

Some Results on the Target Set Selection Problem

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Abstract

In this paper we consider a fundamental problem in the area of viral marketing, called TARGET SET SELECTION problem. We study the problem when the underlying graph is a block-cactus graph, a chordal graph or a Hamming graph. We show that if G is a block-cactus graph, then the TARGET SET SELECTION problem can be solved in linear time, which generalizes Chen's result [2] for trees, and the time complexity is much better than the algorithm in [1] (for bounded treewidth graphs) when restricted to block-cactus graphs. We show that if the underlying graph G is a chordal graph with thresholds $\theta(v) \leq 2$ for each vertex v in G , then the problem can be solved in linear time. For a Hamming graph G having thresholds $\theta(v) = 2$ for each vertex v of G , we precisely determine an optimal target set S for (G, θ) . These results partially answer an open problem raised by Dreyer and Roberts [3].

Key words: target set selection, viral marketing, tree, block graph, block-cactus graph, chordal graph, Hamming graph, social networks, diffusion of innovations.

1 Introduction and preliminaries

A *graph* G consists of a set $V(G)$ of *vertices* together with a set $E(G)$ of unordered pairs of vertices called *edges*. We use uv for an edge $\{u, v\}$. Two vertices u and v are *adjacent* to each other if $uv \in E(G)$. In this paper, all graphs are finite and have no loops or multiple edges. For $S \subseteq V(G)$, the *subgraph of G induced by S* is the graph $G[S]$ with vertex set S and edge set $\{uv \in E(G) : u, v \in S\}$. Denote by $G - S$ the subgraph of G induced by $V(G) \setminus S$ and, for convenience, we write $G - v$ for $G - \{v\}$ when $v \in V(G)$. The *neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V(G) :$

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$uv \in E\}$. The *degree* $d_G(v)$ of v is defined by $d_G(v) = |N_G(v)|$. The *distance* $d_G(x, y)$ of two vertices x and y in G is defined to be the length of the shortest path from x to y in G . A *complete graph* is a graph in which every two distinct vertices are adjacent. The complete graph on n vertices is denoted by K_n . The *n -cycle* is the graph C_n with $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$. The *n -path* is the graph P_n with $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$.

The topology of a person-to-person recommendation social network is usually modeled by a graph G in which the vertices $V(G)$ represent customers, and edges $E(G)$ connect people to their friends. Consider the following scenario: A company wish to market a new product. The company has at hand a description of the social network G formed among a sample of potential customers. The company wants to target key potential customers $S \subseteq V(G)$ of the social network and persuade them into adopting the new product by handing out free samples. We assume that individuals in S will be convinced to adopt the new product after they receive a free sample, and the friends of customers in S would be persuaded into buying the new product, which in turn will recommend the product to other friends. The company hopes that by word-of-mouth effects, convinced vertices in S can trigger a cascade of further adoptions, and many customers will ultimately be persuaded. This advertising technique of spreading commercial message via social networks G is called *viral marketing* by analogy with computer viruses. But now how to find a good set of potential customers S to target?

In general, each vertex v is assigned a threshold value $\theta(v)$. The thresholds represent the different latent tendencies of vertices (customers) to buy the new product when their neighbors (friends) do. To be precise, let G be a connected undirected graph equipped with *thresholds* $\theta : V(G) \rightarrow \mathbb{Z}$. Denote by (G, θ) the social network G equipped with thresholds θ . When θ is a constant function such that $\theta(v) = k$ for all vertices v , (G, θ) will be written as (G, k) for short. Vertices v of G are in one of two states, active or inactive, which indicate whether v is persuaded into buying the new product. We call a vertex v *active* if it has been convinced to adopt the new product and assume that vertex v becomes active if $\theta(v)$ of its neighbors have adopted the new product.

In this paper we consider the following repetitive process, called *activation process* in (G, θ) starting at *target set* $S \subseteq V(G)$, which unfolds in discrete steps. Initially (at time 0), set all vertices in S to be active (with all other vertices inactive). After that, at each time step, the states of vertices are updated according to following rule:

Parallel updating rule: All inactive vertices v that have at least $\theta(v)$ already-

active neighbors become active.

The activation process terminates when no more vertices can get activated. Let $[S]_\theta^G$ denote the set of vertices that are active at the end of the process. If $F \subseteq [S]_\theta^G$, then we say that the target set S *influences* F in (G, θ) . Notice that if v has threshold $\theta(v) > d_G(v)$ and $v \in [S]_\theta^G$ for some target set S , then it must be $v \in S$. We also note that, according to our rule, if an inactive vertex v has threshold $\theta(v) \leq 0$ at time step t , then it becomes active automatically at the next time step. We are interested in the following optimization problem:

TARGET SET SELECTION: Finding a target set S of smallest possible size that influences all vertices in the social network (G, θ) , that is $[S]_\theta^G = V(G)$.

We define $\text{min-seed}(G, \theta)$ to be the minimum size of a target set that guarantees that all vertices in (G, θ) are eventually active at the end of the activation process, that is, $\text{min-seed}(G, \theta) = \min\{|S| : S \subseteq V(G) \text{ and } [S]_\theta^G = V(G)\}$. For $S \subseteq V(G)$, if $[S]_\theta^G = V(G)$ and $|S| = \text{min-seed}(G, \theta)$, then we call S an *optimal target set* for (G, θ) .

Domingos and Richardson [5] considered TARGET SET SELECTION problem in a probabilistic setting and presented heuristic solutions. Kempe, Kleinberg, and Tardos [9] considered probabilistic thresholds, called linear threshold model, and focused on the maximization version of the TARGET SET SELECTION problem – for any given k , find a target set S of size k to maximize the expected number of active vertices at the end of the activation process. They showed that this problem is NP-hard and proved that a hill-climbing algorithm can efficiently obtain an approximation solution that is 63% of optimal.

In this paper we only consider the TARGET SET SELECTION problem with deterministic, explicitly given, thresholds. In 2002, Peleg [11] showed that this problem is NP-hard for majority thresholds, that is $\theta(v) = \lceil d_G(v)/2 \rceil$ for each vertex v in G . In 2009, Dreyer and Roberts [3] showed that the problem is NP-hard for constant thresholds – given a fixed $k \geq 3$, $\theta(v) = k$ for each vertex v in G , and Chen [2] proved that it is NP-hard for bounded bipartite graphs G with thresholds at most 2.

In general, the TARGET SET SELECTION problem is not just NP-hard but also extremely hard to approximate. Kempe, Kleinberg, and Tardos [9] showed that a maximization version of TARGET SET SELECTION with constant thresholds cannot be approximated within any non-trivial factor, unless $P = NP$. In 2009, Chen [2] proved that given any n -vertices regular graph with thresholds $\theta(v) \leq 2$ for any vertex v , the TARGET SET SELECTION problem can not be approximated within the ratio of $O(2^{\log^{1-\epsilon} n})$, for any fixed constant $\epsilon > 0$, unless $NP \subseteq DTIME(n^{\text{poly log}(n)})$.

Very little is known about $\text{min-seed}(G, \theta)$ for specific classes of graphs G . Dreyer and Roberts [3] showed that when G is a tree, the TARGET SET SELECTION problem can be solved in linear time for constant thresholds. Chen [2] showed that when the underlying graph is a tree, the problem can be solved in polynomial-time under a general threshold model. In 2010, Ben-Zwi, Hermelin, Lokshtanov and Newman [1] showed that for n -vertices graph G with treewidth bounded by ω , the TARGET SET SELECTION problem can be solved in $n^{O(\omega)}$ time. In [3, 6], $\text{min-seed}(G, \theta)$ is computed for paths, cycles and for different kinds of grids G under constant threshold model.

The objective of this paper is to study the TARGET SET SELECTION problem when the underlying graph is a block-cactus graph, a chordal graph or a Hamming graph. In Section 2, we show that if G is a block-cactus graph, then the problem can be solved in linear time, which generalizes Chen's result [2] for trees, and the time complexity is much better than the algorithm in [1] (for bounded treewidth graphs) when restricted to block-cactus graphs. In Section 3, we show that if the underlying graph G is a chordal graph with thresholds $\theta(v) \leq 2$ for each vertex v in G , then the TARGET SET SELECTION problem can be solved in linear time. Our results partially answer an open problem raised by Dreyer and Roberts at the end of their paper [3]. In Section 4, for a Hamming graph G having thresholds $\theta(v) = 2$ for each vertex v of G , we precisely determine an optimal target set S for (G, θ) .

In order to study $\text{min-seed}(G, \theta)$ we introduce a sequential version of the above activation process, called *sequential activation process*, which employs the following rule instead of the parallel updating rule:

Sequential updating rule: At each time step t , exactly one of inactive vertices that have at least $\theta(v)$ already-active neighbors becomes active.

The proof of the following lemma is straightforward and so is omitted. In the sequel, Lemma 1 will be used without explicit reference to it.

Lemma 1 *For a social network (G, θ) , an optimal target set under sequential updating rule is also an optimal target set under parallel updating rule, and vice versa.*

Let \mathcal{P} be a sequential activation process on (G, θ) starting out from a target set S . In this process, if v_1, v_2, \dots, v_r is the order that vertices in $[S]_\theta^G \setminus S$ are convinced, then (v_1, v_2, \dots, v_r) is called the *convinced sequence* of \mathcal{P} , and we say that target set S has a convinced sequence (v_1, v_2, \dots, v_r) on (G, θ) .

2 Block-cactus graphs

A vertex v of a graph is called a *cut-vertex* if removal of v and all edges incident to it increases the number of connected components. A *block* of a graph G is a maximal connected induced subgraph of G that has no cut-vertices. A graph G is a *block graph* if every block of G is a complete graph. A block B of a graph G is called a *pendent block* of G if B has at most one cut-vertex of G . A graph G is a *block-cactus graph* if every block of G is either a complete graph or a cycle. Let v be a cut-vertex of G . If $G - v$ consists of two disjoint graphs W_1 and W_2 and let G_i ($i = 1, 2$) be the subgraph of G induced by $\{v\} \cup V(W_i)$, then G is called the *vertex-sum* at v of the two graphs G_1 and G_2 , and denoted by $G = G_1 \oplus_v G_2$.

In the following Theorem 2, let $G_1 \oplus_v G_2$ be a social network equipped with threshold function θ . Let θ_1 be a threshold function of $G_1 - v$ which is the same as the function θ , except that $\theta_1(x) = \theta(x) - 1$ for every $x \in N_{G_1}(v)$. Let S_1 be an optimal target set for $(G_1 - v, \theta_1)$ that maximizes the cardinality of the set $N_{G_1}(v) \cap [S_1]_{\theta}^{G_1}$, where, by slight abuse of notation, θ also means the threshold function of G_1 by restricting the threshold θ of $G_1 \oplus_v G_2$ to the set $V(G_1)$. Let θ_2 be a threshold function of G_2 which is the same as the function θ , except that $\theta_2(v) = \theta(v) - |N_{G_1}(v) \cap [S_1]_{\theta}^{G_1}|$. Let S_2 be an optimal target set for (G_2, θ_2) . Now, with the definitions and notation introduced in this paragraph, we prove the following theorem.

Theorem 2 $S_1 \cup S_2$ is an optimal target set for $(G_1 \oplus_v G_2, \theta)$.

Proof. Consider a sequential activation process in $(G_1 \oplus_v G_2, \theta)$ starting at target set $S_1 \cup S_2$. Clearly $N_{G_1}(v) \cap [S_1]_{\theta}^{G_1} \subseteq [S_1]_{\theta}^{G_1 \oplus_v G_2}$, and hence $V(G_2) \subseteq [S_1 \cup S_2]_{\theta}^{G_1 \oplus_v G_2}$, which implies $V(G_1) \subseteq [S_1 \cup S_2]_{\theta}^{G_1 \oplus_v G_2}$. That is the target set $S_1 \cup S_2$ influences all vertices in $(G_1 \oplus_v G_2, \theta)$. To prove the theorem it remains to show that $|S_1| + |S_2| = \min\text{-seed}(G_1 \oplus_v G_2, \theta)$.

Let S be an optimal target set for $(G_1 \oplus_v G_2, \theta)$ that minimizes the size of the set $S \cap V(G_1 - v)$. Since $(S \cap V(G_1 - v)) \cup \{v\}$ influences all vertices in (G_1, θ) , we have that $S \cap V(G_1 - v)$ influences all vertices in $(G_1 - v, \theta_1)$. It now follows that $|S \cap V(G_1 - v)| = |S_1|$ since if not, then we have $|S \cap V(G_1 - v)| \geq |S_1| + 1$, and hence $(S_1 + v) \cup (S \cap V(G_2))$ is an optimal target set for $(G_1 \oplus_v G_2, \theta)$, a contradiction to the choice of S . Therefore $S \cap V(G_1 - v)$ is an optimal target set for $(G_1 - v, \theta_1)$. By the choice of S_1 , we see that $|N_{G_1}(v) \cap [S \cap V(G_1 - v)]_{\theta}^{G_1}| \leq |N_{G_1}(v) \cap [S_1]_{\theta}^{G_1}|$. This implies that $S_1 \cup [S \cap V(G_2)]$ is an optimal target set for $(G_1 \oplus_v G_2, \theta)$, and hence $S \cap V(G_2)$ influences all vertices in (G_2, θ_2) , which implies $|S \cap V(G_2)| \geq |S_2|$. We conclude that $|S_1| + |S_2| = |S \cap V(G_1 - v)| + |S_2| \leq |S \cap V(G_1 - v)| + |S \cap V(G_2)| = |S|$. Therefore $S_1 \cup S_2$ is an optimal target set for $(G_1 \oplus_v G_2, \theta)$. \blacksquare

Corollary 3 $\text{min-seed}(G_1 \oplus_v G_2, \theta) = \text{min-seed}(G_1 - v, \theta_1) + \text{min-seed}(G_2, \theta_2)$.

Lemma 4 *Let v be a vertex in the social network (G, θ) . If $G \in \{K_n, C_n\}$, then an optimal target set S for $(G - v, \theta_1)$ that maximizes the size of the set $N_G(v) \cap [S]_\theta^G$ can be found in linear time, where θ_1 is the threshold function of $G - v$ which is the same as the function θ , except that $\theta_1(x) = \theta(x) - 1$ for every $x \in N_G(v)$. Moreover the size of the set $N_G(v) \cap [S]_\theta^G$ can also be determined in linear time.*

Proof. Let \mathcal{F} be the set of optimal target sets S for $(G - v, \theta_1)$ such that S maximizes the size of the set $N_G(v) \cap [S]_\theta^G$.

We first consider the case that $G = K_n$. Let $V(G - v) = \{v_1, v_2, \dots, v_{n-1}\}$ such that $\theta_1(v_1) \leq \theta_1(v_2) \leq \dots \leq \theta_1(v_{n-1})$. Let S be an optimal target set for $(G - v, \theta_1)$. Since any two vertices v_i, v_{i+1} in $G - v$ have $N_{G-v}(v_i) = N_{G-v}(v_{i+1})$ and $\theta_1(v_1) \leq \dots \leq \theta_1(v_{n-1})$, we give the following simple observation without proof.

Observation *If $v_i \in S$ and $v_{i+1} \notin S$, then $(S \setminus \{v_i\}) \cup \{v_{i+1}\}$ is an optimal target set for $(G - v, \theta_1)$ and $|(S \setminus \{v_i\}) \cup \{v_{i+1}\}|_\theta^G \geq |[S]_\theta^G|$.*

Since G is a complete graph, the above observation says that if $\text{min-seed}(G - v, \theta_1) = s$, then the target set $\{v_{n-1}, v_{n-2}, \dots, v_{n-s}\} \in \mathcal{F}$. Moreover, such a target set has a convinced sequence $(v_1, v_2, \dots, v_{n-s-1})$ on $(G - v, \theta_1)$. Now we are in a position to show that **Algorithm K** outputs an optimal target set S for $(G - v, \theta_1)$ such that $S \in \mathcal{F}$.

In steps 2-3 of the algorithm we see that $\text{min-seed}(G - v, \theta_1) \geq |\{v_i : \theta_1(v_i) > n - 2 \text{ and } 1 \leq i \leq n - 1\}| = \ell$. In steps 4-8, we want to find the value s such that $\{v_{n-1}, v_{n-2}, \dots, v_{n-\ell}\} \cup \{v_{n-\ell-1}, v_{n-\ell-2}, \dots, v_{n-s}\} \in \mathcal{F}$. During the i th iteration of the for loop in step 4, we have $\{v_1, v_2, \dots, v_{i-1}\} \subseteq [\{v_{n-1}, v_{n-2}, \dots, v_{n-s}\}]_{\theta_1}^{G-v}$. In step 6, when $\theta_1(v_i) > s + i - 1$, in order to influence vertex v_i in $(G - v, \theta_1)$ we need to add another $\theta_1(v_i) - (s + i - 1)$ vertices to the set $\{v_{n-1}, v_{n-2}, \dots, v_{n-s}\}$. Note that in step 5 we have $\theta_1(v_i) \leq n - 2$. It follows that after step 5 and before step 6 we have $n - (s + [\theta_1(v_i) - (s + i - 1)]) > i$. Therefore in step 7 if $n - s = i + 1$, then it must be $\text{min-seed}(G - v, \theta_1) = s$, and hence $\{v_{n-1}, v_{n-2}, \dots, v_{n-s}\} \in \mathcal{F}$. Clearly, the time complexity of **Algorithm K** takes linear time, where the bucket sort algorithm is used to sort vertices by their thresholds.

Let S be the output of the **Algorithm K** and $|S| = s$. Let $V(G) \setminus S = \{u_1, u_2, \dots, u_{n-s}\}$ such that $\theta(u_1) \leq \theta(u_2) \leq \dots \leq \theta(u_{n-s})$. Let $U = \{i : \theta(u_i) > s + i - 1 \text{ and } 1 \leq i \leq n - s\}$. We define the value r by

$$r = \begin{cases} \min U - 1, & \text{if } U \neq \emptyset; \\ n - s, & \text{if } U = \emptyset. \end{cases}$$

Since G is a complete graph, it can be seen that $[S]_\theta^G = S \cup \{u_1, u_2, \dots, u_r\}$. Therefore the size of the set $N_G(v) \cap [S]_\theta^G$ can also be determined in linear time.

Algorithm K

Begin

```

1   $s \leftarrow 0$ ;
2  for  $i = 1$  to  $n - 1$  do if  $\theta_1(v_i) > n - 2$  then  $s \leftarrow s + 1$ ;
3   $\ell \leftarrow s$ ;
4  for  $i = 1$  to  $n - \ell - 1$  do
5  begin
6    if  $\theta_1(v_i) > s + i - 1$  then  $s \leftarrow s + [\theta_1(v_i) - (s + i - 1)]$ ;
7    if  $n - s = i + 1$  then STOP and output  $S = \{v_{n-1}, v_{n-2}, \dots, v_{n-s}\}$ ;
8  end

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End.

Finally, consider the remaining case that $G = C_n$. Let $E(G) = \{vv_1, vv_{n-1}\} \cup \{v_i v_{i+1} : 1 \leq i \leq n - 2\}$. Thus $V(G - v) = \{v_1, v_2, \dots, v_{n-1}\}$. Let \mathcal{H} be the set of optimal target sets S for $(G - v, \theta_1)$. First we consider the following Algorithm C which computes S_1 and S_2 . Clearly, $S_1 \subseteq S$ for each $S \in \mathcal{H}$ and $S_2 \subseteq [S_1]_{\theta_1}^{G-v}$.

Algorithm C

Begin

```

1  Find the set  $S_1 = \{v_i : \theta_1(v_i) > d_{G-v}(v_i) \text{ and } 1 \leq i \leq n - 1\}$ .
2  for  $i = 1$  to  $n - 1$  do if  $v_i \notin S_1$  then  $\theta_1(v_i) \leftarrow \theta_1(v_i) - |N_{G-v}(v_i) \cap S_1|$ ;
3  for  $i = 1$  to  $n - 2$  do if  $v_i \notin S_1$  and  $\theta_1(v_i) \leq 0$  then  $\theta_1(v_{i+1}) \leftarrow \theta_1(v_{i+1}) - 1$ ;
4  for  $i = n - 1$  downto  $2$  do if  $v_i \notin S_1$  and  $\theta_1(v_i) \leq 0$  then  $\theta_1(v_{i-1}) \leftarrow \theta_1(v_{i-1}) - 1$ ;
5  Find the set  $S_2 = \{v_i \notin S_1 : \theta_1(v_i) \leq 0 \text{ and } 1 \leq i \leq n - 1\}$ .
6  output  $S_1$  and  $S_2$ ;
7  output  $\theta_1$ ;

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End.

In the sequel, let S_1, S_2, θ_1 be the outputs of the Algorithm C. Now let $G - v - S_1 - S_2$ have exactly r connected components P_1, P_2, \dots, P_r . Denote by ℓ_i the value $\min\{k : v_k \in V(P_i), 1 \leq k \leq n - 1\}$. We assume that $\ell_1 < \ell_2 < \dots < \ell_r$. For each $1 \leq i \leq r$, we note that P_i is a path and all vertices w in P_i have $\theta_1(w) \in \{1, 2\}$, moreover the two end-vertices w_1, w_2 of P_i have $\theta_1(w_1) = \theta_1(w_2) = 1$. Let $V(P_1) = \{v_a, v_{a+1}, \dots, v_{a+b}\}$ and $V(P_r) = \{v_c, v_{c+1}, \dots, v_{c+d}\}$ for some integers a, b, c, d .

Case 1. $r = 1$. Let $\{u \in V(P_1) : \theta_1(u) = 2\} = \{v_{i_1}, v_{i_2}, \dots, v_{i_q}\}$ such that $i_1 < i_2 < \dots < i_q$. If $q = 0$, then $S_1 \cup \{v_a\}, S_1 \cup \{v_{a+b}\} \in \mathcal{H}$. Clearly either

$S_1 \cup \{v_a\} \in \mathcal{F}$ or $S_1 \cup \{v_{a+b}\} \in \mathcal{F}$. It follows that we can compute $[S_1 \cup \{v_a\}]_\theta^G$ and $[S_1 \cup \{v_{a+b}\}]_\theta^G$ to find a desired set S in \mathcal{F} . When $q = 2t$ for some $t \in \mathbb{Z}^+$, let $U_1 = \{v_a\} \cup \{v_{i_2}, v_{i_4}, \dots, v_{i_{2t}}\}$ and $U_2 = \{v_{i_1}, v_{i_3}, \dots, v_{i_{2t-1}}\} \cup \{v_{a+b}\}$. It can be seen that either $S_1 \cup U_1 \in \mathcal{F}$ or $S_1 \cup U_2 \in \mathcal{F}$. One can compute $[S_1 \cup U_1]_\theta^G$ and $[S_1 \cup U_2]_\theta^G$ to find a desired set S in \mathcal{F} . When $q = 2t - 1$ for some $t \in \mathbb{Z}^+$, let $U = \{v_{i_1}, v_{i_3}, \dots, v_{i_{2t-1}}\}$. Clearly $S_1 \cup U \in \mathcal{F}$.

Case 2. $r \geq 2$. It suffices to assume that $r = 3$, that is $G - v - S_1 - S_2$ has exactly 3 connected components P_1, P_2, P_3 and $\ell_1 < \ell_2 < \ell_3$. Let $\{u \in V(P_1) : \theta_1(u) = 2\} = \{v_{i_1}, v_{i_2}, \dots, v_{i_q}\}$ such that $i_1 < i_2 < \dots < i_q$. Let $\{u \in V(P_2) : \theta_1(u) = 2\} = \{v_{j_1}, v_{j_2}, \dots, v_{j_s}\}$ such that $j_1 < j_2 < \dots < j_s$. Let $\{u \in V(P_3) : \theta_1(u) = 2\} = \{v_{k_1}, v_{k_2}, \dots, v_{k_\ell}\}$ such that $k_1 < k_2 < \dots < k_\ell$. It suffices to consider the case that $q = 2t, s = 2t' - 1, \ell = 2t''$ for some integers t, t', t'' (the remaining cases follow similar arguments as above). let $U_1 = \{v_a\} \cup \{v_{i_2}, v_{i_4}, \dots, v_{i_{2t}}\}$, $U_2 = \{v_{j_1}, v_{j_3}, \dots, v_{j_{2t'-1}}\}$, and $U_3 = \{v_{k_1}, v_{k_3}, \dots, v_{k_{2t''-1}}\} \cup \{v_{c+d}\}$. It can be seen that $S_1 \cup U_1 \cup U_2 \cup U_3 \in \mathcal{F}$.

Concerning the running time of the above algorithm, it is clear that it is linear time. Which completes the proof of the lemma. \blacksquare

Now Theorem 5 follows from Theorem 2 and Lemma 4 immediately.

Theorem 5 *If G is a block-cactus graph, then an optimal target set for (G, θ) can be found in linear time.*

3 Chordal graphs

A graph is called *chordal* if it does not have an induced cycle of length greater than three. A vertex v in G is called *simplicial* if the subgraph of G induced by the neighbors of v is complete. Let $\sigma = [v_1, v_2, \dots, v_n]$ be an ordering of $V(G)$. We say that σ is a *perfect elimination order* if each v_i is a simplicial vertex of the subgraph $G[v_i, v_{i+1}, \dots, v_n]$. In 1965, Fulkerson and Gross [7] showed that every chordal graph has a perfect elimination order. In [12, 13] it was shown that if G is a chordal graph, then there is a linear time algorithm which receives the adjacency sets of G and outputs a perfect elimination order σ of $V(G)$. For nonadjacent vertices u and v of a graph G , a subset $S \subseteq V(G)$ is called a *u - v separator* if the removal of S from G separates u and v into distinct connected components. If no proper subset of S is a u - v -separator, then S is called a *minimal u - v separator*.

Lemma 6 ([4]) *Every chordal graph G has a simplicial vertex. Moreover, if G is not complete, then it has two nonadjacent simplicial vertices.*

Lemma 7 ([7]) *For nonadjacent vertices u and v of a chordal graph G , if S is a minimal u - v separator of G , then S induces a complete subgraph of G .*

Lemma 8 *For $t \geq 2$, let G be a t -connected chordal graph with $\theta(x) \leq t$ for all vertices x . If $S \subseteq V(G)$ induces a complete subgraph of size t in G , then the target set S influences all vertices in (G, θ) .*

Proof. Without loss of generality, we may assume that G is not complete. Let $|V(G)| = n$. To prove this theorem, we want to demonstrate a sequence of distinct vertices $[v_1, v_2, \dots, v_\ell]$ in G such that $G - \{v_1, v_2, \dots, v_\ell\}$ is a complete graph that contains all vertices of S . Moreover, for $1 \leq i \leq \ell$, vertex v_i is adjacent to at least t vertices in the graph $G - \{v_1, v_2, \dots, v_i\}$. It is clear that if such a sequence exists, then the target set S influences all vertices in (G, θ) , since $\theta(x) \leq t$ for all vertices x in G .

To construct such a sequence, by Lemma 6, we can pick a simplicial vertex v_1 of G such that $v_1 \notin S$. Note that $G - v_1$ is t -connected, since otherwise there is a set $U \subseteq V(G - v_1)$ with $|U| \leq t - 1$ such that $G - v_1 - U$ is disconnected. By Lemma 6 it follows that $G - U$ is disconnected, a contradiction to G is t -connected. Next, if $G - v_1$ is not complete, then by Lemma 6 again, we can pick a simplicial vertex v_2 of $G - v_1$ such that $v_2 \notin S$. It can also be seen that $G - v_1 - v_2$ is t -connected. If we continue in this way, we eventually have a desired sequence of distinct vertices $[v_1, v_2, \dots, v_\ell]$ such that the graph $G - \{v_1, v_2, \dots, v_i\}$ is t -connected for each $i \in \{1, 2, \dots, \ell - 1\}$ and $G - \{v_1, v_2, \dots, v_\ell\}$ is a complete graph that contains all vertices of S . Which completes the proof of the lemma. ■

Theorem 9 *Suppose that G is a t -connected chordal graph with $t \geq 2$. (a) $\text{min-seed}(G, t) = t$. (b) If $\theta(x) \leq t$ for each vertex x of G and $\theta(v) < t$ for some vertex v . then $\text{min-seed}(G, \theta) < t$.*

Proof. (a) By Lemma 6, the fact that G is a t -connected chordal graph implies that G contains a complete subgraph H of t vertices. By Lemma 8, we see that the target set $V(H)$ influences all vertices in the social network (G, t) , and hence $\text{min-seed}(G, t) \leq t$. Note that an inactive vertex v in (G, t) become active only if v has at least t already-active neighbors. It follows that $\text{min-seed}(G, t) \geq t$, which completes the proof of part (a).

(b) If v is adjacent to all other vertices of G , then, by Lemma 6, $G - v$ contains a complete subgraph H of size $t - 1$, since $G - v$ has a simplicial vertex and G is t -connected. It follows that, by Lemma 8, the target set $V(H)$ influences all vertices in (G, θ) , and hence $\text{min-seed}(G, \theta) < t$. Now consider the case that v is not adjacent to

some vertex u in G . Clearly there is a minimal v - u separator S such that v adjacent to all vertices of S . Note that $|S| \geq t$, since G is t -connected. Let $S' \subseteq S$ with $|S'| = t - 1$. By Lemma 7, $S' \cup \{v\}$ induces a complete subgraph of size t in G . It follows that, by Lemma 8 and the fact that $\theta(v) \leq t - 1$, the target set S' influences all vertices of (G, θ) . We conclude that $\text{min-seed}(G, \theta) < t$. \blacksquare

Corollary 10 *Let G be a 2-connected chordal graph with thresholds $\theta(v) \leq 2$ for every vertex v of G . Then $\text{min-seed}(G, \theta) = 2$ if and only if $\theta(v) = 2$ for each vertex v of G .*

In the sequel, for convenience, we write $\mathcal{S} \propto (G, \theta)$ to mean that the target set \mathcal{S} influences all vertices in (G, θ) . The following simple fact, which we state without proof, will be used implicitly and frequently in Lemma 12.

Claim 11 *Let v be a vertex in the social network (G, θ) and let θ_1 be the threshold function of $G - v$ which is the same as the function θ , except that $\theta_1(x) = \theta(x) - 1$ for every $x \in N_G(v)$. Then for $S \subseteq V(G - v)$, we have $S \propto (G - v, \theta_1)$ if and only if $S \cup \{v\} \propto (G, \theta)$.*

We state Lemma 12 using the same notation and conventions as in Claim 11.

Lemma 12 *Let G be a 2-connected chordal graph with thresholds $\theta(u) \leq 2$ for every $u \in V(G)$. For a vertex v in G , let \mathcal{F} be the set of optimal target sets S for $(G - v, \theta_1)$ such that S maximizes the size of the set $N_G(v) \cap [S]_\theta^G$. Let $I = \{u \in V(G - v) : \theta_1(u) \leq 0\}$, $J = \{u \in V(G) : \theta(u) < 2\}$ and $J_0 = \{u \in V(G) : \theta(u) \leq 0\}$. Let \mathcal{P}_1 (resp. \mathcal{Q}_1) be the property that there are two distinct vertices $x, y \in I$ (resp. $x, y \in J_0$) such that $d_G(x, y) \leq 2$. Let \mathcal{P}_2 (resp. \mathcal{Q}_2) be the property that there is an edge xy in $G - v$ (resp. G) with $x \in I$ (resp. $x \in J_0$) and $\theta_1(y) = 1$ (resp. $\theta(y) = 1$). Then we have:*

- (a) *If $I \cap N_G(v) \neq \emptyset$, then $\emptyset \in \mathcal{F}$.*
- (b) *If $I \cap N_G(v) = \emptyset$ and \mathcal{P}_1 holds, then $\emptyset \in \mathcal{F}$.*
- (c) *If $I \cap N_G(v) = \emptyset$ and \mathcal{P}_2 holds, then $\emptyset \in \mathcal{F}$.*
- (d) *If $J = \emptyset$, then $\{x\} \in \mathcal{F}$ and $[\{x\}]_\theta^G = \{x\}$ for every $x \in N_G(v)$.*
- (e) *If $J \neq \emptyset$, $I \cap N_G(v) = \emptyset$ and neither \mathcal{P}_1 nor \mathcal{P}_2 holds, then $\{x\} \in \mathcal{F}$ and $[\{x\}]_\theta^G = V(G)$ for every vertex x adjacent to some vertex $w \in J$.*
- (f) *If \mathcal{Q}_1 or \mathcal{Q}_2 holds, then $[\emptyset]_\theta^G = V(G)$.*
- (g) *If neither \mathcal{Q}_1 nor \mathcal{Q}_2 holds, then $[\emptyset]_\theta^G = J_0$.*

Proof. (a) Let $w \in I \cap N_G(v)$. By the facts $vw \in E(G)$, $\theta(w) \leq 1$ and by Lemma 8, we see that $\{v\} \propto (G, \theta)$, and hence $\emptyset \propto (G - v, \theta_1)$.

(b) Clearly $\theta(x) \leq 0$ and $\theta(y) \leq 0$. Since $d_G(x, y) \leq 2$, either $xy \in E(G)$ or $x, y \in N_G(z)$ for some vertex z . In both cases, by Lemma 8, we see that $[\emptyset]_\theta^G = V(G)$. Thus by Claim 11, $\emptyset \propto (G - v, \theta_1)$.

(c) Since $x \notin N_G(v)$, it can be seen that $[\{v\}]_\theta^G \supseteq \{x, y, v\}$. By Lemma 8, it follows that $\{v\} \propto (G, \theta)$, and hence $\emptyset \propto (G - v, \theta_1)$.

(d) For each $x \in N_G(v)$, by Lemma 8, $\{x, v\} \propto (G, \theta)$, and hence $\{x\} \propto (G - v, \theta_1)$. Clearly $\text{min-seed}(G - v, \theta) \geq 1$. It follows that $\{x\}$ is an optimal target set for $(G - v, \theta_1)$. Since $\theta(u) = 2$ for each $u \in V(G)$, we have $||S|_\theta^G| = 1$ for any optimal target set S for $(G - v, \theta_1)$. Therefore $\{x\} \in \mathcal{F}$ and $[\{x\}]_\theta^G = \{x\}$.

(e) Note that $I \cap N_G(v) = \emptyset$ implies that $\theta(y) = 2$ for each $y \in N_G(v)$. We claim that $\{v\}$ can not influence all vertices in (G, θ) . If not, then it must be that either \mathcal{P}_1 or \mathcal{P}_2 holds, a contradiction. Thus $\text{min-seed}(G - v, \theta_1) \geq 1$. Now let $w \in J$ and $x \in N_G(w)$. Note that $x \neq v$. Clearly $[\{x\}]_\theta^G \supseteq \{x, w\}$ and hence, by Lemma 8, $\{v, x\} \propto (G, \theta)$. It follows that, by Claim 11, $\{x\} \propto (G - v, \theta_1)$. Moreover, we have $[\{x\}]_\theta^G = V(G)$, and hence $\{x\} \in \mathcal{F}$. This completes the proof of (e).

Finally, by similar arguments as in the proofs of (c), (d) and (e), it is easy to prove (f) and (g), so we omit the proofs of (f) and (g). \blacksquare

Using the same notation and conventions as in Claim 11 and Lemma 12, we state and prove the following theorem.

Theorem 13 *If G is a chordal graph with thresholds $\theta(x) \leq 2$ for each vertex x in G , then an optimal target set for (G, θ) can be found in linear time.*

Proof. Let G_1 be a block of G which contains exactly one cut vertex v of G . If G is not 2-connected, then G can be written as the following form: $G = G_1 \oplus_v G_2$, where G_2 is an induced subgraph of G and is also chordal. To prove the theorem, we omit the easy case $G_1 = K_2$, which follows from Lemma 4. We only consider the case that G_1 is a 2-connected chordal graph. By using Lemma 12, we can in linear time in terms of the size of G_1 find an optimal target sets S_1 for $(G_1 - v, \theta_1)$ such that S_1 maximizes the size of the set $N_{G_1}(v) \cap [S_1]_\theta^{G_1}$ and compute $|N_{G_1}(v) \cap [S_1]_\theta^{G_1}|$.

Next, we want to find an optimal target set S_2 for (G_2, θ_2) , where θ_2 is a threshold function of G_2 which is the same as the function θ , except that $\theta_2(v) = \theta(v) - |N_{G_1}(v) \cap [S_1]_\theta^{G_1}|$. If G_2 is a 2-connected chordal graph, then S_2 can be found in linear time in terms of the size of G_2 by using Lemma 8 and Corollary 10, and hence an optimal target set $S_1 \cup S_2$ for (G, θ) can be found in linear time by using Theorem 2.

If G_2 has a cut vertex v' and a pendent block G_{21} such that $G_2 = G_{21} \oplus_{v'} G_{22}$, then we can repeat the arguments in the previous paragraphs and use Theorem 2 to

find the desired S_2 in linear time in terms of the size of G_2 , and hence an optimal target set for (G, θ) can be found in linear time. \blacksquare

4 Hamming graphs

Given two graphs G and H , their *Cartesian product* is the graph $G \square H$ with vertex set $V(G) \times V(H)$ and edge set $\{(g, h)(g', h') : gg' \in E(G) \text{ with } h = h', \text{ or } g = g' \text{ with } hh' \in E(H)\}$. The Cartesian product is commutative and associative (see page 29 of [8]). A *Hamming graph* is a Cartesian product of nontrivial complete graphs, i.e., of the form $K_{n_1} \square K_{n_2} \square \cdots \square K_{n_t}$ for some integers $n_1, \dots, n_t \geq 2, t \geq 1$, which is also denoted as $\prod_{i=1}^t K_{n_i}$. Note that $\prod_{i=1}^t K_{n_i}$ has vertex set $V(K_{n_1}) \times V(K_{n_2}) \times \cdots \times V(K_{n_t})$.

Let $u = (u_1, \dots, u_t)$ and $v = (v_1, \dots, v_t)$ be two vertices of $\prod_{i=1}^t K_{n_i}$. The *Hamming distance* $H(u, v)$ between u and v is the number of coordinate positions in which u and v differ. Note that there is an edge between u and v if and only if $H(u, v) = 1$. For $S_1, S_2 \subseteq V(\prod_{i=1}^t K_{n_i})$, denote by $d(S_1, S_2)$ the value $\min\{H(u, v) : u \in S_1, v \in S_2\}$. Let $[i, j]$ denote the set of integers k such that $i \leq k \leq j$. For $A \subseteq [1, t]$, if $u_i = v_i$ for all $i \in A$, then we write $u_{|A} = v_{|A}$. Let u_A denote the set of vertices x in $\prod_{i=1}^t K_{n_i}$ such that $x_{|A} = u_{|A}$. The proof of the following claim is straightforward and hence omitted.

Claim 14 *Let u, v, w be three distinct vertices of $\prod_{i=1}^t K_{n_i}$ and $u_{|A} = v_{|A}$ for some set $A \subseteq [1, t]$. If w is adjacent to both u and v , then $w_{|A} = u_{|A} = v_{|A}$.*

Lemma 15 *Suppose $G = (V, E)$ is the Hamming graph $\prod_{i=1}^t K_{n_i}$. Let $x, y \in V$, $i, j \in [1, t]$ and $A, B \subseteq [1, t]$. The following properties hold.*

- (a) *If $xy \in E$ and $x_i \neq y_i$, then $[x_A \cup \{y\}]_2^G = x_{A \setminus \{i\}}$.*
- (b) $[x_A \cup x_B]_2^G = x_{A \cap B}$.
- (c) *If $xy \in E$ and $x_i \neq y_i$, then $[x_A \cup y_B]_2^G = x_{(A \cap B) \setminus \{i\}}$.*
- (d) *If $H(x, y) = 2$, $x_i \neq y_i$, $x_j \neq y_j$ and $i \neq j$, then $[x_A \cup y_B]_2^G = x_{(A \cap B) \setminus \{i, j\}}$.*
- (e) *If $d(x_A, y_B) \geq 3$, then $[x_A \cup y_B]_2^G = x_A \cup y_B$.*

Proof. (a) First let us consider the case of $i \notin A$. From Claim 14 and the fact $x_{|A} = y_{|A}$, we see that $[x_A \cup \{y\}]_2^G = x_A$. Now we consider the remaining case $i \in A$. To prove this case it suffices to consider the case that $i = 1$ and $A = [1, j]$. We want to prove, by induction on j , that $[x_{[1, j]} \cup \{y\}]_2^G = x_{[2, j]}$ for $j = t, t-1, \dots, 1$. For $j = t$, we see that $[x_{[1, j]} \cup \{y\}]_2^G = [\{x, y\}]_2^G$. Since $x_{[2, t]} = y_{[2, t]}$, it follows from Claim 14 that if $w \in [\{x, y\}]_2^G$ then $w_{[2, t]} = x_{[2, t]} = y_{[2, t]}$, and hence $w \in x_{[2, t]}$. That is $[\{x, y\}]_2^G \subseteq x_{[2, t]}$. Since any vertex in $x_{[2, t]} \setminus \{x, y\}$ is adjacent to both x and y , it follows that $[\{x, y\}]_2^G \supseteq x_{[2, t]}$. Therefore $[\{x, y\}]_2^G = x_{[2, t]}$.

Next, we assume that $[x_{[1,j]} \cup \{y\}]_2^G = x_{[2,j]}$ holds for some $j \in [2, t]$. From this induction hypothesis it follows that $x_{[2,j]} \subseteq [x_{[1,j-1]} \cup \{y\}]_2^G$. For any vertex w in $x_{[2,j-1]}$, either $w \in x_{[2,j]} \cup x_{[1,j-1]}$ or w is adjacent to at least one vertex in $x_{[2,j]}$ and at least one vertex in $x_{[1,j-1]}$. Thus $x_{[2,j-1]} \subseteq [x_{[1,j-1]} \cup \{y\}]_2^G$. On the other hand, by the fact $x_{[2,j-1]} = y_{[2,j-1]}$ and Claim 14, we also see that $x_{[2,j-1]} \supseteq [x_{[1,j-1]} \cup \{y\}]_2^G$. Therefore $[x_{[1,j-1]} \cup \{y\}]_2^G = x_{[2,j-1]}$, this completes the proof of Lemma 15(a).

(b) Since for any $i \in A \setminus B$, there exists a vertex $y \in x_B$ such that $xy \in E$ and $x_i \neq y_i$, by Lemma 15(a), it follows that $[x_A \cup x_B]_2^G \supseteq x_{A \setminus (A \setminus B)} = x_{A \cap B}$. We note that if a vertex w is adjacent to at least two vertices in $x_A \cup x_B$, then, by Claim 14, it must be the case that $w|_{A \cap B} = x|_{A \cap B}$. Therefore $[x_A \cup x_B]_2^G \subseteq x_{A \cap B}$. We conclude that $[x_A \cup x_B]_2^G = x_{A \cap B}$.

(c) Lemma 15(a) shows that $[x_A \cup y_B]_2^G \supseteq [x_A \cup \{y\}]_2^G = x_{A \setminus \{i\}}$, and hence $[x_A \cup y_B]_2^G = [x_{A \setminus \{i\}} \cup y_B]_2^G$, since $x_A \subseteq x_{A \setminus \{i\}}$. It follows that $[x_A \cup y_B]_2^G = [y_{A \setminus \{i\}} \cup y_B]_2^G = y_{(A \setminus \{i\}) \cap B} = x_{(A \cap B) \setminus \{i\}}$, by Lemma 15(b) and the fact that $x_{A \setminus \{i\}} = y_{A \setminus \{i\}}$.

(d) Without loss of generality, consider only the case $\{i, j\} = \{1, 2\}$. Clearly there exist two vertices w, z in G such that $(w_1, w_2) = (y_1, x_2)$, $(z_1, z_2) = (x_1, y_2)$ and $w|_{[3,t]} = z|_{[3,t]} = x|_{[3,t]} = y|_{[3,t]}$. Since w and z are each adjacent to both x and y , we see that $x_A \cup y_B$ influences $\{w, z\}$ in the social network $(G, 2)$. It follows that $[x_A \cup y_B]_2^G \supseteq [x_A \cup \{w\} \cup \{z\}]_2^G$ and $[x_A \cup y_B]_2^G \supseteq [y_B \cup \{w\} \cup \{z\}]_2^G$. By using Lemma 15(a) twice, we see that $[x_A \cup \{w\} \cup \{z\}]_2^G \supseteq x_{A \setminus \{1,2\}}$, and hence $[y_B \cup \{w\} \cup \{z\}]_2^G \supseteq y_{B \setminus \{1,2\}}$. By the fact $y_{B \setminus \{1,2\}} = x_{B \setminus \{1,2\}}$ and using Lemma 15(b), we get that $[x_A \cup y_B]_2^G \supseteq [x_{A \setminus \{1,2\}} \cup x_{B \setminus \{1,2\}}]_2^G = x_{(A \cap B) \setminus \{1,2\}}$. Since $(x_A \cup y_B) \subseteq x_{(A \cap B) \setminus \{1,2\}}$, we conclude that $[x_A \cup y_B]_2^G = x_{(A \cap B) \setminus \{1,2\}}$.

(e) For a vertex w in $V \setminus (x_A \cup y_B)$, by Claim 14, we see that w cannot be adjacent to two distinct vertices in x_A (resp. y_B). Note that since $d(x_A, y_B) \geq 3$ there is no vertex w in $V \setminus (x_A \cup y_B)$ that is adjacent to one vertex in x_A and is also adjacent to one vertex in y_B . This completes the proof of (e). \blacksquare

For $\mathcal{U} \subseteq [1, t]$, let $\overline{\mathcal{U}}$ denote the set $[1, t] \setminus \mathcal{U}$. Using the notation and results in Lemma 15, we immediately obtain the following:

Claim 16 (a) If $x_A \cap y_B \neq \emptyset$, then $[x_A \cup y_B]_2^G = x_{A \cap B}$.

(b) If $d(x_A, y_B) = 1$, then $[x_A \cup y_B]_2^G = x_{A \cap B \cap \overline{\{i\}}}$ for some $i \in [1, t]$.

(c) If $d(x_A, y_B) = 2$, then $[x_A \cup y_B]_2^G = x_{A \cap B \cap \overline{\{i,j\}}}$ for some $i, j \in [1, t]$.

Theorem 17 Suppose $G = (V, E)$ is the Hamming graph $\prod_{i=1}^t K_{n_i}$ and S is a non-empty set of vertices.

(a) There exist vertices $x^1, x^2, \dots, x^k \in V$ and sets $A_1, A_2, \dots, A_k \subseteq [1, t]$ such that $[S]_2^G = \cup_{i=1}^k x_{A_i}^i$ with $d(x_{A_i}^i, x_{A_j}^j) \geq 3$ for any $1 \leq i < j \leq k$.

(b) If $[S]_2^G = \cup_{i=1}^k x_{A_i}^i$ for some vertices x^1, \dots, x^k in V and some sets $A_1, \dots, A_k \subseteq [1, t]$ with $d(x_{A_i}^i, x_{A_j}^j) \geq 3$ for any $1 \leq i < j \leq k$, then the following inequality holds:

$$\sum_{i=1}^k |A_i| \geq (2+t)k - 2|S|. \quad (\star)$$

Proof. (a) Note that $S = \cup_{x \in S} x_{[1,t]}$ and $[S' \cup S^*]_2^G = [[S']_2^G \cup [S^*]_2^G]$ for any $S', S^* \subseteq V$. By using several times Claim 16 and Lemma 15(e), we can get vertices $x^1, x^2, \dots, x^k \in V$ and sets $A_1, A_2, \dots, A_k \subseteq [1, t]$ such that $[S]_2^G = \cup_{i=1}^k x_{A_i}^i$ with $d(x_{A_i}^i, x_{A_j}^j) \geq 3$ for any $1 \leq i < j \leq k$.

(b) To prove this part we use induction on the size of S . When $|S| = 1$ (say $S = \{x^1\}$), since in this scenario $[S]_2^G = x_{[1,t]}^1$, it can be seen that the inequality (\star) clearly holds. Now assume that the statement of Theorem 17(b) holds for any $S \subseteq V$ having $|S| < \ell$.

When $|S| = \ell \geq 2$, the proof is divided into cases according to the value of k .

Case 1. $k = 1$. In this case, pick $x \in S$ and let $S' = S \setminus \{x\}$. Note that S' is not empty. By Theorem 17(a) we see that there are vertices $y^1, y^2, \dots, y^r \in V$ and sets $B_1, B_2, \dots, B_r \subseteq [1, t]$ such that $[S']_2^G = \cup_{i=1}^r y_{B_i}^i$ having $d(y_{B_i}^i, y_{B_j}^j) \geq 3$ for any $1 \leq i < j \leq r$. We have $x_{A_1}^1 = [S]_2^G = [x_{[1,t]} \cup [S']_2^G]_2^G = [x_{[1,t]} \cup (\cup_{i=1}^r y_{B_i}^i)]_2^G$. Then from Claim 16 and Lemma 15(e) we see that $A_1 = (\cap_{i=1}^r B_i) \cap \overline{\mathcal{U}}$ for some set $\mathcal{U} \subseteq [1, t]$ having $|\mathcal{U}| \leq 2r$. Since $|S'| < \ell$, by the induction hypothesis, we have $\sum_{i=1}^r |B_i| \geq (2+t)r - 2|S'|$. It follows that $|A_1| = t - |(\cup_{i=1}^r \overline{B_i}) \cup \mathcal{U}| \geq t - \sum_{i=1}^r (t - |B_i|) - 2r \geq t - rt + (2+t)r - 2|S'| - 2r = (2+t) - 2|S'|$. Thus inequality (\star) holds in this case.

Case 2. $k > 1$. In this case, let $S^* = S \cap x_{A_1}^1$ and $S' = S \setminus S^*$. Note that S^* and S' are not empty. Clearly $[S^*]_2^G = x_{A_1}^1$ and $[S']_2^G = \cup_{i=2}^k x_{A_i}^i$. By the induction hypothesis we see that $|A_1| \geq (2+t) - 2|S^*|$ and $\sum_{i=2}^k |A_i| \geq (2+t)(k-1) - 2|S'|$. It follows immediately that inequality (\star) holds in this case. This completes the proof of the theorem. \blacksquare

Theorem 18 *If $G = (V, E)$ is the Hamming graph $\prod_{i=1}^t K_{n_i}$, then $\text{min-seed}(G, 2) = 1 + \lceil \frac{t}{2} \rceil$.*

Proof. Note that $V = V(K_{n_1}) \times V(K_{n_2}) \times \dots \times V(K_{n_t})$. For each $i = 1, 2, \dots, t$, pick two distinct vertices $x_i, y_i \in V(K_{n_i})$. Let $x = (x_1, x_2, \dots, x_t)$. For $1 \leq j \leq t$, let $p^j = (p_1^j, \dots, p_t^j)$ be a vertex in V such that $p_i^j = x_i$ when $i \neq j$, and $p_j^j = y_j$. For $1 \leq j \leq t-1$, let $q^j = (q_1^j, \dots, q_t^j)$ be a vertex in V such that $q_i^j = x_i$ when $i \notin \{j, j+1\}$, and $q_j^j = y_j, q_{j+1}^j = y_{j+1}$.

First, we want to show that $1 + \lceil \frac{t}{2} \rceil$ is an upper bound for $\text{min-seed}(G, 2)$. The proof is divided into two cases according to the parity of t .

Case 1. $t = 2\ell$. Let $S = \{p^1, p^2\} \cup \{q^3, q^5, q^7, \dots, q^{t-1}\}$. By Lemma 15(d) it can be seen that $[\{p^1, p^2\}]_2^G = p_{[3,t]}^1 = x_{[3,t]}$, $[\{p^1, p^2, q^3\}]_2^G = [x_{[3,t]} \cup q_{[1,t]}^3]_2^G = x_{[5,t]}$, and $[\{p^1, p^2, q^3, q^5\}]_2^G = [x_{[5,t]} \cup q_{[1,t]}^5]_2^G = x_{[7,t]}$. Continue in this way, we obtain $[S]_2^G = [x_{[t-1,t]} \cup q_{[1,t]}^{t-1}]_2^G = x_\emptyset = V$, which means that $\text{min-seed}(G, 2) \leq |S| = \ell + 1 = 1 + \lceil \frac{t}{2} \rceil$.

Case 2. $t = 2\ell + 1$. Let $S = \{p^1, p^2, p^3\} \cup \{q^4, q^6, q^8, \dots, q^{t-1}\}$. By Lemma 15(d) and the same arguments as above, we obtain $[S]_2^G = [x_{[t-1,t]} \cup q_{[1,t]}^{t-1}]_2^G = V$, and hence $\text{min-seed}(G, 2) \leq |S| = \ell + 2 = 1 + \lceil \frac{t}{2} \rceil$.

To show that $1 + \lceil \frac{t}{2} \rceil$ is also a lower bound for $\text{min-seed}(G, 2)$, let S be an optimal target set for $(G, 2)$. Since $[S]_2^G = V = x_\emptyset$, by Theorem 17(b), we have $|\emptyset| \geq (2+t) - 2|S|$, that is $|S| \geq 1 + \frac{t}{2}$. Which completes the proof of the theorem. ■

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