating sets. Thus, in here, we assume that the block graph G with at least one cut-vertex. Let r be the given vertex in G and we want to determine

whether r is contained in every $\gamma_{pr}(G)$ -set. By Lemma 2, it is enough to assume that r is a cut-vertex of G .

Our idea is to prune the original graph G into a small block graph \tilde{G} such that the given vertex r is contained in all minimum paired-dominating sets of G if and only if it is contained in all minimum paired-dominating sets of \tilde{G} . To do this, we first need a vertex ordering and follow this ordering we can prune the original graph. For a vertex $v \in V(G)$ and a block B , the distance of v and B , denoted by $d(v, B)$, is defined as the maximum of $d(u, v)$ for $u \in V(B)$. We say a block B is *farthest from v* if $d(v, B)$ is maximum over all blocks. Note that B is an end block if B is farthest from r . To find the vertex ordering, in here, we need to define a vertex ordering connected operation. Let $S = x_1, x_2, \dots, x_s$ be a vertex ordering and $T = u_1, u_2, \dots, u_t$ be another vertex ordering. We use $S + T$ to denote a new vertex ordering $x_1, x_2, \dots, x_s, u_1, u_2, \dots, u_t$. Beginning with a block farthest from r and working recursively inward, we can find a vertex order v_1, v_2, \dots, v_n as follows.

Procedure VO

$S = \emptyset$; (S is a vertex ordering.)

Let r be a cut-vertex of G ;

While ($G \neq \emptyset$) do

 If (G is a complete graph) then

 Let $V(G) = \{u_1, u_2, \dots, u_a = r\}$. $S = S + u_1, u_2, \dots, u_a$;

$G = G - \{u_1, u_2, \dots, u_a\}$;

 else

 Let B be an end block farthest from r with $V(B) = \{u_1, u_2, \dots, u_b, x\}$, where x is the cut-vertex in B . $S = S + u_1, u_2, \dots, u_b$;

$G = G - \{u_1, u_2, \dots, u_b\}$;

 endif

enddo

Output S .

Let $v_1, v_2, \dots, v_n = r$ be the vertex ordering of a block graph G which is obtained by procedure VO. We define the following notations:

(a) $F_G(v_i) = v_j$, $j = \max\{k \mid v_i v_k \in E, k > i\}$. v_j is called the *father* of v_i and v_i is a *child* of v_j . Obviously, v_j must be a cut-vertex in G . We use $F(v_i)$ for $F_G(v_i)$ if there is no ambiguity.

(b) $C_G(v_i) = \{v_j \mid F_G(v_j) = v_i\}$.

(c) For a block graph G , we define a rooted tree $T(G)$, whose vertex set is $V(G)$, and uv is an edge of $T(G)$ if and only if $F_G(u) = v$. The root of $T(G)$ is r . Moreover let T_v be a subtree of $T(G)$ rooted at v . Every vertex in T_v except v is a descendant of v . For a vertex $v \in V(G)$, $D_G(v)$ denotes the vertex set consisting of the descendants of v in $T(G)$ and $D_G[v] = D_G(v) \cup \{v\}$.

That is, $D_G[v] = V(T_v)$.

Except the vertex ordering, we also need a labeling function $l(v) : V \rightarrow \{\emptyset, r_1, r_2\}$ of each vertex v to help us to determine which vertices can be pruned. At first, $l(v) = \emptyset$ for every vertex $v \in V$.

The following procedure can prune a big block graph G into a small block graph \tilde{G} such that r is contained in all minimum paired-dominating sets of G if and only if r is contained in all minimum paired-dominating sets of \tilde{G} .

Procedure PRUNE. Prune a given block graph into a small block graph.

Input A block graph with at least one cut-vertex and a vertex ordering v_1, v_2, \dots, v_n obtained by procedure VO. For every vertex v , $l(v) = \emptyset$.

Output A smaller block graph.

Method

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 $S = \emptyset;$ 
For  $i = 1$  to  $n - 1$  do
  If  $(v_i \notin S)$  then
    If  $(l(v_i) = \emptyset \text{ and there is no child } v \text{ such that } l(v) = r_1 \text{ or } l(v) = r_2)$  then
       $l(F(v_i)) = r_1;$ 
    else if  $(v_i \text{ satisfies the conditions of Lemma 3 or Lemma 4 or Lemma 5})$  then
       $G = G - D_G[v_i];$ 
      If  $(d(v_i) = 2 \text{ and } |V(B_1)| = |V(B_2)| = 2 \text{ and } C_G(F(v_i)) = \{v_i\})$  then
        (Where  $B_1$  and  $B_2$  are same as those in Lemma 4)
         $S = S \cup \{F(v_i)\};$ 
      endif
    else if  $(v_i \text{ satisfies the conditions of Lemma 6})$  then
       $G = G - (D_G(v_i) - V(B')),$  where  $B'$  is same as  $B'$  in Lemma 6.
    else if  $(v_i \text{ satisfies the conditions of Lemma 7 or Lemma 8})$  then
       $G = G - (D_G(v_i) - D_G[u]),$  where  $u$  is same as  $u$  in Lemma 7 and Lemma 8.
       $l(v_i) = l(u) = r_2; \quad (*)$ 
      If  $(d(v_i, r) = 2 \text{ and } |V(B_1)| = |V(B_2)| = 2 \text{ and } C_G(F(v_i)) = \{v_i\})$  then
        (Where  $B_1$  and  $B_2$  are same as those in Lemma 8)
         $S = S \cup \{F(v_i)\};$ 
      endif
    endif
  endif
endif
endfor
Output  $G$ .
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Next, we will prove the correctness of procedure PRUNE. Let G_i be a subgraph of the original graph G after v_i is considered and $G_0 = G$. It is clear that G_i is a block graph for every $1 \leq i \leq n-1$. We define that $C_i(v) = C_G(v) \cap V(G_i)$, $D_i(v) = D_G(v) \cap V(G_i)$ and $D_i[v] = D_G[v] \cap V(G_i)$ for $0 \leq i \leq n-1$. Note that at the i -th loop, the pruning vertices, for example say $D_G[v_i]$, are $D_{i-1}[v_i]$ as G is updated at each step, i.e., $G = G_{i-1}$ at this time. It is enough to prove that r is contained in all $\gamma_{pr}(G_{i-1})$ -set if and only if r is contained in all $\gamma_{pr}(G_i)$ -set for $1 \leq i \leq n-1$. If $G_i = G_{i-1}$ for some i , then it is obviously true. When v_i is considered, let $R_j = \{v \mid v \in V(G_{i-1}) \text{ and } l(v) = r_j\}$ for $j = 1, 2$.

Lemma 3 *When v_i is a considering vertex such that $d(r, v_i) \geq 3$. If $l(v_i) = \emptyset$, $(R_1 \cup R_2) \cap C_{i-1}(v_i) \neq \emptyset$ and $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ has a perfect matching, then r is contained in all $\gamma_{pr}(G_{i-1})$ -set if and only if r is contained in all $\gamma_{pr}(G_i)$ -set, where $G_i = G_{i-1} - D_{i-1}[v_i]$.*

Proof Let $D_1 = R_1 \cap C_{i-1}(v_i)$, $D_2 = R_2 \cap D_{i-1}(v_i)$ and $D = D_1 \cup D_2$. In details, $D_1 = \{u_1, u_2, \dots, u_a\}$ and $D_2 = \{x_1, y_1, \dots, x_b, y_b\}$, where $x_j y_j \in E$ and $F(y_j) = x_j$ for $1 \leq j \leq b$ for $1 \leq j \leq b$ (see the line indicated (*) in the procedure PRUNE.) Then we obtain the following claim.

Claim 1 $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$.

Proof Any $\gamma_{pr}(G_i)$ -set can be extended to a PDS of G_{i-1} by adding D . Thus $\gamma_{pr}(G_{i-1}) \leq \gamma_{pr}(G_i) + |D|$. For converse, let S be a $\gamma_{pr}(G_{i-1})$ -set. If $y_j \notin S$, then $|D_{i-1}(y_j) \cap S| \geq 2$ and $S - D_{i-1}(y_j) \cup \{y_j, z_j\}$, where z_j is a child of y_j , is also a $\gamma_{pr}(G_{i-1})$ -set. Thus we may assume $y_j \in S$ and w_j be its paired vertex. If $x_j \notin S$, then $S - \{w_j\} \cup \{x_j\}$ is also a $\gamma_{pr}(G_{i-1})$ -set. If $x_j \in S$ and $w_j \neq x_j$, let x'_j is the paired vertex of x_j . Then $x'_j = v_i$, otherwise $S - \{w_j, x'_j\}$ is a smaller PDS of G_{i-1} . It is a contradiction. If $N(v_i) \subseteq S$, then $S - \{v_i, w_j\}$ is a smaller PDS of G_{i-1} . Thus there is a neighbor v'_i of v_i such that $v'_i \notin S$. In this case, $S - \{w_j\} \cup \{v'_i\}$ is also a $\gamma_{pr}(G_{i-1})$ -set. Therefore, we may assume that $D_2 \subseteq S$ and every vertex in D_2 is paired with another vertex in D_2 .

With the similar argument, we may assume that $D_1 \subseteq S$. Let u'_j is the paired vertex of u_j and $CC = \{u'_j \mid u'_j \notin D_1\}$. If $CC = \emptyset$, then we do nothing. If $CC \neq \emptyset$ and $v_i \notin CC$, then $S - CC$ is a smaller PDS of G_{i-1} , a contradiction. Thus we assume $v_i \in CC$ and it is paired with u_1 . Since $G_{i-1}[D_1]$ has a perfect matching, there must be a vertex in D_1 , say u_2 , such that $u'_2 \in CC$. If $N(v_i) \subseteq S$, then $S - \{u'_2, v_i\}$ is a smaller PDS of G_{i-1} . Thus there exists a neighbor v'_i of v_i such that $v'_i \notin S$. In this case, $S - CC \cup \{v_i, v'_i\}$ is a $\gamma_{pr}(G_{i-1})$ -set. Up to now, we may assume that $D \subseteq S$ and every vertex in D is paired with another vertex in D .

If $v_i \notin S$, then $S - D$ is a PDS of G_i . Thus $\gamma_{pr}(G_i) \leq |S| - |D| = \gamma_{pr}(G_{i-1}) - |D|$. Therefore $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$. If $v_i \in S$, let v'_i be its paired vertex. If $v'_i \in C_{i-1}(v_i)$, then there exists a neighbor v''_i of v_i such that $v''_i \notin S \cup C_{i-1}(v_i)$. Otherwise $S - \{v_i, v'_i\}$ is a smaller PDS of

G_{i-1} . Thus $S - \{v'_i\} \cup \{v''_i\}$ is a $\gamma_{pr}(G_{i-1})$ -set. So we assume that $v'_i \notin C_{i-1}(v_i)$. If $N(v'_i) \subseteq S$, then $S - \{v_i, v'_i\}$ is a smaller PDS of G_{i-1} . Thus there is a neighbor v''_i of v'_i such that $v''_i \notin S$, in this case, $S - \{v_i\} \cup \{v''_i\}$ is a $\gamma_{pr}(G_{i-1})$ -set not containing v_i . We may assume S is such a $\gamma_{pr}(G_{i-1})$ -set. Then $S - D$ is a PDS of G_i . Thus $\gamma_{pr}(G_i) \leq |S| - |D| = \gamma_{pr}(G_{i-1}) - |D|$. Therefore $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$. \square

If there is a $\gamma_{pr}(G_i)$ -set S' such that $r \notin S'$, then let $S = S' \cup D$. By Claim 1, S is a $\gamma_{pr}(G_{i-1})$ -set and $r \notin S$. Therefore, if r is contained in all $\gamma_{pr}(G_{i-1})$ -set, then r is contained in all $\gamma_{pr}(G_i)$ -set.

For converse, let S be an arbitrary $\gamma_{pr}(G_{i-1})$ -set and $PD = S \cap D_{i-1}[v_i]$.

Claim 2 $|D| \leq |PD| \leq |D| + 2$.

Proof It is obvious that $|PD| \geq |D|$. Next, we prove $|PD| \leq |D| + 2$. Let v'_i be the father of v_i , i.e., $F(v_i) = v'_i$, and B is a block of G_{i-1} containing v_i and v'_i . We discuss it according to the order of B .

Case 1: $V(B) = \{v_i, v'_i\}$

If $|PD| \geq |D| + 4$ and $|PD|$ is even, then $v'_i, v''_i \notin S$, where v''_i is the father of v'_i . Otherwise, $S - PD \cup D$ is a smaller PDS of G_{i-1} . However, $S - PD \cup D \cup \{v'_i, v''_i\}$ is also a smaller PDS of G_{i-1} , a contradiction. If $|PD| \geq |D| + 3$ and $|PD|$ is odd, then v_i and v'_i are paired in S . If $N(v'_i) \subset S$, then $S - PD - \{v'_i\} \cup D$ is a smaller PDS of G_{i-1} . Thus there is a neighbor w of v'_i such that $w \notin S$, then $S - PD \cup D \cup \{w\}$ is also a smaller PDS of G_{i-1} . It is a contradiction.

Case 2: $V(B) \neq \{v_i, v'_i\}$

Let w be another vertex in $V(B)$. If $|PD| \geq |D| + 4$ and $|PD|$ is even, then $w, v'_i \notin S$, then $S - PD \cup D \cup \{v'_i, w\}$ is a smaller PDS of G_{i-1} . If $|PD| \geq |D| + 3$ and $|PD|$ is odd, then $v_i \in S$. If w is the paired vertex of v_i , then there exists a neighbor w' of w such that $w' \notin S$. However, $S - PD \cup D \cup \{w'\}$ is a smaller PDS of G_{i-1} . It is a contradiction. If v'_i is the paired vertex of v_i , with the same argument to Case 1, we can also get a contradiction. \square

By Claim 2, we have $|D| \leq |PD| \leq |D| + 2$. We discuss the following cases according to the size of PD .

Case 1: $|PD| = |D| + 2$

In this case, $(N(v_i) \cap V(G_i)) \cap S = \emptyset$. If $|N(v_i) \cap V(G_i)| \geq 2$, then let $S' = S - PD \cup \{w', w''\}$, where $w', w'' \in N(v_i) \cap V(G_i)$. By claim 1, S' is a $\gamma_{pr}(G_i)$ -set. Then $r \in S'$. Since $d(v_i, r) \geq 3$, then $r \in S$. If $|N(v_i) \cap V(G_i)| = 1$, then $F(v_i), F(F(v_i)) \notin S$, then let $S' = S - PD \cup \{F(v_i), F(F(v_i))\}$. By Claim 1, S' is a $\gamma_{pr}(G_i)$ -set. Then $r \in S'$. Since $d(v_i, r) \geq 3$, then $r \in S$.

Case 2: $|PD| = |D| + 1$

In this case, $v_i \in S$, let \tilde{v} be its paired vertex. If $N(\tilde{v}) \subseteq S$, then $S - PD - \{\tilde{v}\} \cup D$ is a smaller PDS of G_{i-1} . Thus there is a neighbor w of \tilde{v} such that $w \notin S$. Let $S' = S - PD \cup \{w\}$. by Claim 1, S' is a $\gamma_{pr}(G_i)$ -set. Then $r \in S'$. Since $d(v_i, r) \geq 3$, then $r \in S$.

Case 3: $|PD| = |D|$

In this case, let $S' = S - PD$. Then by Claim 1, S' is a $\gamma_{pr}(G_i)$ -set. Then $r \in S'$. Thus $r \in S$. \square

Lemma 4 *When v_i is a considering vertex such that $d(r, v_i) = 2$. Let B_1 be the block containing v_i and $F(v_i)$, and let B_2 be the block containing $F(v_i)$ and r . Suppose $l(v_i) = \emptyset$, $(R_1 \cup R_2) \cap C_{i-1}(v_i) \neq \emptyset$ and $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ has a perfect matching. If G_{i-1} satisfies one of the following conditions:*

- (1) $|V(B_1)| \geq 3$;
- (2) $|V(B_1)| = 2$ and $C_{i-1}(F(v_i)) \neq \{v_i\}$;
- (3) $|V(B_1)| = 2$, $C_{i-1}(F(v_i)) = \{v_i\}$ and $|V(B_2)| \geq 3$.

Then r is contained in all $\gamma_{pr}(G_{i-1})$ -set if and only if r is contained in all $\gamma_{pr}(G_i)$ -set, where $G_i = G_{i-1} - D_{i-1}[v_i]$.

Proof We still use the notations in Lemma 3. With the same argument to Claim 1, $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$.

If there is a $\gamma_{pr}(G_i)$ -set S' such that $r \notin S'$, then let $S = S' \cup D$. Thus S is a $\gamma_{pr}(G_{i-1})$ -set and $r \notin S$. Therefore, if r is contained in all $\gamma_{pr}(G_{i-1})$ -set, then r is contained in all $\gamma_{pr}(G_i)$ -set.

For converse, let S be an arbitrary $\gamma_{pr}(G_{i-1})$ -set and $PD = S \cap D_{i-1}[v_i]$. With the similar argument to Claim 2, $|D| \leq |PD| \leq |D| + 2$. We discuss the following case according to the size of PD .

Case 1: $|PD| = |D| + 2$

If $|V(B_1)| \geq 3$, then let w be a vertex in $V(B_1)$ other than v_i and $F(v_i)$. Then $w, F(v_i) \notin S$. Let $S' = S - PD \cup \{w, F(v_i)\}$. Then S' is a $\gamma_{pr}(G_i)$ -set. Since any new added vertex is not r , then $r \in S$. If $|V(B_1)| = 2$ and $C_{i-1}(F(v_i)) \neq \{v_i\}$, let w be a child of $F(v_i)$ other than v_i . It is obvious that $r, F(v_i) \notin S$. If $w \notin S$, then $S' = S - PD \cup \{F(v_i), w\}$ is a $\gamma_{pr}(G_i)$ -set. If $w \in S$ and w' is its paired vertex, then there is a neighbor w'' of w' such that $w'' \notin S$. Then $S' = S - PD \cup \{F(v_i), w''\}$ is a $\gamma_{pr}(G_i)$ -set. Thus $r \notin S'$. It contradicts that r is contained in all $\gamma_{pr}(G_i)$ -set. If $|V(B_1)| = 2$, $C_{i-1}(F(v_i)) = \{v_i\}$ and $|V(B_2)| \geq 3$, let w be a vertex in $V(B_2)$ other than $F(v_i)$ and r . Then $\{r, F(v_i), w\} \cap S = \emptyset$. Let $S' = S - PD \cup \{w, F(v_i)\}$. Then S' is a $\gamma_{pr}(G_i)$ -set. However, $r \notin S'$. It contradicts that r is contained in all $\gamma_{pr}(G_i)$ -set.

Case 2: $|PD| = |D| + 1$

In this case, $v_i \in S$. Let v'_i be the paired vertex of v_i , then $v'_i \in V(B_1)$. Suppose $|V(B_1)| \geq 3$. If $v'_i \neq F(v_i)$ and $F(v_i) \notin S$, then $S' = S - PD \cup \{F(v_i)\}$ is a $\gamma_{pr}(G_i)$ -set. If $v'_i \neq F(v_i)$ and $F(v_i) \in S$, then v'_i is a cut-vertex of G_{i-1} . Otherwise, $S - PD - \{v'_i\} \cup D$ is a smaller PDS

of G_{i-1} . It is impossible that $C_{i-1}(v'_i) \subseteq S$. Thus there is a child w of v'_i such that $w \notin S$. $S' = S - PD \cup \{w\}$ is a $\gamma_{pr}(G_i)$ -set. If $v'_i = F(v_i)$, let w be a vertex in $V(B_1)$ other than v_i and $F(v_i)$. If $w \notin S$, then $S' = S - PD \cup \{w\}$ is a $\gamma_{pr}(G_i)$ -set. If $w \in S$, then w is a cut-vertex. If its paired vertex $w' \in C_{i-1}(w)$, then there is a neighbor w'' of w' such that $w'' \notin S$. $S' = S - PD \cup \{w''\}$ is a $\gamma_{pr}(G_i)$ -set. If $w \in S$ and its paired vertex $w' \in V(B_1)$, then w' is also a cut-vertex. It is impossible that $C_{i-1}(w) \subseteq S$, i.e., there is a child w'' of w such that $w'' \notin S$. $S' = S - PD \cup \{w''\}$ is a $\gamma_{pr}(G_i)$ -set. In any case, $r \in S'$. On the other hand, any new added vertex is not r . So $r \in S$.

Suppose $|V(B_1)| = 2$ and $C_{i-1}(F(v_i)) \neq \{v_i\}$. In this case, v_i and $F(v_i)$ are paired in S . Let w be a child of $F(v_i)$ other than v_i . If $w \notin S$, then $S' = S - PD \cup \{w\}$ is a $\gamma_{pr}(G_i)$ -set. If $w \in S$, let w' be its paired vertex. Then there is a neighbor w'' of w' such that $w'' \notin S$. $S' = S - PD \cup \{w''\}$ is a $\gamma_{pr}(G_i)$ -set. In any case, $r \in S'$. Since any new added vertex is not r , thus $r \in S$.

Suppose $|V(B_1)| = 2$, $C_{i-1}(F(v_i)) = \{v_i\}$ and $|V(B_2)| \geq 3$. In this case, v_i and $F(v_i)$ are paired in S . Moreover, $r \notin S$, otherwise $S - PD - \{F(v_i)\} \cup D$ is a smaller PDS of G_{i-1} . Let w be a vertex in $V(B_2)$ other than $F(v_i)$ and r . It is obvious that $w \notin S$, then $S' = S - PD \cup \{w\}$ is a $\gamma_{pr}(G_i)$ -set. Thus $r \notin S'$. It contradicts that r is contained in all $\gamma_{pr}(G_i)$ -set.

Case 3: $|PD| = |D|$

In this case, $S' = S - PD$ is a $\gamma_{pr}(G_i)$ -set. Then $r \in S$ due to $r \in S'$. \square

If $d(v_i, r) = 2$, $|V(B_1)| = 2$, $C_{i-1}(F(v_i)) = \{v_i\}$, $|V(B_2)| = 2$ and v_i satisfies other conditions in Lemma 4, then we can not prune G_{i-1} . We call B_2 the first kind of TYPE-1 block containing r .

Lemma 5 *When v_i is a considering vertex such that $d(r, v_i) = 1$. Let B be the block containing v_i and r . Suppose $l(v_i) = \emptyset$, $(R_1 \cup R_2) \cap C_{i-1}(v_i) \neq \emptyset$ and $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ has a perfect matching. If $|V(B)| \geq 4$ or $|V(B)| = 3$ and every vertex in $V(B)$ is cut-vertex, then r is contained in all $\gamma_{pr}(G_{i-1})$ -set if and only if r is contained in all $\gamma_{pr}(G_i)$ -set, where $G_i = G_{i-1} - D_{i-1}[v_i]$.*

Proof We still use the notations in Lemma 3. With the same argument to Claim 1, $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$.

If there is a $\gamma_{pr}(G_i)$ -set S' such that $r \notin S'$, then let $S = S' \cup D$. Thus S is a $\gamma_{pr}(G_{i-1})$ -set and $r \notin S$. Therefore, if r is contained in all $\gamma_{pr}(G_{i-1})$ -set, then r is contained in all $\gamma_{pr}(G_i)$ -set.

For converse, let S be an arbitrary $\gamma_{pr}(G_{i-1})$ -set and $PD = S \cap D_{i-1}[v_i]$. With the similar argument to Claim 2, $|D| \leq |PD| \leq |D| + 2$.

Suppose $|PD| = |D| + 2$, then $N(v_i) \cap V(B) \cap S = \emptyset$. Otherwise, $S - PD \cup D$ is a smaller PDS of G_{i-1} . Thus $r \notin S$. If $|V(B)| \geq 4$, let w_1 and w_2 be two vertices other than v_i and

r . In this case, $S' = S - PD \cup \{w_1, w_2\}$ is a $\gamma_{pr}(G_i)$ -set. However, $r \notin S'$. It contradicts that r is contained in all $\gamma_{pr}(G_{i-1})$ -set. If $|V(B)| = 3$ and every vertex in $V(B)$ is cut-vertex, let w be another vertex in $V(B)$ other than v_i and r . If there is a child w_1 of w such that $w_1 \notin S$, then $S - PD \cup \{w, w_1\}$ is a $\gamma_{pr}(G_i)$ -set not containing r . It is also a contradiction. Otherwise, take any child of w , say w_1 . Suppose w_2 is the paired vertex of w_1 . If $N(w_2) \subset S$, then $S - PD - \{w_2\} \cup D \cup \{w\}$ is a smaller PDS of G_i . Thus there is a neighbor w_3 of w_2 such that $w_3 \notin S$. Then $S' = S - PD \cup D \cup \{w, w_3\}$ is a $\gamma_{pr}(G_i)$ -set not containing r . It is still a contradiction.

Suppose $|PD| = |D| + 1$, then $v_i \in S$. If r is paired with v_i , then we have done. If $|V(B)| \geq 4$, let w_1 and w_2 are two vertices other than v_i and r . We assume w_1 is the paired vertex of v_i . If $w_2 \notin S$, then $S' = S - PD \cup \{w_2\}$ is a $\gamma_{pr}(G_i)$ -set. If $w_2 \in S$, let w_3 be its paired vertex. If w_2 is not a cut-vertex, then $S - PD - \{w_2\} \cup D$ is a smaller PDS of $\gamma_{pr}(G_i)$ -set. Thus w_2 is a cut-vertex. If $w_3 \in C_{i-1}(w_2)$, then there is a neighbor w_4 of w_3 such that $w_4 \notin S$. $S' = S - PD \cup \{w_4\}$ is a $\gamma_{pr}(G_i)$ -set. If $w_3 \in V(B)$, then w_3 is also a cut-vertex and there is a child w_4 of w_3 such that $w_4 \notin S$. $S' = S - PD \cup \{w_4\}$ is a $\gamma_{pr}(G_i)$ -set. If $|V(B)| = 3$ and every vertex in $V(B)$ is cut-vertex, let w be another vertex in $V(B)$ other than v_i and r . In this case, w is the paired vertex of v_i . If there is a child w_1 of w such that $w_1 \notin S$, then $S' = S - PD \cup \{w_1\}$ is a $\gamma_{pr}(G_i)$ -set. Otherwise, take any child of w , say w_1 , and w_2 is its paired vertex. If $N(w_2) \subseteq S$, then $S - PD - \{w_2\} \cup D$ is a smaller PDS of G_{i-1} . Thus there is a neighbor w_3 of w_2 such that $w_3 \notin S$. Then $S' = S - PD \cup \{w_3\}$ is a $\gamma_{pr}(G_i)$ -set. In any case, $r \in S'$. However, any new added vertex is not r . Thus $r \in S$.

If $|PD| = |D|$, then $S' = S - PD$ is a $\gamma_{pr}(G_i)$ -set. Thus $r \in S$ due to $r \in S'$. \square

If $d(v_i, r) = 1$, $|V(B)| = 3$, there is a vertex in $V(B)$ which is not cut-vertex and v_i satisfies other conditions in Lemma 5, then we can not prune G_{i-1} . We call B the second kind of TYPE-1 block containing r . If $d(v_i, r) = 1$, $|V(B)| = 2$ and v_i satisfies other conditions in Lemma 5, we call B the first kind of TYPE-2 block containing r .

Lemma 6 *When v_i is a considering vertex such that $l(v_i) = r_1$. Let B' is an end block containing v_i in G_{i-1} . If $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ has a perfect matching, then r is contained in all $\gamma_{pr}(G_{i-1})$ -set if and only if r is contained in all $\gamma_{pr}(G_i)$ -set, where $G_i = G_{i-1} - (D_{i-1}(v_i) - V(B'))$.*

Proof Let $D_1 = R_1 \cap C_{i-1}(v_i)$, $D_2 = R_2 \cap D_{i-1}(v_i)$ and $D = D_1 \cup D_2$. Similar to Claim 1, we obtain the following claim.

Claim 3 $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$.

If there is a $\gamma_{pr}(G_i)$ -set S' such that $r \notin S'$, then let $S = S' \cup D$. By Claim 3, S is a $\gamma_{pr}(G_{i-1})$ -set and $r \notin S$. Therefore, if r is contained in all $\gamma_{pr}(G_{i-1})$ -set, then r is contained in all $\gamma_{pr}(G_i)$ -set.

For converse, let S be an arbitrary $\gamma_{pr}(G_{i-1})$ -set and $PD = S \cap (D_{i-1}(v_i) - V(B'))$.

Claim 4 $|D| \leq |PD| \leq |D| + 1$.

Proof If $|PD| \geq |D| + 2$ and $|PD|$ is even. Since $|V(B') \cap S| \geq 1$, then either $v_i \in S$ or $y \in S$, where $y \in V(B') - \{v_i\}$. $S - PD \cup D$ is a smaller PDS of G_{i-1} . It is a contradiction.

If $|PD| \geq |D| + 3$ and $|PD|$ is odd. In this case, $v_i \in S$ and its paired vertex $v \in C_{i-1}(v_i) - V(B')$. Let $x \in V(B') - \{v_i\}$, then $x \notin S$. $S - PD \cup D \cup \{x\}$ is a smaller PDS of G_{i-1} . It is a contradiction. \square

If $|PD| = |D| + 1$, then $v_i \in S$ and its paired vertex $v \in C_{i-1}(v_i) - V(B')$. Let $S' = S - PD \cup \{x\}$, where $x \in V(B') - \{v_i\}$. By Claim 3, S' is a $\gamma_{pr}(G_i)$ -set. Then $r \in S'$. Since $x \neq r$, $r \in S$.

If $|PD| = |D|$. Since $V(B') \cap S \neq \emptyset$, Thus $S' = S - PD$ is a PDS of G_i . By Claim 3, S' is also a $\gamma_{pr}(G_i)$ -set. Thus $r \in S$ due to $r \in S'$. \square

Lemma 7 When v_i is a considering vertex such that $d(r, v_i) \geq 3$ and $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ has not a perfect matching, let M be the maximum matching in $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ and $u \in (R_1 \cap C_{i-1}(v_i)) - V(M)$. Then r is contained in all $\gamma_{pr}(G_{i-1})$ -set if and only if r is contained in all $\gamma_{pr}(G_i)$ -set, where $G_i = G_{i-1} - (D_{i-1}(v_i) - D_{i-1}[u])$.

Proof Let $D_1 = R_1 \cap C_{i-1}(v_i)$ and $D_2 = R_2 \cap D_{i-1}(v_i)$. Take one child of each vertex in $D_1 - V(M) - \{u\}$ to construct vertex set D'_1 . $D = D_1 \cup D_2 \cup D'_1 - \{u\}$. Then we obtain the following claim.

Claim 5 $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$.

Proof Any $\gamma_{pr}(G_i)$ -set can be extended to a PDS of G_{i-1} by adding D . Thus $\gamma_{pr}(G_{i-1}) \leq \gamma_{pr}(G_i) + |D|$.

For converse, let S be a $\gamma_{pr}(G_{i-1})$ -set. With the same argument to Claim 1, $D_2 \subset S$ and every vertex in D_2 is paired with another vertex in D_2 . Moreover, we may assume $D_1 \subset S$. Let $CC = \{x \mid x \notin D_1, x \text{ is paired with one vertex in } D_1\}$. Since M is a maximum matching of $G_{i-1}[D_1]$. Thus $|CC| \geq |D_1| - |V(M)| = |D'_1| + 1$. If $v_i \notin S$, then $S - CC \cup D'_1 \cup \{v_i\}$ is also a $\gamma_{pr}(G_{i-1})$ -set. If $v_i \in S$ and v_i is paired with one vertex in D_1 , then $S - CC \cup D'_1 \cup \{v_i\}$ is also a $\gamma_{pr}(G_{i-1})$ -set. If $v_i \in S$ and v_i is not paired with any vertex in D_1 , let v be its paired vertex. Then $v \notin C_{i-1}(v_i)$, otherwise, $S - CC - \{v\} \cup D'_1$ is a smaller PDS of G_{i-1} . Thus $v \in V(B)$, where B is a block containing v_i and $F(v_i)$. If $N(v) \subseteq S$, then $S - CC - \{v\} \cup D'_1$ is a smaller PDS of G_{i-1} . Thus there is a neighbor v' of v such that $v' \notin S$. Then $S - CC \cup D'_1 \cup \{v'\}$ is also a $\gamma_{pr}(G_{i-1})$ -set. Therefore, we may assume $D_1 \cup D'_1 \cup \{v_i\} \subseteq S$ and they are paired each other. Since u is the paired vertex of v_i , $S - D$ is a PDS of G_i . Therefore, $\gamma_{pr}(G_i) \leq |S - D| = |S| - |D| = \gamma_{pr}(G_{i-1}) - |D|$.

So $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$. \square

If there is a $\gamma_{pr}(G_i)$ -set S' such that $r \notin S'$, then let $S = S' \cup D$ if $u \in S'$ or $v_i \in S'$ and otherwise, let $S = S' - D_{i-1}[u] \cup \{u, v_i\} \cup D$. By claim 5, S is a $\gamma_{pr}(G_{i-1})$ -set and $r \notin S$. Therefore, if r is contained in all $\gamma_{pr}(G_{i-1})$ -set, then r is contained in all $\gamma_{pr}(G_i)$ -set.

For converse, let S be an arbitrary $\gamma_{pr}(G_{i-1})$ -set and $PD = (D_{i-1}(v_i) - D_{i-1}[u]) \cap S$. We obtain the following claim.

Claim 6 $|D| \leq |PD| \leq |D| + 1$

Proof It is obvious that $|PD| \geq |D|$. Suppose $|PD| \geq |D| + 2$ and $|PD|$ is even. If $v_i \in S$, then $S - PD \cup D$ is a smaller PDS of G_{i-1} . If $v_i \notin S$, then $S - D_{i-1}[v_i] \cup D \cup \{v_i, u\}$ is a smaller PDS of G_{i-1} . It is a contradiction. Suppose $|PD| \geq |D| + 3$ and $|PD|$ is odd. In this case, one of vertices v_i, u is in S such that its paired vertex is in $D_{i-1}(v_i) - D_{i-1}[u]$. If v_i is such a vertex, then $|D_{i-1}[v_i] \cap S| \geq |D| + 4$. $S - D_{i-1}[v_i] \cup D \cup \{u, v_i\}$ is a smaller PDS of G_{i-1} . It is a contradiction. If u is such a vertex and $v_i \notin S$, then $S - PD \cup D \cup \{v_i\}$ is a smaller PDS of G_{i-1} . It is also a contradiction. If u is such a vertex and $v_i \in S$, then the paired vertex of v_i is not a child of v_i . Let v be its paired vertex. If $N(v) \subset S$, then $S - PD - \{v\} \cup D$ is a smaller PDS of G_{i-1} . Thus there is a neighbor v' of v such that $v' \notin S$. However, $S - PD \cup D \cup \{v'\}$ is also a smaller PDS of G_{i-1} . It is also a contradiction. \square

Suppose $|PD| = |D| + 1$. If $v_i \in S$ and its paired vertex is in $D_{i-1}(v_i) - D_{i-1}[u]$, then $|D_{i-1}[v_i] \cap S| \geq |D| + 2$. Let $S' = S - D_{i-1}[v_i] \cup \{u, v_i\}$. By Claim 5, S' is a $\gamma_{pr}(G_i)$ -set. If $u \in S$ and its paired vertex is in $D_{i-1}(v_i) - D_{i-1}[u]$. If $v_i \notin S$, then $S' = S - PD \cup \{v_i\}$ is a $\gamma_{pr}(G_i)$ -set by Claim 5. If $v_i \in S$, let v be its paired vertex. Then $v \in V(G_i)$ and there is a neighbor v' of v such that $v' \notin S$. $S' = S - PD \cup D \cup \{v'\}$ is a $\gamma_{pr}(G_i)$ -set. In any case, $r \in S'$. Since $d(r, v_i) \geq 3$, any new added vertex is not r . thus $r \in S$.

Suppose $|PD| = |D|$. If $v_i \notin S$, then $S' = S - D_{i-1}[v_i] \cup \{v_i, u\}$ is a $\gamma_{pr}(G_i)$ -set. If $v_i \in S$, then $S' = S - PD$ is a $\gamma_{pr}(G_i)$ -set. In any case, $r \in S'$. Since $d(v_i, r) \geq 3$, then $r \in S$. \square

Similar to Lemma 4 and Lemma 5, we can obtain the following lemma. The detail of the proof is omitted in here.

Lemma 8 *When v_i is a considering vertex such that $d(r, v_i) \leq 2$ and $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ has not a perfect matching, let M be the maximum matching in $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ and $u \in R_1 \cap C_{i-1}(v_i) - V(M)$. Let B_1 be a block containing v_i and $F(v_i)$ and B_2 be a block containing $F(v_i)$ and $F(F(v_i))$ if exists. If G_{i-1} satisfies one of the following conditions:*

- (1) $d(v_i, r) = 2$ and $|V(B_1)| \geq 3$;
- (2) $d(v_i, r) = 2$, $|V(B_1)| = 2$ and $C_{i-1}(F(v_i)) \neq \{v_i\}$;

- (3) $d(v_i, r) = 2$, $|V(B_1)| = 2$, $C_{i-1}(F(v_i)) = \{v_i\}$ and $|V(B_2)| \geq 3$;
- (4) $d(v_i, r) = 1$ and $|V(B_2)| \geq 4$;
- (5) $d(v_i, r) = 1$, $|V(B_2)| = 3$ and every vertex in $V(B_2)$ is cut-vertex.

Then r is contained in all $\gamma_{pr}(G_{i-1})$ -set if and only if r is contained in all $\gamma_{pr}(G_i)$ -set, where $G_i = G_{i-1} - (D_{i-1}(v_i) - D_{i-1}[u])$.

If $d(v_i, r) = 2$, $|V(B_1)| = 2$, $C_{i-1}(F(v_i)) = \{v_i\}$, $|V(B_2)| = 2$ and v_i satisfies other conditions in Lemma 8, then we can not prune G_{i-1} . We call B_2 the first kind of TYPE-3 block containing r . If $d(v_i, r) = 1$, $|V(B_2)| = 3$ and there is a vertex in $V(B_2)$ which is not cut-vertex and v_i satisfies other conditions in Lemma 8, then we still can not prune G_{i-1} . We call B_2 the second kind of TYPE-3 block containing r . If $d(v_i, r) = 1$, $|V(B_2)| = 2$ and v_i satisfies other conditions in Lemma 8, we call B_2 the second kind of TYPE-2 block containing r .

Summarizing the above lemmas, we have

Theorem 1 *Let G be a block graph with at least one cut-vertex and let \tilde{G} be the output of procedure PRUNE. Then r is contained in all minimum paired-dominating sets of G if and only if r is contained in all minimum paired-dominating sets of \tilde{G} .*

3 Algorithm

In this section, we will give some judgement rules to determine whether r is contained in all minimum paired-dominating sets of \tilde{G} , where \tilde{G} is the output of procedure PRUNE. Let $\tilde{R}_j = \{v \mid v \in V(\tilde{G}) \text{ and } l(v) = r_j\}$ for $j = 1, 2$. For $v \in V(\tilde{G})$, define $C_{\tilde{G}}(v) = C_G(v) \cap V(\tilde{G})$, $D_{\tilde{G}}(v) = D_G(v) \cap V(\tilde{G})$ and $D_{\tilde{G}}[v] = D_G[v] \cap V(\tilde{G})$.

According to lemmas in section 2, we can divide blocks containing r in \tilde{G} into the following categories (suppose B is a block containing r in \tilde{G} . Some examples of each category are shown in Fig. 1.):

- $L_1 = \{B \mid B \text{ is an end block with } |V(B)| = 2\};$ $L_2 = \{B \mid B \text{ is an end block with } |V(B)| \geq 3\};$
- $L_3 = \{B \mid B \text{ is a TYPE-1 block}\};$ $L_4 = \{B \mid B \text{ is a TYPE-2 block}\};$
- $L_5 = \{B \mid B \text{ is a TYPE-3 block}\};$
- $L_6 = \{B \mid |\tilde{R}_1 \cap (V(B) - \{r\})| \text{ is odd and } \tilde{R}_2 \cap V(B) = \emptyset\};$
- $L_7 = \{B \mid |\tilde{R}_1 \cap (V(B) - \{r\})| \neq 0 \text{ is even and } \tilde{R}_2 \cap V(B) = \emptyset\};$
- $L_8 = \{B \mid |\tilde{R}_1 \cap (V(B) - \{r\})| \text{ is odd and } \tilde{R}_2 \cap V(B) \neq \emptyset\};$
- $L_9 = \{B \mid |\tilde{R}_1 \cap (V(B) - \{r\})| \text{ is even and } \tilde{R}_2 \cap V(B) \neq \emptyset\}.$

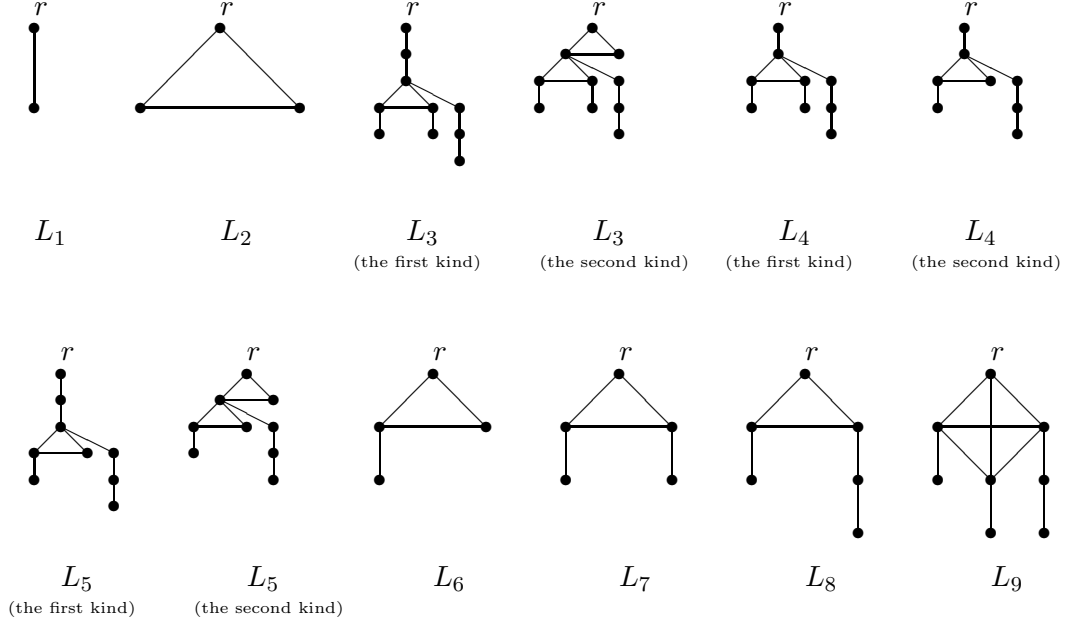


Fig. 1. Some examples of nine categories of blocks containing r in \tilde{G}

In order to simply the proof of judgement rules, we define $D(B)$ for any block $B \in \bigcup_{i=3}^9 L_i$ as follows:

- (1): If $B \in L_3$, then $|V(B)| = 2$ or $|V(B)| = 3$ and there is a vertex in $V(B)$ that is not cut-vertex. If $|V(B)| = 2$, then B is the first kind. Let u be the child of r in $V(B)$ and v be the child of u . If $|V(B)| = 3$, then B is the second kind. Let v be the child of r in $V(B)$ and v is a cut-vertex. In any case, $\tilde{G}[\tilde{R}_1 \cap C_{\tilde{G}}(v)]$ has a perfect matching. $D(B) = (\tilde{R}_1 \cap C_{\tilde{G}}(v)) \cup (\tilde{R}_2 \cap D_{\tilde{G}}(v))$.
- (2): If $B \in L_4$, then $|V(B)| = 2$. Let v be the child of r in $V(B)$. If B is the first kind, then $\tilde{G}[\tilde{R}_1 \cap C_{\tilde{G}}(v)]$ has a perfect matching. Let $D(B) = (\tilde{R}_1 \cap C_{\tilde{G}}(v)) \cup (\tilde{R}_2 \cap D_{\tilde{G}}(v))$. Otherwise, let M be the maximum matching in $\tilde{G}[\tilde{R}_1 \cap C_{\tilde{G}}(v)]$. Take one child of each vertex in $(\tilde{R}_1 \cap C_{\tilde{G}}(v)) - V(M) - \{w\}$ to construct D' , where $w \in \tilde{R}_1 \cap C_{\tilde{G}}(v) - V(M)$. $D(B) = (\tilde{R}_1 \cap C_{\tilde{G}}(v)) \cup D' \cup (\tilde{R}_2 \cap D_{\tilde{G}}(v)) \cup \{v, w\}$.
- (3): If $B \in L_5$, then $|V(B)| = 2$ or $|V(B)| = 3$ and there is a vertex in $V(B)$ that is not cut-vertex. If B is the first kind, let u be the child of r in $V(B)$ and v be the child of u . If B is the second kind, let v be the child of r in $V(B)$ and v is a cut-vertex. In any case, $\tilde{G}[\tilde{R}_1 \cap C_{\tilde{G}}(v)]$ has not a perfect matching. $D(B)$ is defined same as the second kind of (2).
- (4): If $B \in L_6 \cup L_8$, let $CC = \bigcup_{v \in V(B)} D_{\tilde{G}}[v]$. $D(B) = ((\tilde{R}_1 \cup \tilde{R}_2) \cap CC) \cup \{w\}$, where w is a child of some vertex in $\tilde{R}_1 \cap CC$.
- (5): If $B \in L_7 \cup L_9$, let $CC = \bigcup_{v \in V(B)} D_{\tilde{G}}[v]$. $D(B) = (\tilde{R}_1 \cup \tilde{R}_2) \cap CC$.

Lemma 9 *Let \tilde{G} be a output of procedure PRUNE, then r is contained in all minimum paired-dominating sets of \tilde{G} if and only if \tilde{G} satisfies one of the following conditions:*

- (1) $|L_1| \geq 1$;
- (2) $|L_1| = 0$ and $|L_2| \geq 2$;
- (3) $|L_1| = 0$, $|L_2| = 1$ and $|L_3 \cup L_6 \cup L_8| \geq 1$;
- (4) $|L_1| = 0$, $|L_2| = 0$ and $|L_3| \geq 2$;
- (5) $|L_1| = 0$, $|L_2| = 0$, $|L_3| = 1$ and $|L_6 \cup L_8| \geq 1$.

Proof If $|L_1| \geq 1$, then r is a support vertex in \tilde{G} , and hence r is contained in all minimum paired-dominating sets of \tilde{G} . Thus in the following discussion, we assume $|L_1| = 0$.

Case 1: $|L_2| \geq 2$

In this case, r is contained in at least two end block with order at least three, say B_1 and B_2 are two such blocks. Let S be an arbitrary $\gamma_{pr}(\tilde{G})$ -set. If $r \notin S$, then $|V(B_i) \cap S| \geq 2$ for $i = 1, 2$. Then $S - V(B_1) - V(B_2) \cup \{r, x\}$, where x is a vertex in $V(B_1) - \{r\}$, is a smaller PDS of \tilde{G} , a contradiction. Thus $r \in S$.

Case 2: $|L_2| = 1$ and $|L_3 \cup L_6 \cup L_8| \geq 1$

Let $B' \in L_2$ and S be an arbitrary $\gamma_{pr}(\tilde{G})$ -set not containing r . It is obvious $|V(B') \cap S| \geq 2$. If $|L_3| \geq 1$, let $B \in L_3$. If B is the first kind, let u be a child of r in $V(B)$. Since $r \notin S$, $|D_{\tilde{G}}[u] \cap S| \geq 2 + |D(B)|$. However, $S - D_{\tilde{G}}[u] - V(B') \cup D(B) \cup \{r, u\}$ is a smaller PDS of \tilde{G} . If B is the second kind, let w be a vertex in $V(B)$ which is not cut-vertex and u be another vertex. Since $r \notin S$, $|(D_{\tilde{G}}[u] \cup \{w\}) \cap S| \geq |D(B)| + 2$. Then $S - D_{\tilde{G}}[u] - V(B') - \{w\} \cup D(B) \cup \{r, u\}$ is a smaller PDS of \tilde{G} , a contradiction. Thus $r \in S$.

If $|L_6 \cup L_8| \geq 1$, let $B \in L_6 \cup L_8$. $CC = \bigcup_{v \in V(B)} D_{\tilde{G}}[v]$. Since $r \notin S$, $|CC \cap S| \geq |D(B)|$. However, $S - CC - V(B') \cup D(B) \cup \{r\} - \{w\}$, where $w \in D(B)$ and $l(w) = \emptyset$, is a smaller PDS of \tilde{G} , a contradiction. Thus $r \in S$.

Case 3: $|L_2| = 1$ and $|L_3 \cup L_6 \cup L_8| = 0$

Let $B' \in L_2$ and $y, z \in V(B') - \{r\}$. Since r is a cut-vertex, So $L_4 \cup L_5 \cup L_7 \cup L_9 \neq \emptyset$. Let S' be a vertex set by collecting $D(B)$ for any $B \in L_4 \cup L_5 \cup L_7 \cup L_9$. It is obvious that $S' \cup \{y, z\}$ is a $\gamma_{pr}(\tilde{G})$ -set. However, $r \notin S$.

Case 4: $|L_2| = 0$ and $|L_3| \geq 2$

Let $B_1, B_2 \in L_3$ and S be an arbitrary $\gamma_{pr}(\tilde{G})$ -set. Suppose $r \notin S$. For B_j ($j = 1, 2$), let $CC_j = \bigcup_{v \in V(B_j)} D_{\tilde{G}}[v]$. Since $r \notin S$, $|CC_j \cap S| \geq |D(B_j)| + 2$ for $j = 1, 2$. However, $S - CC_1 - CC_2 \cup D(B_1) \cup D(B_2) \cup \{r, u\}$, where u is a child of r in $V(B_1)$, is a smaller PDS of \tilde{G} , a contradiction. Thus $r \in S$.

Case 5: $|L_2| = 0$, $|L_3| = 1$ and $|L_6 \cup L_8| \geq 1$

Let $B_1 \in L_3$ and $B_2 \in L_6 \cup L_8$. Suppose S be an arbitrary $\gamma_{pr}(\tilde{G})$ -set and $r \notin S$. For B_j ($j = 1, 2$),

let $CC_j = \bigcup_{v \in V(B_j)} D_{\tilde{G}}[v]$. Since $r \notin S$, $|CC_1 \cap S| \geq |D(B_1)| + 2$ and $|CC_2 \cap S| \geq |D(B_2)|$. However, $S - CC_1 - CC_2 \cup D(B_1) \cup D(B_2) \cup \{r\} - \{w\}$, where $w \in D(B_2)$ and $l(w) = \emptyset$, is a smaller PDS of \tilde{G} , a contradiction. Thus $r \in S$.

Case 6: $|L_2| = 0$, $|L_3| = 1$ and $|L_6 \cup L_8| = 0$

Let $B \in L_3$. If B is the first kind, let u be the child of r in $V(B)$ and v be the child of u . If B is the second kind, let $\{u, v\} = V(B) - \{r\}$. Let S' be a vertex set by collecting $D(B^*)$ for any $B^* \in L_4 \cup L_5 \cup L_7 \cup L_9$. Let $S = S' \cup D(B) \cup \{u, v\}$. Then it is obvious S is a $\gamma_{pr}(\tilde{G})$ -set. However, $r \notin S$.

Case 7: $|L_2| = |L_3| = 0$

Let B be any block containing r , then $B \in L_4 \cup L_5 \cup L_6 \cup L_7 \cup L_8 \cup L_9$. Let S' be a vertex set by collecting $D(B)$ for any $B \in L_4 \cup L_5 \cup L_6 \cup L_7 \cup L_8 \cup L_9$. If $L_6 \cup L_7 \cup L_8 \cup L_9 \neq \emptyset$, then S' is a γ_{pr} -set of \tilde{G} . However, $r \notin S'$. Thus we may assume $L_6 \cup L_7 \cup L_8 \cup L_9 = \emptyset$. Then $B \in L_4 \cup L_5$. If there is a block $B \in L_4 \cup L_5$ which is the second kind of TYPE-2 or TYPE-3 block, then S' is still a γ_{pr} -set of \tilde{G} not containing r . Thus we may assume that $B \in L_4 \cup L_5$ and B is the first kind of TYPE-2 or TYPE-3 block. If there is a block $B \in L_5$, let u be the child of r in $V(B)$ and v is the child of u . Let w be the paired vertex in $D(B)$ and w' be the child of w . Then $S = S' \cup \{u, w'\}$ is a $\gamma_{pr}(\tilde{G})$ -set of \tilde{G} . However, $r \notin S$. Then $B \in L_4$ for any block B and B is the first kind of TYPE-2 block. Let v be the child of r in $V(B)$. If there is a child w of v such that $l(w) = r_1$. Let w' be the child of w . Then $S = S' \cup \{v_1, w'\}$ is a $\gamma_{pr}(\tilde{G})$ -set not containing r . Thus we may assume every child w of v satisfies $l(w) = r_2$. Let w' be the child of w such that $l(w') = r_2$ and let w'' be the child of w' . Take $S = S' \cup \{v, w''\}$. It is obvious that S is a $\gamma_{pr}(\tilde{G})$ -set not containing r . \square

Now we are ready to present the algorithm to determine whether r is contained in all minimum paired-dominating sets of G .

Algorithm VIAMPDS. Determine whether the cut-vertex r of a block graph G is contained in all minimum paired-dominating sets of G

Input. A block graph G with at least one cut-vertex and a cut-vertex r . The vertex ordering obtained by procedure VO.

Output. True or False

Method

Let \tilde{G} be the output of procedure PRUNE with input G .

Let $L_1 = \{B \mid B \text{ is an end block with } |V(B)| = 2\};$

$L_2 = \{B \mid B \text{ is an end block with } |V(B)| \geq 3\};$

$L_3 = \{B \mid B \text{ is a TYPE-1 block}\};$

$L_6 = \{B \mid B \text{ is a block such that } |\tilde{R}_1 \cap (V(B) - \{r\})| \text{ is odd and } \tilde{R}_2 \cap V(B) = \emptyset\};$

$L_8 = \{B \mid B \text{ is a block such that } |\tilde{R}_1 \cap (V(B) - \{r\})| \text{ is odd and } \tilde{R}_2 \cap V(B) \neq \emptyset\}.$
 $(B \text{ is a block containing } r)$
 If $(|L_1| \geq 1)$ then
 Return Ture;
 else if $(|L_2| \geq 2)$ then
 Return Ture;
 else if $(|L_2| = 1 \text{ and } |L_3 \cup L_6 \cup L_8| \geq 1)$ then
 Return Ture;
 else if $(|L_2| = 0 \text{ and } |L_3| \geq 2)$ then
 Return Ture;
 else if $(|L_2| = 0 \text{ and } |L_3| = 1 \text{ and } |L_6 \cup L_8| \geq 1)$ then
 Return Ture;
 else
 Return False;
 endif
 end

Theorem 2 *Algorithm VIAMPDS can determine whether the give cut-vertex of a block graph G with at least one cut-vertex is contained in all minimum paired-dominating sets in linear-time $O(n + m)$, where $n = |V(G)|$ and $m = |E(G)|$.*

Proof By Theorem 1, r is contained in all minimum paired-dominating sets of G if and only if r is contained in all minimum paired-dominating sets of \tilde{G} , where \tilde{G} is the output of procedure PRUNE with input G . Moreover, by Lemma 9, the judgement rules in algorithm VIAMPDS can determine whether r is contained in all minimum paired-dominating sets of \tilde{G} . On the other hand, every vertex and edge is used in a constant times in algorithm VIAMPDS. Thus the theorem follows. \square

4 Conclusion

In this paper, we give a linear-time algorithm VIAMPDS to determine whether the given vertex is contained in all minimum paired-dominating sets of a block graph. Furthermore, the algorithm VIAMPDS can be used to determine the set of vertices contained in all minimum paired-dominating sets of a blocks graph in polynomial time. Finally, we would like to point out that if changing the pruning rules and judgement rules, our method is also available to determine whether a given vertex is contained in all minimum (total) dominating sets of a block graph.

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