THE WEIGHTED SITTING CLOSER TO FRIENDS THAN ENEMIES PROBLEM IN THE LINE

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Abstract. The weighted *Sitting Closer to Friends than Enemies* (SCFE) problem is to find an injection of the vertex set of a given weighted graph into a given metric space so that, for every pair of incident edges with different weight, the end vertices of the heavier edge are closer than the end vertices of the lighter edge. The *Seriation* problem is to find a simultaneous reordering of the rows and columns of a symmetric matrix such that the entries are monotone nondecreasing in rows and columns when moving towards the diagonal. If such a reordering exists, it is called a *Robinson* ordering. In this work, we establish a connection between the SCFE problem and the Seriation problem. We show that if the *extended adjacency matrix* of a given weighted graph G has no Robinson ordering then G has no injection in $\mathbb{R}$ that solves the SCFE problem. On the other hand, if the extended adjacency matrix of G has a Robinson ordering, we construct a polyhedron that is not empty if and only if there is an injection of the vertex set of G in $\mathbb{R}$ that solves the SCFE problem. As a consequence of these results, we conclude that deciding the existence of (and constructing) such an injection in $\mathbb{R}$ for a given *complete* weighted graph can be done in polynomial time. On the other hand, we show that deciding if an *incomplete* weighted graph has such an injection in $\mathbb{R}$ is NP-Complete.

1. Introduction

Consider a data set. The task is to construct a graphic representation of the data set so that similarities between data points are graphically expressed. To complete this task, the only information available is a *similarity matrix* of the data set, i.e., a symmetric, square matrix whose entry ij contains a similarity measure between data points i and j (the larger the value the more similar the data points are). Hence, the task is to draw all data points in a *paper* so that for every three data points i, j, and k, if i is at least as similar to j than k, then i should be placed closer in the drawing to j than k. In colloquial words, for each data point j, the farther the other data points are, the less similar they are to j.

A slightly simpler version of this problem, introduced in [12], has been studied under the name of the Sitting Closer to Friends than Enemies (SCFE) problem. The SCFE problem uses signed graphs as an input. Therefore, the similarity matrix has entries 1 and -1, representing similarity and dissimilarity, or friendship and enmity between the data points, from where the problem obtains its name. The

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SCFE problem has been studied in the real line [12, 8, 18] and in the circumference [2] (which means that the *paper* is the real line or the circumference). In both cases, the real line and the circumference, it has been shown that deciding the existence of such an injection for a given signed graph is NP-Complete. Nevertheless, in both cases again, when the problem is restricted to complete signed graphs there exists a characterization of the families of complete signed graphs that admit a solution for the SCFE problem and it can be decided in polynomial time [12, 2]. Therefore, a natural next step is to consider the case when similarities range in an extended set of values.

The SCFE problem in the line seems to be closely related to the *Seriation* problem. Liiv in [16] defines the Seriation problem as "an exploratory data analysis technique to reorder objects into a sequence along a one-dimensional continuum so that it best reveals regularity and pattering among the whole series". Seriation has applications in archaeology [19], data visualization [4], exploratory analysis [11], bioinformatics [25], and machine learning [9], among others. Liiv in [16] presents an interesting survey on seriation, matrix reordering and its applications. The first important contribution of this document is to show that the SCFE and the Seriation problems are different. Indeed, we show that seriation is a necessary condition to solve the SCFE problem, but it is not sufficient.

To continue with our exposition, in Section 2 we introduce the notation and definitions used along the document. The rest of the document is organized as follows. In Section 3, we present the state of the art and contextualize our contributions. In Section 5, we present the characterization of weighted graphs with an injection in $\mathbb{R}$ that satisfies the restrictions of the SCFE problem. Furthermore, we present the results related with complete weighted graphs. In Section 6, we present the results regarding incomplete weighted graphs. We conclude in Section 7 with some final remarks and future work.

2. Notation and Definitions

We use standard notation. A graph is denoted by $G = (V, E)$. We consider only undirected graphs, without parallel edges and loopless. The set of vertices of G is V and the set of edges is E, a set of 2-elements subsets of V. We use n and m to denote $|V|$ and $|E|$, respectively. Two distinct vertices u and v in V are said to be *neighbors* if $\{u, v\} \in E$. In that case, we say that they are connected by an edge which is denoted by $\{u, v\}$. A graph is said to be *complete* if every pair of distinct vertices is connected by an edge, otherwise, we say that it is *incomplete*.

In this document, we work with weighted graphs. We denote by $w : E \to \mathbb{R}^+$ a positive real valued function that assigns $w(\{u, v\})$, a positive real value, to the edge $\{u, v\}$ in E. For our purposes, we consider that w is a similarity measure, i.e., for any $\{u, v\} \in E$ the value $w(\{u, v\})$ measures how similar u and v are. We consider that the larger the similarity measure is, the more similar the vertices are. It is worth mentioning that the fact that the weights are positive is just a choice made for simplicity. Actually, if we have negative weights, one can translate all weights by a constant and obtain only positive weights. All our results are still valid if we remove this assumption.

Let $(\mathcal{M}, d)$ be a metric space. A *drawing* of a graph $G = (V, E)$ into $\mathcal{M}$ is an injection $D : V \to \mathcal{M}$. We define a certain type of drawings that capture the requirements of the SCFE problem.

Definition 2.1. Let $G = (V, E)$ be a graph, and $w : E \to \mathbb{R}^+$ be a positive function on E. Let $(\mathcal{M}, d)$ be a metric space. We say that a drawing D of G into $\mathcal{M}$ is *valid distance* if, for all pair $\{t, u\}$, $\{t, v\}$ of incident edges in E such that $w(\{t, u\}) > w(\{t, v\})$,

$$d(D(t), D(u)) < d(D(t), D(v)).$$

In colloquial words, a drawing is valid distance, or simply *valid*, when it places vertices t and u strictly closer than t and v in $\mathcal{M}$ whenever t and u have a strictly larger similarity measure than t and v. Now, the weighted SCFE problem in its most general presentation is defined as follows.

Definition 2.2. Given a weighted graph G and a metric space $\mathcal{M}$, the weighted SCFE problem in $\mathcal{M}$ is to decide whether G has a valid drawing in $\mathcal{M}$, and, in case of a positive answer, find one.

In this document, we focus our attention on the case when the metric space is the real line, i.e., we consider the metric space to be the set of real values $\mathbb{R}$ with the Euclidean distance.

Since we present a matrix oriented analysis, we introduce the next two matrix related definitions. Given a matrix A, the entry in the *i-th* row and *j-th* column of A is denoted by A_{ij}. For every weighted graph $G = (V, E)$ and an ordering π of the vertex set V, we define $A^\pi(G)$, the *similarity matrix* of G ordered according to π, as follows. Let π_i be the *i-th* element of V according to π, then:

$$A^\pi(G)_{ij} = \begin{cases} * & \text{if } i \neq j \text{ and } \{\pi_i, \pi_j\} \notin E, \\ w(\{\pi_i, \pi_j\}) & \text{if } i \neq j \text{ and } \{\pi_i, \pi_j\} \in E, \\ \max_{e \in E} w(e) & \text{if } i = j. \end{cases}$$

This matrix is also known as the extended weighted *adjacency matrix* of G. The *i-th* row (and *i-th* column) contains the similarities between vertex π_i and the rest of the vertices of G. We may use only A^π when the graph G is contextually clear or only A when G and π are contextually clear. Note that any similarity matrix of any weighted graph is symmetric since w is symmetric. A similarity matrix of a complete weighted graph does not have entries with the symbol $*$. In that case, we say that a similarity matrix is *complete*, otherwise we say that it is *incomplete*.

W. S. Robinson in [22] introduced Robinsonian matrices. A complete similarity matrix A is said to be *Robinson* if its entries are monotone nondecreasing in rows and columns when moving towards the diagonal, i.e., if for all integers $1 \leq i < j \leq n$,

$$A_{ij} \leq \min\{A_{ij-1}, A_{i+1j}\}.$$

Equivalently, a complete similarity matrix A is Robinson if for all integers $1 \leq i < l \leq n$ and $j, k \in [i, l]$:

$$A_{il} \leq \min\{A_{ij}, A_{kl}\}.$$

On the other hand, a complete similarity matrix A is *Robinsonian* if its rows and columns can be reordered simultaneously by a permutation π such that A^π is Robinson.

The Robinson matrix definition can be naturally extended to incomplete matrices. In that case, a similarity matrix is *Robinson* if its specified entries are monotone nondecreasing in rows and columns when moving towards the diagonal, i.e., an incomplete similarity matrix A is Robinson if for all integers $1 \leq i < l \leq n$ and $j, k \in [i, l]$, such that $A_{il} \neq *$, $A_{ij} \neq *$ and $A_{kl} \neq *$,

$$A_{il} \leq \min\{A_{ij}, A_{kl}\}.$$

Again, we say that a similarity matrix is *Robinsonian* if its rows and columns can be simultaneously reordered in such a way that it passes to be Robinson. Finally, we say that an ordering π of the vertex set V of a weighted graph $G = (V, E)$ is Robinson if $A^{\pi}(G)$ is Robinson.

3. Context, Related Work, and Our Contributions

Robinsonian matrices were defined by W. S. Robinson in [22] in a study on how to order chronologically archaeological deposits. The *Seriation* problem introduced in the same work is to decide whether the similarity matrix of a data set is Robinsonian and reorder it as a Robinson matrix if possible. Recognition of complete Robinsonian matrices has been studied by several authors. Mirkin et al. in [17] presented an $O(n^4)$ recognition algorithm, where $n \times n$ is the size of the matrix. On the other hand, using divide and conquer techniques, Chepoi et al. in [5] introduced an $O(n^3)$ recognition algorithm. Later, Préa and Fortin in [20] provided an $O(n^2)$ optimal recognition algorithm for complete Robinsonian matrices using PQ trees.

Using the relationship between Robinsonian matrices and unit interval graphs presented in [21], Monique Laurent and Matteo Seminaroti in [13] introduced a recognition algorithm for Robinsonian matrices that uses Lex-BFS, whose time complexity is $O(L(m + n))$, where m is the number of nonzero entries in the matrix, and L is the number of different values in the matrix. Later in [14], the same authors presented a recognition algorithm with time complexity $O(n^2 + nm \log n)$ that uses similarity-first search, an extension of breadth-first search for weighted graphs. Again, using the relationship between Robinsonian matrices and unit interval graphs, Laurent et al. in [15] gave a characterization of Robinsonian matrices via forbidden patterns.

The Seriation problem also has been studied as an optimization problem. Given an $n \times n$ matrix D, *seriation in the presence of errors* is to find a Robinsonian matrix R that minimizes the error defined as: $\max ||D_{ij} - R_{ij}||$ over all i and j in $\{1, 2, 3, \ldots, n\}$. Chepoi et al. in [6] proved that seriation in the presence of errors is an NP-Hard problem. Later in [7], Chepoi and Seston gave a factor 16 approximation algorithm. Fortin in [10] surveyed the challenges for Robinsonian matrix recognition.

The SCFE problem was first introduced by Kermarrec and Thraves in [12]. Besides the introduction of the SCFE problem, the authors of [12] also characterized the set of complete signed graphs with a valid drawing in $\mathbb{R}$ and presented a polynomial time recognition algorithm. Later, Cygan et al. in [8] proved that the SCFE problem is NP-Complete if it is not restricted to complete signed graphs. Moreover, they gave a different characterization of the complete signed graphs with a valid drawing in $\mathbb{R}$. Actually, the authors of [8] proved that a complete signed graph has a valid drawing in $\mathbb{R}$ if and only if its positive subgraph is a unit interval graph. The SCFE problem in the real line also was studied as an optimization problem by Pardo et al. in [18]. In that work, the authors defined as an error a violation of the inequality in Definition 2.1 and provided optimization algorithms that construct a drawing attempting to minimize the number of errors.

The SCFE problem also has been studied for different metric spaces. First, Benitez et al. in [2] studied the SCFE problem in the circumference. The authors of that work proved that the SCFE problem in the circumference is NP-Complete and gave a characterization of the complete signed graphs with a valid drawing. Indeed, they showed that a complete signed graph has a valid drawing in the circumference if and only if its positive subgraph is a proper circular arc graph. Later, Becerra in [1] studied the SCFE problem in trees. The main result of her work was to prove that a complete signed graph G has a valid drawing in a tree if and only if its positive subgraph is strongly chordal.

Spaen et al. in [23] studied the SCFE problem from a different perspective. They studied the problem of finding $L(n)$, the smallest dimension k such that any signed graph on n vertices has a valid drawing in $\mathbb{R}^k$, with respect to the Euclidean distance. They showed that $\log_5(n-3) \leq L(n) \leq n-2$.

Our Contributions. Our first contribution is to show that the Seriation and the SCFE problems are not the same. Indeed, we show that the SCFE problem implies a stronger condition than the Seriation problem. In Lemma 4.1, we show that if a weighted graph $G = (V, E)$ has a valid drawing in $\mathbb{R}$, there is a Robinson ordering of V. Nevertheless, in Lemma 4.2, we show that there is a weighted graph $G = (V, E)$ with a Robinson ordering of V, but G does not have a valid drawing in $\mathbb{R}$.

On the other hand, for each weighted graph G with a Robinson ordering of its vertex set, we construct a polyhedron defined by an inequality system $M(G)\mathbf{x} \leq \mathbf{b}$ which we use to provide a characterization of the set of weighted graphs with a valid drawing in $\mathbb{R}$. Indeed, we show in Theorem 5.2 that a weighted graph G has a valid drawing in $\mathbb{R}$ if and only if its polyhedron defined by $M(G)\mathbf{x} \leq \mathbf{b}$ is not empty.

Our first result applied to complete weighted graphs allows us to conclude in Corollary 5.6 that given a complete weighted graph G, determining whether G has a valid drawing in $\mathbb{R}$, and finding one if applicable, can be done in polynomial time.

On the other hand, when the weighted graph is not complete, the previous result does not apply anymore. In Corollary 6.2, we state that recognition of incomplete Robinsonian matrices is NP-complete. Furthermore, using results shown in [8] by Cygan et al., we conclude, under the assumption of the Exponential Time Hypothesis, the nonexistence of a subexponential-time algorithm that determines if an incomplete similarity matrix is Robinsonian. Therefore, the construction of the polyhedron cannot be done in polynomial time (unless P=NP). Nevertheless, in Section 6, we provide a recognition algorithm of $n \times n$ incomplete Robinsonian matrices with time complexity $O(n \cdot 2^{2n})$.

4. Robinson Orderings and Valid Distance Drawings

In this section we connect Robinson orderings and valid distance drawings. We start this section showing that, if a weighted graph has a valid drawing in $\mathbb{R}$ its similarity matrix is Robinsonian. Therefore, having a Robinsonian similarity matrix is a necessary condition to have a valid distance drawing in $\mathbb{R}$.

Lemma 4.1. *Let $G = (V, E)$ be a weighted graph. If G has a valid distance drawing in $\mathbb{R}$, then V has a Robinson ordering.*

Proof. Let $G = (V, E)$ be a weighted graph with weight function w. Let $D : V \to \mathbb{R}$ be a valid distance drawing of G in $\mathbb{R}$. The valid drawing D determines an ordering on the set of vertices V. Indeed, for u and v in V, we say that $u <_D v$ if $D(u) < D(v)$. We show that $A(G)$ is Robinson when it is written using the ordering determined by D for its rows and columns.

Enumerate V according to the ordering determined by D. Consider any integers i, j, k, l such that $1 \leq i < l \leq n$, $j, k \in [i, l]$, and $A(G)_{il} \neq *$, $A(G)_{ij} \neq *$ and $A(G)_{kl} \neq *$. First, we point our that $A(G)_{il} \leq A(G)_{ij}$, since D is valid distance, and $d(D(i), D(l)) > d(D(i), D(j))$. Equivalently, since D is valid distance and $d(D(i), D(l)) > d(D(k), D(l))$, we have $A(G)_{il} \leq A(G)_{kl}$. Therefore, $A(G)_{il} \leq \min\{A(G)_{ij}, A(G)_{kl}\}$. In conclusion, $A(G)$, the similarity matrix of G written according to the ordering determined by D, is Robinson. Hence, the ordering determined by D is Robinson.

□

Nevertheless, having a Robinson similarity matrix is not enough to have a valid distance drawing in $\mathbb{R}$. In the next, lemma we show the existence of a weighted graph with a Robinson ordering of its vertices but without valid distance drawing in $\mathbb{R}$.

Lemma 4.2. *There exists a complete weighted graph G with Robinson similarity matrix but without a valid distance drawing in $\mathbb{R}$.*

Proof. Let G be the complete weighted graph with vertex set $\{a, b, c, d, e\}$ and similarity matrix

$$A(G) = \begin{bmatrix} 5 & 2 & 2 & 1 & 1 \\ 2 & 5 & 3 & 2 & 1 \\ 2 & 3 & 5 & 4 & 1 \\ 1 & 2 & 4 & 5 & 5 \\ 1 & 1 & 1 & 5 & 5 \end{bmatrix}$$

written with rows and columns ordered as a, b, c, d, e. $A(G)$ is Robinson, nevertheless, we will show by contradiction that G does not have a valid drawing in $\mathbb{R}$.

Assume that G has a valid drawing D in $\mathbb{R}$. Since the order a, b, c, d, e of the rows and columns of $A(G)$ is the only one that makes $A(G)$ Robinson, D has to be such that

$$D(a) < D(b) < D(c) < D(d) < D(e). \tag{4.1}$$

Since D is a valid drawing, the following inequalities hold:

$$D(b) - D(a) > D(c) - D(b) \tag{4.2}$$

$$D(e) - D(b) > D(b) - D(a) \tag{4.3}$$

$$D(c) - D(b) > D(d) - D(c) \tag{4.4}$$

$$D(e) - D(c) > D(c) - D(a) \tag{4.5}$$

$$D(d) - D(c) > D(e) - D(d). \tag{4.6}$$

Without loss of generality, assume that $D(a) = 0$. Then, from inequalities (4.1) and (4.2) we obtain:

$$D(b) < D(c) < 2D(b). \tag{4.7}$$

On the other hand, from inequalities (4.5) and (4.6), we obtain $2D(c) < D(e) < 2D(d) - D(c)$, which implies:

$$3D(c) < 2D(d). \tag{4.8}$$

Finally, inequality (4.4) is equivalent to $2D(d) < 4D(c) - 2D(b)$, which, together with (4.8), implies $2D(b) < D(c)$. But, the last inequality contradicts inequality (4.7). □

5. The Weighted SCFE Problem in the line

The goal of this section is to find a solution for the weighted SCFE problem in the real line. We transform the weighted SCFE problem in the real line into the problem of finding a point in a convex polyhedron. Actually, given a weighted graph G, we construct a convex polyhedron defined by an inequality system $M(G)\mathbf{x} \leq \mathbf{b}$, where each point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in the convex polyhedron is a valid drawing of G in $\mathbb{R}$. Indeed, for any given $\mathbf{x}$ in the polyhedron, each variable x_i represents the position of vertex i in $\mathbb{R}$ for that valid drawing. Therefore, finding a point in the polyhedron is equivalent to find a valid drawing for G in $\mathbb{R}$.

We first remark that if a given weighted graph G has a valid drawing in $\mathbb{R}$, it actually has an infinite number of them. Indeed, given a valid drawing in $\mathbb{R}$ for a weighted graph G, one can obtain a different valid drawing for the same graph by summing or multiplying each vertex position by any positive constant. The second case (when each position is multiplied by a positive constant) is important for us, because it allows us to state the following lemma.

Lemma 5.1. *Let $G = (V, E)$ be a weighted graph with a valid distance drawing in $\mathbb{R}$. Then, for any $\epsilon > 0$ there exists a valid distance drawing D_ϵ of G in $\mathbb{R}$ such that:*

$$\min_{u,v \in V} |D_\epsilon(u) - D_\epsilon(v)| \geq \epsilon.$$

Proof. Let G be a weighted graph with a valid drawing D in $\mathbb{R}$. We consider without loss of generality that v is labeled according to the ordering determined by D, i.e., $1 <_D 2 <_D 3 <_D \ldots <_D n$. Consider any $\epsilon > 0$. Let $\delta = \min_{1 \leq i < n} D(i+1) - D(i)$ be the minimum distance between two consecutive vertices in the drawing. Multiply every $D(i)$ by ϵ/δ. Therefore, we obtain a new valid drawing D_ϵ defined as $D_\epsilon(i) = \epsilon D(i)/\delta$, such that $\min_{u,v \in V} |D_\epsilon(u) - D_\epsilon(v)| \geq \epsilon$.

□

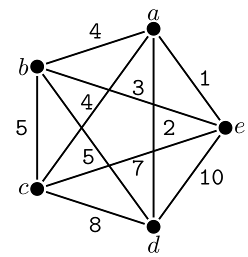
Now, we proceed with the construction of the matrix $M(G)$ and the vector $\mathbf{b}$ of the inequality system $M(G)\mathbf{x} \leq \mathbf{b}$ that defines our polyhedron. By Lemma 4.1, the ordering of the vertex set of G defined by a valid drawing is Robinson. Hence, we may assume that $A(G)$ is Robinson.

If we want to construct a valid drawing D in $\mathbb{R}$ for G, the vertices should be ordered in the same way as the rows and columns of $A(G)$. Hence, if the *i-th* row (or column) of $A(G)$ contains the similarities of vertex $i \in \{1, 2, 3, \ldots, n\}$, then D has to be so that $D(1) < D(2) < \cdots < D(n)$. Therefore, we want $x_1 < x_2 < \cdots < x_n$. Now, considering Lemma 5.1, we write the following set of restrictions for any $\epsilon > 0$:

$$x_i - x_{i+1} \leq -\epsilon, \quad \forall i \in \{1, 2, 3, \ldots, n-1\}. \tag{5.1}$$

These restrictions are called *ordering restrictions*. Lemma 5.1 allows us to pick any $\epsilon > 0$ for these restrictions. We set a value for ϵ, and use that value for the inequalities we define below.

On the other hand, each row of $A(G)$ provides two types of restrictions. We call these restrictions *right with respect to left* and *left with respect to right* restrictions. Right with respect to left restrictions are obtained as follows. For each row j



(a)

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 10 & 4 & 4 & 2 & 1 \\ 4 & 10 & 5 & 5 & 3 \\ 4 & 5 & 10 & 8 & 7 \\ 2 & 5 & 8 & 10 & 10 \\ 1 & 3 & 7 & 10 & 10 \end{bmatrix} \end{array}$$

(b)

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & -2 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

(c)

FIGURE 1. Example of a complete weighted graph, its similarity matrix, and its corresponding matrix of restrictions. Subfigure (a) shows a complete weighted graph. Subfigure (b) shows its Robinson similarity matrix. It also shows the order of the vertices in which the similarity matrix is written. Subfigure (c) shows the restriction matrix for the weighted graph in Subfigure (a). In the first 4 rows appear the ordering restrictions. Rows five and six show the right with respect to left and left with respect to right restrictions for vertex b. Rows seven and eight show right with respect to left restrictions for vertices c and d, respectively.

and for every index $k > j$, let $i(k)$ be the largest index such that $i(k) < j$ and $A(G)_{ji(k)} < A(G)_{jk}$. Therefore, since $A(G)_{ji(k)} < A(G)_{jk}$, vertices j and k are more similar between them than vertices j and $i(k)$. Hence, in any valid drawing D it must occur $D(k) - D(j) < D(j) - D(i(k))$. We transform this strict inequality into the following restriction for a sufficiently small $\epsilon > 0$:

$$x_{i(k)} - 2x_j + x_k \leq -\epsilon, \quad \forall j \in \{2, 3, \ldots, n-1\} \text{ and } \forall k > j. \tag{5.2}$$

Left with respect to right restrictions are symmetrical to the previous restriction. For each row j and for every index $i < j$, let $k(i)$ be the smallest index such that $j < k(i)$ and $A(G)_{ji} > A(G)_{jk(i)}$. Therefore, since $A(G)_{ji} > A(G)_{jk(i)}$, vertices i and j are more similar between them than vertices j and $k(i)$. Hence, in any valid drawing D, it must occur $D(j) - D(i) < D(k(i)) - D(j)$. We transform this strict inequality into the following restriction for a sufficiently small $\epsilon > 0$:

$$-x_i + 2x_j - x_{k(i)} \leq -\epsilon, \quad \forall j \in \{2, 3, \ldots, n-1\} \text{ and } \forall i < j. \tag{5.3}$$

It is worth mentioning that some of the inequalities described in equations (5.2) and (5.3) may be obtained from inequalities presented in Equation (5.1) and different inequalities described in equations (5.2) and (5.3). Hence, some restrictions may be redundant. In an attempt to keep the presentation of this document clean and simple, we omit a discussion in this regard. It is worth mentioning though that it does not impact the results of this document.

Given a weighted graph G with n vertices, the *matrix of restrictions* of G (or the *matrix of coefficients* of G), denoted by $M(G)$, is the matrix that includes the $n-1$ ordering restrictions, the at most $(n-2)(n-1)/2$ right with respect to left restrictions, and the at most $(n-2)(n-1)/2$ left with respect to right restrictions. In total, the matrix $M(G)$ has $h \leq (n-1)^2$ rows and n columns. On the other hand, the vector $\mathbf{b}$ is a $h \times 1$ vector with a $-\epsilon$ in every entry. An example of a weighted graph, its Robinson similarity matrix, and its corresponding matrix of restrictions is given in Figure 1.

Now, we show that for any weighted graph G with Robinson similarity matrix, the convex polyhedron defined by $M(G)\mathbf{x} \leq \mathbf{b}$ is not empty if and only if G has a valid drawing in $\mathbb{R}$.

Theorem 5.2. *Let G be a weighted graph with Robinson similarity matrix $A(G)$. Let $M(G)$ be the $h \times n$ matrix of restrictions of G obtained from $A(G)$. Let $\mathbf{b}$ be the $h \times 1$ vector with $-\epsilon < 0$ in every entry. Then, G has a valid distance drawing in $\mathbb{R}$ if and only if the polyhedron defined by $M(G)\mathbf{x} \leq \mathbf{b}$ is not empty.*

Proof. Let G be a weighted graph with valid distance drawing in $\mathbb{R}$. Let D be a valid drawing of G in $\mathbb{R}$. Label the vertices of G according to the order determined by D, i.e., the left most vertex in D is vertex 1, the next vertex is vertex 2 and so on until vertex n. By construction of $M(G)\mathbf{x} \leq \mathbf{b}$, for any $\epsilon > 0$, D can be scaled to a valid drawing D' such that the vector $(D'(1), D'(2), \ldots, D'(n))$ belongs to the polyhedron $M(G)\mathbf{x} \leq \mathbf{b}$.

On the other hand, assume that the polyhedron $M(G)\mathbf{x} \leq \mathbf{b}$ is not empty. Let $x = (x_1, x_2 \ldots, x_n)$ be a point in $M(G)\mathbf{x} \leq \mathbf{b}$. Label the vertices of G according to the columns of its Robinson similarity matrix $A(G)$, i.e., vertex i is the vertex corresponding to the *i-th* column of $A(G)$. Now, consider the drawing D of G in $\mathbb{R}$ defined as follows: $D(i) = x_i$ for all $1 \leq i \leq n$.

We show now that D is valid distance. Assume that D is not a valid distance drawing. Therefore, there exist three vertices i, j and k such that $A_{ij} < A_{ik}$, but $|D(i) - D(j)| \leq |D(i) - D(k)|$. Note that the last inequality is not valid if $D(i) < D(k) < D(j)$ or if $D(j) < D(k) < D(i)$, therefore, these cases are discarded. If $D(i) < D(j) < D(k)$ or $D(k) < D(j) < D(i)$, there is a contradiction since $A_{ij} < A_{ik}$, and, in that case, $A(G)$ would not be Robinson.

Assume that $D(j) < D(i) < D(k)$. Therefore, $|D(i) - D(j)| \leq |D(i) - D(k)|$ becomes $D(i) - D(j) \leq D(k) - D(i)$, or equivalently, $0 \leq D(j) - 2D(i) + D(k)$. Nevertheless, since $A_{ij} < A_{ik}$, the right with respect to left restriction $x_j - 2x_i + x_k \leq -\epsilon$ is included in $M(G)\mathbf{x} \leq \mathbf{b}$. Therefore, since D comes from a point in $M(G)\mathbf{x} \leq \mathbf{b}$, $D(j) - 2D(i) + D(k) \leq -\epsilon$, which is a contradiction since $\epsilon > 0$.

If we assume now $D(k) < D(i) < D(j)$, then $|D(i) - D(j)| \leq |D(i) - D(k)|$ becomes $0 \leq -D(k) + 2D(i) - D(j)$. Nevertheless, since $A_{ij} < A_{ik}$, the left with respect to right restriction $-x_k + 2x_i - x_j \leq -\epsilon$ is included in $M(G)\mathbf{x} \leq \mathbf{b}$. By equivalent arguments than before, we achieve a contradiction.

Therefore, the condition $|D(i) - D(j)| \leq |D(i) - D(k)|$ is not possible, and hence, D is a valid distance drawing. □

If the valid drawings are restricted to be nonnegative, then the SCFE problem can be treated as a linear program. Because, if the polyhedron defined by the inequality system $M(G)\mathbf{x} \leq \mathbf{b}$ is not empty, there is always a point $\mathbf{x}$ in that polyhedron with $x_1 = 0$. Therefore, the SCFE problem is equivalent to find $\min x_1$ subject to $M(G)\mathbf{x} \leq \mathbf{b}$, and nonnegative $\mathbf{x}$.

The last Theorem is stated for a weighted graph with a Robinson similarity matrix. It is well known that a Robinsonian matrix may have many different Robinson orderings. Therefore, the reader may wonder which of these many Robinson orderings is the one that we can use to apply Theorem 5.2. In the next part of this section, we answer that question for complete weighted graphs. Indeed, we show that any Robinson ordering will provide the same answer when Theorem 5.2 is applied.

Now we show that the existence of a valid distance drawing in $\mathbb{R}$ corresponding to a Robinson ordering is consistent among all Robinson orderings of the vertex set of a complete weighted graph. In other words, we show that, given a complete weighted graph $G = (V, E)$, if there is a Robinson ordering π of V and a valid distance drawing D of G such that the ordering induced by D is equal to π, then, for any Robinson ordering σ of V there exists a valid distance drawing Σ of G such that the ordering induced by Σ is equal to σ.

Given a Robinson ordering π of V, we say that π *has a valid drawing* if there is a valid distance drawing D of $G = (V, E)$ in $\mathbb{R}$ such that the ordering induced by D is equal to π.

To prove our result, we use the fact that all Robinson orderings for a complete similarity matrix can be represented by a PQ-tree (see [20, 3]). A PQ-tree on a set V is a tree that represents a set of permutations of V. The nodes of a PQ-tree are of three types: leaves, that represent the elements of V, P nodes, and Q nodes. The children of a P node are not ordered, and any permutation of them is allowed. The children of a Q node are ordered and that order can only be reversed. Hence, given a PQ-tree $\mathcal{T}$, we obtain a permutation of V represented by $\mathcal{T}$ by applying one of the operations allowed to each P and Q nodes, and then looking at the leaves to find the resultant permutation. Let say π is a permutation obtained in this way. If we modify the operation applied to one node α of $\mathcal{T}$ and maintain all the other nodes equal, we say that the new permutation σ *is obtained from* π *by modifying* α.

For a node α of a PQ-tree $\mathcal{T}$, we denote $\mathcal{T}(\alpha)$ the subtree of $\mathcal{T}$ with root α and by S_α the set of leaves of $\mathcal{T}(\alpha)$. A node of a PQ-tree is said to be *basic* if all its children are leaves. Préa et al. in [20] show that for every node α, and for every Robinson ordering π represented in a PQ-tree $\mathcal{T}$, S_α is *consecutive* according to π, i.e., $S_\alpha = \{\pi_{l+1}, \pi_{l+2}, \ldots, \pi_{l+r}\}$ for some $0 \leq l$ and $0 \leq r \leq n - l$, where π_i denotes the *i-th* vertex according to π. In addition, we denote by $S^I_\alpha := \pi_{l+1}$ and $S^R_\alpha := \pi_{l+r}$ the first and the last elements of S_α, respectively. On the other hand, Préa et al. also show that

$$w(\{u, x\}) = w(\{v, x\}) \quad \forall u, v \in S_\alpha \text{ and } x \in V \setminus S_\alpha. \tag{5.4}$$

We show now that operations on a Q-node of the PQ-tree maintain the characteristic of having a valid distance drawing.

Lemma 5.3. *Let $G = (V, E)$ be a complete weighted graph, $\mathcal{T}$ be the PQ-tree that represents all Robinson orderings of V, and α be a Q-node of $\mathcal{T}$. Let π and σ be two Robinson orderings of V such that σ is obtained from π by applying the operation associated to α (i.e., reversing the children of α). Then, π has a valid distance drawing if and only if σ has a valid distance drawing.*

Proof. Let G, $\mathcal{T}$, α, π, and σ be as in the statement of the lemma. Assume that π has a valid drawing D. We show that σ also has a valid drawing. It is worth noticing that this sense of the equivalence is enough to show the lemma, since the opposite sense is shown by exchanging π and σ and repeating the analysis.

Let $\pi_1, \pi_2, \ldots, \pi_n$ be the vertex set of G ordered according to π. The children of α can be P nodes, Q nodes or leaves. Without loss of generality, we assume that all of them are P nodes or Q nodes, since a leaf can be seen as a Q node with a single children that is the leaf. Let $\beta_1, \beta_2, \dots, \beta_h$ be α's children ordered according to π. We use the following notation for the elements of S_{β_t}:

$$\{\pi_{l+R_{t-1}+1}, \pi_{l+R_{t-1}+2}, \dots, \pi_{l+R_{t-1}+r_t} = \pi_{l+R_t}\},$$

where $R_0 = 0$ and $R_t = R_{t-1} + r_t$, and $|S_{\beta_t}| = r_t$.

Let $\tilde\beta_1, \tilde\beta_2, \dots, \tilde\beta_h$ be α's children ordered according to σ. We use the following notation for the elements of $S_{\tilde\beta_t}$:

$$\{\sigma_{l+\tilde R_{t-1}+1}, \sigma_{l+\tilde R_{t-1}+2}, \dots, \sigma_{l+\tilde R_{t-1}+\tilde r_t} = \sigma_{l+\tilde R_t}\},$$

where $\tilde R_0 = 0$ and $\tilde R_t = \tilde R_{t-1} + \tilde r_t$ where $|S_{\tilde\beta_t}| = \tilde r_t = r_{h-t+1}$.

Note that $\tilde R_t = R_h - R_{h-t}$, $R_t = \tilde R_h - \tilde R_{h-t}$ and $\tilde R_h = R_h$. In addition, $S_{\tilde\beta_t} = S_{\beta_{h-t+1}}$ and $S_{\beta_t} = S_{\tilde\beta_{h-t+1}}$. In particular

$$S^I_{\tilde\beta_t} = \pi_{l+R_{t-1}+1} = S^I_{\beta_{h-t+1}} = \sigma_{l+\tilde R_{h-t}+1}$$

and

$$S^R_{\tilde\beta_t} = \pi_{l+R_t} = S^R_{\beta_{h-t+1}} = \sigma_{l+\tilde R_{h-t+1}}.$$

Hence,

$$\begin{aligned} x \in S_{\tilde\beta_t} &\iff x \in \{\sigma_{l+\tilde R_{t-1}+1}, \dots, \sigma_{l+\tilde R_t}\} \\ &\iff x \in S_{\beta_{h-t+1}} \\ &\iff x \in \{\pi_{l+R_{h-t}+1}, \dots, \pi_{l+R_{t-h+1}}\}. \end{aligned}$$

Assume that $\sigma_i = x$ is the j-element of $\tilde\beta_t$, then

$$\sigma_i = \sigma_{l+\tilde R_{t-1}+j} \Longrightarrow i = l + \tilde R_{t-1} + j \Longrightarrow j = i - l - \tilde R_{t-1}.$$

On the other hand, x is the j-element of β_{h-t+1}, i.e., $x = \pi_{l+R_{h-t}+j}$. Hence:

$$\sigma_i = \pi_{l+R_{h-t}+j} = \pi_{l+R_{h-t}+i-l-\tilde R_{t-1}} = \pi_{i+R_{h-t}-R_h+R_{h-t+1}}.$$

Therefore, σ is as follows:

$$\sigma_i = \begin{cases} \pi_i & \text{if } i \in [1, l], \\ \pi_{i+R_{h-t}-R_h+R_{h-t+1}} & \text{if } i \in [l + \tilde R_{t-1} + 1, l + \tilde R_t],\ t \in \{1, \dots, h\}, \\ \pi_i & \text{if } i \in [l + \tilde R_h + 1, n]. \end{cases}$$

We denote by $f : \{1, \dots, n\} \to \{1, \dots, n\}$ the implicit bijection defined above in $\sigma_i = \pi_{f(i)}$. Next, we recursively define the drawing Σ such that $\Sigma(\sigma_{l+1}) = D(\pi_{l+1})$ and $\forall i \in \{l+2, \dots, n\}$:

$$d(\Sigma(\sigma_i), \Sigma(\sigma_{i-1})) = \begin{cases} d(D(\pi_{f(i)}), D(\pi_{f(i)-1})) & \text{if } \sigma_i \in S_{\tilde\beta_t} \setminus \{S^I_{\tilde\beta_t}\}, \\ & \text{and } t \in \{1, \dots, h\} \\ d(D(S^R_{\beta_{h-t+2}}), D(S^I_{\beta_{h-t+1}})) & \text{if } \sigma_i = S^I_{\tilde\beta_t}, \\ & \text{and } t \in \{2, \dots, h\}. \end{cases}$$

Thus, we define the following drawing Σ that induces σ:

$$\Sigma(\sigma_i) =$$
$$\begin{cases} D(\pi_i) & \text{if } i \in [1, l], \\ \Sigma(\sigma_{i-1}) + D(\pi_i) - D(\pi_{i-1}) & \text{if } i = l+1, \\ \Sigma(\sigma_{i-1}) + D(\pi_{f(i)}) - D(\pi_{f(i)-1}) & \text{if } i \in [l + \tilde{R}_{t-1} + 2, l + \tilde{R}_t], \\ & \text{and } t \in [1, h], \\ \Sigma(\sigma_{i-1}) + D(\pi_{f(l+\tilde{R}_{t-2}+1)}) - D(\pi_{f(l+\tilde{R}_t)}) & \text{if } i = l + \tilde{R}_{t-1} + 1, \\ & \text{and } t \in [2, h], \\ D(\pi_i) & \text{if } i \in [l + \tilde{R}_h + 1, n]. \end{cases}$$

Observe that:

(1) $\Sigma(\sigma_{l+1}) = \Sigma(\sigma_l) + D(\pi_{l+1}) - D(\pi_l) = D(\pi_l) + D(\pi_{l+1}) - D(\pi_l) = D(\pi_{l+1})$,
(2) $i = l + \tilde{R}_{t-1} + 1 \Longleftrightarrow \sigma_i = S^I_{\tilde{\beta}_t}$, $\pi_{f(l+\tilde{R}_{t-2}+1)} = S^I_{\tilde{\beta}_{t-1}}$ and $\pi_{f(l+\tilde{R}_t)} = S^R_{\tilde{\beta}_t}$.

We show now that Σ is a valid distance drawing for G that induces σ as an ordering. By contradiction, assume that there exists a triplet σ_i, σ_j, σ_k that breaks Definition 2.1, for some values $1 \le i < j < k \le n$. If the three elements of the triplet do not belong to S_α, their positions according to Σ do not change. Therefore, the distances between them do not change. Hence, they cannot violate Definition 2.1. On the other hand, if the three elements of the triplet belong to S_{β_t} for some $1 \le t \le h$, as we have seen in the previous paragraph, they maintain their distances. Therefore, they cannot violate Definition 2.1.

Assume that only one element of the triplet belongs to S_α. For instance, $\sigma_i \in S_\alpha$, while $\sigma_j \notin S_\alpha$ and $\sigma_k \notin S_\alpha$. On the other hand, assume that $d(\Sigma(\sigma_i), \Sigma(\sigma_j)) \ge d(\Sigma(\sigma_j), \Sigma(\sigma_k))$, while $w(\sigma_i, \sigma_j) > w(\sigma_j, \sigma_k)$. Since σ_i and σ_{l+1} belong to S_α and $\sigma_j \notin S_\alpha$, Equation (5.4) implies $w(\sigma_{l+1}, \sigma_j) = w(\sigma_i, \sigma_j)$. Since, $d(\Sigma(\sigma_{l+1}), \Sigma(\sigma_j)) \ge d(\Sigma(\sigma_i), \Sigma(\sigma_j))$, the triplet $\sigma_{l+1}, \sigma_j, \sigma_k$ also breaks Definition 2.1.

Now,

$$\begin{aligned} d(\Sigma(\sigma_{l+1}), \Sigma(\sigma_j)) = \Sigma(\sigma_j) - \Sigma(\sigma_{l+1}) \qquad &= D(\pi_j) - D(\pi_{l+1}) \\ &= d(D(\pi_{l+1}), D(\pi_j)), \end{aligned}$$

and

$$d(\Sigma(\sigma_j), \Sigma(\sigma_k)) = \Sigma(\sigma_k) - \Sigma(\sigma_j) = D(\pi_k) - D(\pi_j) = d(D(\pi_j), D(\pi_k)).$$

Therefore, $d(D(\pi_{l+1}), D(\pi_j)) \ge d(D(\pi_j), D(\pi_k))$. On the other hand,

$$w(\sigma_{l+1}, \sigma_j) = w(\pi_{l+R_{h-1}+1}, \pi_j),$$

since $\sigma_l = \pi_{l+R_{h-1}+1}$ and $\sigma_j = \pi_j$. But, Equation (5.4) implies

$$w(\pi_{l+R_{h-1}+1}, \pi_j) = w(\pi_{l+1}, \pi_j),$$

since $\pi_{l+R_{h-1}+1}$ and π_l belong to S_α, and $\pi_j \notin S_\alpha$. In conclusion,

$$w(\pi_{l+1}, \pi_j) = w(\pi_{l+R_{h-1}+1}, \pi_j) = w(\sigma_l, \sigma_j) > w(\sigma_j, \sigma_k) = w(\pi_j, \pi_k).$$

Hence, the triplet π_{l+1}, π_i, π_k breaks Definition 2.1 in D, which is a contradiction since D is valid distance. The analysis of all cases when one element of the triplet belongs to S_α are equivalent to this analysis by showing that either $\sigma_{l+1}, \sigma_j, \sigma_k$ or $\sigma_{l+R_h}, \sigma_j, \sigma_k$ also breaks Definition 2.1. We omit them to simplify the presentation.

Assume now that two elements of the triplet belong to S_α. Say σ_i and σ_j belong to S_α, while σ_k does not belong to S_α. Assume as well that

$$d(\Sigma(\sigma_i), \Sigma(\sigma_j)) \geq d(\Sigma(\sigma_j), \Sigma(\sigma_k)),$$

while $w(\sigma_i, \sigma_j) > w(\sigma_j, \sigma_k)$.

Since σ is Robinson, we have that:

$$w(\sigma_{l+1}, \sigma_{l+R_h}) \geq w(\sigma_{l+1}, \sigma_j) \geq w(\sigma_i, \sigma_j),$$

and

$$w(\sigma_j, \sigma_k) \geq w(\sigma_{l+R_h}, \sigma_k).$$

Therefore, $w(\sigma_{l+1}, \sigma_{l+R_h}) > w(\sigma_{l+R_h}, \sigma_k)$. On the other hand,

$$d(\Sigma(\sigma_{l+1}), \Sigma(\sigma_{l+R_h})) \geq d(\Sigma(\sigma_{l+1}), \Sigma(\sigma_j)) \geq d(\Sigma(\sigma_i), \Sigma(\sigma_j))$$

and

$$d(\Sigma(\sigma_j), \Sigma(\sigma_k)) \geq d(\Sigma(\sigma_{l+R_h}), \Sigma(\sigma_k)).$$

Hence,

$$d(\Sigma(\sigma_{l+1}), \Sigma(\sigma_{l+R_h})) \geq d(\Sigma(\sigma_{l+R_h}), \Sigma(\sigma_k))$$

and the triplet $\sigma_{l+1}, \sigma_{l+R_h}, \sigma_k$ also breaks Definition 2.1. Now,

$$\begin{aligned} d(\Sigma(\sigma_{l+1}), \Sigma(\sigma_{l+R_h})) &= \Sigma(\sigma_{l+R_h}) - \Sigma(\sigma_{l+1}) \\ &= D(\pi_{l+R_h}) - D(\pi_{l+1}) \\ &= d(D(\pi_{l+1}), D(\pi_{l+R_h})), \end{aligned}$$

and

$$\begin{aligned} d(\Sigma(\sigma_{l+R_h}), \Sigma(\sigma_k)) &= \Sigma(\sigma_k) - \Sigma(\sigma_{l+R_h}) \\ &= D(\pi_k) - D(\pi_{l+R_h}) \\ &= d(D(\pi_{l+R_h}), D(\pi_k)). \end{aligned}$$

Therefore,

$$d(D(\pi_{l+1}), D(\pi_{l+R_h})) \geq d(D(\pi_{l+R_h}), D(\pi_k).$$

On the other hand, applying repeatedly Equation (5.4) we obtain:

$$\begin{aligned} w(\pi_{l+R_h}, \pi_{l+1}) &= w(\sigma_{l+1}, \sigma_{l+R_h}) \\ &> w(\sigma_{l+R_h}, \sigma_k) \\ &= w(\pi_{l+1}, \pi_k) \\ &= w(\pi_{l+R_h}, \pi_k). \end{aligned}$$

Therefore, the triplet $\pi_{l+1}, \pi_{l+R_h}, \pi_k$ breaks Definition 2.1. Which is a contradiction because D is valid distance. The analysis of all cases when two elements of the triplet belong to S_α are equivalent to this analysis. We omit them to simplify the presentation.

Finally, assume that the three elements of the triplet belong to S_α. These three vertices cannot all be in $S_{\tilde{\beta}_t}$ for any t since, in this case, the distances between them according to Σ remain the same as in D. Therefore, they could not break Definition (2.1). Assume that σ_i and σ_j belong to $S_{\tilde{\beta}_t}$ and σ_k belongs to $S_{\tilde{\beta}_p}$ with $1 \leq p < t \leq h$. Assume as well that $d(\Sigma(\sigma_i), \Sigma(\sigma_j)) \geq d(\Sigma(\sigma_j), \Sigma(\sigma_k))$, while $w(\sigma_i, \sigma_j) > w(\sigma_j, \sigma_k)$.

It is worth noticing, that σ inverts the ordering of the children of α. Indeed, using the relationship $\tilde{\beta}_t = \beta_{h-t+1}$, we have:

$$S_{\tilde{\beta}_t} = \{\sigma_{l+\tilde{R}_{t-1}+1}, \dots, \sigma_{l+\tilde{R}_t}\} = S_{\beta_{h-t+1}} = \{\pi_{l+R_{h-t}+1}, \dots, \pi_{l+R_{h-t+1}}\}$$

and

$$S_{\tilde{\beta}_p} = \{\sigma_{l+\tilde{R}_{p-1}+1}, \dots, \sigma_{l+\tilde{R}_p}\} = S_{\beta_{h-p+1}} = \{\pi_{l+R_{h-p}+1}, \dots, \pi_{l+R_{h-p+1}}\}$$

By Equation (5.4)

$$w(\sigma_i, \sigma_j) > w(\sigma_j, \sigma_k) = w(\sigma_k, \sigma_{l+\tilde{R}_{t-1}+1}) = w(\sigma_{l+\tilde{R}_{t-1}+1}, \sigma_{l+\tilde{R}_p}).$$

Since D is a valid drawing,

$$d(D(\pi_i), D(\pi_j)) < d(D(\pi_{l+R_{h-t}+1}), D(\pi_{l+R_{h-p+1}})).$$

Thus,

$$\begin{aligned} d(\Sigma(\sigma_i), \Sigma(\sigma_j)) &= d(D(\pi_i), D(\pi_j)) < d(D(\pi_{l+R_{h-t}+1}), D(\pi_{l+R_{h-p+1}})) \\ &= d(\Sigma(\sigma_{l+\tilde{R}_t}), \Sigma(\sigma_{l+\tilde{R}_{p-1}+1})) \leq d(\Sigma(\sigma_j), \Sigma(\sigma_k)), \end{aligned}$$

which is a contradiction. The analysis of all cases when the three elements of the triplet belong to S_α are equivalent to this analysis. We omit them for simplicity of the presentation. □

Now, we show that any operation on a P-node of the PQ-tree maintains the characteristic of having a valid distance drawing.

Lemma 5.4. *Let $G = (V, E)$ be a complete weighted graph, $\mathcal{T}$ be the PQ-tree that represents all Robinson orderings of V, and α be a P-node of $\mathcal{T}$. Let π and σ be two Robinson orderings of V such that σ is obtained from π via a permutation of the children of α. Then, π has a valid drawing if and only if σ has a valid drawing.*

Proof. We use Lemma 5.3 in this proof, since this lemma is a particular case of the previous lemma.

Let G, $\mathcal{T}$, α, π, and σ be as in the statement of the lemma. Assume that π has a valid drawing D. We will show that σ also has a valid drawing. It is worth noticing that, as in the previous lemma, this sense of the equivalence is enough to show the lemma.

The children of α can be P nodes, Q nodes or leaves. Without loss of generality, we assume that all of them are P nodes or Q nodes, since a leaf can be seen as a Q node with a single children that is the leaf. Let $\beta_1, \beta_2, \dots, \beta_h$ be α's children ordered according to π. It is enough for us to consider that σ is obtained via a transposition of two consecutive children of α, since any permutation of S_α can be expressed as a product of transpositions of consecutive elements in S_α. Let β_t and β_{t+1} be the two children to be transposed that produce σ. Now, let $\mathcal{T}'$ be a PQ-tree obtained from $\mathcal{T}$ by replacing β_t and β_{t+1} with a Q node that has as children β_t and β_{t+1}. Now, it is worth noticing that $\mathcal{T}'$ encodes π and σ. Furthermore, Lemma 5.3 implies that if π has a valid drawing, then σ also has a valid drawing, and the proof is completed. □

With these two lemmas together we can state the result.

Theorem 5.5. *Let $G = (V, E)$ be a complete weighted graph and π be any Robinson ordering of V. If π has a valid drawing, all Robinson orderings of V have a valid drawing.*

Since complete Robinsonian matrices can be recognized in time $O(n^2)$, it is possible to construct the matrix $M(G)$ in polynomial time when G is complete. Therefore, we can state the following corollary.

Corollary 5.6. Let G be a complete weighted graph. Deciding whether G has a valid drawing in $\mathbb{R}$ can be done in polynomial time. Moreover, a valid drawing for G in $\mathbb{R}$ can be computed also in polynomial time if such drawing exists.

6. The Weighted SCFE Problem for Incomplete Weighted Graphs

The construction presented in the previous section can also be applied to incomplete weighted graphs. The only requirement is that the matrix A is presented as a Robinson matrix. Nevertheless, we will see in this section that, if the condition of being complete is not requested for the weighted graph, it is not possible to determine in polynomial time whether its similarity matrix is Robinsonian or not, unless P=NP. Despite this bad result, we present an exponential-time algorithm to recognize incomplete Robinsonian matrices. Hence, once this recognition has been done, we can apply the tools developed in the previous section to solve the weighted SCFE problem. It is worth noticing that, in the case of incomplete weighted graphs, we do not have a result equivalent to Theorem 5.5. Hence, we cannot guarantee that after applying the methodology developed in the previous section to incomplete weighted graphs, we will obtain a definitive answer.

Cygan et al. in [8] proved the NP-Completeness of the particular case of the SCFE problem where the weight in the edges can only take values $+1$ or -1. On the other hand, Kermarrec and Thraves proved in [12] the following theorem rephrased in our own words.

Theorem 6.1. *[Lemmas 3 and 4 in* [12]*] Let $G = (V, E)$ be a weighted graph and $w : E \to \{+1, -1\}$ be its weight function. Then, G has a valid distance drawing in $\mathbb{R}$ if and only if there exists an ordering π of V that verifies the following two conditions:*

(1) *For all $i < j < k$ such that $\{\pi_i, \pi_k\}$ and $\{\pi_j, \pi_k\}$ belong to E,*
$$w(\{\pi_j, \pi_k\}) = -1 \quad \Longrightarrow \quad w(\{\pi_i, \pi_k\}) = -1.$$
(2) *For all $i < j < k$ such that $\{\pi_i, \pi_k\}$ and $\{\pi_i, \pi_j\}$ belong to E,*
$$w(\{\pi_i, \pi_j\}) = -1 \quad \Longrightarrow \quad w(\{\pi_i, \pi_k\}) = -1.$$

In other words, a weighted graph $G = (V, E)$ with weights $+1$ or -1 has a valid distance drawing in the line if and only if there exists an ordering π of V such that $A^\pi(G)$ is Robinson. Hence, using these two results, we can state the following corollary.

Corollary 6.2. Let G be an incomplete weighted graph and $A(G)$ be its similarity matrix. Deciding whether $A(G)$ is Robinsonian or not is a NP-Complete problem.

Besides this negative result, Cygan et al. also proved in [8] the existence of a constant $C > 0$ such that no algorithm solves the SCFE problem in the line in time $O(2^{C(n+m)})$, unless the Exponential Time Hypothesis[1] (ETH) fails. Therefore, we conclude that, under the assumption of the ETH, it is impossible to have

[1]The Exponential Time Hypothesis states that there exists a constant $C > 0$ such that no algorithm solving the 3-CNF-SAT problem in $O(2^{CN})$ exists, where N denotes the number of variables in the input formula.

a subexponential-time algorithm that determines if the similarity matrix of an incomplete weighted graph is Robinsonian.

On the positive side, using similar ideas to those presented in [8], we present an exponential-time algorithm that decides if a given incomplete similarity matrix is Robinsonian or not. We start with the following definition.

Definition 6.3. Let A be an incomplete $n \times n$ similarity matrix. Let $\{V, U\}$ be a bipartition of the set $\{1, 2, \dots, n\}$. We say that an element $u \in U$ is *good* for V if for all $k \in V$ and $p \in U$

$$A_{ku} \neq * \wedge A_{kp} \neq * \implies A_{ku} \geq A_{kp},$$

and

$$A_{pu} \neq * \wedge A_{pk} \neq * \implies A_{pu} \geq A_{pk}.$$

Now, we state the following result.

Lemma 6.4. *Let A be an incomplete $n \times n$ similarity matrix. A is Robinsonian if and only if there exists an ordering $\pi = \pi_1, \pi_2, \dots, \pi_n$ of the set $\{1, 2, \dots, n\}$ such that for every $1 \leq j \leq n-1$, the element π_{j+1} is good for $\{\pi_1, \dots, \pi_j\}$.*

Proof. Let A be Robinsonian and $\pi = \pi_1, \pi_2, \dots, \pi_n$ be a Robinson ordering of its rows and columns. Therefore, A^π is Robinson. We show by contradiction that the ordering π satisfies the conditions of the lemma. Assume for instance that the first condition is broken for some $1 \leq j \leq n-1$. Therefore, there exists $1 \leq k \leq j$ and $j+1 < p \leq n$ such that: $A_{\pi_k \pi_{j+1}} \neq *$, $A_{\pi_k \pi_p} \neq *$, and $A_{\pi_k \pi_{j+1}} < A_{\pi_k \pi_p}$. Since, $A_{\pi_k \pi_{j+1}} = A^\pi_{kj+1}$, $A_{\pi_k \pi_p} = A^\pi_{kp}$, and $j+1 < p$, we obtain a contradiction with the fact that A^π is Robinson. A similar conclusion can be drawn if we assume that the second condition of the lemma is broken for some $1 \leq j \leq n-1$. Therefore, the necessary condition for A to be Robinsonian holds.

Assume now that there exists an ordering $\pi = \pi_1, \pi_2, \dots, \pi_n$ such that for every $1 \leq j \leq n-1$, the element π_{j+1} is good for the set $\{\pi_1, \dots, \pi_j\}$. We show that A^π is Robinson. Consider integers i and l such that $1 \leq i < l \leq n$. Consider now any integer j in $[i, l]$. Since the element π_j is good for the set $\{\pi_1, \pi_2, \dots, \pi_{j-1}\}$, the first condition of Definition 6.3 implies that if $A^\pi_{ij} \neq *$ and $A^\pi_{il} \neq *$, then $A^\pi_{ij} \geq A^\pi_{il}$. Equivalently, since the element π_k is good for the set $\{\pi_1, \pi_2, \dots, \pi_{k-1}\}$, if we consider an integer $k \in [i, l]$, the second condition of Definition 6.3 implies that if $A^\pi_{kl} \neq *$ and $A^\pi_{il} \neq *$, then $A^\pi_{kl} \geq A^\pi_{il}$. Therefore, for any $1 \leq i < l \leq n$ and $j, k \in [i, l]$, we have that $A^\pi_{il} \leq \min\{A^\pi_{ij}, A^\pi_{kl}\}$. Hence, A^π is Robinson. □

Finally, to determine if an incomplete similarity matrix A is Robinsonian we use the algorithm presented in [8] which proceeds as follows: construct a directed graph $H = (S, F)$ where the vertex set S is formed by all subsets of $\{1, 2, \dots, n\}$, and for every $X, Y \subseteq \{1, 2, \dots, n\}$ there exists an arc from X to Y if $Y \setminus X = \{i\}$ and i is a good element for X. The arc $(X, Y) \in F$ is labeled by i. Thus, the existence of an ordering π such that A^π is Robinson is equivalent to the existence of a directed path from a vertex that is a singleton of $\{1, 2, \dots, n\}$ to the vertex $\{1, 2 \dots, n\}$. The existence of such directed path in H can be determined in time $O(n \cdot 2^{2n})$, using repeatedly the single source shortest path algorithm presented in [24] for all the n vertices representing a singleton. Furthermore, the Robinson ordering π is determined by the ordering of the labels along that directed path.

7. Final Remarks

Interestingly, in this work we show that the Seriation and the SCFE problems are not the same. Nevertheless, there are cases in which they are equivalent. For instance, an exhaustive analysis shows that if a weighted graph has at most four vertices then its similarity matrix is Robinsonian if and only if it has a valid drawing in $\mathbb{R}$. Whereas, in the proof of Lemma 4.2, we present a weighted graph with five vertices where seriation is not sufficient.

The Seriation and the SCFE problems are also equivalent if the number of different weights is not too big. Indeed, Theorem 6.1 states that when the weight function can take only two values, seriation and the SCFE problem are equivalent. Nevertheless, in the proof of Lemma 4.2, we exhibit an example of a weighted graph with five different weights where seriation is not enough. This final remark rises an interesting question, when this separation between the Seriation and the SCFE problems occurs?. Is the Seriation problem equivalent to the SCFE problem when the graph has four different weights?.

On the other hand, in Theorem 5.5 we have established that, when the weighted graph is complete, if one Robinson ordering has a valid distance drawing, then all Robinson orderings have one. Such a result is crucial to show that the SCFE problem has a polynomial time algorithm when the input is a complete weighted graph. To prove this Theorem, we use the fact that there is a PQ-tree that encodes all Robinson orderings for a complete weighted graph. We do not have a result like that for incomplete weighted graphs. Hence, it is not clear the existence of a result for incomplete weighted graphs equivalent to Theorem 5.5. We believe that this is a really interesting problem that remains open. Indeed, is it possible to encode all Robinson Orderings for an incomplete weighted graph in a PQ-tree?.

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