# Approximation algorithm for the Multicovering Problem 

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#### Abstract

Let $\mathcal{H}=(V, \mathcal{E})$ be a hypergraph with maximum edge size $\ell$ and maximum degree $\Delta$. For given numbers $b_{v} \in \mathbb{N}_{\geq 2}, v \in V$, a set multicover in $\mathcal{H}$ is a set of edges $C \subseteq \mathcal{E}$ such that every vertex $v$ in $V$ belongs to at least $b_{v}$ edges in $C$. set multicover is the problem of finding a minimum-cardinality set multicover. Peleg, Schechtman and Wool conjectured that for any fixed $\Delta$ and $b:=\min _{v \in V} b_{v}$, the problem of set multicover is not approximable within a ratio less than $\delta:=\Delta-b+1$, unless $\mathcal{P}=\mathcal{N} \mathcal{P}$. Hence it's a challenge to explore for which classes of hypergraph the conjecture doesn't hold. We present a polynomial time algorithm for the set multicover problem which combines a deterministic threshold algorithm with conditioned randomized rounding steps. Our algorithm yields an approximation ratio of $\max \left\{\frac{148}{149} \delta,\left(1-\frac{(b-1) e^{\frac{\delta}{4}}}{94 \ell}\right) \delta\right\}$. Our result not only improves over the approximation ratio presented by Srivastav et al (Algorithmica 2016) but it's more general since we set no restriction on the parameter $\ell$. Moreover we present a further polynomial time algorithm with an approximation ratio of $\frac{5}{6} \delta$ for hypergraphs with $\ell \leq(1+\epsilon) \bar{\ell}$ for any fixed $\epsilon \in\left[0, \frac{1}{2}\right]$, where $\bar{\ell}$ is the average edge size. The analysis of this algorithm relies on matching/covering duality due to Ray-Chaudhuri (1960), which we convert into an approximative form. The second performance disprove the conjecture of peleg et al for a large subclass of hypergraphs.


Keywords: Integer linear programs, hypergraphs, approximation algorithm, randomized rounding, set cover and set multicover, $\mathbf{k}$-matching .

## 1 Introduction

The set multicover problem is a fundamental covering issue that widely explored in the theory of optimization. A nicely formulation of this problem may given by the notion of hypergraphs which offer tools to deal with sets.
A hypergraph is a pair $\mathcal{H}=(V, \mathcal{E})$, where $V$ is a finite set and $\mathcal{E} \subseteq 2^{V}$ is a family of some subsets of $V$. We call the elements of $V$ vertices and the elements of $\mathcal{E}$ (hyper-)edges. Further let $n:=|V|, m:=|\mathcal{E}|$. W.l.o.g. let the elements of $V$ be enumerated as $v_{1}, v_{2}, \ldots, v_{n}$. Let $\ell$ be the maximum edge size, $\bar{\ell}=$ $\frac{1}{m} \sum_{j=1}^{m}\left|E_{j}\right|$ the average edge size and let $\Delta$ be the maximum vertex degree, where the degree of a vertex is the number of edges containing that vertex. If for every $E \in \mathcal{E} ;|E|=\ell$ than the hypergraph is called uniform.
Let $\mathbf{b}:=\left(b_{1}, b_{2}, \ldots, b_{n}\right) \in \mathbb{N}_{\geq 2}^{n}$ be given. If a vertex $v_{i}, i \in[n]$, is contained in at least $b_{i}$ edges of some subset $C \subseteq \mathcal{E}$, we say that the vertex $v_{i}$ is fully covered by $b_{i}$ edges in $C$. A set multicover in $\mathcal{H}$ is a set of edges $C \subseteq \mathcal{E}$ such that every vertex $v_{i}$ in $V$ is fully covered by $b_{i}$ edges in $C$. set multicover problem is the task of finding a set multicover of minimum cardinality. Note that the usual set cover problem, which is known to be NP-hard [12, is a special case with $b_{i}=1$ for all $i \in[n]$. Furthermore Peleg, Schechtman and Wool conjectured that for any fixed $\Delta$ and $b:=\min _{i \in[n]} b_{i}$ the problem cannot be approximated by a ration smaller than $\delta:=\Delta-b+1$ unless $\mathcal{P}=\mathcal{N} \mathcal{P}$. Hence it remained an open problem whether an approximation ration of $\beta \delta$ with $\beta<1$ constant can be proved. We say that an algorithm $A$ is an approximation algorithm for set multicover problem with performance ratio $\alpha$, if for each instance $I$ of size $n$ of set multicover problem, $A$ runs in polynomial time in $n$ and returns a value $|A(I)|$ such that $|A(I)| \leq \alpha \cdot$ Opt, where Opt is the cardinality of a minimum set multicover.
The set multicover problem can also be formulated as an integer linear program as follows

$$
\min \left\{\sum_{j=1}^{m} x_{j} ; A x \geq \mathbf{b}, x \in\{0,1\}^{m}\right\} \quad(\operatorname{ILP}(\Delta, \mathbf{b}))
$$

where $A=\left(a_{i j}\right)_{i \in[n], j \in[m]} \in\{0,1\}^{n \times m}$ is the vertex-edge incidence matrix of $\mathcal{H}$ and $\mathbf{b}=\left(b_{1}, b_{2}, \ldots, b_{n}\right) \in \mathbb{N}_{\geq 2}^{n}$ is the given integer vector.
The linear programming relaxation $\operatorname{LP}(\Delta, \mathbf{b})$ of $\operatorname{ILP}(\Delta, \mathbf{b})$ is given by relaxing the integrality constraints to $x_{j} \in[0,1]$ for all $j \in[m]$. Let $x^{*} \in[0,1]^{m}$ be an optimal solution of $\operatorname{LP}(\Delta, \mathbf{b})$ than $\mathrm{Opt}^{*}=\sum_{j=1}^{m} x_{j}^{*}$ is the value of the optimal solution to $\operatorname{LP}(\Delta, \mathbf{b})$, We have Opt* $\leq$ Opt.
Related Work. The set cover problem $(b=1)$ has been over decades intensively explored. Several deterministic approximation algorithms are exhibited for this problem [18|10|14], all with approximation ratios $\Delta$. On the other hand Khot and Regev in [13] proved that the problem cannot be approximated within factor $\delta-\epsilon=\Delta-\epsilon$ assuming that unique games conjecture is true. Furthermore Johnson 11 and Lovász [16] gave a greedy algorithm with performance ratio $H(\ell)$, where $H(\ell)=\sum_{i=1}^{\ell} \frac{1}{i}$ is the harmonic number. Notice that $H(\ell) \leq 1+$ $\ln (\ell)$. For hypergraphs with bounded $\ell$, Duh and Frer 3] used the technique
called semi-local optimization improving $H(\ell)$ to $H(\ell)-\frac{1}{2}$. In contrast to set cover problem it is less known for the case $b \geq 2$. Let give a brief summary of the known approximability results. In paper [21], Vazirani using dual fitting method extended the result of Lovász [16] for $b \geq 1$. Later Fujito et al. [7] improved the algorithm of Vazirani and achieved an approximation ratio of $H(\ell)-\frac{1}{6}$ for $\ell$ bounded. Hall and Hochbaum (9] achieved by a greedy algorithm based on LP duality an approximation ratio of $\Delta$. By a deterministic threshold algorithm Peleg, Schechtman and Wool in 1997 [19|20] improved this result and gave an approximation ratio $\delta$. They were also the first to propose an approximation algorithm for the setmulticover problem with approximation ratio below $\delta$, namely a randomized rounding algorithm with performance ratio $\left(1-\left(\frac{c}{n}\right)^{\frac{1}{\delta}}\right) \cdot \delta$ for a small constant $c>0$. However, their ratio is depending on $n$ and asymptotically tends to $\delta$. A randomized algorithm of hybrid type was later given by Srivastav et al [6]. Their algorithm achieves for hypergraphs with $l \in \mathcal{O}\left(\max \left\{(n b)^{\frac{1}{5}}, n^{\frac{1}{4}}\right\}\right)$ an approximation ratio of $\left(1-\frac{11(\Delta-b)}{72 l}\right) \cdot \delta$ with constant probability.
Our Results. The main contribution of our paper is the combination of a deterministic threshold-based algorithm with conditioned randomized rounding steps. The idea is to algorithmically discard instances that can be handled deterministically in favor of instances for which we obtain a constant factor approximation less than $\delta$ using a randomized strategy.
In the following we give a brief overview of the method. First we give some fundamentals results based on the LP relaxation with threshold that allows us to come up with an approximation ratio strictly less than $\delta$ and use this results for the first algorithm. This is an extension of an algorithm by Hochbaum 10 for the set cover problem and the vertex cover problem.
Let $\left(x_{1}^{*}, \cdots, x_{m}^{*}\right)$ be the optimal solution of the $\operatorname{LP}(\Delta, \mathbf{b})$. We define $C_{1}:=$ $\left\{E_{j} \in \mathcal{E} \left\lvert\, x_{j}^{*} \geq \frac{2}{\delta+1}\right.\right\}, C_{2}:=\left\{E_{j} \in \mathcal{E} \left\lvert\, \frac{1}{\delta} \leq x_{j}^{*}<\frac{2}{\delta+1}\right.\right\}$ and $C_{3}:=\left\{E_{j} \in \mathcal{E} \mid 0<\right.$ $\left.x_{j}^{*}<\frac{2}{\delta+1}\right\}$. It follows that $C_{1} \cap C_{2}=\emptyset$ and $C_{1} \cup C_{2}$ is a feasible set multicover. Our first algorithm is designed as a cascade of a deterministic and a randomized rounding step followed by greedy repairing. The threshold type algorithm first solves the relaxed $\operatorname{LP}(\Delta, \mathbf{b})$ problem and then picks all the edges corresponding to variables with fractional values at least $\frac{2}{\delta+1}$ to the output set. Depending on the cardinality of sets $C_{1}$ and $C_{2}$, we use LP-rounding with randomization for the hyperedges of the set $C_{3}$. Every edge of $C_{3}$ is independently added to the output set with probability $\frac{\delta+1}{2} x_{j}^{*}$. To guarantee feasibility, we proceed with a repairing step. Our algorithm is an extension of an example given in 4|5|6|8|9|20 for the vertex cover, partial vertex cover and set multicover problem in graphs and hypergraphs.
The methods used in this paper rely on an application of the ChernoffHoeffding bound technique for sums of independent random variables and are based on estimating the variance of the summed random variables for invoking the Chebychev-Cantelli inequality.
We give a detailed analysis of the first algorithm in which we explore the cases by comparing the cardinality of the two sets $C_{1}$ and $C_{2}$ and the relative cardi-
nality of $C_{1}$ with respect to Opt*. Our algorithm yields a performance ratio of $\max \left\{\frac{148}{149} \delta,\left(1-\frac{(b-1) e^{\frac{\delta}{4}}}{94 \ell}\right) \delta\right\}$. This ratio means a constant factor less than $\delta$ for many settings of the parameters $\delta, b$ and $\ell$. Further it is asymptotically better than the former approximation ratios due to Peleg et al and Srivastav et al. Furthermore we consider the problem in hypergraphs $\mathcal{H}=(V, \mathcal{E})$ with $\ell \leq(1+\epsilon) \bar{\ell}$ and do not assume that $\ell$ and $\Delta$ are constants. We give a polynomial-time approximation algorithm with an approximation ratio of $\frac{5}{6}\left(1-\frac{1}{2 \ell}\right) \delta$ for any fixed $\epsilon \in\left[0, \frac{1}{2}\right]$. The main progress is that our approximation ratio of at most $\frac{5}{6} \delta$. Hence we disprove the conjecture of peleg et al for a large and important class of hypergraphs. Note that uniform hypergraphs fulfill the condition $\ell \leq(1+\epsilon) \bar{\ell}$.

Fundamental results and approximations for set multicover problem

| Hypergraph | Approximation ratio |
| :---: | :---: |
| - | $H(\ell)[21]$ |
| bounded $\ell$ | $H(\ell)-\frac{1}{6}[7]$ |
| - | $\delta[9[20]$ |
| - | $\left(1-\left(\frac{c}{n}\right)^{\frac{1}{\delta}}\right) \cdot \delta$ where $c>0$ is a constant. [19] |
| $l \in \mathcal{O}\left(\max \left\{(n b)^{\frac{1}{5}}, n^{\frac{1}{4}}\right\}\right)$ | $\left(1-\frac{11(\Delta-b)}{72 \ell}\right) \cdot \delta[6]$ |
| - | $\max \left\{\frac{148}{149} \delta,\left(1-\frac{(b-1) e^{\frac{\delta}{4}}}{94 \ell}\right) \delta\right\} \quad$ (this paper) $)$ |
| $\ell \leq(1+\epsilon) \bar{\ell}$, for $\epsilon \in\left[0, \frac{1}{2}\right]$ | $\frac{5}{6}\left(1-\frac{1}{2 \ell}\right) \delta($ this paper $)$ |

Outline of the paper. In Section 2 we give all the definitions and tools needed for the analysis of the performed results. Section 3 we present a randomized algorithm of hybrid type and its analysis. Section 4 we give a deterministic algorithm based on matching/covering duality and its analysis. Finally we sketch some open questions.

## 2 Definitions and preliminaries

Let $\mathcal{H}=(V, \mathcal{E})$ be a hypergraph, $V$ and $\mathcal{E}$ is the set of vertices and hyperedges respectively. For every vertex $v \in V$ we define the vertex degree of $v$ as $d(v):=$ $|\{E \in \mathcal{E} \mid v \in E\}|$ and $\Gamma(v):=\{E \in \mathcal{E} \mid v \in E\}$ the set of edges incident to $v$. The maximum vertex degree is $\Delta:=\max _{v \in V} d(v)$. Let $l$ denote the maximum cardinality of a hyperedge from $\mathcal{E}$. It is convenient to order the vertices and edges, i.e., $V=\left\{v_{1}, \ldots, v_{n}\right\}$ and $\mathcal{E}=\left\{E_{1}, \ldots, E_{m}\right\}$, and to identify the vertices and edges with their indices.
set multicover problem:
Let $\mathcal{H}=(V, \mathcal{E})$ be a hypergraph and $\left(b_{1}, b_{2}, \ldots, b_{n}\right) \in \mathbb{N}_{\geq 2}^{n}$. We call $C \subseteq \mathcal{E}$ a set multicover if every vertex $i \in V$ is contained in at least $b_{i}$ hyperedges of $C$. set multicover is the problem of finding a set multicover with minimum cardinality.
For the later analysis we will use the following ChernoffHoeffding Bound inequality for a sum of independent random variables:

Theorem 1 (see [17]). Let $X_{1}, \ldots, X_{n}$ be independent $\{0,1\}$-random variables. Let $X=\sum_{i=1}^{n} X_{i}$. For every $0<\beta \leq 1$ we have

$$
\operatorname{Pr}[X \geq(1+\beta) \cdot \mathbb{E}(X)] \leq \exp \left(-\frac{\beta^{2} \mathbb{E}(X)}{3}\right)
$$

A further useful concentration theorem we will use is the Chebychev-Cantelli inequality:

Theorem 2 (see [18], page 64). Let $X$ be a non-negative random variable with finite mean $\mathbb{E}(X)$ and variance $\operatorname{Var}(X)$. Then for any $a>0$ it holds that

$$
\operatorname{Pr}(X \leq \mathbb{E}(X)-a) \leq \frac{\operatorname{Var}(X)}{\operatorname{Var}(X)+a^{2}}
$$

Definition 1. let $\mathbf{k} \in \mathbb{N}_{0}^{n}$. A k-matching in $\mathcal{H}$ is a set $M \subseteq \mathcal{E}$ such that in no vertex $v_{i}$ of $\mathcal{H}$ more than $k_{i}, i \in[n]$ hyperedges from $M$ are incident. The $\mathbf{k}$ matching problem is to find a maximum cardinality $\mathbf{k}$-matching. $\nu_{\mathbf{k}}(\mathcal{H})$ denotes this maximum cardinality, and is called the $\mathbf{k}$-matching number of $\mathcal{H}$.

We need the following duality theorem from combinatorics:
Theorem 3 (Ray-Chaudhuri, 1960 [2]). Consider a hypergraph $\mathcal{H}=(V, \mathcal{E})$ with vertex degree $d(v)$ for every $v \in V$. let $\mathbf{b}, \mathbf{k} \in \mathbb{N}_{0}^{n}$ such that for every $v \in V$, $b_{v}+k_{v}=d(v)$. A subset of hyperedges $M_{\mathbf{k}}$ is a $\mathbf{k}$-matching in $\mathcal{H}$ if and only if the subset $S_{\mathbf{b}}:=\mathcal{E} \backslash M_{\mathbf{k}}$ is a $\mathbf{b}$-set multicover in $\mathcal{H}$. Furthermore $M_{\mathbf{k}}$ is of maximum cardinality if and only if $S_{\mathbf{b}}$ is of minimum cardinality.

Remark 1. Note that Theorem 3 holds also for hypergraphs with multi-sets i.e., hypergraphs with multiple hyperedges.

## 3 The randomized rounding algorithm

Let $\mathcal{H}=(V, \mathcal{E})$ be a hypergraph with maximum vertex degree $\Delta$ and maximum edge size $\ell$. An integer, linear programming formulation of set multicover
problem is the following:

$$
\begin{aligned}
& \min \sum_{j=1}^{m} x_{j}, \\
\operatorname{ILP}(\Delta, \mathbf{b}): \quad & \sum_{j=1}^{m} a_{i j} x_{j} \geq b_{i} \quad \text { for all } i \in[n], \\
& x_{j} \in\{0,1\} \quad \text { for all } j \in[m],
\end{aligned}
$$

where $A=\left(a_{i j}\right)_{i \in[n], j \in[m]} \in\{0,1\}^{n \times m}$ is the vertex-edge incidence matrix of $\mathcal{H}$ and $\mathbf{b}=\left(b_{1}, b_{2}, \ldots, b_{n}\right) \in \mathbb{N}_{\geq 2}^{n}$ is the given integer vector. We define $b:=$ $\min _{i \in[n]} b_{i}$ and $\delta=\Delta-b+1$.
The linear programming relaxation $\operatorname{LP}(\Delta, \mathbf{b})$ of $\operatorname{ILP}(\Delta, \mathbf{b})$ is given by relaxing the integrality constraints to $x_{j} \in[0,1]$ for all $j \in[m]$. Let Opt resp. Opt ${ }^{*}$ be the value of an optimal solution to $\operatorname{ILP}(\Delta, \mathbf{b}) \operatorname{resp} \operatorname{LP}(\Delta, \mathbf{b})$. Let $\left(x_{1}^{*}, \cdots, x_{m}^{*}\right)$ be the optimal solution of the $\operatorname{LP}(\Delta, \mathbf{b}) . \mathrm{So} \mathrm{Opt}^{*}=\sum_{j=1}^{m} x_{j}^{*}$ and Opt ${ }^{*} \leq \mathrm{Opt}$. The next lemma shows that the $b_{i}$ greatest values of the LP variables correspondent to the incident edges for any vertex $v_{i}$ are all greater than or equal to $\frac{1}{\delta}$.

Lemma 1 (see $[\mathbf{2 0}]$ ). Let $b_{i}, d, \Delta, n \in \mathbb{N}$ with $2 \leqslant b_{i} \leqslant d-1 \leqslant \Delta-1, i \in[n]$. Let $x_{j} \in[0,1], j \in[d]$, such that $\sum_{j=1}^{d} x_{j} \geqslant b_{i}$. Then at least $b_{i}$ of the $x_{j}$ fulfill the inequality $x_{j} \geqslant \frac{1}{\delta}$.
Our second lemma shows that the $b_{i}-1$ greatest values of the LP variables correspondent to the incident edges for any vertex $v_{i}$ are all greater than or equal to $\frac{2}{\delta+1}$ and with lemma 1 we summarize about the $b_{i}$ greatest values of the LP variables correspondent to the incident edges for any vertex $v_{i}$.
Lemma 2. Let $b_{i}, d, \Delta, n \in \mathbb{N}$ with $2 \leqslant b_{i} \leqslant d-1 \leqslant \Delta-1, i \in[n]$. Let $x_{j} \in[0,1], j \in[d]$, such that $\sum_{j=1}^{d} x_{j} \geqslant b_{i}$. Then at least $b_{i}-1$ of the $x_{j}$ fulfill the inequality $x_{j} \geqslant \frac{2}{\delta+1}$ and exists an element $x_{j}$ distinct of them all who fulfill the inequality $x_{j} \geqslant \frac{1}{\delta}$.
Proof. W.L.O.G we suppose $x_{1} \geq x_{2} \geq \cdots \geq x_{b_{i}} \geq \cdots \geq x_{d}$.
Hence $b_{i}-2 \geq \sum_{j=1}^{b_{i}-2} x_{j}$ and $\left(d-b_{i}+2\right) x_{b_{i-1}} \geq \sum_{j=b_{i}-1}^{d} x_{j}$
Then

$$
\begin{aligned}
b_{i}-2+(\Delta-b+2) x_{b_{i}-1} & \geq b_{i}-2+\left(\Delta-b_{i}+2\right) x_{b_{i}-1} \\
& \geq b_{i}-2+\left(d-b_{i}+2\right) x_{b_{i}-1} \\
& \geq \sum_{j=1}^{b_{i}-2} x_{j}+\sum_{j=b_{i}-1}^{d} x_{j}=\sum_{j=1}^{d} x_{j} \\
& \geq b_{i}
\end{aligned}
$$

So we have $x_{b_{i}-1} \geq \frac{2}{\delta+1}$
Since for all $j \in\left[b_{i}-1\right] ; x_{j} \geq x_{b_{i}-1}$ then for all $j \in\left[b_{i}-1\right] ; x_{j} \geq \frac{2}{\delta+1}$.
Furthermore by lemma 1 and the assumption on the orders of the variables $x_{j}$, for all $j \in\left[b_{i}\right]$ we have $x_{j} \geq \frac{1}{\delta}$ and particularly $x_{b_{i}} \geq \frac{1}{\delta}$.

Corollary 1. Let $\mathcal{H}$ be a hypergraph and $\left(x_{1}^{*}, \cdots, x_{m}^{*}\right)$ be the optimal solution of the $L P(\Delta, \mathbf{b})$. Then $C:=\left\{E_{j} \in \mathcal{E} \left\lvert\, x_{j}^{*} \geq \frac{1}{\delta}\right.\right\}$ is a set multicover such that $|C|<\delta$ Opt.

Proof. Clearly with lemma 1, $C$ is a feasible set multicover.
Let $C_{1}:=\left\{E_{j} \in \mathcal{E} \left\lvert\, x_{j}^{*} \geq \frac{2}{\delta+1}\right.\right\}$ and $C_{2}:=\left\{E_{j} \in \mathcal{E} \left\lvert\, \frac{1}{\delta} \leq x_{j}^{*}<\frac{2}{\delta+1}\right.\right\}$.
Note that $C=C_{1} \cup C_{2}$ and $C_{1} \cap C_{2}=\emptyset$, so we have $\left|C_{1}\right|+\left|C_{2}\right|=|C|$.
Let $E_{j} \in C_{1}$ Then $x_{j}^{*} \geq \frac{2}{\delta+1}$. Since $\delta>1$ we have $\frac{2 \delta}{\delta+1}>1$
Hence

$$
\delta \mathrm{Opt}^{*}=\sum_{j=1}^{m} \delta x_{j}^{*} \geq \sum_{E_{j} \in C_{1}} \delta x_{j}^{*}+\sum_{E_{j} \in C_{2}} \delta x_{j}^{*}
$$

From this we can immediately deduce

$$
\begin{equation*}
\delta \mathrm{Opt}^{*} \geq \frac{2 \delta}{\delta+1}\left|C_{1}\right|+\left|C_{2}\right| \tag{1}
\end{equation*}
$$

Hence

$$
\delta \mathrm{Opt}^{*}>\left|C_{1}\right|+\left|C_{2}\right|=|C|
$$

Then

$$
|C|<\delta \mathrm{Opt}
$$

### 3.1 The algorithm

In this section we present an algorithm with conditioned randomized rounding based on the properties satisfied by the two sets $C_{1}$ and $C_{2}$.
Let us give a brief explanation of the ingredients of the algorithm SET bMULTICOVER.
We start with an empty set C, which will be extended to a feasible set multicover.
First we solve the LP-relaxation $\operatorname{LP}(\Delta, \mathbf{b})$ in polynomial time. Let $\alpha=\frac{(b-1) \delta e^{\frac{\delta}{4}}}{47 \ell}$ and $t=73$. The rest of the action depends on the following two cases.

- If $\left|C_{1}\right| \geq \alpha \cdot$ Opt $^{*}$ or $t\left|C_{1}\right| \geq\left|C_{2}\right|:$ we pick in the cover $C$ all edges of the two sets $\left|C_{1}\right|$ and $\left|C_{2}\right|$.
Recall that by lemma the $C=C_{1} \cup C_{2}$ is a feasible set multicover.
- If $\left|C_{1}\right|<\alpha \cdot \mathrm{Opt}^{*}$ and $t\left|C_{1}\right|<\left|C_{2}\right|$ : we use LP-rounding with randomization on the edges of the set $C_{3}$, every edge of $C_{3}$ is independently picked in the cover with probability $\frac{\delta+1}{2} x_{j}^{*}$. To guarantee a feasible cover we proceed for a step of repairing.

```
Algorithm 1: SET b-MULTICOVER
    Input : a hypergraph \(\mathcal{H}=(V, \mathcal{E})\) with maximum degree \(\Delta\) and maximum
                hyperedge size \(\ell\).
                    Let \(b_{i} \in \mathbb{N}_{\geq 2}\) for \(i \in[n] ; b:=\min _{i \in[n]} b_{i}\) and \(\delta=\Delta-b+1\).
```

    Output: A set multicover \(C\)
        1. Initialize \(C:=C_{1}=C_{2}=\emptyset\). Set \(\lambda=\frac{\delta+1}{2} ; \alpha=\frac{(b-1) \delta e^{\frac{\delta}{4}}}{47 \ell}\) and \(t=73\).
    2. Obtain an optimal solution \(x^{*} \in[0,1]^{m}\) by solving the \(\operatorname{LP}(\Delta, \mathbf{b})\) relaxation.
    3. Set \(C_{1}:=\left\{E_{j} \in \mathcal{E} \left\lvert\, x_{j}^{*} \geq \frac{1}{\lambda}\right.\right\}, C_{2}:=\left\{E_{j} \in \mathcal{E} \left\lvert\, \frac{1}{\lambda}>x_{j}^{*} \geq \frac{1}{\delta}\right.\right\}\)
        and \(C_{3}:=\left\{E_{j} \in \mathcal{E} \left\lvert\, 0<x_{j}^{*}<\frac{1}{\lambda}\right.\right\}\).
    4. Take all edges of the set \(C_{1}\) in the cover \(C\).
    5. if \(\left|C_{1}\right| \geq \alpha \cdot\) Opt \(^{*}\) or \(t\left|C_{1}\right| \geq\left|C_{2}\right|\) then return \(C=C_{1} \cup C_{2}\). Else
        (a) (Randomized Rounding) For all edges \(E_{j} \in C_{3}\) include the edge \(E_{j}\) in
        the cover \(C\), independently for all such \(E_{j}\), with probability \(\lambda x_{j}^{*}\).
        (b) (Repairing) Repair the cover \(C\) (if necessary) as follows: Include arbitrary
        edges from \(C_{3}\), incident to vertices \(i \in[n]\) not covered by \(b_{i}\) edges, to \(C\)
        until all vertices are fully covered.
            (c) Return the cover \(C\).
    
### 3.2 Analysis of the algorithm

Case $\left|\mathbf{C}_{\mathbf{1}}\right| \geq \alpha \cdot$ Opt* or $\mathbf{t}\left|\mathbf{C}_{\mathbf{1}}\right| \geq\left|\mathbf{C}_{\mathbf{2}}\right|$.
Theorem 4. Let $\mathcal{H}$ be a hypergraph with maximum vertex degree $\Delta$ and maximum edge size $\ell$. Let $\alpha=\frac{(b-1) \delta e^{\frac{\delta}{4}}}{47 \ell}$ and $t=73$ as defined in algorithm 1, if $\left|C_{1}\right| \geq$ $\alpha \cdot \mathrm{Opt}^{*}$ or $t\left|C_{1}\right| \geq\left|C_{2}\right|$, the algorithm 1 achieve a factor of $\left(1-\frac{(b-1) e^{\frac{\delta}{4}}}{94 \ell}\right) \delta$ or $\frac{148}{149} \delta$ respectively.

Proof. Case $\left|\mathbf{C}_{\mathbf{1}}\right| \geq \alpha \cdot$ Opt $^{*}$. With the definition of the sets $C_{1}$ and $C_{2}$ we have

$$
\begin{aligned}
\delta \mathrm{Opt}^{*}=\sum_{j=1}^{m} \delta x_{j}^{*} & \geq \sum_{E_{j} \in C_{1}} \delta x_{j}^{*}+\sum_{E_{j} \in C_{2}} \delta x_{j}^{*} \\
& \geq \frac{2 \delta}{\delta+1}\left|C_{1}\right|+\left|C_{2}\right| \\
& \geq \frac{2 \delta}{\delta+1}\left|C_{1}\right|+\left(|C|-\left|C_{1}\right|\right) \\
& \geq \frac{\delta-1}{\delta+1}\left|C_{1}\right|+|C| \\
& \delta \geq 3 \frac{1}{2}\left|C_{1}\right|+|C| \\
& \geq \frac{1}{2} \alpha \cdot \mathrm{Opt}^{*}+|C|
\end{aligned}
$$

Hence

$$
|C| \leq\left(1-\frac{(b-1) e^{\frac{\delta}{4}}}{94 \ell}\right) \delta \cdot \mathrm{Opt}^{*}
$$

Case $t\left|\mathbf{C}_{\mathbf{1}}\right| \geq\left|\mathbf{C}_{\mathbf{2}}\right|$. We have

$$
t\left|C_{1}\right| \geq|C|-\left|C_{1}\right|
$$

Therefore

$$
\left|C_{1}\right| \geq \frac{1}{t+1}|C|
$$

Next with we have

$$
\begin{aligned}
\delta \mathrm{Opt}^{*} & \geq \frac{2 \delta}{\delta+1}\left|C_{1}\right|+\left(|C|-\left|C_{1}\right|\right) \\
& \geq \frac{\delta-1}{\delta+1}\left|C_{1}\right|+|C| \\
& \geq \frac{\delta-1}{\delta+1} \times \frac{1}{t+1}|C|+|C| \\
& \stackrel{\delta \geq 3}{2 t+2}|C|+|C|
\end{aligned}
$$

Then

$$
\begin{aligned}
|C| & \leq \frac{1}{1+\frac{1}{2 t+2}} \cdot \delta \mathrm{Opt}^{*} \\
& \leq \frac{148}{149} \cdot \delta \mathrm{Opt}^{*}
\end{aligned}
$$

Case $\left|\mathbf{C}_{\mathbf{1}}\right|<\alpha \cdot$ Opt $^{*}$ and $\mathbf{t}\left|\mathbf{C}_{\mathbf{1}}\right|<\left|\mathbf{C}_{\mathbf{2}}\right|$.
Let $X_{1}, \ldots, X_{m}$ be $\{0,1\}$-random variables defined as follows:

$$
X_{j}= \begin{cases}1 & \text { if the edge } E_{j} \text { was picked into the cover before repairing } \\ 0 & \text { otherwise }\end{cases}
$$

Note that the $X_{1}, \ldots, X_{m}$ are independent for a given $x^{*} \in[0,1]^{m}$. For all $i \in[n]$ we define the $\{0,1\}$ - random variables $Y_{i}$ as follows:

$$
Y_{i}= \begin{cases}1 & \text { if the vertex } v_{i} \text { is fully covered before repairing } \\ 0 & \text { otherwise }\end{cases}
$$

We denote $X:=\sum_{j=1}^{m} X_{j}$ and $Y:=\sum_{i=1}^{n} Y_{i}$ respectively the cardinality of the cover and the cardinality of vertices fully covered before the step of repairing. At this step by lemma 2 one more edge for each vertex is at most needed to be fully covered. The cover denoted by C obtained by the algorithm 1 is bounded by

$$
\begin{equation*}
|C| \leq X+n-Y \tag{2}
\end{equation*}
$$

Our goal by the next lemma is to estimate the expectation of the random variable $X$ so is the expectation and variance of the random variable $Y$ for the proof of the theorem 5. This is a restriction of Lemma 4 in [6] to the last case in algorithm 1

Lemma 3. Let $l$ and $\Delta$ be the maximum size of an edge resp. the maximum vertex degree, not necessarily constants. Let $\alpha>0, t>0$ and $\lambda=\frac{\delta+1}{2}$ as in Algorithm 1. In case $\left|C_{1}\right|<\alpha \cdot$ Opt* $^{*}$ and $t\left|C_{1}\right|<\left|C_{2}\right|$ we have
(i) $\mathbb{E}(Y) \geq\left(1-e^{-\lambda}\right) n$.
(ii) $\operatorname{Var}(Y) \leq n^{2}\left(1-\left(1-e^{-\lambda}\right)^{2}\right)$.
(iii) $1+\frac{t}{2}<\mathbb{E}(X) \leq \lambda \mathrm{Opt}^{*}$.
(iv) $\frac{(b-1) n}{\alpha \ell}<\mathrm{Opt}^{*}$.

Proof. (i) Let $i \in[n], r=d(i)-b_{i}+1$. If $\left|C_{1} \cap \Gamma\left(v_{i}\right)\right| \geq b_{i}$, then the vertex $v_{i}$ is fully covered and $\operatorname{Pr}\left(Y_{i}=0\right)=0$. Otherwise we get by Lemma $2\left|C_{1} \cap \Gamma\left(v_{i}\right)\right|=$ $b_{i}-1$ and $\sum_{E_{j} \in\left(\Gamma\left(v_{i}\right) \cap C_{3}\right)} x_{j}^{*} \geq 1$. Therefore

$$
\begin{aligned}
\operatorname{Pr}\left(Y_{i}=0\right) & =\prod_{E_{j} \in\left(\Gamma\left(v_{i}\right) \cap C_{3}\right)}\left(1-\lambda x_{j}^{*}\right) \\
& \leq \prod_{E_{j} \in\left(\Gamma\left(v_{i}\right) \cap C_{3}\right)} e^{-\lambda x_{j}^{*}}=e^{-\lambda \sum_{E_{j} \in\left(\Gamma\left(v_{i}\right) \cap C_{3}\right)} x_{j}^{*}} \\
& \leq e^{-\lambda} .
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbb{E}(Y) & =\sum_{i=1}^{n} \operatorname{Pr}\left(Y_{i}=1\right)=\sum_{i=1}^{n}\left(1-\operatorname{Pr}\left(Y_{i}=0\right)\right) \\
& \geq \sum_{i=1}^{n}\left(1-e^{-\lambda}\right) \\
& \geq\left(1-e^{-\lambda}\right) n .
\end{aligned}
$$

(ii) We have

$$
\begin{aligned}
Y=\sum_{i=1}^{n} Y_{i} & \Rightarrow Y \leq n \\
& \Rightarrow Y^{2} \leq n^{2} \\
& \Rightarrow \mathbb{E}\left(Y^{2}\right) \leq n^{2}
\end{aligned}
$$

Then

$$
\begin{aligned}
\operatorname{Var}(Y) & =\mathbb{E}\left(Y^{2}\right)-\mathbb{E}(Y)^{2} \leq n^{2}-\left(1-e^{-\lambda}\right)^{2} n^{2} \\
& \leq n^{2}\left(1-\left(1-e^{-\lambda}\right)^{2}\right)
\end{aligned}
$$

(iii) By using the LP relaxation and the definition of the sets $C_{1}$ and $C_{3}$, and since $\lambda x_{j}^{*} \geq 1$ for all $E_{j} \in C_{1}$, we get

$$
\begin{aligned}
\mathbb{E}(X) & =\left|C_{1}\right|+\sum_{E_{j} \in C_{3}} \lambda x_{j}^{*} \\
& \leq \lambda \sum_{E_{j} \in C_{1}} x_{j}^{*}+\lambda \sum_{E_{j} \in C_{3}} x_{j}^{*} \\
& \leq \lambda \sum_{E_{j} \in \mathcal{E}} x_{j}^{*} \\
& \leq \lambda \mathrm{Opt}^{*} .
\end{aligned}
$$

Now we can get a lower bound for the expectation of $X$

$$
\begin{aligned}
\mathbb{E}(X)=\left|C_{1}\right|+\sum_{E_{j} \in C_{3}} \lambda x_{j}^{*} & \stackrel{C_{2} \subset C_{3}}{\geq}\left|C_{1}\right|+\lambda \sum_{E_{j} \in C_{2}} x_{j}^{*} \\
& \geq\left|C_{1}\right|+\lambda \sum_{E_{j} \in C_{2}} \frac{1}{\delta} \\
& \geq\left|C_{1}\right|+\frac{\lambda}{\delta}\left|C_{2}\right| \\
& >\left|C_{1}\right|+\frac{1}{2}\left|C_{2}\right| \\
t\left|C_{1}\right|<\left|C_{2}\right| & \left|C_{1}\right|+\frac{t}{2}\left|C_{1}\right| \\
& >\left(1+\frac{t}{2}\right)\left|C_{1}\right| \\
& \gg 1+\frac{t}{2}
\end{aligned}
$$

Therefore

$$
1+\frac{t}{2}<\mathbb{E}(X)
$$

(iv) Let us consider $\tilde{\mathcal{H}}$ the subhypergraph induced by $C_{1}$ in witch degree equality gives

$$
\sum_{i \in V} d(i)=\sum_{E_{j} \in C_{1}}\left|E_{j}\right|
$$

Since the minimum vertex degree in the subhypergraph $\tilde{\mathcal{H}}$ is $b-1$ with $b:=$ $\min _{i \in[n]} b_{i}$, we have

$$
(b-1) n \leq \sum_{i \in V} d(i)=\sum_{E \in C_{1}}\left|E_{j}\right| \leq \ell\left|C_{1}\right|
$$

Therefore

$$
\frac{(b-1) n}{\ell} \leq\left|C_{1}\right|
$$

With $\left|C_{1}\right|<\alpha \cdot$ Opt $^{*}$ we obtain

$$
\frac{(b-1) n}{\alpha \ell}<\mathrm{Opt}^{*}
$$

Theorem 5. Let $\mathcal{H}$ be a hypergraph with maximum vertex degree $\Delta$ and maximum edge size $\ell$. Let $\alpha=\frac{(b-1) \delta e^{\frac{\delta}{4}}}{47 \ell}$ and $t=73$ as in algorithm 1. In case $\left|C_{1}\right|<\alpha \cdot$ Opt* and $t\left|C_{1}\right|<\left|C_{2}\right|$, the algorithm 1 returns a set multicover $C$ such that

$$
|C|<\frac{15 \delta+14}{20} \cdot \mathrm{Opt}
$$

with probability greater than 0.53 .

Proof. Suppose $\left|C_{1}\right|<\alpha \cdot$ Opt $^{*}$ such that $\alpha=\frac{(b-1) \delta e^{\frac{\delta}{4}}}{47 \ell}$. With $\gamma=2 n e^{-\frac{\lambda}{2}}$ we have

$$
\begin{aligned}
& \operatorname{Pr}\left(Y \leq n\left(1-e^{-\lambda}\right)-\gamma\right) \leq \operatorname{Pr}(Y \leq \mathbb{E}(Y)-\gamma) \\
& \leq \frac{\operatorname{Th} 2}{} \frac{\operatorname{Var}(Y)}{\operatorname{Var}(Y)+\gamma^{2}} \\
& \leq \frac{1}{1+\frac{\gamma^{2}}{\operatorname{Var}(Y)}} \\
& \leq \frac{1}{1+\frac{\gamma^{2}}{n^{2}\left(1-\left(1-e^{-\lambda}\right)^{2}\right)}} \\
& \leq \frac{1}{1+\frac{\gamma^{2}}{n^{2}\left(2 e^{-\lambda}-e^{-2 \lambda}\right)}} \\
& 2 n^{2} e^{-\lambda} \\
& \leq \frac{1}{1+2} \\
& \leq \frac{1}{3}
\end{aligned}
$$

Choosing $\beta=\frac{2}{5}$ and $t=73$ we obtain

$$
\begin{aligned}
& \operatorname{Pr}\left(X \geq \frac{7}{10}(\delta+1) \mathrm{Opt}^{*}\right)=\operatorname{Pr}\left(X \geq(1+\beta) \cdot \frac{\delta+1}{2} \mathrm{Opt}^{*}\right) \\
& =\operatorname{Pr}\left(X \geq(1+\beta) \cdot \lambda \mathrm{Opt}^{*}\right) \\
& \stackrel{\text { Lem }{ }^{3}{ }^{\text {iiii) }}}{ } \operatorname{Pr}(X \geq(1+\beta) \mathbb{E}(X)) \\
& \stackrel{\text { Th }}{\leq 1} \exp \left(-\frac{\beta^{2} \mathbb{E}(X)}{3}\right) \\
& \stackrel{\text { Lem }}{\leq}{ }^{\text {(iii) }} \exp \left(-\frac{\beta^{2}\left(1+\frac{1}{2} t\right)}{3}\right) \\
& \leq \quad \exp (-2) \text {. }
\end{aligned}
$$

Therefore it holds that
$\operatorname{Pr}\left(X \leq \frac{7}{10}(\delta+1)\right.$ Opt $^{*}$ and $\left.Y \geq n\left(1-e^{-\lambda}\right)-\gamma\right) \geq 1-\left(\frac{1}{3}+\exp (-2)\right)$.
Since we have

$$
\begin{aligned}
n e^{-\lambda}+\gamma & =n e^{-\lambda}+2 n e^{-\frac{\lambda}{2}} \\
& \leq 3 n e^{-\frac{\lambda}{2}} \\
& =3 n e^{-\frac{\delta+1}{4}} \\
& \leq \frac{\delta}{20} \cdot \frac{n(b-1)}{\ell} \cdot \frac{47 \ell}{(b-1) \delta e^{\frac{\delta}{4}}} \\
& \leq \frac{\delta}{20} \cdot \frac{n(b-1)}{\alpha \ell} \\
\left.\operatorname{Lem} \frac{3}{\leq} i v\right) & \frac{\delta}{20} \cdot \mathrm{Opt}^{*}
\end{aligned}
$$

Then

$$
\begin{aligned}
\operatorname{Pr}\left(|C| \leq\left(\frac{15 \delta+14}{20}\right) \cdot \mathrm{Opt}^{*}\right) & =\operatorname{Pr}\left(|C| \leq\left(\frac{7}{10}(\delta+1)+\frac{\delta}{20}\right) \cdot \mathrm{Opt}^{*}\right) \\
& \stackrel{22}{\geq} \operatorname{Pr}\left(X+n-Y \leq\left(\frac{7}{10}(\delta+1)+\frac{\delta}{20}\right) \cdot \mathrm{Opt}^{*}\right) \\
& \geq \operatorname{Pr}\left(X \leq\left(\frac{7}{10}(\delta+1)\right) \cdot \mathrm{Opt}^{*} \text { and } n-Y \leq \frac{\delta}{20} \cdot \mathrm{Opt}^{*}\right) \\
& \geq \operatorname{Pr}\left(X \leq\left(\frac{7}{10}(\delta+1)\right) \cdot \mathrm{Opt}^{*} \text { and } Y \geq n-\frac{\delta}{20} \cdot \mathrm{Opt}^{*}\right) \\
& \geq \operatorname{Pr}\left(X \leq\left(\frac{7}{10}(\delta+1)\right) \cdot \mathrm{Opt}^{*} \text { and } Y \geq n-n e^{-\lambda}-\gamma\right) \\
& \geq \operatorname{Pr}\left(X \leq\left(\frac{7}{10}(\delta+1)\right) \cdot \mathrm{Opt}^{*} \text { and } Y \geq n\left(1-e^{-\lambda}\right)-\gamma\right) \\
& \geq 1-\left(\frac{1}{3}+\exp (-2)\right) \\
& \geq 0.53
\end{aligned}
$$

Remark 2. The analysis of the algorithm 1 gave three upper bounds for our cover $C$ in cardinality .
Since $\frac{15 \delta+14}{20}$ is smaller than $\left(1-\frac{(b-1) e^{\frac{\delta}{4}}}{94 \ell}\right) \delta$ and $\frac{148}{149} \delta$ for $\delta \geq 3$, we get an approximation ratio of $\max \left\{\frac{148}{149} \delta,\left(1-\frac{(b-1) e^{\frac{\delta}{4}}}{94 \ell}\right) \delta\right\}$.
As mentioned above our performed guaranty improves over the ratio presented by Srivastav et al [6], and this without restriction on the parameter $\ell$.
Namely, for $\delta \geq 24$ we have

$$
\begin{aligned}
e^{\frac{\delta}{4}}>\frac{11 \times 94}{72}(\delta-1) & \Rightarrow \frac{11(\delta-1)}{72 \ell}<\frac{e^{\frac{\delta}{4}}}{94 \ell} \\
& \stackrel{b-1 \geq 1}{\Rightarrow} \frac{11(\Delta-b)}{72 \ell}<\frac{(b-1) e^{\frac{\delta}{4}}}{94 \ell} \\
& \Rightarrow\left(1-\frac{(b-1) e^{\frac{\delta}{4}}}{94 \ell}\right) \delta<\left(1-\frac{11(\Delta-b)}{72 \ell}\right) \delta
\end{aligned}
$$

## 4 The $\frac{5}{6}\left(1-\frac{1}{2 \ell}\right) \delta$-Approximation for the set multicover problem

The designed algorithm in this section and its analysis relies on the lemma below. We denote by $\bar{\Delta}:=\frac{1}{n} \cdot \sum_{v \in V} \operatorname{deg}(v)$ the average vertex degree in $\mathcal{H}$ and by $\bar{b}:=$ $\frac{1}{n} \cdot \sum_{v \in V} b_{v}$ the average of the vector $\mathbf{b}=\left(b_{1}, \cdots, b_{n}\right)$.

Lemma 4. Let $\mathcal{H}=(V, \mathcal{E})$ be a hypergraph and $\boldsymbol{b} \in \mathbb{N}^{V}$ and $\boldsymbol{k} \in \mathbb{N}_{0}^{V}$ such that $b_{v}+k_{v}=\operatorname{deg}(v)$ for any $v \in V$. Then $\nu_{k}(\mathcal{H}) \leq\left(\frac{\bar{\Delta} \ell}{\bar{b}} \frac{\bar{\ell}}{}-1\right)$ Opt.

Proof. Let $S^{*}$ be an optimal set multicover of $\mathcal{H}$. By definition we have

$$
\begin{equation*}
m=\frac{n \bar{\Delta}}{\bar{\ell}} \tag{3}
\end{equation*}
$$

and by double-counting for the pairs $(v, E)$ with $v \in E$ and $E \in S^{*}$ we get

$$
\begin{equation*}
n \bar{b}=\sum_{v \in V} b_{v} \leq \ell\left|S^{*}\right| \tag{4}
\end{equation*}
$$

so we get
$\nu_{k}(\mathcal{H})=m-\mathrm{Opt}=\left(\frac{\mathrm{m}}{\left|\mathrm{S}^{*}\right|}-1\right) \cdot \mathrm{Opt} \stackrel{(\sqrt{3})}{=}\left(\frac{\mathrm{n} \bar{\Delta}}{\ell\left|\mathrm{S}^{*}\right|}-1\right) \cdot \mathrm{Opt} \stackrel{[4]}{\leq}\left(\frac{\bar{\Delta} \ell\left|\mathrm{S}^{*}\right|}{\mathrm{b} \ell\left|\mathrm{S}^{*}\right|}-1\right) \cdot \mathrm{Opt}$.
Let $\mathcal{M}$ be an approximation algorithm for the $k$-matching problem with approximation guarantee $0<r \leq 1$.

We will use Theorem 3 to construct a set multicover in $\mathcal{H}$. let $\mathbf{k} \in \mathbb{N}_{0}^{n}$ with components $k_{i}=d\left(v_{i}\right)-b_{i}$ for all $i \in[n]$. Theorem 3 says that if we can find a k-matching $M$ in $\mathcal{H}$, then $S:=\mathcal{E} \backslash M$ is a set multicover in $\mathcal{H}$.

```
Algorithm 2: set multicover Algorithm
    Input : A hypergraph \(\mathcal{H}=(V, \mathcal{E})\) with \(|V|=n, \mathbf{b} \in \mathbb{N}^{n}\).
    Output: A set multicover \(S\).
    1. For every \(v_{i} \in V\) set \(k_{i}:=d\left(v_{i}\right)-b_{i}\) and \(\mathbf{k}:=\left(k_{1}, \ldots, k_{n}\right)\).
    2. Execute Algorithm \(\mathcal{M}\) on input \(\mathcal{H}\) and \(\mathbf{k}\) and return a \(\mathbf{k}\)-matching \(M\).
    3. Set \(S:=\mathcal{E} \backslash M\).
    4. Return \(S\).
```

Theorem 6. The algorithm 圆 returns a set multicover with an approximation guarantee of $(1-r) \cdot \frac{\bar{\Delta} \cdot \ell}{\bar{b} \cdot \bar{\ell}}+r \leq \frac{\bar{\Delta} \cdot \ell}{\bar{b} \cdot \bar{\ell}}$. Where $r$ is the approximation ratio given by any algorithm to find a k-matching.

Note that $\Delta \geq b$ is implicitly assumed since otherwise the problem is unfeasible; indeed, we assume $\operatorname{deg}_{\mathcal{H}}(v) \geq b_{v}$ for all $v$.

Proof. For the solution $S$ of Algorithm 2 the following holds:

$$
\begin{array}{rlrl}
|S| & =|\mathcal{E}|-|M| & & \text { operation of algorithm } \\
& =|\mathcal{E}|-\mathrm{Opt}+\mathrm{Opt}-|\mathrm{M}| & & \text { Ray-Chaudhuri } \\
& =\mathrm{Opt}+\nu_{\mathbf{k}}(\mathcal{H})-|\mathrm{M}| & & \text { approximation guarantee of } \mathcal{M} \\
& \leq \mathrm{Opt}+\nu_{\mathbf{k}}(\mathcal{H})-\mathrm{r} \cdot \nu_{\mathbf{k}}(\mathcal{H}) & \\
& \leq \mathrm{Opt}+(1-\mathrm{r}) \cdot \nu_{\mathbf{k}}(\mathcal{H}) & \\
& \leq \mathrm{Opt}+(1-\mathrm{r}) \cdot\left(\frac{\bar{\Delta} \ell}{\overline{\mathrm{b}}} \overline{\bar{\ell}}-1\right) \cdot \mathrm{Opt} & & \text { by Lemma } 4 \\
& =\left(\frac{\bar{\Delta} \ell}{\bar{b}} \frac{\bar{\ell}}{\ell}-r\left(\frac{\bar{\Delta} \ell}{\bar{b}} \bar{\ell}-1\right)\right) \cdot \mathrm{Opt}=\left(\frac{\bar{\Delta} \ell}{\overline{\mathrm{b}}} \bar{\ell}(1-\mathrm{r})+\mathrm{r}\right) \cdot \mathrm{Opt}
\end{array}
$$

Corollary 2. Let $\mathcal{H}=(V, \mathcal{E})$ with $\ell \leq(1+\epsilon) \bar{\ell}$ for any fixed $\epsilon \in\left[0, \frac{1}{2}\right]$. The algorithm 2 returns a set multicover with an approximation guarantee of $\frac{5}{6} \delta$.

Proof. We have for all $b \geq 3$

$$
5 b(b-1) \leq(b+2)(5 b-9) \stackrel{\Delta \geq b+2}{\leq} \Delta(5 b-9)
$$

Hence $9 \Delta \leq 5 b(\Delta-b+1)$ and therewith $\frac{\Delta}{b} \leq \frac{5}{9} \delta$.
Now let consider the class of hypergraphs with $\ell \leq(1+\epsilon) \bar{\ell}$ for any fixed $\epsilon \in\left[0, \frac{1}{2}\right]$. Let set $r(\mathcal{H})=\frac{1}{\ell}$ than we get

$$
\begin{aligned}
\frac{\bar{\Delta} \ell}{\bar{b} \bar{\ell}}(1-r(\mathcal{H}))+r(\mathcal{H}) & \leq \frac{\Delta}{b}(1+\epsilon)\left(1-\frac{1}{l}\right)+\frac{1}{l} \\
& \leq\left(\frac{5}{9}(1+\epsilon)\left(1-\frac{1}{l}\right)+\frac{1}{l \delta}\right) \cdot \delta \\
& \delta \geq 3 \\
& \left.\leq \frac{15}{18}\left(1-\frac{1}{l}\right)+\frac{1}{3 l}\right) \cdot \delta \\
& \leq\left(\frac{5}{6}-\frac{1}{2 l}\right) \cdot \delta \\
& \leq \frac{5}{6} \delta
\end{aligned}
$$

The assumption $r(\mathcal{H})=\frac{1}{\ell}$ may be proved by a simple greedy analysis that we present in the following section.

### 4.1 An $\frac{1}{\ell}$-approximation algorithm for the k-matching problem

In this section we present a greedy algorithm that constructs a k-matching in $\mathcal{H}$ with an approximation ratio of $\frac{1}{\ell}$.

```
Algorithm 3: k-Matching Algorithm
    Input : A hypergraph \(\mathcal{H}=(V, \mathcal{E})\) with \(|V|=n\) and \(|\mathcal{E}|=m, \mathbf{k} \in \mathbb{N}_{0}^{n}\).
    Output: A k-matching \(M\) in \(\mathcal{H}\).
        . Initialize \(M:=\emptyset\). Consider any ordering \(E_{1}, E_{2}, \cdots, E_{m}\) of the edges in \(\mathcal{E}\).
        for \(i=1,2, \cdots, m\) do
        if \(M \cup\left\{E_{i}\right\}\) is a \(\mathbf{k}\)-matching then
            set \(M:=M \cup\left\{E_{i}\right\}\).
        Return \(M\).
```

Theorem 7. Algorithm 3 constructs a k-matching $M$ in $\mathcal{H}$ with $|M| \geq \frac{1}{\ell} \nu_{\mathbf{k}}(\mathcal{H})$.
Proof. It is clear by construction that $M$ is a k-matching. Set $N:=|M|$ and let $M=\left\{f_{1}, \cdots, f_{N}\right\}$. Let $\left|M^{*}\right|$ be a maximum $\mathbf{k}$-matching in $\mathcal{H}$. We compare the cardinality of $M$ with the cardinality of $M^{*}$ by iteratively adding all $M$ edges into $M^{*}$ and removing some $M^{*}$-edges in order to fulfill the k-matching condition. Suppose we have arrived at edge $f_{i}, i \in[N]$. If $f_{i} \in M^{*}$, we keep $f_{i}$ in $M^{*}$. Otherwise, for every $v \in f_{i}$, for which the $\mathbf{k}$-matching conditions in $M^{*} \cup\left\{f_{i}\right\}$ is violated, remove one $M^{*}$-edge incident in $v$. Thus at most $\ell M^{*}$ edge are removed in this step. Define $r_{i}$ as the number of removed $M^{*}$-edges in step $i$, if $f_{i} \notin M^{*}$, and $r_{i}:=1$, if $f_{i} \in M^{*}$. Trivially $r_{i} \leq \ell$ for all $i \in[N]$. Furthermore, $\sum_{i \in[N]} r_{i}=\left|M^{*}\right|$. Assume for a moment that $\sum_{i \in[N]} r_{i} \leq\left|M^{*}\right|-1$. Then some $M^{*} \backslash M$ edges have not been removed. Let $E_{t}$ be such an edge. Since $M \cup\left\{E_{t}\right\}$ is a k-matching, the algorithm should have included $E_{t}$ into $M$, which
is a contradiction. Summation over the number of iterations gives

$$
|M|=\sum_{i \in[N]} 1 \geq \sum_{i \in[N]} \frac{r_{i}}{\ell}=\frac{1}{\ell} \sum_{i \in[N]} r_{i}=\frac{1}{\ell}\left|M^{*}\right|=\frac{1}{\ell} \nu_{\mathbf{k}}(\mathcal{H}) .
$$

Remark 3. We note that,
i) our approximation ratio of $\frac{1}{\ell}$ for $\mathbf{k}$-matching problem improves over the ratio of $\frac{1}{\ell+1}$ presented by Krysta [15]. This to our knowledge the best achieved result for the problem without restrictions on either on $\mathbf{k}$ nor on the instance.
ii) Theorem 7 will be used for the construction of a $\mathbf{k}$-matching in this paper. It is also possible to use other $\mathbf{k}$-matching approximation algorithms.

## 5 Future Work

We believe now that the conjecture of Peleg et all holds in general setting. Hence proving the truly of the conjecture remains a big challenge for our future works.

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