The Partition Method for Poset-free Families

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Abstract

Given a finite poset P, let $\mathrm{La}(n, P)$ denote the largest size of a family of subsets of an n-set that does not contain P as a (weak) subposet. We employ a combinatorial method, using partitions of the collection of all full chains of subsets of the n-set, to give simpler new proofs of the known asymptotic behavior of $\mathrm{La}(n, P)$, as $n \to \infty$, when P is the r-fork $\mathcal{V}_r$, the four-element N poset $\mathcal{N}$, and the four-element butterfly-poset $\mathcal{B}$.

Dedicated to Gerard Chang on the occasion of his 60th birthday

1 Largest Families Without a Forbidden Poset

We study how large can a family of subsets of an n-set be that avoids containing a given finite poset P as a subposet. For two posets $P = (P, \leq)$ and $P' = (P', \leq')$, we say that P contains P' as a *weak subposet* if there exists an injection $f\colon P' \to P$ that preserves the partial ordering, meaning that whenever $u \leq' v$ in P', we have $f(u) \leq f(v)$ in P [13]. Throughout the paper, when we say subposet, we mean weak subposet.

Let $[n] := \{1, \ldots, n\}$, and let the Boolean lattice $\mathcal{B}_n = (2^{[n]}, \subseteq)$ denote the power set of $[n]$ with the set-inclusion relation. We consider families $\mathcal{F} \subseteq 2^{[n]}$. Then $\mathcal{F}$ can be viewed as a subposet of $\mathcal{B}_n$. If $\mathcal{F}$ contains no subposet P, we say $\mathcal{F}$ is *P-free.* We are interested in determining the largest size of a P-free family of subsets of $[n]$, denoted $\mathrm{La}(n, P)$.

The original result of this type is Sperner's Theorem [14] on the maximum size of antichain in the Boolean lattice. An antichain is a family of subsets such that no subset is included in another, which we can view as a poset that contains no two-element chain as a subposet. Erdős [5] generalized this to give the largest size $\mathrm{La}(n, \mathcal{P}_k)$ of a family that does not contain a chain of size k, the *path poset* $\mathcal{P}_k$ consisting of k totally ordered elements $A_1 < A_2 < \cdots < A_k$.

Katona brought the attention of many researchers to the generalization of this problem to posets P besides chains $\mathcal{P}_k$. In every case that is solved, $\mathrm{La}(n, P)$ is asymptotic to a constant times $\binom{n}{\lfloor\frac{n}{2}\rfloor}$, as $n \to \infty$, and, moreover, the constant is an integer. Griggs and Lu conjectured that this is true for every P, though it remains stubbornly unresolved, even for some posets with as few as 4 elements:

Conjecture 1.1 *[9] For any poset P, define*

$$\pi(P) := \lim_{n\to\infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor\frac{n}{2}\rfloor}}.$$

Then $\pi(P)$ exists and it is an integer determined by P.

Among the studies over the last fifteen years we mention the cases that P is $\mathcal{V}_r$ [15], the butterfly $\mathcal{B}$ [4], the bipartite poset $\mathcal{K}_{r,s}$ [3], the poset $\mathcal{N}$ [7], the crown $\mathcal{O}_{2k}$ [9], any tree poset [1], the diamond $\mathcal{D}_k$ [8], and harps [8]. See [8] for a survey of the subject. The methods employed by these researchers include the LYM inequality, the cyclic permutation method, linear programming, and probabilistic arguments. Our goal in this paper is to illustrate a strategy we introduced in [8], which we call the Partition Method, by presenting simpler new proofs of several of the results listed above.

2 The Lubell Function and the Partition Method

We say that a family of subsets of $[n]$ is a *full (maximal) chain* if it consists of $n+1$ subsets $\emptyset \subset \{i_1\} \subset \{i_1, i_2\} \subset \cdots \subset [n]$. Fix some family $\mathcal{F} \subseteq 2^{[n]}$. Let $\mathscr{C} := \mathscr{C}_n$ denote the collection of all $n!$ full chains in the Boolean lattice $\mathcal{B}_n$. A method used by Katona *et al.* involves counting the number of full chains that meet $\mathcal{F}$. Here we collect information about the average number of times chains $\mathcal{C} \in \mathscr{C}$ meet $\mathcal{F}$, which can be used to give an upper bound on $|\mathcal{F}|$. The *height* of a family $\mathcal{F}$ is

$$h(\mathcal{F}) := \max_{\mathcal{C}\in\mathscr{C}} |\mathcal{F} \cap \mathcal{C}|.$$

We consider what we call the *Lubell function* of $\mathcal{F}$, which is

$$\bar{h}(\mathcal{F}) = \bar{h}_n(\mathcal{F}) := \mathrm{ave}_{\mathcal{C}\in\mathscr{C}} |\mathcal{F} \cap \mathcal{C}|.$$

This is the expected value $\mathrm{E}(|\mathcal{F} \cap \mathcal{C}|)$ over a random full chain $\mathcal{C}$ in $\mathcal{B}_n$. Then $\bar{h}(\mathcal{F})$ is essentially the function at the heart of Lubell's elegant proof of Sperner's Theorem [12, 6] with the observation:

Lemma 2.1 *[8] Let $\mathcal{F}$ be a collection of subsets of $[n]$. Then*

$$\bar{h}(\mathcal{F}) = \sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}}.$$

Lubell's proof uses the simple facts that $|\mathcal{A} \cap \mathcal{C}| \leq 1$ for any antichain $\mathcal{A}$ and that $\binom{n}{k}$ is maximized by taking $k = \lfloor \frac{n}{2} \rfloor$, to derive Sperner's Theorem that $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. By similar reasoning for general families $\mathcal{F}$ we obtain a general upper bound on $|\mathcal{F}|$. The idea is that $\bar{h}(\mathcal{F})$ can be viewed as weighted sum where each set F has weight $1/\binom{n}{|F|}$. To maximize $|\mathcal{F}|$, the sets in the family must have weights as small as possible, which is close to the middle level in the Boolean lattice.

In the following lemma, $\Sigma(n, m)$ is the sum of the m middle binomial coefficients in n. The family of m middle sizes of subsets is denoted by $\mathcal{B}(n, m)$. Depending on the parity of $m + n$, either there is a unique family as $\mathcal{B}(n, m)$ or there are two possible families as $\mathcal{B}(n, m)$. However, the size of $\mathcal{B}(n, m)$ is always $\Sigma(n, m)$.

Lemma 2.2 *[8] Let $\mathcal{F}$ be a collection of subsets of $[n]$. If $\bar{h}(\mathcal{F}) \leq m$, for real number $m > 0$, then*

$$|\mathcal{F}| \leq m\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Moreover, if m is an integer, then

$$|\mathcal{F}| \leq \Sigma(n, m),$$

and equality holds if and only if $\mathcal{F} = \mathcal{B}(n, m)$ (when $n+m$ is odd), or if $\mathcal{F} = \mathcal{B}(n, m-1)$ together with any $\binom{n}{(n-m)/2}$ subsets of sizes $(n-m)/2$ or $(n+m)/2$ (when $n+m$ is even).

The value of the Lubell function of a family $\mathcal{F}$ bounds its size. However, it can be difficult to obtain a good bound on $\bar{h}(\mathcal{F})$ for P-free families $\mathcal{F}$. We have discovered that a "partition method" can be fruitful [8]. Specifically, we partition the set $\mathscr{C}$ of full chains into blocks $\mathscr{C}(i)$, and then bound the average size $|\mathcal{F} \cap \mathcal{C}|$ over full chains $\mathcal{C} \in \mathscr{C}(i)$, for each i separately. The principle is that the average size $|\mathcal{F} \cap \mathcal{C}|$ over all full chains $\mathcal{C}$ is at most the maximum over i of the average over block $\mathscr{C}(i)$. The partition depends on the family $\mathcal{F}$.

In next sections, we present different partitions on the set of full chains for several posets P that were studied before. Before that, we include a technical proposition on estimating the sum of small binomial coefficients that is well-known in information theory and probability theory. If a poset P is nontrivial (not an antichain), then the largest size of P-free families is $\Omega(\binom{n}{\lfloor \frac{n}{2} \rfloor})$. The main body of a large P-free family must be near the middle levels in the Boolean lattice since the total number of sets far from the middle is small compared to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proposition 2.3 *[9] For $k = 2\sqrt{n \log n}$, we have the following estimation of the sum of binomial coefficients:*

$$2\sum_{i=0}^{\lfloor \frac{n}{2} - k \rfloor} \binom{n}{i} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} O\left(\frac{1}{n^{3/2}}\right).$$

For our problems it means we may restrict our consideration to P-free families containing subsets of sizes restricted to the interval $[\frac{n}{2} - 2\sqrt{n \log n}, \frac{n}{2} + 2\sqrt{n \log n}]$.

3 Families Containing no Forks

Let $\mathcal{V}_{r+1}$ denote the $(r+1)$-fork poset, that is an antichain on r elements with an extra element smaller than every element in the antichain. The $\mathcal{V}_{r+1}$-free families were first studied by Thanh [15]. His upper bound was improved by De Bonis and Katona [3].

Using the partition method we get an upper bound for $\mathrm{La}(n, \mathcal{V}_{r+1})$ similar to the result of De Bonis and Katona. The error term they obtained is $O(\frac{1}{n^2})$, which is better than ours, but our method is much simpler for showing that $\pi(\mathcal{V}_{r+1}) = 1$.

Theorem 3.1 *For the $(r+1)$-fork $\mathcal{V}_{r+1}$,*

$$\mathrm{La}(n, \mathcal{V}_{r+1}) \leq \left(1 + \frac{2r}{n} + O\left(\frac{\sqrt{\log n}}{n^{3/2}}\right)\right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Proof. Given a $\mathcal{V}_{r+1}$-free $\mathcal{F} \subset 2^{[n]}$, let k be some positive quantity that will be determined later. Split $\mathcal{F}$ into $\mathcal{F}^k$ and $\mathcal{F}'$ such that $\frac{n}{2} - k \leq |F| \leq \frac{n}{2} + k$ for each $F \in \mathcal{F}^k$. We concentrate on $\mathcal{F}^k$, in view of Proposition 2.3. Apply what we call the *min partition* on the set of full chains, meaning that for nonempty subsets $A \subsetneq [n]$, the block $\mathscr{C}_A$ consists of full chains satisfying $\min \mathcal{F}^k \cap \mathcal{C} = A$, while $\mathscr{C}(\emptyset)$ is the collection of full chains containing no sets in $\mathcal{F}^k$. We compute $\mathrm{ave}_{\mathcal{C} \in \mathscr{C}_A} |\mathcal{F}^k \cap \mathcal{C}|$ for each block $\mathscr{C}_A$. There are at most r sets in $\mathcal{F}^k \cap (A, [n]]$, and each such set B contributes $1/\binom{n-|A|}{|B|-|A|} \leq 1/\binom{n-|A|}{1}$ to $\mathrm{ave}_{\mathcal{C} \in \mathscr{C}_A} |\mathcal{F}^k \cap \mathcal{C}|$. This gives us the following inequality:

$$\begin{aligned}
\bar{h}(\mathcal{F}^k) &\leq \max_{\mathscr{C}_A} \left(\sum_{B \in \mathcal{F}^k \cap (A,[n])} 1 + \frac{1}{\binom{n-|A|}{|B|-|A|}} \right) \\
&\leq \max_{\mathscr{C}_A} \left(1 + \frac{r}{n - |A|}\right) \\
&\leq \max_{|A| \in [\frac{n}{2}-k, \frac{n}{2}+k]} \left(1 + \frac{r}{n - |A|}\right) \\
&\leq 1 + \frac{r}{\frac{n}{2} - k} \\
&\leq 1 + \frac{2r}{n} \left(1 + \frac{2k}{n} + \left(\frac{2k}{n}\right)^2 + \cdots \right) \\
&\leq 1 + \frac{2r}{n} + C\frac{k}{n^2} \quad \text{whenever } k \text{ is } o(n).
\end{aligned}$$

Letting $k = 2\sqrt{n \log n}$, we get

$$|\mathcal{F}^k| \leq \left(1 + \frac{2r}{n} + O\left(\frac{\sqrt{\log n}}{n^{3/2}}\right)\right) \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

and

$$|\mathcal{F}| = |\mathcal{F}^k| + |\mathcal{F}'| \leq \left(1 + \frac{2r}{n} + O\left(\frac{\sqrt{\log n}}{n^{3/2}}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

for any $\mathcal{V}_{r+1}$-free family $\mathcal{F}$. $\square$

Figure 1: $\mathcal{V}_{r+1}$ and $\mathcal{N}$.

4 Families Containing no $\mathcal{N}$

We say that four distinct sets A, B, C, and D form an $\mathcal{N}$ if they satisfy $A \subset B$, $C \subset B$ and also $C \subset D$. Our first task is to study the structure of $\mathcal{N}$-free families. Thinking in terms of the Hasse diagram, we say two sets E and F in a family $\mathcal{F}$ are *connected* if there exist sets $E = E_1$, E_2,..., $E_k = F$ all in $\mathcal{F}$ such that either $E_i \subseteq E_{i+1}$ or $E_{i+1} \subseteq E_i$. A (connected) *component* of $\mathcal{F}$ is a subfamily such that any two sets in it are connected, while any other set not in it is not connected to a set in the component. Let $\mathcal{F}$ be an $\mathcal{N}$-free family and denote its connected components by $\mathcal{F}_1$, $\mathcal{F}_2$,..., $\mathcal{F}_m$. These components must each be one of the following types: $\mathcal{P}_3$, $\mathcal{V}_r^*$(for some $r \geq 0$), and Λ_s(for some $s \geq 2$) where $\mathcal{V}_r^*$ means $r+1$ subsets such that $A \subset B_i$ for $i = 1, ..., r$ and $\{B_1, \ldots, B_r\}$ is an antichain, and Λ_s is a dual (up-side-down) $\mathcal{V}_s^*$. The poset $\mathcal{V}_r^*$ is also called *induced r-fork* in $\mathcal{F}$.

We give an upper bound for $\mathrm{La}(n, \mathcal{N})$ by partitioning the set of full chains and computing the ave$|\mathcal{F} \cap \mathcal{C}|$. It is similar to the proof of Griggs and Katona [7], who used linear programming technique to estimate $\mathrm{La}(n, \mathcal{N})$. As with our last theorem, the error term in our result is a little weaker than their $O(\frac{1}{n^2})$ but the method is relatively simple and straightforward to show $\pi(\mathcal{N}) = 1$.

Theorem 4.1 *For the poset $\mathcal{N}$,*

$$\mathrm{La}(n, \mathcal{N}) \leq \left(1 + \frac{2}{n} + O\left(\frac{\sqrt{\log n}}{n^{3/2}}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Proof. Let $\mathcal{F} \subset 2^{[n]}$ be $\mathcal{N}$-free. First suppose $\mathcal{F}$ contains $\mathcal{P}_3$ with $A \subset B \subset C$. Let $[A, C]$ be the interval of S with $A \subseteq S \subseteq C$. Note that except for A, B, and C, no set in $\mathcal{F}$ contains or is contained in any set in $[A, C]$. Then we see that replacing C by $(C - B) \cup A$ does not produce an $\mathcal{N}$, while eliminating the $\mathcal{P}_3$. By repeating the substitution, we obtain an $\mathcal{N}$-free family as large as $\mathcal{F}$, which is also $\mathcal{P}_3$-free. From now on, we only consider $\mathcal{N}$-free families whose components consisting of $\mathcal{V}_r^*$ or Λ_s. For such a family $\mathcal{F}$, we partition it into $\mathcal{F}^k$ and $\mathcal{F}'$ as in the last section. Let $\mathcal{F}_1$,..., $\mathcal{F}_m$ be the components of $\mathcal{F}^k$. Define $\mathscr{C}(\mathcal{F}_i)$ to be the collection of full chains that contain some set in $\mathcal{F}_i$ and $\mathscr{C}(\emptyset)$ be the collection of full chains do not contain any set in $\mathcal{F}^k$. No full chain contains two sets in different components, else the two components would be connected. Therefore, $\{\mathscr{C}(\mathcal{F}_i)\}_{i=1}^m \cup \{\mathscr{C}(\emptyset)\}$ is a partition of $\mathscr{C}$. We bound $\bar{h}(\mathcal{F}_k)$ by evaluating $\mathrm{ave}_{\mathcal{C}\in\mathscr{C}(\mathcal{F}_i)}|\mathcal{F}^k\cap\mathcal{C}|$.

Let us compute $\mathrm{ave}_{\mathcal{C}\in\mathscr{C}(\mathcal{F}_i)}|\mathcal{F}^k\cap\mathcal{C}|$ for $\mathcal{F}_i = \mathcal{V}_r^*$. It is trivial that if $r = 0$, then $\mathrm{ave}_{\mathcal{C}\in\mathscr{C}(\mathcal{F}_i)}|\mathcal{F}^k\cap\mathcal{C}| = 1$ Let $\mathcal{F}_i = \{A, B_1, \ldots, B_r\}$, $A \subset B_i$, be a component of $\mathcal{F}$, which is a $\mathcal{V}_r^*$ for some $r \geq 1$. Because $\{B_1, \ldots, B_r\}$ is an antichain, a full chain $\mathcal{C} \in \mathscr{C}(\mathcal{F}_i)$ either contains exactly one of sets in $\mathcal{F}_i$, or it contains two sets, both A and some B_j. Let the sizes of $B_1, \ldots, B_r$ and A be $b_1 \ldots, b_r$, and a, respectively. We have

$$\begin{aligned}
\mathrm{ave}_{\mathcal{C}\in\mathscr{C}(\mathcal{F}_i)}|\mathcal{F}^k\cap\mathcal{C}| &= \frac{1}{|\mathscr{C}(\mathcal{F}_i)|}\left(\sum_{\mathcal{C}:|\mathcal{F}_i\cap\mathcal{C}|=1} 1 + \sum_{\mathcal{C}:|\mathcal{F}_i\cap\mathcal{C}|=2} 2\right) \\
&= \frac{1}{|\mathscr{C}(\mathcal{F}_i)|}\left(\sum_{\mathcal{C}\in\mathscr{C}(\mathcal{F}_i)} 1 + \sum_{\mathcal{C}:|\mathcal{F}_i\cap\mathcal{C}|=2} 1\right) \\
&= 1 + \frac{1}{|\mathscr{C}(\mathcal{F}_i)|}\sum_{1\leq j\leq r} a!(b_j-a)!(n-b_j)! \\
&\leq 1 + \frac{a!(b_1-a)!(n-b_1)! + \cdots + a!(b_r-a)!(n-b_r)!}{b_1!(n-b_1)! + \cdots + b_r!(n-b_r)!} \\
&\leq 1 + \max_{1\leq j\leq r}\left\{\frac{a!(b_j-a)!(n-b_j)!}{b_j!(n-b_j)!}\right\} \\
&= 1 + \max_{1\leq j\leq r}\left\{\frac{1}{\binom{b_j}{a}}\right\} \\
&\leq 1 + \max_{1\leq j\leq r}\left\{\frac{1}{b_j}\right\} \\
&\leq 1 + \frac{1}{\frac{n}{2}-k} \\
&\leq 1 + \frac{2}{n} + C\,\frac{k}{n^2} \quad \text{whenever } k \text{ is } o(n).
\end{aligned}$$

Let $k = 2\sqrt{n\log n}$, then

$$\mathrm{ave}_{\mathcal{C}\in\mathscr{C}(\mathcal{F}_i)}|\mathcal{F}^k\cap\mathcal{C}| \leq 1 + \frac{2}{n} + O\left(\frac{\sqrt{\log n}}{n^{3/2}}\right)$$

as $\mathcal{F}_i = \mathcal{V}_r$ for any r. This is also an upper bound for $\mathrm{ave}_{\mathcal{C}\in\mathscr{C}(\mathcal{F}_i)}|\mathcal{F}^k\cap\mathcal{C}|$ if $\mathcal{F}_i = \Lambda_s$. Therefore,

$$\bar{h}(\mathcal{F}^k) \leq 1 + \frac{2}{n} + O\left(\frac{\sqrt{\log n}}{n^{3/2}}\right).$$

Hence, we have

$$|\mathcal{F}| = |\mathcal{F}^k| + |\mathcal{F}'| \leq \left(1 + \frac{2}{n} + O\left(\frac{\sqrt{\log n}}{n^{3/2}}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

for any $\mathcal{N}$-free family $\mathcal{F}$. □

5 Families Containing no Butterfly

The butterfly poset $\mathcal{B}$ consists of two antichains $\{A_1, A_2\}$ and $\{B_1, B_2\}$ such that $A_i \leq B_j$ for $i, j = 1, 2$. The result of De Bonis, Katona, and Swanepoel on butterfly-free families [4] can be deduced using our partition method. Their elegant proof employed the famous *cyclic permutation method* of Katona.

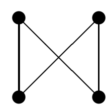

Figure 2: Butterfly poset $\mathcal{B}$.

Theorem 5.1 *For $n \geq 3$, $\mathrm{La}(n, \mathcal{B}) = \Sigma(n, 2)$. Furthermore, for $n \neq 4$, any $\mathcal{B}$-free family $\mathcal{F}$ with $|\mathcal{F}| = \Sigma(n, 2)$ must be $\mathcal{B}(n, 2)$. The largest $\mathcal{B}$-free families of subsets of $[4]$ are $\mathcal{B}(4, 2)$ and $\mathcal{F} = \{\{1\}, \{4\}, \{1, 2, 3\}, \{2, 3, 4\}\} \cup \binom{[4]}{2}$, up to relabelling the elements of $[4]$.*

Proof. The family $\mathcal{B}(n, 2)$ does not contain $\mathcal{B}$ as a subposet since no two sets of size k can simultaneously contain two different $(k-1)$-sets. Hence $\mathrm{La}(n, \mathcal{B}) \geq \Sigma(n, 2)$. We need to show $\Sigma(n, 2)$ is also an upper bound.

To begin, we claim that for $n \geq 3$ we have a $\mathcal{B}$-free $\mathcal{F}$ with $|\mathcal{F}| = \mathrm{La}(n, \mathcal{B})$ such that $\emptyset, [n] \notin \mathcal{F}$. For suppose $\emptyset \in \mathcal{F}$. If there exists some $S \in \binom{[n]}{1}$ not in $\mathcal{F}$, replacing $\emptyset$ by S does not produce $\mathcal{B}$ in the new family. We can do such a swap if there is some room in $\binom{[n]}{1}$. We cannot do the swap only when $\mathcal{F}$ contains $\{\emptyset\} \cup \binom{[n]}{1}$. Then the remaining sets in $\mathcal{F} \setminus (\{\emptyset\} \cup \binom{[n]}{1})$ must be mutually disjoint, else consider $a \in E \cap F$ for $E, F \in \mathcal{F} \setminus (\{\emptyset\} \cup \binom{[n]}{1})$. The four sets E, F, $\{a\}$, and $\emptyset$ would form a butterfly in $\mathcal{F}$, which is not allowed. Moreover, all sets in $\mathcal{F} \setminus (\{\emptyset\} \cup \binom{[n]}{1})$ have sizes at least two. So the number of such sets is $|\mathcal{F} \setminus (\{\emptyset\} \cup \binom{[n]}{1})| \leq \lfloor \frac{n}{2} \rfloor$. Hence, $|\mathcal{F}| \leq 1 + n + \lfloor \frac{n}{2} \rfloor < \Sigma(n, 2)$ for $n \geq 3$, a contradiction. The dual argument allows us to swap $[n]$ in $\mathcal{F}$ by some $(n-1)$-subset if $[n] \in \mathcal{F}$.

We now focus on $\mathcal{B}$-free families $\mathcal{F}$ containing neither $\emptyset$ nor $[n]$. When $n = 3$, we only consider $\mathcal{F} \subseteq \mathcal{B}(3, 2)$, so $\mathrm{La}(3, \mathcal{B}) = \Sigma(3, 2)$. When $n \geq 4$, note that for any full chain $\mathcal{C}$, we have $|\mathcal{F} \cap \mathcal{C}| \leq 3$, since $\mathcal{B}$ is a subposet of $\mathcal{P}_4$. Let us bound $\bar{h}(\mathcal{F})$ using the following partition of $\mathscr{C}$: Consider the subfamily $\mathcal{M}$ of $\mathcal{F}$ such that for every $M \in \mathcal{M}$, there exist two sets A and B in $\mathcal{F}$ with $A \subset M \subset B$. Let $\mathscr{C}(M)$ be the collection of full chains containing $M \in \mathcal{M}$, and let $\mathscr{C}(\emptyset)$ be the collection of full chains which contain no sets in $\mathcal{M}$. The collections $\{\mathscr{C}(M)\}_{M\in\mathcal{M}}$ together with $\mathscr{C}(\emptyset)$ form a partition of $\mathscr{C}$, since if a full chain contains distinct M_1 and M_2 in $\mathcal{M}$, then we have either $A \subset M_1 \subset M_2 \subset B$ or $A \subset M_2 \subset M_1 \subset B$ for some sets A and B in $\mathcal{F}$, both containing $\mathcal{B}$ as a subposet.

Next let us compute $\mathrm{ave}_{\mathcal{C}\in\mathscr{C}(\emptyset)}|\mathcal{F}\cap\mathcal{C}|$ and $\mathrm{ave}_{\mathcal{C}\in\mathscr{C}(M)}|\mathcal{F}\cap\mathcal{C}|$ for each $M \in \mathcal{M}$. Clearly, if a full chain $\mathcal{C}$ does not contain a set in $\mathcal{M}$, then $|\mathcal{F}\cap\mathcal{C}| \le 2$, which gives $\mathrm{ave}_{\mathcal{C}\in\mathscr{C}(\emptyset)}|\mathcal{F}\cap\mathcal{C}| \le 2$. For any $M \in \mathcal{M}$, there uniquely exist sets $A, B \in \mathcal{F}$ with $A \subset M \subset B$, or one can find a butterfly in $\mathcal{F}$. Note that $0 < |A| < |M| < |B| < n$. Thus,

$$\begin{aligned}\mathrm{ave}_{\mathcal{C}\in\mathscr{C}(M)}|\mathcal{F}\cap\mathcal{C}| &= 1 + \frac{1}{\binom{n-|M|}{|B|}} + \frac{1}{\binom{|M|}{|A|}} \\ &\le 1 + \frac{1}{n-|M|} + \frac{1}{|M|} \\ &\le 1 + \frac{1}{2} + \frac{1}{2} = 2,\end{aligned}$$

and equality holds if and only if $n = 4$ and $|M| = 2$. Immediately, we get $\mathrm{La}(n, \mathcal{B}) \le \Sigma(n, 2)$ since $\bar{h}(\mathcal{F}) \le 2$.

We still need to obtain all maximum-sized $\mathcal{B}$-free families. For $n = 3$, one can easily run through all cases to get that the only possibility is $\mathcal{B}(3, 2)$.

We next deal with $n \ge 5$. Does there exist any $\mathcal{B}$-free family $\mathcal{F}$ with $|\mathcal{F}| = \Sigma(n, 2)$, that contains $\emptyset$ or $[n]$? If $\mathcal{F}$ contains both $\emptyset$ and $[n]$, we know that we can replace them by some 1-set S and $(n-1)$-set T not in $\mathcal{F}$ to obtain a new $\mathcal{B}$-free family $\mathcal{F}'$ with $\bar{h}(\mathcal{F}') \le 2$. Therefore, $\bar{h}(\mathcal{F} \setminus \{\emptyset, [n]\}) = \bar{h}(\mathcal{F}' \setminus \{S, T\}) \le 2 - \frac{2}{n}$. However, if $\bar{h}(\mathcal{F} \setminus \{\emptyset, [n]\}) \le 2 - \frac{2}{n}$, then the size of $\mathcal{F} \setminus \{\emptyset, [n]\}$ cannot reach $\Sigma(n, 2) - 2$ when $n \ge 5$. This is because any family containing $\Sigma(n, 2) - 2$ sets has Lubell function value at least $2 - 2/\binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$, which is achieved by selecting the sets as close to the middle levels as possible. Thus, any $\mathcal{B}$-free family of the largest size cannot have both $\emptyset$ and $[n]$, if $n \ge 5$. Suppose $\mathcal{F}$ contains $\emptyset$ but not $[n]$ (the case $\mathcal{F}$ contains $[n]$ but not $\emptyset$ is similar). We see $\bar{h}(\mathcal{F} \setminus \{\emptyset\}) \le 2 - \frac{1}{n}$, which is again impossible since $\Sigma(n, 2) - 1$ sets have a total weight at least $2 - 2/\binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$. Hence a largest $\mathcal{B}$-free family contains neither $\emptyset$ nor $[n]$ for $n \ge 5$.

Furthermore, our previous inequality implies that if $\mathcal{F}$ contains $\mathcal{P}_3 : A \subset M \subset B$, then $\bar{h}(\mathcal{F}) < 2$ for $n \ge 5$, since one of the blocks has $\mathrm{ave}_{\mathcal{C}\in\mathscr{C}(M)}|\mathcal{F} \cap \mathcal{C}| < 2$ and others have average at most two. So the largest family must be a $\mathcal{P}_3$-free family and hence $\mathcal{F} = \mathcal{B}(n, 2)$, by Erdős's Theorem [5] on $\mathcal{P}_k$-free families.

It remains to treat the more complicated case of $n = 4$, for which we know $\mathrm{La}(4, \mathcal{B}) = 10$. If $\mathcal{F}$ contains both $\emptyset$ and $[4]$, then $\mathcal{F}' = \mathcal{F} \setminus \{\emptyset, [4]\}$ cannot contain any chain. Hence $|\mathcal{F}'| \le \Sigma(4, 1) = \binom{4}{2}$. Then $|\mathcal{F}| \le 8$, and it is not a largest $\mathcal{B}$-free family.

Suppose $\mathcal{F}$ contains $\emptyset$ but not $[4]$. We assume that some set $\{a\}$ is not in $\mathcal{F}$ by our early argument. By the maximality, $|(\mathcal{F} \setminus \{\emptyset\}) \cup \{\{a\}\}| = 10$. Since $\bar{h}((\mathcal{F} \setminus \{\emptyset\}) \cup \{\{a\}\}) \le 2$, we conclude that $(\mathcal{F} \setminus \{\emptyset\}) \cup \{\{a\}\}$ contains $\binom{[4]}{2}$ and four sets in $\binom{[4]}{1} \cup \binom{[4]}{3}$ by viewing the Lubell function as the weighted sum of sets. Note that if $\emptyset \in \mathcal{F}$ and $\binom{[4]}{2} \subset \mathcal{F}$ then there is no 1-set in $\mathcal{F}$, or we can find four sets forming $\mathcal{B}$ in $\mathcal{F}$. However, if $\mathcal{F}$ contains two 3-sets, then their intersection is a 2-set. These sets together form $\mathcal{B}$ in $\mathcal{F}$ as well. Hence, $\emptyset \notin \mathcal{F}$. Dually, $\mathcal{F}$ cannot contain $[4]$ but not $\emptyset$. We conclude that for $n = 4$ a largest $\mathcal{B}$-free family must be contained in $\mathcal{B}(4,3)$.

When $\mathcal{F} \subset \mathcal{B}(4,3)$, then $|\mathcal{F}| = 10$ and $\bar{h}(\mathcal{F}) \leq 2$ imply that $\mathcal{F}$ contains $\binom{[4]}{2}$ and four sets in $\binom{[4]}{1} \cup \binom{[4]}{3}$. If the four sets are in the same level, then we done. Assume that one of them is in $\binom{[4]}{1}$ and three of them are in $\binom{[4]}{3}$. Let $\{a\}$ be the 1-set in $\mathcal{F}$. Then only one of the three sets $\{a,b,c\}$, $\{a,b,d\}$, and $\{a,c,d\}$ is in $\mathcal{F}$. Otherwise, $\mathcal{F}$ contains $\mathcal{B}$ as a subposet. This forces $|\mathcal{F}| < 10$, a contradiction. For the same reason we can say $\mathcal{F}$ cannot contain three 1-sets and one 3-set. So $\mathcal{F}$ must contain two sets in $\binom{[4]}{1}$ and two sets in $\binom{[4]}{3}$. Suppose that $\{a\}$ and $\{b\}$ are in $\mathcal{F}$. Then neither $\{a,b,c\}$ nor $\{a,b,d\}$ can be in $\mathcal{F}$. The last possible family is $\mathcal{F} = \{\{a\},\{b\},\{a,c,d\},\{b,c,d\}\} \cup \binom{[4]}{2}$. One can verify that this is a $\mathcal{B}$-free family of the largest size in addition to $\mathcal{B}(4,2)$. $\square$

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