The complexity of total edge domination and some related results on trees

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Abstract: For a graph $G = (V, E)$ with vertex set V and edge set E, a subset F of E is called an *edge dominating set* (resp. a *total edge dominating set*) if every edge in $E \backslash F$ (resp. in E) is adjacent to at least one edge in F, the minimum cardinality of an edge dominating set (resp. a total edge dominating set) of G is the *edge domination number* (resp. *total edge domination number*) of G, denoted by $\gamma'(G)$ (resp. $\gamma'_t(G)$). In the present paper, we prove that the total edge domination problem is NP-complete for bipartite graphs with maximum degree 3. We also design a linear-time algorithm for solving this problem for trees. Finally, for a graph G, we give the inequality $\gamma'(G) \leqslant \gamma'_t(G) \leqslant 2\gamma'(G)$ and characterize the trees T which obtain the upper or lower bounds in the inequality.

Keywords: Edge domination; Total edge domination; NP-completeness; Linear-time algorithm; Trees

1   Introduction

Dominating problems have been subject of many studies in graph theory, and have many applications in operations research, e.g., in resource allocation and network routing, as well as in coding theory. There are many variants of domination, we mainly fucus on the total edge domination which is a variant of edge domination. Edge domination is introduced by Mitchell and Hedetniemi [7] and is related to telephone switching network [6]. Edge domination is also related to the approximation of the vertex cover problem, since an independent edge dominating set is a matching [3].

In this paper we in general follow [1] for natation and graph theory terminology. All graphs considered here are finite, undirected, connected, have no loops or multiple edges. Let $G = (V, E)$ be a graph with vertex set V and edge set E. A subset F of E is called an *edge dominating set* (abbreviated for *ED-set*) of G if every edge not in F is adjacent to at least one edge in F. The edge domination number, denoted by $\gamma'(G)$, is the minimum cardinality of an ED-set of G. An ED-set of G with cardinality $\gamma'(G)$ is called a $\gamma'(G)$-*set*. The edge domination problem has been studied by several authors for example [2, 4, 11, 13]. Yannakakis and Gavril [13] showed that, the edge domination problem is NP-complete even when graphs are planar or bipartite of maximum degree 3, but solvable for trees and claw-free chordal graphs.

The concept of the total edge domination, a variant of edge domination, was introduced by Kulli and Patwari [5]. A subset F_t of E is called a *total edge dominating set*

(abbreviated for *TED-set*) of G if every edge is adjacent to at least one edge in F_t. The *total edge domination number*, denoted by $\gamma_t'(G)$, is the minimum cardinality of a TED-set of G. A TED-set of G with cardinality $\gamma_t'(G)$ is called a $\gamma_t'(G)$-*set*. Zhao et al. proved [14] that the total edge domination problem is NP-complete for planar graphs with maximum degree three, and for undirected path graphs and also constructed a linear algorithm for total edge domination problem in trees by a label method. For more study on total edge domination, see for example references [8, 9, 10].

As far as we know, there is no discussion on the complexity of total edge domination problem for bipartite graphs. For this reason, we prove that the total edge domination problem is NP-complete for bipartite graphs with maximum degree 3. We also design another linear time algorithm for computing $\gamma_t'(T)$ of a tree T by the dynamic programming method, different from the algorithm in [14]. Kulli et al. [5] gave the lower bound of the total edge domination number for a graph G: $\gamma'(G) \leqslant \gamma_t'(G)$, it is obvious that $\gamma_t'(G) \leqslant 2\gamma'(G)$. So, for any graph G, $\gamma'(G) \leqslant \gamma_t'(G) \leqslant 2\gamma'(G)$. In this paper, we show that the bounds are sharp and characterize trees achieving the lower or upper bound.

Notation. Let $G = (V, E)$ be a graph. For $v \in V$, denote by $N_G(v)$ the *open neighborhood* of v in G, i.e., $N_G(v) = \{u \in V \mid uv \in E\}$, by $deg_G(v)$ the size of $N_G(v)$ called the *degree* of v, and by $E_G(v)$ the set of all the edges of G incident with v, i.e., $E_G(v) = \{e \in E \mid v \text{ is incident with } e\}$. Similarly, for $e \in E$, denote by $N_G(e)$ the *open neighbourhood* of e in G, i.e., $N_G(e) = \{e' \in E \mid e' \text{ is adjacent to } e\}$ and by $N_G[e] = N_G(e) \cup \{e\}$ the *closed neighbourhood* of e. For two vertices $u, v \in V$, the *distance* $d_G(u, v)$ is defined as the length of a shortest path between u and v in G. We define the shorter distance between vertex w and one endpoint of edge e as the *distance* between w and e, denoted by $d_G(w, e)$. The maximum distance among all pairs of vertices is called the *diameter* of G, denoted by $diam(G)$. If there is no ambiguity in the sequel, the subscript in the notation is omitted.

A *leaf* of a graph G is a vertex of degree one and a *support vertex* (resp. *strong support vertex*) of G is a vertex adjacent to a leaf (resp. adjacent to at least two leaves). A *leaf edge* (or *pendant edge*) of G is an edge with one leaf as an endpoint. Consider one vertex of a tree as special, called the *root* of this tree. A tree with the fixed root is a *rooted tree*. For a vertex v of a rooted tree T with root r, a neighbour of v away from r is called a *child*. For a positive integer k, a *star* $S_{1,k}$ is a tree that contains exactly one non-leaf vertex called a *center vertex* and k leaves. A *double star* is a tree that contains exactly two non-leaf vertices called *center vertices*.

2 The result on NP-completeness

In this section, we are going to prove that the total edge domination problem is NP-complete for bipartite graphs with maximum degree 3. To prove that a problem P is NP-complete, it is enough to prove that $P \in \mathcal{NP}$ and to show that a known NP-complete problem is reducible to the problem P in polynomial time. The known NP-complete problem used in our reduction is the SAT-3 restricted problem as follows:

SAT-3 RESTRICTED PROBLEM (SAT-3 RES) [12].
Instance: A set of clauses $C_1, C_2, \ldots, C_p$ containing only variables, with at most three literals per clause, such that every variable occurs two times and its negation once.
Question: Is there a truth assignment of zeros and ones to the variables satisfying all the clauses?

The decision total edge domination problem is stated as follows:

Instance: A graph $G = (V, E)$ and a positive integer $k \leqslant |E|$.
Question: Does G have a total edge dominating set of size at most k?

Now we can state our main result in this section.

Theorem 2.1. *The total edge domination problem for bipartite graphs with maximum degree 3 is NP-complete.*

Proof. The reduction is from the SAT-3 restricted problem. Consider a set of clauses $\{C_1, \ldots, C_p\}$ with variables $x_1, x_2, \ldots, x_n$ as input for the SAT-3 restricted problem. Now we construct a graph $G = (V, E)$. For any $1 \leqslant l \leqslant p$, there are two adjacent vertices, say d_l and d'_l, corresponding to the clause C_l, denoted by G_l. For any $1 \leqslant i \leqslant n$, there is a subgraph of G, which is a disjoint union of three paths $a_i a_{i,0} a_{i,1} a_{i,2}$, $b_i b_{i,0} b_{i,1} b_{i,2}$, $c_i c_{i,0} c_{i,1} c_{i,2}$, and two edges $a_i c_i$ and $c_i b_i$, corresponding to the variable x_i, denoted by G_{x_i} (see Fig. 1). For any clause C_l, if $x_i \in C_l$, then we connect d_l to one of vertices $a_{i,0}$ and $b_{i,0}$ to ensure that $d(a_{i,0}) = 3$ and $d(b_{i,0}) = 3$ (from conditions in SAT-3 RES); if $\overline{x_i} \in C_l$, then we connect d'_l to $c_{i,0}$ (for an example, see Fig. 1). It is obvious that G is bipartite, coloring vertices with white and black, shown as Fig. 1. We will show that there is a truth assignment of zeros and ones to the variables satisfying all clauses $\{C_1, C_2, \ldots, C_p\}$ if and only if G has a total edge dominating set of size $6n$.

Figure 1: The induced subgraph of G by G_{l_1}, G_{x_i}, G_{x_j} and G_{x_k}; for example, $C_{l_1} = (x_i \overline{x_j} x_k)$.

Necessity: Given a satisfying assignment of the clauses, define a set F of edges as follows (assume that x_i is in two clauses C_{l_1} and C_{l_3}, and $\overline{x_i}$ is in clause C_{l_2}):
$$F = \{a_{i,0} d_{j_1}, a_{i,0} a_{i,1}, b_{i,0} d_{j_3}, b_{i,0} b_{i,1}, c_i c_{i,0}, c_{i,0} c_{i,1} \mid x_i = 1\}$$
$$\cup \{a_i a_{i,0}, a_{i,0} a_{i,1}, b_i b_{i,0}, b_{i,0} b_{i,1}, c_{i,0} d'_{j_2}, c_{i,0} c_{i,1} \mid x_i = 0\},$$
(see Fig. 2). It is obvious that F is a TED-set of size $6n$.

Figure 2: The construction of a TED-set F on G_{x_i}, represented by the thick edges.

Conversely, we assume that G has a TED-set F of size $6n$. For any $1 \leqslant i \leqslant n$, in view of leaf edges $a_{i,1} a_{i,2}, b_{i,1} b_{i,2}, c_{i,1} c_{i,2}$, F must contain three edges $a_{i,0} a_{i,1}, b_{i,0} b_{i,1}, c_{i,0} c_{i,1}$ and its respective adjacent edges. Thus the subgraph G_{x_i} contains exactly 6 edges in F. For the convenience of proof, we assume that x_i is contained in clauses C_{l_1} and C_{l_2}, and $\overline{x_i}$ is contained in clause C_{l_3}.

Case 1. $c_i c_{i,0} \notin F$.

In this case, F must contain $a_i a_{i,0}, b_i b_{i,0}$ and we may assume that the edge adjacent to $c_{i,0} c_{i,1}$ in F is $c_{i,0} d'_{l_3}$, otherwise we can add $c_{i,0} d'_{l_3}$ into F by deleting $c_{i,1} c_{i,2}$ from F.

Case 2. $c_i c_{i,0} \in F$.

Similar to Case 1, we can assume that $a_{i,0} d_{l_1}, b_{i,0} d_{l_2} \in F$.

Therefore, regardless of whether F contains $c_i c_{i,0}$, we can always give a special total edge dominating set F of size $6n$. We define a truth assignment τ by, if $c_i c_{i,0} \in F$, setting $x_i = 1$ and $x_i = 0$, otherwise. Since F is a TED-set constructed as above, at least one edge in F is adjacent to $d_l d'_l$ for every l (note that $d_l d'_l \notin F$). Consequently τ satisfies all clauses.

The degree of vertices except for d_l and d'_l in G constructed above is at most 3, but if $C_l = x_{i_1} x_{i_2} x_{i_3}$ (resp., $\overline{x_{i_1}} \ \overline{x_{i_2}} \ \overline{x_{i_3}}$), then $d_G(d_l) = 4$ (resp., $d_G(d'_l) = 4$). Then we use a tricky technique: (1) replace H shown as Fig. 3(a) for $d_l d'_l$ and, (2) replace the three edges connecting the vertices a, b, c corresponding to variables and d_l (resp. d'_l) with the three edges connecting a, b, c and x, y, z in H, respectively, say ax, by, cz.

(a) H contains exactly 9 edges in F when none of the three edges $\{ax, by, cz\}$ belongs to F;

(b) H contains exactly 8 edges in F when ax is in F.

Figure 3: The graph H and the construction of a TED-set F on H, represented by the thick edges.

It is easy to show by a straightforward case analysis that: for a TED-set F of G,

(1). if none of the three edges $\{ax, by, cz\}$ belongs to F, then F contains at least nine edges from H, see Fig. 3(a).

(2). if one of three edges $\{ax, by, cz\}$ is in F, say ax, then F contains at least eight edges from H, see Fig. 3(b).

Especially, let s be the number of 3-literal clauses which satisfies that the literals contained are all positive or all negative. Then we can similarly show that there is a truth assignment of zeros and ones to the variables satisfying all clauses $\{C_1, C_2, \ldots, C_p\}$ if and only if G has a total edge dominating set of size $6n + 8s$. $\qquad \square$

From the proof of Theorem 2.1, the graph constructed has a girth of at least 10.

Corollary 2.1. *The total edge domination problem for bipartite graphs of girth at least 10 with maximum degree 3 is NP-complete.*

Proof. The notations are as in the proof of Theorem 2.1. By the construction of G, there are no edges among G_l's (or H) and among G_{x_i}'s. So a cycle C is either in H (note that there is no cycles in G_l or G_{x_i})or formed by going through $G_{l_1}, G_{x_{i_1}}, G_{l_2}, G_{x_{i_2}}, \ldots, G_{l_k}, G_{x_k}, G_{l_1}$ ($k \geqslant 2$); in the second case the intersection of C and G_{x_i} contains at least three edges and so the length of C is at least $5k \geqslant 10$. Note that the girth of H is more than 12. $\qquad \square$

3 A linear-time algorithm for trees

In this section, we work on a linear-time algorithm for finding the total edge domination number of a tree by using the dynamic programming method.

First, we define some sets and some parameters. Let T be a tree with an edge e. We define:

$$\mathcal{F}_1(T, e) := \{F \mid F \text{ is a TED-set of } T \text{ with } e \in F\};$$
$$\mathcal{F}_0(T, e) := \{F \mid F \text{ is a TED-set of } T \text{ with } e \notin F\};$$
$$\mathcal{F}_{\bar{1}}(T, e) := \{F \mid F \text{ is an ED-set of } T \text{ with a unique isolated edge } e \text{ in } F\};$$
$$\mathcal{F}_{\bar{0}}(T, e) := \{F \mid F \text{ is a TED-set of } T - e, \text{but } e \text{ is not dominated by } F\}.$$

It is easily obtained

Lemma 3.1. *Let e be a leaf edge of tree T. Then*
$\mathcal{F}_1(T, e) \neq \emptyset$ *if and only if $T \neq K_2$;*
$\mathcal{F}_0(T, e) \neq \emptyset$ *if and only if T has at least 3 edges;*
$\mathcal{F}_{\bar{1}}(T, e) \neq \emptyset$ *(resp. $\mathcal{F}_{\bar{0}}(T, e) \neq \emptyset$) if and only if $T \setminus N[e]$ has no K_2 as components.*

We denote

$$\gamma'_1(T, e) := \min\{ |F| \mid F \in \mathcal{F}_1(T, e)\};$$
$$\gamma'_0(T, e) := \min\{ |F| \mid F \in \mathcal{F}_0(T, e)\};$$
$$\gamma'_{\bar{1}}(T, e) := \min\{ |F| \mid F \in \mathcal{F}_{\bar{1}}(T, e)\};$$
$$\gamma'_{\bar{0}}(T, e) := \min\{ |F| \mid F \in \mathcal{F}_{\bar{0}}(T, e)\}.$$

By convention, if a set is empty, then we set the value as infinity. For example, if $\mathcal{F}_{\bar{0}}(T, e) = \emptyset$, then we set $\gamma'_{\bar{0}}(T, e) = \infty$. We can define $F \in \mathcal{F}_1(T, e)$ (resp. $\mathcal{F}_0(T, e)$, $\mathcal{F}_{\bar{1}}(T, e)$, $\mathcal{F}_{\bar{0}}(T, e)$) of minimum cardinality as a $\gamma'_1(T, e)$ (resp. $\gamma'_0(T, e)$, $\gamma'_{\bar{1}}(T, e)$, $\gamma'_{\bar{0}}(T, e)$)-set of T. We give some inequality relationships among four values defined as above.

Lemma 3.2. *Let T be a tree with an edge e. If $\mathcal{F}_1(T, e), \mathcal{F}_0(T, e), \mathcal{F}_{\bar{1}}(T, e)$ and $\mathcal{F}_{\bar{0}}(T, e)$ are non-empty sets, then*
(1) $\gamma'_1(T, e) \leqslant \gamma'_0(T, e) + 1$;
(2) $\gamma'_1(T, e) \leqslant \gamma'_{\bar{1}}(T, e) + 1$;
(3) $\gamma'_1(T, e) \leqslant \gamma'_{\bar{0}}(T, e) + 2$;
(4) $\gamma'_{\bar{1}}(T, e) \leqslant \gamma'_{\bar{0}}(T, e) + 1$.

Proof. Let e' be any edge in $N(e)$.

(1) Let $F \in \mathcal{F}_0(T, e)$. Then there exists an edge $e'' \in F$ adjacent to e and further $F + e$ is a TED set of T containing e. Therefore $\gamma'_1(T, e) \leqslant \gamma'_0(T, e) + 1$.

(2) Let $F_0 \in \mathcal{F}_{\bar{1}}(T, e)$. Then $e \in F_0$ and $N(e) \cap F_0 = \emptyset$ by the definition of $\mathcal{F}_{\bar{1}}(T, e)$. $F_0 + e'$ is a TED-set of containing e. Therefore $\gamma'_{t,1}(T, e) \leqslant \gamma'_{\bar{1}}(T, e) + 1$.

(3) Let $F_1 \in \mathcal{F}_{\bar{0}}(T, e)$. Then $N[e] \cap F_1 = \emptyset$ by the definition of $\mathcal{F}_{\bar{0}}(T, e)$. $F_1 + e + e'$ is a TED-set of T containing e. Thus $\gamma'_1(T, e) \leqslant \gamma'_{t,\bar{0}}(T, e) + 2$.

(4) Let $F_2 \in \mathcal{F}_{\bar{0}}(T, e)$. Then $F_2 + e$ is an ED-set of T with a unique isolated edge e by the definition of $\mathcal{F}_{\bar{1}}(T, e)$. Thus $\gamma'_{\bar{1}}(T, e) \leqslant \gamma'_{t,\bar{0}}(T, e) + 1$. $\qquad\square$

Before giving the dynamic programming algorithm, we designed an edge data structure as follows.

Root the tree T at any leaf, say r. The *height*, denoted by h, of T is the maximum distance between r and all other vertices of T. The *level i* $(0 \leqslant i \leqslant h)$ is the set of vertices of T with a distance i from r.

For such a rooted tree T of order $n+1$, let us label the edges of T as $1, 2, \ldots, n$. We go through every level from h to 1. For each i, $1 \leqslant i \leqslant h$, we traverse the edges connecting the vertices on i and $i-1$ in any order, from left to right. We list the fathers of all edges

of T (the edge numbered n has no father by writing father $[n] = 0$), so we can use a data structure called an *edge parent array* to represent T. Let e^0 be a non-leaf edge in rooted tree T, u the endpoint of e^0 away from the root. Denote by $N_c(e^0)$ the set of neighbors of e^0 with endpoints u, called *children neighbors* of e^0, say $\{e^1, e^2, \ldots, e^q\}$ for some integer q. For $0 \leqslant j \leqslant q$, let T^j be the component containing e^j of $T \setminus (\{e^0, e^1, \ldots, e^q\} \setminus \{e^j\})$.

Theorem 3.1. *Let T be a rooted tree with a non-leaf edge e^0 and $N_c(e^0) = \{e_1, e_2, \ldots, e_q\}$ for some integer $q \geqslant 1$. For $0 \leqslant j \leqslant q$, T^j are defined as above, and denote*

$$\theta_j := \min\{\gamma_1'(T^j, e^j), \gamma_0'(T^j, e^j), \gamma_{\bar{1}}'(T^j, e^j), \gamma_{\bar{0}}'(T^j, e^j)\};$$
$$A_1 := \{j \in \{1, 2, \ldots, q\} | \theta_j = \gamma_1'(T^j, e^j)\};$$
$$A_2 := \{j \in \{1, 2, \ldots, q\} | \theta_j = \gamma_0'(T^j, e^j)\};$$
$$A_3 := \{j \in \{1, 2, \ldots, q\} | \theta_j = \gamma_{\bar{1}}'(T^j, e^j)\};$$
$$A_4 := \{j \in \{1, 2, \ldots, q\} | \theta_j = \gamma_{\bar{0}}'(T^j, e^j)\}.$$

Then

(1). $\gamma_1'(T, e^0) = \begin{cases} \min\{\gamma_1'(T^0, e^0), \gamma_{\bar{1}}'(T^0, e^0)\} + \sum_{j=1}^{q} \theta_j, & \text{if } A_1 \cup A_3 \neq \emptyset; \\ \min\{\gamma_1'(T^0, e^0), \gamma_{\bar{1}}'(T^0, e^0) + 1\} + \sum_{j=1}^{q} \theta_j, & \text{if } A_1 \cup A_3 = \emptyset. \end{cases}$

(2). $\gamma_0'(T, e^0) = \begin{cases} \min\{\gamma_0'(T^0, e^0), \gamma_{\bar{0}}'(T^0, e^0)\} + \sum_{j=1}^{q} \theta_j, & \text{if } A_1 \neq \emptyset \text{ or } |A_3| \geqslant 2; \\ \min\{\gamma_0'(T^0, e^0), \gamma_{\bar{0}}'(T^0, e^0)\} + \sum_{j=1}^{q} \theta_j + 1, & \begin{array}{l} \text{if } A_1 = \emptyset \text{ and } |A_3| = 1 \text{ or} \\ A_1 = A_3 = \emptyset, \ A_2 \neq \emptyset, \ A_4 \neq \emptyset; \end{array} \\ \min\{\gamma_0'(T^0, e^0), \gamma_{\bar{0}}'(T^0, e^0) + 1\} + \sum_{j=1}^{q} \theta_j, & \text{if } A_1 = A_3 = A_4 = \emptyset; \\ \min\{\gamma_0'(T^0, e^0), \gamma_{\bar{0}}'(T^0, e^0)\} + \sum_{j=1}^{q} \theta_j + 1, & \begin{array}{l} \text{if } A_1 = A_2 = A_3 = \emptyset, \text{ and there is } j \in A_4 \\ \text{such that } \gamma_1'(T^j, e^j) - \gamma_{\bar{0}}'(T^j, e^j) = 1; \end{array} \\ \min\{\gamma_0'(T^0, e^0), \gamma_{\bar{0}}'(T^0, e^0)\} + \sum_{j=1}^{q} \theta_j + 2, & \begin{array}{l} \text{if } A_1 = A_2 = A_3 = \emptyset \text{ and any } j \in A_4, \\ \gamma_1'(T^j, e^j) - \gamma_{\bar{0}}'(T^j, e^j) = 2. \end{array} \end{cases}$

(3). $\gamma_{\bar{1}}'(T, e^0) = \gamma_{\bar{1}}'(T^0, e^0) + \sum_{j=1}^{q} \min\{\gamma_0'(T^j, e^j), \gamma_{\bar{0}}'(T^j, e^j)\};$

(4). $\gamma_{\bar{0}}'(T, e^0) = \gamma_{\bar{0}}'(T^0, e^0) + \sum_{j=1}^{q} \gamma_0'(T^j, e^j).$

Proof. For the convenience, for $0 \leqslant j \leqslant q$, we define $F_{T^j} = F_T \cap T^j$ for an edge subset F_T of T and thus $|F_T| = \sum_{j=0}^{q} |F_{T^j}|$. Especially, for $0 \leqslant j \leqslant q$, if F_T is a TED-set of T, then $F_{T^j} \in \mathcal{F}_1(T^j, e^j) \cup \mathcal{F}_0(T^j, e^j) \cup \mathcal{F}_{\bar{1}}(T^j, e^j) \cup \mathcal{F}_{\bar{0}}(T^j, e^j)$ by the definition. Denote $\overline{N_c}(e^0) = N(e^0) \setminus N_c(e^0)$.

(1). Let F_T be a $\gamma_1'(T, e^0)$-set.
Case 1.1. $\overline{N_c}(e^0) \cap F_T \neq \emptyset$.

In this case, the restriction F_{T^0} of F_T on T^0 is a TED-set of T^0, further a $\gamma_1'(T^0, e^0)$-set. For any j $(1 \leqslant j \leqslant q)$, F_{T^j} is a set of size θ_j in $\mathcal{F}_1(T^j, e^j) \cup \mathcal{F}_0(T^j, e^j) \cup \mathcal{F}_{\bar{1}}(T^j, e^j) \cup \mathcal{F}_{\bar{0}}(T^j, e^j)$ by the definition of F_{T^j}. So

$$\gamma_1'(T, e^0) = \gamma_1'(T^0, e^0) + \sum_{j=1}^{q} \theta_j.$$

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Case 1.2. $\overline{N_c}(e^0) \cap F_T = \emptyset$.

In this case, $F_{T^0} \in \mathcal{F}_{\bar{1}}(T^0, e^0)$. Thus

$$\gamma_1'(T, e^0) \geqslant \gamma_{\bar{1}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j. \tag{1}$$

In order to connect e^0 in F_T, there exists some $1 \leqslant j \leqslant q$ such that $e^j \in F_{T^j}$.

Subcase 1.2.1. $A_1 \cup A_3 \neq \emptyset$, say, $j_1 \in A_1$.

We take any $\gamma_{\bar{1}}'(T^0, e^0)$-set B^0 and $\gamma_1'(T^{j_1}, e^{j_1})$-set B^{j_1}. For any $j \neq j_1$ ($1 \leqslant j \leqslant q$), we choose an edge set B^j of size θ_j in $\mathcal{F}_1(T^j, e^j) \cup \mathcal{F}_0(T^j, e^j) \cup \mathcal{F}_{\bar{1}}(T^j, e^j) \cup \mathcal{F}_{\bar{0}}(T^j, e^j)$. Then $\cup_{j=0}^{q} B^j$ is a TED-set of T of size $\gamma_{\bar{1}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j$ satisfying $|N(e_0) \cap (\cup_{j=0}^{q} B^j)| \geqslant 1$. Combined with (1), we have

$$\gamma_1'(T, e^0) = \gamma_{\bar{1}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j.$$

Subcase 1.2.2. $A_1 \cup A_3 = \emptyset$, i.e., $A_1 = \emptyset$ and $A_3 = \emptyset$.

In this subcase, equality does not hold in Eq. (1). If $A_2 \neq \emptyset$, combined with Lemma 3.1, Lemma 3.2 (1) and $A_1 = \emptyset$, for any $j \in A_2$, $\gamma_1'(T^j, e^j) = \gamma_0'(T^j, e^j) + 1 = \theta_j + 1$. Otherwise, If $A_2 = \emptyset$, then $A_4 = \{1, 2, \ldots, q\}(\neq \emptyset)$. Combined with Lemma 3.2 (4) and $A_3 = \emptyset$, for any $j \in A_4$, $\gamma_{\bar{1}}'(T^j, e^j) = \gamma_{\bar{0}}'(T^j, e^j) + 1 = \theta_j + 1$. Similar to Subcase 1.2.1, whatever which case it is, we can construct a TED-set of T of size $\gamma_{\bar{1}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j + 1$ satisfying $|N(e_0) \cap (\cup_{j=0}^{q} B^j)| \geqslant 1$. So

$$\gamma_1'(T, e^0) = \gamma_{\bar{1}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j + 1.$$

(2). Let F_T be a $\gamma_0'(T, e^0)$-set.

If $\overline{N_c}(e^0) \cap F_T \neq \emptyset$, then the restriction F_{T^0} of F_T on T^0 is a TED-set of T^0, further a $\gamma_0'(T^0, e^0)$-set. So

$$\gamma_0'(T, e^0) \geqslant \gamma_0'(T^0, e^0) + \sum_{j=1}^{q} \theta_j. \tag{2}$$

If $\overline{N_c}(e^0) \cap F_T = \emptyset$, then the restriction F_{T^0} of F_T on T^0 belongs to $\mathcal{F}_{\bar{0}}(T^0, e^0)$, further a $\gamma_{\bar{0}}'(T^0, e^0)$-set. So

$$\gamma_0'(T, e^0) \geqslant \gamma_{\bar{0}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j. \tag{3}$$

Case 2.1. $A_1 \neq \emptyset$, say $j_1 \in A_1$.

We take any $\gamma_0'(T^0, e^0)$-set in the case of $\overline{N_c}(e^0) \cap F_T \neq \emptyset$ and any $\gamma_{\bar{0}}'(T^0, e^0)$-set in the case of $\overline{N_c}(e^0) \cap F_T = \emptyset$. Denoted by B^0, any $\gamma_1'(T^{j_1}, e^{j_1})$-set B^{j_1}, and for any $j \neq j_1$ ($1 \leqslant j \leqslant q$), an edge set B^j of size θ_j in $\mathcal{F}_1(T^j, e^j) \cup \mathcal{F}_0(T^j, e^j) \cup \mathcal{F}_{\bar{1}}(T^j, e^j) \cup \mathcal{F}_{\bar{0}}(T^j, e^j)$. Thus $\cup_{j=0}^{q} B^j$ is a TED-set of T of size $\gamma_0'(T^0, e^0) + \sum_{j=1}^{q} \theta_j$ in the case of $\overline{N_c}(e^0) \cap F_T \neq \emptyset$ or $\gamma_{\bar{0}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j$ in the case of $\overline{N_c}(e^0) \cap F_T = \emptyset$ satisfying $|N(e_0) \cap (\cup_{j=0}^{q} B^j)| \neq 0$. Combined with (2) and (3), we have

$$\gamma_0'(T, e^0) = \min\{\gamma_0'(T^0, e^0), \gamma_{\bar{0}}'(T^0, e^0)\} + \sum_{j=1}^{q} \theta_j.$$

Case 2.2. $A_1 = \emptyset$ and $A_3 \neq \emptyset$.

If $|A_3| \geqslant 2$, then, for $j_1, j_2 \in A_2$, we can take a $\gamma_{\overline{1}}'(T^{j_1}, e^{j_1})$-set B^{j_1} and a $\gamma_1'(T^{j_2}, e^{j_2})$-set B^{j_2}. The others B^0 and B^j for $1 \leqslant j \leqslant q$ and $j \neq j_1, j_2$ are taken as Subcase 2.1. Similarly, we can obtain

$$\gamma_0'(T, e^0) = \min\{\gamma_0'(T^0, e^0), \gamma_{\overline{0}}'(T^0, e^0)\} + \sum_{j=1}^{q} \theta_j.$$

If $|A_3| = 1$, say $A_3 = \{j_3\}$, then neither Eq. (2) nor Eq. (3) take equality in this case. According to Lemma 3.2 (2) and $A_1 = \emptyset$, $\gamma_1'(T^{j_3}, e^{j_3}) = \gamma_{\overline{1}}'(T^{j_3}, e^{j_2}) + 1 = \theta_{j_3} + 1$. We take a $\gamma_1'(T^{j_3}, e^{j_3})$-set B^{j_3}. The others B^0 and B^j for $1 \leqslant j \leqslant q$ and $j \neq j_3$ are taken as Subcase 2.1. Thus $\cup_{j=0}^{q} B^j$ is a TED-set of T of size $\gamma_0'(T^0, e^0) + \sum_{j=1}^{q} \theta_j + 1$ in the case of $\overline{N_c}(e^0) \cap F_T \neq \emptyset$ or $\gamma_{\overline{0}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j + 1$ in the case of $\overline{N_c}(e^0) \cap F_T = \emptyset$ satisfying $|N(e_0) \cap (\cup_{j=0}^{q} B^j)| \neq 0$. So

$$\gamma_0'(T, e^0) = \min\{\gamma_0'(T^0, e^0), \gamma_{\overline{0}}'(T^0, e^0)\} + \sum_{j=1}^{q} \theta_j + 1.$$

Case 2.3. $A_1 = A_3 = \emptyset$ and $A_2 \neq \emptyset$.

If $A_4 = \emptyset$, i.e., $A_2 = \{1, 2, \ldots, q\}$, and $\overline{N_c}(e^0) \cap F_T \neq \emptyset$, then we take any $\gamma_0'(T^0, e^0)$-set B^0. For $1 \leqslant j \leqslant q$, we take any $\gamma_0'(T^j, e^j)$-set B^j. Thus $\cup_{j=0}^{q} B^j$ is a TED-set of T of size $\gamma_0'(T^0, e^0) + \sum_{j=1}^{q} \theta_j$.

If $A_4 = \emptyset$ and $\overline{N_c}(e^0) \cap F_T = \emptyset$, then equality does not hold in Eq. (3). By Lemma 3.2 (1) and $A_1 = \emptyset$, for any $j \in A_2, \gamma_1'(T^j, e^j) = \gamma_0'(T^j, e^j) + 1 = \theta_j + 1$. We take a $\gamma_1'(T^{j_1}, e^{j_1})$-set B^{j_1} for some $1 \leqslant j_1 \leqslant q$ and others B^j for any $0 \leqslant j \leqslant q$ and $j \neq j_1$ are taken as in Subcase 2.1. Thus $\cup_{j=0}^{q} B^j$ is a TED-set of T of size $\gamma_{\overline{0}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j + 1$.

If $A_4 \neq 0$, then equality does not hold in Eqs. (2) and (3). By Lemma 3.2 (1) and $A_1 = \emptyset$, for any $j \in A_2, \gamma_1'(T^j, e^j) = \gamma_0'(T^j, e^j) + 1 = \theta_j + 1$. We take any $\gamma_1'(T^{j_1}, e^{j_1})$-set B^{j_1} for some $1 \leqslant j_1 \leqslant q$ and the others B^j for any $0 \leqslant j \leqslant q$ and $j \neq j_1$ are taken as in Subcase 2.1. Thus $\cup_{j=0}^{q} B^j$ is a TED-set of T of size $\gamma_0'(T^0, e^0) + \sum_{j=1}^{q} \theta_j + 1$ in the case of $\overline{N_c}(e^0) \cap F_T \neq \emptyset$ or $\gamma_{\overline{0}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j + 1$ in the case of $\overline{N_c}(e^0) \cap F_T = \emptyset$.

So

$$\gamma_0'(T, e^0) = \begin{cases} \min\{\gamma_0'(T^0, e^0), \gamma_{\overline{0}}'(T^0, e^0) + 1\} + \sum_{j=1}^{q} \theta_j, & \text{if } A_4 = \emptyset; \\ \min\{\gamma_0'(T^0, e^0), \gamma_{\overline{0}}'(T^0, e^0)\} + \sum_{j=1}^{q} \theta_j + 1, & \text{if } A_4 \neq \emptyset. \end{cases}$$

Case 2.4. $A_1 = A_2 = A_3 = \emptyset$, i.e., $A_4 = \{1, 2, \ldots, q\}$.

In this case, to obtain a $\gamma_0'(T, e^0)$-set, we need one $\gamma_1'(T^{j'}, e^{j'})$-set or at least two $\gamma_{\overline{1}}'(T^{j''}, e^{j''})$-sets for $1 \leqslant j'' \leqslant q$. So, equality does not hold in Eqs. (2) and (3). By Lemma 3.2 (4), for each $\forall j \in A_4$, we have $\gamma_{\overline{0}}'(T^j, e^j) + 1 \leqslant \gamma_1'(T^j, e^j) \leqslant \gamma_{\overline{0}}'(T^j, e^j) + 2$. If there exists $j_4 \in A_4$ such that $\gamma_1'(T^{j_4}, e^{j_4}) - \gamma_{\overline{0}}'(T^{j_4}, e^{j_4}) = 1$, then we can take a $\gamma_1'(T^{j_4}, e^{j_4})$-set B^{j_4} and the others B^j for $0 \leqslant j \leqslant q$ and $j \neq j^4$ are taken as in Subcase 2.1. Thus $\cup_{j=0}^{q} B^j$ is a TED-set of T of size $\gamma_0'(T^0, e^0) + \sum_{j=1}^{q} \theta_j + 1$ in the case of $\overline{N_c}(e^0) \cap F_T \neq \emptyset$ or $\gamma_{\overline{0}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j + 1$ in the case of $\overline{N_c}(e) \cap F_T = \emptyset$. Otherwise, for all j, $\gamma_1'(T^j, e^j) - \gamma_{\overline{0}}'(T^j, e^j) = 2$. Thus, the left-hand sides in both Eqs. (2) and (3) are at least two more than the right-hand sides. We can take a $\gamma_1'(T^{j_4}, e^{j_4})$-set B^{j_4} and the others B^j for $0 \leqslant j \leqslant q$ and $j \neq j^4$ are taken as in Subcase 2.1. Thus $\cup_{j=0}^{q} B^j$ is a TED-set of T of size $\gamma_0'(T^0, e^0) + \sum_{j=1}^{q} \theta_j + 2$ in the case of $\overline{N_c}(e^0) \cap F_T \neq \emptyset$ or $\gamma_{\overline{0}}'(T^0, e^0) + \sum_{j=1}^{q} \theta_j + 1$ in the case of $\overline{N_c}(e^0) \cap F_T = \emptyset$. Therefore

$$\gamma'_0(T, e^0) = \begin{cases} \min\{\gamma'_0(T^0, e^0), \gamma'_{\overline{0}}(T^0, e^0)\} + \sum_{j=1}^{q} \theta_j + 1, & \text{there is } j \text{ such that } \gamma'_1(T^j, e^j) - \gamma'_{t,\overline{0}}(T^j, e^j) = 1; \\ \min\{\gamma'_0(T^0, e^0), \gamma'_{\overline{0}}(T^0, e^0)\} + \sum_{j=1}^{q} \theta_j + 2, & \text{for any } j \text{ such that } \gamma'_1(T^j, e^j) - \gamma'_{t,\overline{0}}(T^j, e^j) = 2. \end{cases}$$

(3). Let F_T be a $\gamma'_{\overline{1}}(T, e)$-set.

The restriction F_{T^0} of F_T on T^0 belongs to $\mathcal{F}_{\overline{1}}(T^0, e)$, for $1 \leqslant j \leqslant q$, the restriction F_{T^j} of F_T on T^j belongs to $\mathcal{F}_0(F^j, e^j)$ or $\mathcal{F}_{\overline{0}}(F^j, e^j)$, the converse also holds. Therefore

$$\gamma'_{\overline{1}}(T, e) = \gamma'_{\overline{1}}(T^0, e) + \sum_{j=1}^{q} \min\{\gamma'_0(T^j, e^j), \gamma'_{\overline{0}}(T^j, e^j)\}.$$

(4). Let F_T be a $\gamma'_{\overline{0}}(T, e)$-set.

The restriction F_{T^0} of F_T on T^0 belongs to $\mathcal{F}_{\overline{0}}(T^0, e)$, for $1 \leqslant j \leqslant q$, the restriction F_{T^j} of F_T on T^j belongs to $\mathcal{F}_0(F^j, e^j)$, the converse also holds. Therefore

$$\gamma'_{\overline{0}}(T, e) = \gamma'_{\overline{0}}(T^0, e) + \sum_{j=1}^{q} \gamma'_0(T^j, e^j).$$

$\square$

By Theorem 3.1, we give algorithms as follows.

Algorithm 1 Determine the value of $\gamma'_1(T, i')$.

Require: an edge i of a rooted tree T which represent by its edge parent array $[1, 2, 3, \ldots, n]$.
Ensure: $\gamma'_1(T, i')$
1: $i' \leftarrow father(i)$;
2: $N_c(i') \leftarrow children(i')$;
3: $T^0 \leftarrow$ the component containing i' of $T - (N_c(i'))$;
4: **for** each $j \in N_c(i')$ **do**
5:     $T^j \leftarrow$ the component containing j of $T - (N_c(i') + i' - j)$;
6: **end for**
7: **for** each $j \in N_c(i')$ **do**
8:     $\theta_j \leftarrow \min\{\gamma'_1(T^j, j), \gamma'_0(T^j, j), \gamma'_{\overline{1}}(T^j, j), \gamma'_{\overline{0}}(T^j, j)\}$;
9:     $A_1 \leftarrow \{j \in N_c(i') | \theta_j = \gamma'_1(T^j, j)\}$;
10:     $A_3 \leftarrow \{j \in N_c(i') | \theta_j = \gamma'_{\overline{1}}(T^j, j)\}$;
11: **end for**
12: **if** $A_1 \cup A_3 \neq \emptyset$ **then**
13:     $\gamma'_1(T, i') \leftarrow \min\{\gamma'_1(T^0, i'), \gamma'_{\overline{1}}(T^0, i')\} + \sum_{j \in N_c(i')} \theta_j$
14: **else**
15:     $\gamma'_1(T, i') \leftarrow \min\{\gamma'_1(T^0, i'), \gamma'_{\overline{1}}(T^0, i') + 1\} + \sum_{j \in N_c(i')} \theta_j$.
16: **end if**

Algorithm 2 Determine the value of $\gamma'_{\overline{1}}(T, i')$.

Require: an edge i of a rooted tree T which represent by its edge parent array $[1, 2, 3, \ldots, n]$.
Ensure: $\gamma'_{\overline{1}}(T, i')$

    $i' \leftarrow father(i)$;
    $N_c(i') \leftarrow children(i')$;
    $T^0 \leftarrow$ the component containing i' of $T - (N_c(i'))$;
    for each $j \in N_c(i')$ **do**
       $T^j \leftarrow$ the component containing j of $T - (N_c(i') + i' - j)$;
    end for
    $\gamma'_{\overline{1}}(T, i') \leftarrow \gamma'_{\overline{1}}(T^0, i') + \sum_{j \in N_c(i')} \min\{\gamma'_0(T^j, j), \gamma'_{\overline{0}}(T^j, j)\}$

Theorem 3.2. *Algorithm 5 produces the total edge domination number of a tree in linear-time.*

Algorithm 3 Determine the value of $\gamma_0'(T, i')$.

Require: an edge i of a rooted tree T which represent by its edge parent array $[1, 2, 3, \ldots, n]$.

Ensure: $\gamma_0'(T, i')$;

  $N_c(i') \leftarrow children(i')$;

  $T^0 \leftarrow$ the component containing i' of $T - (N_c(i'))$;

  for each $j \in N_c(i')$ **do**

    $T^j \leftarrow$ the component containing j of $T - (N_c(i') + i' - j)$;

  end for

  for each $j \in N_c(i')$ **do**

    $\theta_j \leftarrow \min\{\gamma_1'(T^j, j), \gamma_0'(T^j, j), \gamma_{\overline{1}}'(T^j, j), \gamma_{\overline{0}}'(T^j, j)\}$;

    $A_1 \leftarrow \{j \in N_c(i') | \theta_j = \gamma_1'(T^j, j)\}$;

    $A_2 \leftarrow \{j \in N_c(i') | \theta_j = \gamma_0'(T^j, j)\}$;

    $A_3 \leftarrow \{j \in N_c(i') | \theta_j = \gamma_{\overline{1}}'(T^j, j)\}$;

    $A_4 \leftarrow \{j \in N_c(i') | \theta_j = \gamma_{\overline{0}}'(T^j, j)\}$;

  end for

  if $A_1 \neq \emptyset$ *or* $|A_3| \geqslant 2$ **then**

    $\gamma_0'(T, i') \leftarrow \min\{\gamma_0'(T^0, i'), \gamma_{\overline{0}}'(T^0, i')\} + \sum\limits_{j \in N_c(i')} \theta_j$

  else if $A_1 = \emptyset$, *and* $|A_3| = 1$ *or* $A_3 = \emptyset,\ A_2 \neq \emptyset,\ A_4 \neq \emptyset$ **then**

    $\gamma_0'(T, i') \leftarrow \min\{\gamma_0'(T^0, i'), \gamma_{\overline{0}}'(T^0, i')\} + \sum\limits_{j \in N_c(i')} \theta_j + 1$

  else if $A_1 = A_3 = A_4 = \emptyset$ **then**

    $\gamma_0'(T, i') \leftarrow \min\{\gamma_0'(T^0, i'), \gamma_{\overline{0}}'(T^0, i') + 1\} + \sum\limits_{j \in N_c(i')} \theta_j$

  else if $A_1 = A_2 = A_3 = \emptyset$, *and* $\exists j \in A_4$, *such that* $\gamma_1'(T^j, j) - \gamma_{\overline{0}}'(T^j, j) == 1$ **then**

    $\gamma_0'(T, i') \leftarrow \min\{\gamma_0'(T^0, i'), \gamma_{\overline{0}}'(T^0, i')\} + \sum\limits_{j \in N_c(i')} \theta_j + 1$

  else if $A_1 = A_2 = A_3 = \emptyset$, *and* $\forall j \in A_4, \gamma_1'(T^j, j) - \gamma_{\overline{0}}'(T^j, j) == 2$ **then**

    $\gamma_0'(T, i') \leftarrow \min\{\gamma_0'(T^0, i'), \gamma_{\overline{0}}'(T^0, i')\} + \sum\limits_{j \in N_c(i')} +2$

  end if

Algorithm 4 Determine the value of $\gamma_{\overline{0}}'(T, i')$.

Require: an edge i of an rooted tree T which represent by its edge parent array $[1, 2, 3, \ldots, n]$.

Ensure: $\gamma_1'(T, i')$

  $i' \leftarrow father(i)$;

  $N_c(i') \leftarrow children(i')$;

  $T^0 \leftarrow$ the component containing i' of $T - (N_c(i'))$;

  for each $j \in N_c(i')$ **do**

    $T^j \leftarrow$ the component containing j of $T - (N_c(i') + i' - j)$;

  end for

  $\gamma_{\overline{0}}'(T, i') \leftarrow \gamma_{\overline{0}}'(T^0, i') + \sum\limits_{j \in N_c(i')}^{m} \gamma_0'(T^j, j)$

Algorithm 5 Determine the total edge domination number of a tree.

Require: an edge rooted tree T represent by its edge parent array $[1, 2, 3, \ldots, n]$.
Ensure: a minimum total edge domination number of T.
1: **for** each $i \in [1, n]$ **do**
2: $\quad \gamma'_1(T, 1) \leftarrow \infty; \gamma'_0(T, 1) \leftarrow \infty; \gamma'_{\overline{1}}(T, 1) \leftarrow 1; \gamma'_{\overline{0}}(T, 1) \leftarrow 0;$
3: **end for**
4: **for** each $i \in [1, n-1]$ **do**
5: $\quad i' \leftarrow father(i);$
6: $\quad N_c(i') \leftarrow children(i');$
7: $\quad T^0 \leftarrow$ the component containing i' of $T - (N_c(i'));$
8: $\quad$ **for** each $j \in N_c(i')$ **do**
9: $\quad\quad T^j \leftarrow$ the component containing j of $T - (N_c(i') + i' - j);$
10: $\quad$ **end for**
11: $\quad$ **for** each $j \in N_c(i')$ **do**
12: $\quad\quad \theta_j := \min\{\gamma'_1(T^j, j), \gamma'_0(T^j, j), \gamma'_{\overline{1}}(T^j, j), \gamma'_{\overline{0}}(T^j, j)\};$
13: $\quad\quad A_1 := \{j \in N_c(i') | \theta_j = \gamma'_1(T^j, j)\};$
14: $\quad\quad A_2 := \{j \in N_c(i') | \theta_j = \gamma'_0(T^j, j)\};$
15: $\quad\quad A_3 := \{j \in N_c(i') | \theta_j = \gamma'_{\overline{1}}(T^j, j)\};$
16: $\quad\quad A_4 := \{j \in N_c(i') | \theta_j = \gamma'_{\overline{0}}(T^j, j)\};$
17: $\quad\quad \gamma'_1(T, i')$=Determine the value of $\gamma'_1(T, i').$
18: $\quad\quad \gamma'_0(T, i')$=Determine the value of $\gamma'_0(T, i').$
19: $\quad\quad \gamma'_{\overline{1}}(T, i')$=Determine the value of $\gamma'_{\overline{1}}(T, i').$
20: $\quad\quad \gamma'_{\overline{1}}(T, i')$=Determine the value of $\gamma'_{\overline{0}}(T, i').$
21: $\quad$ **end for**
22: **end for**
23: **return** $\gamma'(T) = \min\{\gamma'_1(T, n), \gamma'_0(T, n), \gamma'_{\overline{1}}(T, n), \gamma'_{\overline{0}}(T, n)\}$

Proof. It is easy to know that the running times of Algorithms 1, 2, 3 and 4 are constant times. Then Algorithm 5, needing to visit each father edge e of T once, and all of the statements within which can be executed in a constant time, so with an adequate data structure the algorithm works in linear-time. $\qquad \square$

4 Characterizing $(\gamma'_t = 2\gamma')$-trees and $(\gamma'_t = \gamma')$-trees

In this section we provide a constructive characterization of trees satisfying $\gamma'_t(T) = 2\gamma'(T)$ and $\gamma'_t(T) = \gamma'(T)$, denoted by $(\gamma'_t = 2\gamma')$-trees and $(\gamma'_t = \gamma')$-trees, respectively.

First, we begin with some properties of specific graphs used in this section.

Example 4.1. *Let T be a star or a double star. Then $\gamma'(T) = 1$ and $\gamma'_t(T) = 2$.*

Example 4.2. *If T is a path with five vertices, then $\gamma'(T) = \gamma'_t(T) = 2$. If T is a path with six vertices, then $\gamma'(T) = 2$ and $\gamma'_t(T) = 3$.*

Theorem 4.1. *Let G be a connected graph of diameter $\geqslant 4$. Then there exists a minimum edge dominating set (resp. a minimum total edge dominating set) D of G such that D contains no leaf edges of G.*

Proof. Suppose to the contrary that each minimum edge dominating set contains some leaf edges and D is a minimum edge dominating set containing least leaf edges. Then for each leaf edge $e \in D$, $N(e) \cap D = \emptyset$, otherwise, $D - e$ is a smaller edge dominating set, a contradiction. Choose one non-leaf edge e' of $N(e)$, then $D' = D - e + e'$ is a new minimum edge dominating set containing less leaf edges than D, a contradiction. Similarly, we can prove the total version. $\qquad \square$

Corollary 4.1. *Let T be a tree with diameter 4. Then $\gamma'_t(T) = \gamma'(T)$.*

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Proof. The induced subgraph of all non-leaf edges in T is a star $S_{1,k}$. In order to dominate all leaf edges, by Theorem 4.1, $E(S_{1,k})$ is a minimum edge dominating set and also a TED-set of T, so $\gamma'_t(T) \leqslant \gamma'(T)$, combined with $\gamma'(T) \leqslant \gamma'_t(T)$, we get $\gamma'_t(T) = \gamma'(T)$. $\qquad\square$

Corollary 4.2. *Let T be a tree with diameter 5. Then $\gamma'_t(T) = \gamma'(T)$ or $\gamma'_t(T) = \gamma'(T)+1$.*

Proof. From the condition, the induced subgraph of all non-leaf edges in T is exactly a double star, say H and two adjacent center vertices r, t. Let D be a minimum edge dominating set of T containing no leaf edges by theorem 4.1 and, e an leaf edge in H, say $e = vr$ or vt. Since, in T, v is incident with at least one leaf edge, D contains e. Thus $(E(H) - rt) \subseteq D$. Combined that $D + rt$ induces an connected subgraph, exactly H, further $D + rt$ is a total edge dominating set of T, so $\gamma'_t(T) = \gamma'(T)$ or $\gamma'_t(T) = \gamma'(T) + 1$. $\qquad\square$

4.1 $(\gamma'_t = 2\gamma')$-trees

In this subsection we provide a constructive characterization of trees T satisfying $\gamma'_t(T) = 2\gamma'(T)$. Note that a star or double star satisfies the condition above. In what follows we consider the trees satisfying the condition other than stars.

Our aim is to describe an inductive procedure of the tree T with $\gamma'_t(T) = 2\gamma'(T)$ by labelling. For the initiated step, for any vertex v of P_4, we give a label C or L to v, denoted by $l(v)$, defined as $l(v) = L$ if v is a leaf of P_4, $l(v) = C$, otherwise. For convenience, we call an edge with both endpoints labelled C as $C - C$ edge.

Let $\mathcal{T}$ be the family of labelled trees T containing the labelled P_4 as the initiated labelled tree, constructed inductively by the two operations $\mathcal{O}_1$, $\mathcal{O}_2$ listed below (i.e., constructing a bigger labelled tree T' from a smaller labelled tree T in $\mathcal{T}$).

Operation $\mathcal{O}_1$: Let $T \in \mathcal{T}$ and v a vertex of T with $l(v) = L$ such that: (1). each vertex labelled C of distance 2 from v is adjacent to a leaf vertex; (2). For any $C - C$ edge wu of distance 1 from v, say v is adjacent to u, either u has a leaf other than v or $N(w) - u$ are all leaves. Construct a bigger tree T' in $\mathcal{T}$ from T and a labelled P_4 by identifying v and a leaf vertex of P_4, labelling the identified vertex as L and keeping the labels of the other vertices unchanged, see Fig. 5(a).

Operation $\mathcal{O}_2$: Let $T \in \mathcal{T}$ and v a vertex of T with $l(v) = C$. Construct a bigger tree T' in $\mathcal{T}$ from T by adding a new vertex u adjacent to v, labelling u as L, keeping the labels of the other vertices unchanged, see Fig. 5(b).

(a) Operation $\mathcal{O}_1$;　　　　　　　　(b) Operation $\mathcal{O}_2$.

Figure 4: Two operations.

From the two operations above, we can get the following simple observations.

Observation 4.1. *Let $T \in \mathcal{T}$. Then*

(1) Each leaf vertex is labelled L and each support vertex is labelled C.

(2) *Exactly one neighbor of each vertex labelled C is labelled C, and the remaining neighbours are labelled L.*

(3) *No two vertices labelled L are adjacent.*

(4) *If one endpoints of a $C - C$ edge has a non-leaf neighbor labelled L, then the other endpoint has one leaf neighbor.*

Lemma 4.1. *Let $T \in \mathcal{T}$ and U the set of edges whose endpoints are labelled C in T. Then U is a $\gamma'(T)$-set.*

Proof. By Observation 4.1 (2) and (3), we know that U is an edge dominating set of T and further each component of the induced subgraph $T[U]$ is K_2. By Observation 4.1 (4) and Theorem 4.1, the size of any edge dominating set is at least $|D|$. Thus, U is a $\gamma'(T)$-set of T. $\qquad\square$

Lemma 4.2. *Let $T \in \mathcal{T}$. Then T is a $(\gamma'_t = 2\gamma')$-tree.*

Proof. We proceed by induction on the size m of the edge set of a tree $T \in \mathcal{T}$. For the initial step, it is obvious that $\gamma'_t(P_4) = 2\gamma'(P_4)$. For the inductive hypothesis, we assume that, for every $\overline{T} \in \mathcal{T}$ of edge size less than m, $\gamma'_t(\overline{T}) = 2\gamma'(\overline{T})$. Let $T \in \mathcal{T}$ with edge size m, and suppose T is obtained from a tree $\overline{T} \in \mathcal{T}$ by one of two operations. We need to prove that $\gamma'_t(T) = 2\gamma'(T)$. Next, we divide two cases to analyze according to which operation is used to construct the tree T from $\overline{T}$.

Case 1. T is obtained from $\overline{T}$ and a labelled $P_4 = u_1 u_2 u_3 u_4$ by Operation 1, i.e., identifying u_1 and $v(\in V(\overline{T}))$, denoted by v the identifying vertex in T.

By Lemma 4.1, we have $\gamma'(T) = \gamma'(\overline{T}) + 1$. Next, we just need to show $\gamma'_t(T) = \gamma'_t(\overline{T}) + 2$.

On the one hand, the union of a $\gamma'_t(\overline{T})$-set of $\overline{T}$ and $\{vu_2, u_2u_3\}$ is a TED-set of T, further $\gamma'_t(T) \leqslant \gamma'_t(\overline{T})+2$. On the other hand, it is sufficient to show that $\gamma'_t(\overline{T})+2 \leq \gamma'_t(T)$. Without loss of generality, let $N_{\overline{T}}(v) = \{v_1, \ldots, v_r\}$ for some positive integer r. For $1 \leqslant i \leqslant r$, from the definition of Operation 1 and Observation 4.1 (3), $l_{\overline{T}}(v) = L$ and $l_{\overline{T}}(v_i) = C$; by Observation 4.1 (2), we denote by w_i $(1 \leqslant i \leqslant r)$ the unique vertex labelled C adjacent to v_i in $\overline{T}$; and by the choice of v in the definition of Operation 1, w_i has one leaf neighbor in $\overline{T}$.

By Theorem 4.1, we let F_t be such a $\gamma'_t(T)$-set that F_t contains no leaf edges. If the restriction $F_t|_{\overline{T}}$ of F_t on $\overline{T}$ is a TED-set of $\overline{T}$, then $\gamma'_t(\overline{T}) + 2 \leq \gamma'_t(T)$. In what follows we assume that $F_t|_{\overline{T}}$ is not a TED-set of $\overline{T}$, then $|E_{\overline{T}}(v) \cap F_t| \leqslant 1$.

If $E_{\overline{T}}(v) \cap F_t = \emptyset$, then $F_t|_{\overline{T}}$ does not dominate some edge incident with v in $\overline{T}$, say vv_i for some integer i, further there is no leaf edge e incident with v_i in T, otherwise F_t does not dominate e in T. By the choice of v in Operation 1, all neighbors of w_i other than v_i are all leaves, a contradiction with the choice of F_t. If $E_{\overline{T}}(v) \cap F_t$ has a unique edge, say vv_i for some i, then $w_iv_i \notin F_t$. Since w_i has a leaf vertex by the choice of v in Operation 1, there is one edge in F_t incident with w_i. Therefore the restriction of $F_t - vv_i + v_iw_i$ on $\overline{T}$ is a TED-set of $\overline{T}$, further $\gamma'_t(\overline{T}) + 2 \leqslant \gamma'_t(T)$.

Case 2. T is obtained from $\overline{T}$ by adding a new vertex u adjacent to v labelled C (i.e., Operation 2).

By Lemma 4.1, we can easily get $\gamma'(T) = \gamma'(\overline{T})$. Then $\gamma'_t(T) \leqslant 2\gamma'(T) = 2\gamma'(\overline{T}) = \gamma'_t(\overline{T}) \leqslant \gamma'_t(T)$, and so $\gamma'_t(T) = 2\gamma'(T)$.

Combined the two cases above, we have $\gamma'_t(T) = 2\gamma'(T)$ for $T \in \mathcal{T}$. $\qquad\square$

Lemma 4.3. *Let T be a tree with $\gamma'_t(T) = 2\gamma'(T)$, F a $\gamma'(T)$-set. Then $N[e] \cap N[e'] = \emptyset$ for any distinct edges $e, e' \in F$.*

Proof. By contradiction. Assume that there exist two edges e, e' in F such that $N[e] \cap N[e'] \neq \emptyset$, say $e'' \in N[e] \cap N[e']$. Now we construct a TED-set S of T from $F + e''$: for any edge $f \in F - e - e'$, adding an edge adjacent to f to $F + e''$. Then $|S| \leqslant 2|F| - 1 = 2\gamma'(T) - 1$, a contradiction. $\qquad\square$

Corollary 4.3. *Let T be a tree with $\gamma'_t(T) = 2\gamma'(T)$, vu and uw two adjacent edges in T. Then v, w and u can't all be support vertices.*

Proof. This follows directly from Lemma 4.3. $\qquad\square$

Lemma 4.4. *Let T be a non-star tree with $\gamma'_t(T) = 2\gamma'(T)$. Then $T \in \mathcal{T}$.*

Proof. We proceed by induction on the edge size of a non-star tree T with $\gamma'_t(T) = 2\gamma'(T)$. For the initial step, if T is a tree with $diam(T) = 3$, then T is a double star with $\gamma'(T) = 1$ and $\gamma'_t(T) = 2$, so we can obtain T from a labelled P_4 by doing a series of Operation $\mathcal{O}_2$. By Corollaries 4.1 and 4.2, if T is a tree with $diam(T) = 4$ or 5, then T does not satisfy $\gamma'_t(T) = 2\gamma'(T)$. In what follows let T be a tree of edge size m and diameter at least 6 with $\gamma'_t(T) = 2\gamma'(T)$. For the inductive hypothesis, we assume that every tree $\overline{T}$ of edge size less than m with $\gamma'_t(\overline{T}) = 2\gamma'(\overline{T})$ is in $\mathcal{T}$.

If a support vertex v has two leaf neighbor in T with $\gamma'_t(T) = 2\gamma'(T)$ and w is one of leaf neighbors of v, then v is still a support vertex in $\overline{T} = T - w$. Combined with Theorem 4.1, a minimum edge dominating set (resp. a minimum total edge set) of $\overline{T}$ containing no leaf edges is exactly a minimum edge edge dominating set (resp. a minimum total edge set) of T containing no leaf edges. So $\gamma'(\overline{T}) = \gamma'(T)$, $\gamma'_t(\overline{T}) = \gamma'_t(T)$. Therefore, $2\gamma'(\overline{T}) = 2\gamma'(T) = \gamma'_t(T) = \gamma'_t(\overline{T})$. By the inductive hypothesis, $\overline{T} \in \mathcal{T}$ with a labeling. By Observation 4.1 (1), the support vertex v is labelled C in $\overline{T}$. Thus we can obtain the tree T by applying Operation $\mathcal{O}_2$ to $\overline{T}$.

Let P be a longest path in T, say $P = v_0 v_1 \ldots v_t$ for some t ($t \geqslant 6$) and denoted by $e_i = v_i v_{i+1}$. If v_2 has a leaf neighbor, say v'_1, let $\overline{T} = T - v'_1$. By Theorem 4.1, $\overline{T}$ has a $\gamma'(\overline{T})$-set (resp. a $\gamma'_t(\overline{T})$-set) containing e_1, which is still a $\gamma'(T)$-set (resp. a $\gamma'_t(T)$-set), so $\gamma'_t(\overline{T}) = \gamma'_t(T) = 2\gamma'(T) = 2\gamma'(\overline{T})$. By the inductive hypothesis, $\overline{T} \in \mathcal{T}$ with a labelling. By Observation 4.1 (1) and (2), the vertices v_1 and v_2 are labelled C in $\overline{T}$. Thus we can obtain the tree T by applying Operation $\mathcal{O}_2$ to $\overline{T}$.

In what follows we assume that each support vertex of T has exactly one leaf neighbor and v_2 is not a support vertex. Let F be a $\gamma'(T)$-set of T containing non-leaf edges by Theorem 4.1, thus $e_1 \in F$. For convenience, we root T at the vertex v_t.

Claim 1. *For every child v of v_3, the subtree of $T - v_3$ containing v is exactly P_3.*

By contradiction. If v is a leaf, then there exists an edge incident with v_3 in F, say e. Note that $e_1 \in F$. But $N[e] \cap N[e_1] \neq \emptyset$, a contradiction with Lemma 4.3. If v has only leaf children, then $vv_3 \in F$. Similarly, we can obtain a contradiction because $N[vv_3] \cap N[e_1] \neq \emptyset$. If v has at least two support children, then $|E(v) \cap F| \geqslant 2$, a contradiction. So v has exactly one support child, combined with the same role of v as v_2 in the choice of P and the assumption, we obtain the claim.

Claim 2. *For a child v'_3 of v_4, the length of a longest path starting at v'_3 in the subtree $T - v_4$ containing v'_3 is not 2.*

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Assume to the contrary that there exists one child v_3' of v_4 such that the length of a longest path P starting at v_3' in the subtree $T - v_4$ containing v_3' is 2, say $P = v_3'v_2'v_1'$. Obviously, $v_3' \neq v_3$ and $v_3'v_2' \in F$. Combined with Lemma 4.3 and $e_1 \in F$, $E(v_4) \cap F = \emptyset$, then e_3 is not dominated by F, a contradiction.

Claim 3. *If there exists a child v_3' of v_4 such that the subtree of $T - v_4$ containing v_3' is P_2, then v_5 has no leaf child.*

Similar to the analysis of Claims 1 and 2, we can show it by contradiction.

Claim 4. *If $diam(T) > 6$ and there exist no children v_3' of v_4 such that the subtree of $T - v_4$ containing v_3' is P_2, then v_4 and v_5 are both support vertices.*

Suppose to the contrary. Since there is no subtree of $T \setminus \{v_4\}$ containing v_3' isomorphic to P_2.

(a) A total edge dominating set F_t of T; (b) a minimum total edge dominating set F_t' of T.

Figure 5: Illstration for Claim 4 in Lemma 4.4.

Let $\{v_3^1, v_3^2, \ldots, v_3^s\}$ be the set of non-leaf children of v_4 for some positive integer s. For any $1 \leqslant i \leqslant l$, combined the assumption and Claim 2, the length of a longest path starting at v_3^i in the subtree $T - v_4$ containing v_3^i is 3. By the symmetry of v_3 and v_3^i and Claim 1, the subtree of $T - v_4$ containing v_3^i is exactly P_4, say $v_3^i v_2^i v_1^i v_0^i$. By the choice of F and Lemma 4.3, for any $1 \leqslant i \leqslant s$, $v_2^i v_1^i \in F$ and $E(v_3^i) \cap F = \emptyset$. So $\{e_4\} \subseteq F$.

If v_4 is not support, let e_7' be the unique edge in $(E_T(v_7) - e_6) \cap F \neq \emptyset$ by $\{e_4\} \subseteq F$ and Lemma 4.3. Now we construct a TED-set F_t of T from $F_0 = F - e_4 + e_5$ by first adding the common neighbor edge e_6 of e_5 and e_7' in F_0, second, for any $1 \leqslant i \leqslant l$, adding $v_2^i v_3^i$ into F_0, and adding a neighbor edge of each edge in $F_0 - \{e_5, e_7'\} + \{v_1^1 v_2^1, v_1^2 v_2^2, \cdots, v_1^s v_2^s\}$ (see Fig. 5(a)). It is obvious that F_t is a TED-set of T and $|F_t| \leqslant 2|F| - 1$, a contradiction.

If v_5 is not support, let A be the set of vertices of distance 2 from v_5 in the subtree of $T - e_4$ containing v_5. For $v \in A$, $|F \cap E(v)| = 1$, say e_v, by $e_4 \in F$ and Lemma 4.3, and denoted by e_v' the unique edge of $E(v)$ of distance 1 from v_5. Note that $e_v' \neq e_v$ because v_5 is of distance 2 from e_v. Now we can construct a TED-set F_t' of T from $F_0' = F - e_4 + e_3$ by first adding e_2 and the set $\{e_v'| v \in A\}$ into F_0', second adding a neighbor edge of each edge in $F_0' - \{e_v | v \in A\} - \{e_1, e_3\}$ (see Fig. 5(b)). Note that $\{e_1, e_2, e_3\} \in F_t'$. It is obvious that F_t' is a TED-set of T and $|F_t'| \leqslant 2|F| - 1$, a contradiction. So we prove Claim 4.

By Claim 1 and the assumption that each support vertex of T has exactly one leaf neighbor and v_2 is not a support vertex, $d(v_1) = d(v_2) = 2$, thus the subgraph induced by $\{v_0, v_1, v_2, v_3\}$ is P_4. Let $\overline{T} = T - \{v_0, v_1, v_2\}$.

Claim 5. $\gamma_t'(\overline{T}) = 2\gamma'(\overline{T})$.

Combined with Lemma 4.3 and $e_1 \in F$, we have $E(v_3) \cap F = \emptyset$, thus the restriction of F on $\overline{T}$ is an ED-set of $\overline{T}$, further $\gamma'(\overline{T}) \leqslant \gamma'(T) - 1$. Combined with the obvious

inequality: $\gamma'_t(T) \leqslant \gamma'_t(\overline{T}) + 2$, we have $2\gamma'(\overline{T}) \leqslant 2(\gamma'(T) - 1) = 2\gamma'(T) - 2 = \gamma'_t(T) - 2 \leqslant \gamma'_t(\overline{T}) \leqslant 2\gamma'(\overline{T})$. Consequently we must have equality throughout this inequality chain. Particularly, we have $\gamma'_t(\overline{T}) = 2\gamma'(\overline{T})$.

By Claim 5 and the inductive hypothesis, $\overline{T} \in \mathcal{T}$ with a labelling. In what follows we show that T is obtained from $\overline{T}$ by Operation $\mathcal{O}_1$ (the identifying vertex is v_3, the role of v). By Claim 1 and Observation 4.1 (1), (2), (3), we have $l(v_3) = L$, $l(v_4) = C$. In the case $l(v_5) = C$, if $diam(T) = 6$, then all neighbors of v_5 other than v_4 are all leaves; if $diam(T) > 6$, then each child of v_4 is labelled L and by Claim 4, v_4 and v_5 have both one leaf neighbor. In the other case $l(v_5) = L$, there is one child v'_3 of v_4 labelling C. Combined with Corollary 4.1 and Claim 2, v'_3 has only leaf children. For the other C-C edges wu of distance 1 from v_3, say v_3 is adjacent to u, i.e., u is the child of v_3, by Claim 1, $N(w) - u$ are all leaves. Combined all cases above, it is obvious that the $C - C$ edge incident with v_4 of distance 1 from v_3 satisfies the condition in Operation $\mathcal{O}_1$. Therefore we can apply Operation $\mathcal{O}_1$ from $\overline{T}$ to obtain the tree T, further, $T \in \mathcal{T}$. $\qquad \square$

As an immediate consequence of Lemmas 4.2 and 4.4, we have

Theorem 4.2. *A non-star tree is a $(\gamma'_t = 2\gamma')$-tree if and only if $T \in \mathcal{T}$.*

4.2 $(\gamma'_t = \gamma')$-trees

In this subsection we provide a constructive characterization of $(\gamma'_t = \gamma')$-trees T, i.e., a tree satisfying $\gamma'_t(T) = \gamma'(T)$. We use edge labelling to describe a procedure of constructing T recursively, which is different from the vertex labelling in the previous subsection. By Example 4.1 and Corollary 4.1, for the initial step, let T be a tree with $diam(T) = 4$, in which each edge is either a leaf edge or a support edge, we label support edges in T with S, leaf edges adjacent to at least two non-leaf-edges with L_2, other leaf edges with L_1.

Let $\mathcal{T}_t$ be the family of edge-labelled trees T that contains edge-labelled trees with diameter 4 and is under the five operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5$ listed below: constructing a bigger tree from a smaller tree in $\mathcal{T}_t$. For convenience, we call an edge labelled S (resp. L_1, L_2) in $T \in \mathcal{T}_t$ an S (resp. L_1, L_2)-edge, and denote by $D(T)$ the set of S-edges. First, according to the label of the associated edges of the vertex v in an edge-labelled tree $T \in \mathcal{T}_t$, we partition the vertex set of T into the following four subsets A_1, A_2, B and C listed below:

$$A_1 := \{v| \text{ Only one } S - \text{edge in } E(v)\};$$
$$A_2 := \{v| \text{ At least two } S - \text{edges in } E(v)\};$$
$$B := \{v| \text{ All edge in } E(v) \text{ are } L_2 - \text{edges}\};$$
$$C := V - A_1 - A_2 - B.$$

(a) $v \in A_1$: exactly one S-edge. (b) $v \in A_2$: at least two S-edges. (c) $v \in B$: all edges in $E(v)$ are L_2-edges. (d) $v \in C$: at least one L_1-edge but no S-edges in $E(v)$.

Figure 6: Vertex partition of $T \in \mathcal{T}$.

Now, we list the five operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5$:

Operation $\mathcal{O}_1$: Let $T \in \mathcal{T}_t$, v a vertex of T belonging to $A_1 \cup A_2$. Construct a bigger tree T' in $\mathcal{T}_t$ from T by adding a new vertex u adjacent to v. If $v \in A_1$, then label vu

as L_1; (by definition, u is in C, A_1, A_2, B are unchanged;) if $v \in A_2$, then label vu as L_2 (note that $u \in B$ and A_1, A_2, C are unchanged), see Fig. 7(a).

Operation $\mathcal{O}_2$: Let $T \in \mathcal{T}_t$, v a vertex of T belonging to A_2. Construct a bigger tree T' in $\mathcal{T}_t$ from T by adding two new adjacent vertices u_1, u_2, connecting v and u_1 and labelling vu_1 as S and u_1u_2 as L_1 (obviously, $u_1 \in A_1$ and $u_2 \in C$), see Fig. 7(b).

Operation $\mathcal{O}_3$: Let $T \in \mathcal{T}_t$, $v \notin A_1$ a vertex of T satisfying, in the case $v \in C$, that each L_1-edge in $E(v)$ is either adjacent to one leaf edge or contained in a $P_4 = vwxy$, whose edges are labelled as L_1, L_1, L_2 consecutively and all edges in $E(x)$ are L_2-edges except wx. Construct a bigger tree T' in $\mathcal{T}_t$ from T by adding a new path $u_1u_2u_3u_4u_5$ to join v and u_2, and labelling u_2u_3, u_3u_4 as S, vu_2, u_1u_2, u_4u_5 as L_1, see Fig. 7(c). (From the definition, $u_2, u_4 \in A_1$, $u_3 \in A_2$, $u_1, u_5 \in C$ and if $v \in B$, then v is moved from B to C.)

Operation $\mathcal{O}_4$: Let $T \in \mathcal{T}_t$, $v \in B$ a vertex of T. Construct a bigger tree T' in $\mathcal{T}_t$ from T by adding a new path $u_1u_2u_3u_4$ to join v and u_1, and labelling vu_1, u_3u_4 as L_1, u_1u_2, u_2u_3 as S, see Fig. 7(d). (Similarly, $u_1, u_3 \in A_1$, $u_2 \in A_2$, $u_4 \in C$, and v is moved from B to C.)

Operation $\mathcal{O}_5$: Let $T \in \mathcal{T}_t$, v a vertex of T. Construct a bigger tree T' in $\mathcal{T}_t$ from T by adding a new path $u_1u_2u_3u_4u_5$ to join v and u_3, and labelling vu_3 as L_2, u_1u_2, u_4u_5 as L_1, u_2u_3, u_3u_4 as S, see Fig. 7(e). (From the definition, $u_2, u_4 \in A_1$, $u_1, u_5 \in C$, $u_3 \in A_2$ and if $v \in B$, then v is moved from B to C.)

(a) Operation $\mathcal{O}_1$: the edge uv is labelled as L_2 if $v \in A_2$ and is labelled as L_1 if $v \in A_1$.

(b) Operation $\mathcal{O}_2$.

(c) Operation $\mathcal{O}_3$.

(d) Operation $\mathcal{O}_4$.

(e) Operation $\mathcal{O}_5$.

Figure 7: Five operations.

From the five operations above, we can get the simple observations as follows.

Observation 4.2. *Let $T \in \mathcal{T}_t$.*

(1) *One endpoint of an L_1-edge is incident with exactly one S-edge, the other endpoint is incident with either non S-edges or at least two S-edges.*

(2) *An L_2-edge is adjacent to at least two S-edges.*

(3) *A leaf edge is labelled L_1 or L_2. Furthermore, a leaf edge adjacent to exactly one non-leaf edge e is labelled L_1 and e is labelled as S.*

(4) *Each edge in T is adjacent to at least one S-edge, and each component of the induced subgraph $T[D(T)]$ is a nontrivial star. Further, $D(T)$ is a total edge dominating set of T.*

Lemma 4.5. *Let $T \in \mathcal{T}_t$. Then $D(T)$ is a $\gamma_t'(T)$-set and T is a $(\gamma_t' = \gamma')$-tree.*

Proof. Let $T \in \mathcal{T}_t$, we first prove that $D(T)$ (simply, D) is a $\gamma_t'(T)$-set. By Observation 4.2 (4), D is a TED-set of T. It is sufficient to find a set L of L_1-edges of size $|D|$ such that each edge in L has exactly one neighbor S-edge. In order to prove that there is such an edge set of each tree T in $\mathcal{T}_t$, we proceed by induction on the size m of the edge set of T. For the initial step, the leaves adjacent to exactly one non-leaf edge of T with diameter 4 construct the required set L. For the inductive step, we assume each tree $\overline{T}$ of size less than m in $\mathcal{T}_t$ has a set $\overline{L}$ of L_1-edges such that each edge in $\overline{L}$ has exactly one neighbor S-edge. Now we divide five cases as follows:

Case 1. T is obtained by applying Operation $\mathcal{O}_1$ from $\overline{T}$ and a vertex u.

In this case, $D(T) = D(\overline{T})$, and let $L = \overline{L}$, which is the desired set for T.

Case 2. T is obtained by applying Operation $\mathcal{O}_2$ from $\overline{T}$ and an edge u_1u_2 in which a vertex v in $\overline{T}$ is adjacent to u_1.

In this case, $D(T)$ is one more S-edge than $D(\overline{T})$. By Observation 4.2 (1), there is no L_1-edges in $\overline{L}$ incident with v. So $\overline{L} \cup \{u_1u_2\}$ is a desired set for T.

Case 3. T is obtained by applying Operation $\mathcal{O}_3$ from $\overline{T}$ and a path $u_1u_2u_3u_4u_5$.

In this case, $D(T)$ is two more edges than $D(\overline{T})$. If $v \in A_2 \cup B$, by the definitions of A_2, B and Observation 4.2 (1), there are no L_1-edges incident with v in $\overline{T}$. So $\overline{L} \cup \{u_1u_2, u_4u_5\}$ is a desired set for T.

When $v \in C$, if there is no L_1-edge incident with v in $\overline{L}$, then $\overline{L} \cup \{u_1u_2, u_4u_5\}$ is a desired set for T. Otherwise, let $e' = vw$ be the L_1-edge in $\overline{L}$, from the definition of Operation $\mathcal{O}_3$, there is one leaf edge e'' incident with w or there exists a $P_4 = vwxy$ in $\overline{T}$, whose edges are labelled as L_1, L_1, L_2 consecutively and all edges in $E(x)$ are L_2-edges except wx, then $(\overline{L} - e') \cup \{u_1u_2, u_4u_5, e''\}$ or $(\overline{L} - e') \cup \{u_1u_2, u_4u_5, wx\}$ is a desired set for T.

Therefore, we can always find a desired set for T in this case.

Case 4. T is obtained by applying Operation $\mathcal{O}_4$ from $\overline{T}$ and a path $u_1u_2u_3u_4$.

In this case, $D(T)$ is two more edges than $D(\overline{T})$. If there is no L_1-edge in $\overline{L}$ adjacent to some edge in $E(v)$, then $\overline{L} \cup \{vu_1, u_3u_4,\}$ is a desired set for T. Otherwise, let wx be the L_1-edge in $\overline{L}$ adjacent to some edge in $E(v)$. By Observation 4.2 (1),(2), without loss of generality, assume $x \in A_1$, then there is an L_1-edge xy in $E(x)$ such that y is either a leaf vertex or only incident with L_2-edges except xy. So $(\overline{L} - wx) \cup \{vu_1, u_3u_4, yx\}$ is a desired set for T.

Hence, we can always find a desired set for T in this case.

Case 5. T is obtained by applying Operation $\mathcal{O}_5$ from $\overline{T}$ and a path $u_1u_2u_3u_4u_5$.

In this case, $D(T)$ is two more edges than $D(\overline{T})$. So $\overline{L} \cup \{u_1u_2, u_4u_5\}$ is a desired edge set for T.

Combined the five cases above, for $T \in \mathcal{T}_t$, we can always find an edge set L collecting L_1-edge such that each edge in L has exactly one neighbor S-edge. Since the edges in L need at least $|L|$ edges to dominate, $\gamma'(T) \geqslant |L| = |D|$. Hence, D is a $\gamma_t'(T)$-set and T is a $(\gamma_t' = \gamma')$-tree $\qquad \square$

Lemma 4.6. *Let T be a $(\gamma_t' = \gamma')$-tree, F_t a $\gamma_t'(T)$-set. Then any component of the induced subgraph $T[F_t]$ is nontrivial star.*

Proof. By contradiction. If there is a $P = v_1v_2v_3v_4$ in $T[F_t]$, then the edges which are dominated by v_2v_3 are also dominated by v_1v_2 or v_3v_3. So $F_t - v_2v_3$ is an edge dominating set with cardinality $|F_t|-1$, a contradiction. Hence every component of $T[F_t]$ is a nontrivial star. $\qquad \square$

Lemma 4.7. *Let T be a $(\gamma'_t = \gamma')$-tree. Then $T \in \mathcal{T}_t$.*

Proof. We proceed by induction on the edge size of a nontrivial tree T satisfying $\gamma'_t(T) = \gamma'(T)$. For the initial step, by Corollary 4.1, a tree T with diameter 4 satisfies $\gamma'_t(T) = \gamma'(T)$ and is in $\mathcal{T}_t$. For the inductive hypothesis, we assume that every tree $\overline{T}$ with $\gamma'_t(\overline{T}) = \gamma'(\overline{T})$ has edge size less than m and $diam(\overline{T}) \geqslant 5$, there exists an edge label such that $\overline{T} \in \mathcal{T}$.

If a support vertex v of T has at least two leaf neighbors, say u and w two of them, then v is still a support vertex in $\overline{T} = T - w$. By Theorem 4.1, any minimum edge dominating set of $\overline{T}$ containing no leaf edges is still an edge dominating set of T. So $\gamma'(T) = \gamma'_t(T) = \gamma'(\overline{T}) \leqslant \gamma'_t(\overline{T}) \leqslant \gamma'_t(T)$ and $\gamma'_t(\overline{T}) = \gamma'(\overline{T})$. Hence, by the inductive hypothesis, $\overline{T} \in \mathcal{T}_t$. By Observation 4.2 (1), (2), (3), uv is an L_1- or L_2-edge and $v \in A_2 \cup A_1$ in $\overline{T}$. We can obtain the tree T by applying Operation $\mathcal{O}_1$ from $\overline{T}$ and a new vertex w, so $T \in \mathcal{T}_t$. We may assume that each support vertex of $(\gamma'_t = \gamma')$-tree T of edge size m has exactly one leaf neighbor, denoted by Assumption 1.

If a support vertex v of T, say w is a leaf neighbor of v, has a support neighbor with degree 2, then let $\overline{T} = T - w$. Similar to the discuss as above, $\gamma'_t(\overline{T}) = \gamma'(\overline{T})$ and by the inductive hypothesis, $\overline{T} \in \mathcal{T}_t$. By Observation 4.2 (3), uv is an S-edge in $\overline{T}$, combined with Observation 4.2 (4), $v \in A_2$. We can obtain the tree T by applying Operation $\mathcal{O}_1$ from $\overline{T}$ and a new vertex w, so $T \in \mathcal{T}_t$. We may assume that there is no support vertex which has a support neighbor of degree 2, denoted by Assumption 2.

If v has at least three support neighbors of degree 2 in T, say $\{u_1, u_2, \ldots, u_l\}$ and $l \geqslant 3$, and set $\overline{T}$ as the tree from T by deleting $\{u_3, u_4, \ldots, u_l\}$ and their respective children, then similar to the discuss as above, $\overline{T} \in \mathcal{T}_t$ and T is obtained from $\overline{T}$ by applying a series of Operation $\mathcal{O}_2$, so $T \in \mathcal{T}_t$. Hence we may assume that every vertex has at most two support neighbors of degree 2, denoted by Assumption 3.

Let F_t be a $\gamma'_t(T)$-set containing non-leaf edges, $P = v_0 v_1 \ldots v_t$ the longest path of T, say the edge $e_i = v_i v_{i+1}$. Obviously, v_1 is a support vertex of degree 2 and each child of v_2 is a support vertex of degree 2. We root T at the vertex v_t.

Since e_0 is a leaf edge, F_t must contain e_1. Combined with Lemma 4.6 and the choice of F_t, it is impossible to contain both e_2 and e_3 in F_t, i.e., $e_2 \in F_t$ and $e_3 \notin F_t$ or $e_2 \notin F_t$ and $e_3 \in F_t$ or $e_2 \notin F_t$ and $e_3 \notin F_t$.

Combined with Assumptions 2 and 3, $d(v_2) = 2$ or 3. Next, we divide two cases according to the degree of v_2.

Case 1. $d(v_2) = 3$.

By Assumptions 1 and 2, v_2 has another support child v'_1 of degree 2, say v'_0 is the child of v'_1.

Subcase 1.1. $e_2 \in F_t$.

In this subcase, we can let $\overline{T} = T - \{v_0, v_1\}$. Combined with Lemma 4.6 and the choice of F_t, the restriction of F_t on $\overline{T}$ is a TED-set of $\overline{T}$, further, $\gamma'_t(\overline{T}) \leqslant \gamma'_t(T) - 1$. Combined with an obvious inequality: $\gamma'(T) \leqslant \gamma'(\overline{T}) + 1$, we have $\gamma'(\overline{T}) + 1 \leqslant \gamma'_t(\overline{T}) + 1 \leqslant \gamma'_t(T) = \gamma'(T) \leqslant \gamma'(\overline{T}) + 1$, and so $\gamma'_t(\overline{T}) = \gamma'(\overline{T})$. By the inductive hypothesis, there is an edge label of $\overline{T}$ such that $\overline{T} \in \mathcal{T}_t$. By Observation 4.2 (3), $v'_1 v_2$ is an S-edge in $\overline{T}$, combined with Observation 4.2 (4), there are at least two S-edges incident with v_2 in $\overline{T}$ in either case, $v_2 \in A_2$. We can obtain the tree T by applying Operation $\mathcal{O}_2$ from $\overline{T}$ and a new edge $v_0 v_1$, so $T \in \mathcal{T}_t$.

Subcase 1.2. $e_2 \notin F_t$.

Let $\overline{T} = T - \{v_0, v_1, v'_1, v'_0, v_2\}$. Since edges $v'_1 v'_0$ and e_0 are leaf edges, combined with Lemma 4.6 and the choice of F_t, the restriction of F_t on $\overline{T}$ is a TED-set of $\overline{T}$, further,

$\gamma_t'(\overline{T}) \leqslant \gamma_t'(T) - 2$.

Combined with an obvious inequality: $\gamma'(T) \leqslant \gamma'(\overline{T}) + 2$, we have $\gamma'(\overline{T}) + 2 \leqslant \gamma_t'(\overline{T}) + 2 \leqslant \gamma_t'(T) = \gamma'(T) \leqslant \gamma'(\overline{T}) + 2$, and so $\gamma_t'(\overline{T}) = \gamma'(\overline{T})$. By the inductive hypothesis, there is an edge label of $\overline{T}$ such that $\overline{T} \in \mathcal{T}_t$. We can obtain the tree T by applying Operation $\mathcal{O}_5$ from $\overline{T}$ and a path $v_0 v_1 v_2 v_1' v_0'$, so $T \in \mathcal{T}_t$.

Case 2. $d(v_2) = 2$.

Since $d(v_2) = 2$, we have $\{e_1, e_2\} \subseteq F_t$ by the choice of F_t. So $(E(v_3) - e_2) \cap F_t = \emptyset$ by Lemma 4.6.

Claim 6. *Let v_2' be a child of v_3 other than v_2. Then v_2' is a leaf vertex.*

By contradiction. v_2' has at most one support child by symmetry and Assumption 3. Then $v_2' v_3$ belongs to F_t by the choice of F_t, a contradiction with Lemma 4.6.

Claim 7. *Let v_4' be any non-leaf child of v_5. Then each subtree $T_{4'}$ of $T - v_4'$ not containing v_5 is isomorphic to one of the graphs in the following figure.*

$(a) \qquad (b) \qquad (c) \qquad (d) \qquad (e) \qquad (f)$

Figure 8: The subgraphs following v_4'.

Let v_3' be the child of v_4' in $T_{4'}$. If the length of a longest path starting at v_3' in $T_{4'}$ is 3, combined with symmetry, Claim 6 and Assumptions 1, 2, 3, then $T_{4'}$ is isomorphic to (a) or (b). If the length of a longest path starting at v_3' in $T_{4'}$ is 2, combined with Assumptions 1, 2, 3, then $T_{4'}$ is isomorphic to (e) or (f). If the length of a longest path starting at v_3' in $T_{4'}$ is 1, by Assumption 1, then $T_{4'}$ is isomorphic to (c). If the length of a longest path starting at v_3' in $T_{4'}$ is 0, then $T_{4'}$ is (d). Therefore, $T_{4'}$ is isomorphic to one of the graphs in the Fig. 8.

If v_4 has one non-leaf child, say v_3'', such that the subtree T_4 of $T - v_4$ containing v_3'' is isomorphic to (e) in Fig. 8, then T_4 is a P_5. Let $\overline{T}$ be the subtree of $T - T_4$. If $v_4 v_3'' \in F_t$, then $F_t - v_4 v_3'' + e_3 - e_2$ is still an ED-set of T of size $|F_t| - 1$, a contradiction. Hence $v_4 v_3'' \notin F_t$ and the restriction of F_t on $\overline{T}$ is a TED-set of $\overline{T}$, further $\gamma_t'(\overline{T}) \leqslant \gamma_t'(T) - 2$. Combined with an obvious inequality: $\gamma'(T) \leqslant \gamma'(\overline{T}) + 2$, we have $\gamma'(\overline{T}) + 2 \leqslant \gamma_t'(\overline{T}) + 2 \leqslant \gamma_t'(T) = \gamma'(T) \leqslant \gamma'(\overline{T}) + 2$, so $\gamma'(\overline{T}) = \gamma_t'(\overline{T})$. By the inductive hypothesis, $\overline{T} \in \mathcal{T}_t$ with an edge labelling. Thus T is obtained from $\overline{T}$ by applying Operation $\mathcal{O}_5$. So $T \in \mathcal{T}_t$. In what follows assume that there is no subtree of $T - v_4$ not containing v_5 isomorphic to (e) in Fig. 8, denoted by Assumption 4.

Claim 8. *If $|E(v_5) \cap F_t| \geqslant 1$ and there is a child, say v_4'', of v_5 such that there is a subtree $T_{4''}$ of $T - v_4''$ not containing v_5 isomorphic to (e) in Fig. 8, then T is obtained from $\overline{T} = T - T_{4''}$ by applying Operation $\mathcal{O}_5$.*

Let v_3''' be the child of v_4'' in $T_{4''}$. Obviously $T_{4''}$ is a P_5. In one case $|E(v_5) \cap F_t| \geqslant 2$, if $v_4'' v_3''' \in F_t$, then $F_t' = F_t - v_4'' v_3''' + v_4'' v_5$ is still a TED-set of T. The restriction of F_t' on $\overline{T}$ is a TED-set of $\overline{T}$, further $\gamma_t'(\overline{T}) \leqslant \gamma_t'(T) - 2$. Similar to the discuss as above, $\overline{T} \in \mathcal{T}_t$. Thus T is obtained from $\overline{T}$ by applying Operation $\mathcal{O}_5$. If $v_4'' v_3''' \notin F_t$, then the restriction of F_t on $\overline{T}$ is a TED-set of $\overline{T}$. Similar to the discuss as above, T is obtained from $\overline{T}$ by applying Operation $\mathcal{O}_5$.

In the other case $|E(v_5) \cap F_t| = 1$, say $e_5' = E(v_5) \cap F_t$. Let x be any non-leaf neighbor of v_5 in T. We claim that $|E(x) \cap F_t| \neq 1$. Indeed, if this is not the case, say $e_x = E(x) \cap F_t$, then $F_t - e_5' - e_x + xv_5$ is an ED-set of T, a contradiction. So $v_4'' v_3''' \notin F_t$. Combined Lemma 4.6, similar to the discuss as above, T is obtained from $\overline{T}$ by applying Operation $\mathcal{O}_5$.

In what follows assume that, if $|E(v_5) \cap F_t| \geqslant 1$ and let v_4' be a child of v_5, there is no subtree of $T - v_4'$ not containing v_5 isomorphic to (e) in Fig. 8, denoted by Assumption 5.

By Claim 7, let $\{v_3^1, v_3^2, \ldots, v_3^w\}$ be the set of children of v_4 such that the subtree of $T - v_4$ containing v_3^i is isomorphic to (a) for $0 \leqslant i \leqslant w$, $\{u_3^1, u_3^2, \ldots, u_3^z\}$ the set of children of v_4 such that the subtree of $T - v_4$ containing u_3^j is isomorphic to (b) in Fig. 8 for $0 \leqslant j \leqslant z$. Combined with the structure of (a) and (b) and Lemma 4.6, we have $|(E(v_3^i) - v_3^i v_4) \cap F_t| = 1$ and $|(E(u_3^j) - u_3^j v_4) \cap F_t| = 1$, say $e_v^i = (E(v_3^i) - v_3^i v_4) \cap F_t$ and $e_u^j = (E(u_3^j) - u_3^j v_4) \cap F_t$ for each i and j. Then,

Claim 9. $w \leqslant 1$. Further, if $w = 1$, then $d(v_4) = 2$.

By contradiction. If $w \geqslant 2$, then $F_t - e_v^1 + e_3 - e_v^2$ is an ED-set of T of size $|F_t| - 1$, a contradiction. So $w \leqslant 1$.

Assume that $d(v_4) \geqslant 3$ when $w = 1$. If $z \neq 0$, then $F_t - e_u^1 + v_4 u_3^1 - e_v^1$ is an ED-set of T of size $|F_t| - 1$, a contradiction. If there is a subtree of $T - v_4$ not containing v_5 isomorphic to one of (c), (d) and (f) in Fig. 8, then $E(v_4) \cap F_t \neq \emptyset$ by the choice of F_t and Lemma 4.6, thus $F_t - e_v^1$ is an ED-set of T of size $|F_t| - 1$, a contradiction. Therefore, if $w = 1$, then $d(v_4) = 2$.

By Claim 9, we have the following two claims.

Claim 10. There is no subtree of $T - v_4$ not containing v_5 isomorphic to (f) in Fig. 8.

By Claim 9, we just need to consider the case $w = 0$. By contradiction. If there is a subtree of $T - v_4$ containing one child, say v_3', of v_4 isomorphic to (f), then $|E(v_3') \cap F_t| \geqslant 2$ and $v_4 v_3' \in F_t$. Thus $F_t - v_4 v_3' - e_2 + e_3$ is an ED-set of T of size $|F_t| - 1$ by Lemma 4.6, a contradiction.

Combined with Assumption 4 and Claim 10, there is no subtree of $T - v_4$ not containing v_5 isomorphic to (e) or (f).

Claim 11. If $w = 1$, then $E(v_4) \cap F_t = \emptyset$. Otherwise, $|E(v_4) \cap F_t| \neq 1$.

By contradiction. Assume $E(v_4) \cap F_t \neq \emptyset$ when $w = 1$, then $F_t - e_2$ is an ED-set of T, a contradiction. Assume $|E(v_4) \cap F_t| = 1$ when $w = 0$, then $e_4 \in F_t$ by Claims 7 and 10. Thus $F_t - e_4 + e_3 - e_2$ is an ED-set of T, a contradiction. Therefore, if $w = 1$, then $E(v_4) \cap F_t = \emptyset$. Otherwise, $|E(v_4) \cap F_t| \neq 1$.

Combined with Claims 7, 10 and 11, if $|E(v_4) \cap F_t| \geqslant 2$, then there is a subtree of $T - v_4$ not containing v_5 isomorphic to graph (c) in Fig. 8, i.e., a $P_2 = uv$, say u is a child of v_4. By Claim 9, the subtree T^a of $T - v_4$ containing v_3 is isomorphic to (b). Let $\overline{T}$ be the subtree of $T - T^a$. By $|E(v_4) \cap F_t| \geqslant 2$, the restriction of F_t on $\overline{T}$ is a TED-set of $\overline{T}$, further $\gamma_t'(\overline{T}) \leqslant \gamma_t'(T) - 2$. Combined with an obvious inequality: $\gamma'(T) \leqslant \gamma'(\overline{T}) + 2$, we have $\gamma'(\overline{T}) + 2 \leqslant \gamma_t'(\overline{T}) + 2 \leqslant \gamma_t'(T) = \gamma'(T) \leqslant \gamma'(\overline{T}) + 2$, and so $\gamma'(\overline{T}) = \gamma_t'(\overline{T})$. By the inductive hypothesis, $\overline{T} \in \mathcal{T}_t$ with an edge labelling. In $\overline{T}$, by Observation 4.2 (3), the leaf edge uv is an L_1-edge and $v_4 u$ is an S-edge. Combined with Observation 4.2 (4), $v_4 \in A_2$. Therefore T is obtained from $\overline{T}$ by applying Operation $\mathcal{O}_3$. So $T \in \mathcal{T}_t$.

If $E(v_4) \cap F_t = \emptyset$, then there is no subtree of $T - v_4$ not containing v_5 isomorphic to (c) or (d). Combined with Claims 7, 9, Assumption 4 and the above analysis, then we may

assume that each subtree of $T - v_4$ not containing v_5 is isomorphic to (a) or (b), denoted by Assumption 6.

By Assumption 6, we can divide two subcases to discuss according to the subtree T_4 of $T - v_4$ containing v_3 is isomorphic to (a) or (b) as follow.

Subcase 2.1. T_4 is isomorphic to (a).

By Claims 9 and 11, $d(v_3) = d(v_4) = 2$ and $E(v_4) \cap F_t = \emptyset$. Obviously, $e_4 \notin F_t$ and $|E(v_5) \cap F_t| \geqslant 1$.

Claim 12. $|E(v_5) \cap F_t| \geqslant 2$.

If $|(E(v_5) - e_4) \cap F_t| = 1$, say $e_5' = E(v_5) \cap F_t$, then $F_t - e_5' + e_4 - e_2$ is an ED-set of T of size $|F_t| - 1$, a contradiction.

Combined with $|E(v_5) \cap F_t| \geqslant 2$ and $E(v_4) \cap F_t = \emptyset$, there is a child other than v_4, say v_4', of v_5 such that $v_5 v_4' \in F_t$. Obviously, v_4' is not a leaf. Then

Claim 13. v_4' *is a support vertex of degree 2.*

We first show that v_4' is a support vertex. Assume to the contrary that v_4' has no leaf children, combined with Claim 7, Assumption 5 and Lemma 4.6, every subtree of $T - v_4'$ not containing v_5 is isomorphic to (a) or (b) in Fig. 8. Obviously $F_t - v_5 v_4'$ is still an ED-set of T of size $|F_t| - 1$, a contradiction. Therefore, v_4' is a support vertex.

If $d(v_4') \geqslant 3$, then we denote by T' a subtree of $T - v_4'$ containing a non-leaf child of v_4'. Combined with Assumption 5 and Claim 7, T' is not isomorphic to (a) or (e) in Fig. 8. By Lemma 4.6, T' is not isomorphic to (c) or (f). Hence T' is isomorphic to (b), say v_3' the non-leaf child of v_4', v_2' the non-leaf child of v_3'. Obviously, $F_t - v_5 v_4' - v_2' v_3' + v_3' v_4'$ is an ED-set of T of size $|F_t| - 1$, a contradiction. So $d(v_4') = 2$.

Since $d(v_2) = d(v_3) = d(v_4) = 2$, the subgraph induced by $\{v_0, v_1, v_2, v_3\}$ is P_4. Let $\overline{T} = T - \{v_0, v_1, v_2, v_3\}$. By $E(v_4) \cap F_t = \emptyset$, the restriction of F_t on $\overline{T}$ is a TED-set of $\overline{T}$, further $\gamma_t'(\overline{T}) \leqslant \gamma_t'(T) - 2$. Combined with an obvious inequality: $\gamma'(T) \leqslant \gamma'(\overline{T}) + 2$, we have $\gamma'(\overline{T}) + 2 \leqslant \gamma_t'(\overline{T}) + 2 \leqslant \gamma_t'(T) = \gamma'(T) \leqslant \gamma'(\overline{T}) + 2$, and so $\gamma'(\overline{T}) = \gamma_t'(\overline{T})$. By the inductive hypothesis, $\overline{T} \in \mathcal{T}_t$ with an edge labelling. By Observation 4.2 (3), e_4 is an L_2- or L_1-edge. Combined with Claim 13 and Observation 4.2 (1), (4), e_4 is an L_2-edge. Since $d_{\overline{T}}(v_4) = 1$, $v_4 \in B$. Therefore T is obtained from $\overline{T}$ by applying Operation $\mathcal{O}_4$, $T \in \mathcal{T}_t$.

Subcase 2.2. T_4 is isomorphic to (b).

In this subcase, let v_4' be any non-leaf child of v_5, by symmetry and Assumption 6, there is no subtree of $T - v_4'$ not containing v_5 isomorphic to (a) in Fig. 8, and each subtree of $T - v_4$ not containing v_5 is isomorphic to (b), i.e., a P_5. Let $\overline{T} = T - T_4$. Since $e_3 \notin F_t$ by Lemma 4.6, the restriction of F_t on $\overline{T}$ is a TED-set of $\overline{T}$, further $\gamma_t'(\overline{T}) \leqslant \gamma_t'(T) - 2$. Combined with an obvious inequality: $\gamma'(T) \leqslant \gamma'(\overline{T}) + 2$, we have $\gamma'(\overline{T}) + 2 \leqslant \gamma_t'(\overline{T}) + 2 \leqslant \gamma_t'(T) = \gamma'(T) \leqslant \gamma'(\overline{T}) + 2$, further $\gamma'(\overline{T}) = \gamma_t'(\overline{T})$. By the inductive hypothesis, $\overline{T} \in \mathcal{T}_t$ with an edge labelling. Combined with the structure of (b) and Observation 4.2 (2), then in $\overline{T}$, all edges connecting v_4 and its children being L_1-edges, so $v_4 \in C$ by Observation 4.2 (1). If in $\overline{T}$, e_4 is an L_2-edge or e_4 is an L_1-edge and adjacent to a leaf edge, then T is obtained from $\overline{T}$ by applying Operation $\mathcal{O}_3$. In what follows we assume that in $\overline{T}$, e_4 is an L_1-edge and adjacent to non-leaf edges, denoted by Assumption 7. Note that in $\overline{T}$, there is only one S-edge in $E_{\overline{T}}(v_5)$, say e_5', and $v_5 \in A_1$ in $\overline{T}$.

By Lemma 4.5, all S-edges in $\overline{T}$ construct a minimum total edge dominating set $D(\overline{T})$. Then, $F_t' = D(\overline{T}) + e_1 + e_2$ is a minimum total edge dominating set of T by

$\gamma_t'(\overline{T}) + 2 = \gamma_t'(T)$. Further, $|E(v_5) \cap F_t'| = 1$, Claim 8 still holds. Then, we have the following claim:

Claim 14. e_5 *is an* L_1*-edge in* $E_{\overline{T}}(v_5)$, *and all edges in* $(E_{\overline{T}}(v_6) - e_5)$ *are* L_2*-edges.*

Let v_4' be any non-leaf child of v_5 other than v_4 and v_3' any child of v_4' in $\overline{T}$. There is no subtree of $\overline{T} - v_4'$ containing v_3' isomorphic to (e) in $\overline{T}$ by Assumption 5 and Claim 8. We claim that the length of a longest path starting at v_3' in the subtree T' of $T - v_4'$ containing v_3' is 3 or 1. If the length of a longest path starting at v_3' in T' is 2 or 0, by Lemma 4.6 and Observation 4.2 (3), (4), then $v_4' \in A_1$ in $\overline{T}$, a contradiction by Observation 4.2 (1) and Lemma 4.6. By symmetry, Claim 7 and Assumptions 1, 2, 3, we know that T' is isomorphic to (b) or (c). Let $\{z_4^1, \ldots, z_4^h\}$ be the set of children of v_5 such that there is a subtree T_4^r of $T - z_4^r$ not containing v_5 is isomorphic to (b) in $\overline{T}$ for $1 \leqslant r \leqslant h$. Let z_3^r be the child of z_4^r in T_4^r. By the structure of (b), then $|E(z_3^r) \cap F_t'| = 1$, say $e^r = (E(z_3^r) - z_3^r z_4^r) \cap F_t'$ for each r. If $E(v_6)$ has an S-edge other than e_5, then $F_t' - \{e^1, \ldots, e^h\} + \{z_3^1 z_4^1, \ldots, z_3^h z_4^h\} - e_5'$ is an ED-set of T of size $|F_t'| - 1$, a contradiction. So all edges in $(E(v_6) - e_5)$ are L_1- or L_2-edges. By Observation 4.2 (2), (4), e_5 is an L_1-edge.

By contradiction. If there is one edge $e_6' = v_6 v_7'$ in $(E_{\overline{T}}(v_6) - e_5)$ is an L_1-edge, then there is exactly one S-edge e_6'' incident with v_7' by Observation 4.2 (1). Thus $F_t' - e_6'' + e_6' - \{e^1, \ldots, e^h\} + \{z_3^1 z_4^1, \ldots, z_3^h z_4^h\} - e_5'$ is an ED-set of T of size $|F_t'| - 1$, a contradiction. Therefore, all edges in $(E_{\overline{T}}(v_6) - e_5)$ are L_2-edges.

Combined with the structure of (b) and Claim 14, all L_1-edges in $(E_{\overline{T}}(v_4) - e_4)$ are adjacent to a leaf edge, and there exist a P_4 starting at v_4, whose edges are labelled as L_1, L_1, L_2 consecutively, and all edges in $(E_{\overline{T}}(v_6) - e_5)$ are L_2-edges. Hence we can obtain T from $\overline{T}$ by applying Operation $\mathcal{O}_3$. $\square$

As an immediate consequence of Lemmas 4.5 and 4.7, we have the following characterization of $(\gamma_t' = \gamma')$-trees.

Theorem 4.3. *A tree is a* $(\gamma_t' = \gamma')$*-tree if and only if* $T \in \mathcal{T}_t$.

5 Acknowledgements

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