Nonnegative partial s-goodness for the equivalence of a 0-1 linear program to weighted linear programming

Meijia Han · **Wenxing Zhu**

Abstract The 0-1 linear programming problem with nonnegative constraint matrix and objective vector e origins from many NP-hard combinatorial optimization problems. In this paper, we consider recovering an optimal solution to the problem from a weighted linear programming. We first formulate the problem equivalently as a sparse optimization problem. Next, we consider the consistency of the optimal solution of the sparse optimization problem and the weighted linear programming problem. In order to achieve this, we establish nonnegative partial s-goodness of the constraint matrix and the weighted vector. Further, we use two quantities to characterize a sufficient condition and necessary condition for the nonnegative partial s-goodness. However, the two quantities are difficult to calculate, therefore, we provide a computable upper bound for one of the two quantities to verify the nonnegative partial s-goodness. Finally, we give three examples to illustrate that our theory is effective and verifiable.

Keywords Integer programming · Nonnegative partial s-goodness · Weighted linear programming · Sparse optimization

1 Introduction

In this paper, we consider the integer programming (IP) with linear inequalities:

$$\min_x \{\sum_{i=1}^{n} x_i : Ax \geq b, \, x \in \{0,1\}^n\}, \tag{1}$$

Meijia Han
Center for Discrete Mathematics and Theoretical Computer Science, Fuzhou University, Fuzhou 350116, China
E-mail: 1099091207@qq.com

Wenxing Zhu
Center for Discrete Mathematics and Theoretical Computer Science, Fuzhou University, Fuzhou 350116, China
E-mail: wxzhu@fzu.edu.cn

where $A \geq 0 \in R^{m \times n}$ is a given matrix, $b \in R^m$ is a given vector. Problem (1) arises from many combinatorial optimization problems, such as the minimum vertex covering [4], the maximum independent set, and the max-cut problems [2], etc. These problems have been proven to be NP-hard, and many exact and heuristics or approximation algorithms have been proposed, including cutting-plane method [9], branch and bound [5] and local branch [13] algorithms, relaxation induced neighbourhood search [6], and objective scaling ensemble approach [14].

It is worth to note that most of the above mentioned algorithms are based on the corresponding linear programming relaxation. Therefore, it is natural to ask under which conditions the integer programming problem (1) is equivalent to its corresponding linear programming relaxation problem:

$$\min_{x}\{\sum_{i=1}^{n} x_i : Ax \geq b, \, 0 \leq x \leq 1\}.$$

At present, there are some theories for ensuring that the above linear programming problem has integral solution.

The first well known theory is totally unimodular (TUM for short), under which it holds that

Theorem 1 *([10]) If A is totally unimodular and b is an integer vector, then the vertices of the polyhedron $P = \{x : Ax \leq b, \, 0 \leq x \leq 1\}$ are integral.*

Another well known theory is the totally dual integrality (TDI for short) proposed by Edmonds and Giles [8], which is a weaker sufficient condition than TUM.

Theorem 2 *([8]) If the linear system $Ax \leq b$ is TDI and b is integer valued, then $P = \{x : Ax \leq b, \, 0 \leq x \leq 1\}$ is an integral polyhedron.*

Note that the linear programming relaxation of problem (1) can be written into the form of linear complementarity problem (LCP):

$$\begin{aligned} q + Mz &\geq 0, \\ z^T(q + Mz) &= 0, \\ z &\geq 0. \end{aligned} \tag{2}$$

where $z \in R^n$ is the vector of variables. Assuming that the solution set of LCP is non-empty, Chandrasekaran et al. [3] gave the class I of integral matrices M if the corresponding LCP has an integer solution for each integral vector q. Utilizing TDI, Dubey and Neogy [7] obtained some new conditions for the existence of an integer solution to LCP with a hidden $Z-$matrix and hidden $K-$matrix. These results are extended from TUM or TDI.

In this paper, we wish to establish a sufficient and necessary condition other than TUM and TDI, such that the integer program (1) is solvable by linear programming relaxation. We consider providing an adjustable weight c, $0 < c \leq 1$, such that the optimal solution of the weighted linear programming relaxation problem

$$\min_{x}\{c^T x : Ax \geq b, 0 \leq x \leq 1\}, \tag{3}$$

is an optimal solution of problem (1).

Let the l_0−norm $\|x\|_0$ be defined as the number of nonzero elements in the vector x. In Section 2 of this paper, we will prove that problem (1) can be written equivalently as the l_0−norm minimization problem

$$\min_x\{\|x\|_0 : Ax \geq b, 0 \leq x \leq 1\}. \tag{4}$$

Therefore, we expect to establish a sufficient condition through the sparse optimization approach, such that problems (4) and (3) have the same unique optimal solution, and thus provides an optimal solution to problem (1).

Eq. (4) is a sparse optimization problem. In this field, the problem

$$\min_x\{\|x\|_0 : A_1x + A_2y = b\} \tag{5}$$

has been studied mostly, for the optimal solution of $\min_x\{\|x\|_1 : A_1x + A_2y = b\}$ to be an optimal solution of the above problem. The proposed well known conditions are the partial restricted isometry property (PRIP) of the parameter δ^r_{s-r} [1], and the partial null space property (PNSP) of order $s - r$ [1]. In [15, 18, 16], the authors provided the k−th order range space property (k-RSP) of the constraint matrix A and a given matrix W, under which the l_0−norm minimization problem has the same unique optimal solution as the problem $\min_x\{\|W_1x + W_2y\|_1 : A_1x + A_2y = b\}$. The condition was further extended to the l_0−norm minimization problem with non-negative constraints [17].

Another condition is the s-goodness of the constraint matrix A [11], under which the unique optimal solution of the l_1-norm minimization problem $\min_x\{\|x + y\|_1 : A_1x + A_2y = b\}$ is exactly an optimal one of problem (5). In [12], the condition was extended for considering the problem

$$\min_x\{\|x\|_1 : A_1x + A_2y = b\}, \tag{6}$$

and the partial s-goodness of the constraint matrix (A_1, A_2) was established, for the exact partial s-sparse optimal solution via the partial l_1−norm minimization.

In this paper, we will propose the nonnegative partial s-goodness condition for problems (3) and (4), such that they have the same unique optimal solution. Specifically, we propose a definition of nonnegative partial s-goodness and its characterization $\gamma_{s,K}(\cdot)$ and $\hat{\gamma}_{s,K}(\cdot)$ with respect to the constraint matrices and the coefficients of the objective function. On this basis, we give a computable upper bound of $\hat{\gamma}_{s,K}(\cdot)$, and thus obtain verifiable sufficient conditions for the nonnegative partial s-goodness. Through which we provide conditions such that the optimal solution of problem (1) can be obtained from problem (3).

This paper is organized as follows. Section 2 gives an example of problem (1), and proves the equivalence between problems (1) and (4). In Section 3, we define nonnegative partial s-goodness and its characterization for a constraint matrix A' and the coefficient c of the objective function in problem (3). Moreover, we derive necessary condition and sufficient condition for (A', c) to be nonnegative partial s-goodness, and discuss efficiently computable upper bound of $\hat{\gamma}_{s,K}(\cdot)$ in Section 4. In Section 5, we give a heuristic algorithm and three examples to show the feasibility of the proposed theory. Finally, Section 6 concludes this paper.

2 Problem reformulation

In this section, we prepare some preliminary work for the consequent research. First, we give an example of the considered problem (1). Then, we show that the considered integer programming problem (1) can be converted equivalently to an l_0-norm minimization problem.

2.1 An example of the considered integer programming problem

We take the maximum independent set problem as an example to show that it is a special form of problem (1). Given an undirected graph $G = (V, E)$, where $V = \{1, \cdots, n\}$ is the set of vertices and E is the set of edges, the maximum independent set problem can be formulated as

$$\max_{x} \sum_{i=1}^{n} x_i$$
$$s.t. \ Ax \leq 1 \tag{7}$$
$$x \in \{0,1\}^n,$$

where A is the adjacency matrix of graph G.

Let $\tilde{x}_i = 1 - x_i$, $i = 1, 2, \cdots, n$. Then problem (7) can be written equivalently as

$$\min_{\tilde{x}} \sum_{i=1}^{n} \tilde{x}_i$$
$$s.t. \ A\tilde{x} \geq 1$$
$$\tilde{x} \in \{0,1\}^n,$$

which is a special form of problem (1).

2.2 Equivalence between integer programming problem and l_0-norm minimization problem

In this subsection, we show the equivalence between optimal solution of the l_0-norm minimization problem (4) and the integer programming problem (1) after a certain operation. First, we have the following result.

Theorem 3 *For any optimal solution x^* of the l_0-norm minimization problem* (4)*, $\hat{x}^* = (\lceil x_1^* \rceil, \lceil x_2^* \rceil, \cdots, \lceil x_n^* \rceil)^T$ is an optimal solution of the integer programming problem* (1) *and the l_0-norm minimization problem* (4) *respectively.*

Proof For any optimal solution x^* of problem (4), let $\hat{x}^* = (\lceil x_1^* \rceil, \lceil x_2^* \rceil, \cdots, \lceil x_n^* \rceil)^T \in \{0,1\}^n$. By noting that $\hat{x}^* \geq x^*$ and $A \geq 0$, we have $A\hat{x}^* \geq Ax^* \geq b$. Hence, $\hat{x}^*$ is a feasible solution of problems (1) and (4) respectively. Further, since $\sum_{i=1}^{n} |x_i^*|_0 = \sum_{i=1}^{n} |\lceil x_i^* \rceil|_0 = \sum_{i=1}^{n} |\hat{x}_i^*|_0$, $\hat{x}^*$ is an optimal solution of problem (4).

Next, for any $x \in \{x \in \{0,1\}^n : Ax \geq b\}$, x is also a feasible solution of problem (4). Since $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} |x_i|_0$, and by $\sum_{i=1}^{n} |x_i^*|_0 = \sum_{i=1}^{n} |\hat{x}_i^*|_0$, we can obtain that $\sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} |\hat{x}_i^*|_0 = \sum_{i=1}^{n} \hat{x}_i^*$. Hence $\hat{x}^*$ is also an optimal solution of problem (1). $\square$

Theorem 3 implies that the following corollary holds.

Corollary 1 *If problem* (4) *has a unique integer optimal solution* $\hat{x}^*$, *then* $\hat{x}^*$ *is also an optimal solution of the integer programming problem* (1).

Conversely, we next prove that an optimal solution of problem (1) is also an optimal solution of problem (4).

Theorem 4 *If x is an optimal solution of problem* (1), *then x is also an optimal solution of problem* (4).

Proof If x is an optimal solution of problem (1), then x is a feasible solution of problem (4), and satisfies that $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} |x_i|_0$. For any optimal solution x^* of problem (4), by Theorem 3, $\hat{x}^* = (\lceil x_1^* \rceil, \lceil x_2^* \rceil, \cdots, \lceil x_n^* \rceil)^T$ is an optimal solution of problems (1) and (4). Hence, $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \hat{x}_i^* = \sum_{i=1}^{n} |\lceil x_i^* \rceil|_0 = \sum_{i=1}^{n} |x_i^*|_0$. So x is also an optimal solution of problem (4). $\square$

By introducing slack variables, letting $A' := [A_1, A_2] \in R^{(m+n) \times (m+2n)}$, where $A_1 = \binom{A}{I} \in R^{(m+n) \times n}$, $A_2 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \in R^{(m+n) \times (m+n)}$, $b' = \binom{b}{1} \in R^{m+n}$, then problem (4) can be rewritten as the form

$$\min_{x,y} \{\|x\|_0 : A_1 x + A_2 y = b', x \geq 0, y \geq 0\}. \tag{8}$$

Correspondingly, the weighted linear programming problem (3) can be written in the form of the partially weighted linear programming problem

$$\min_{x,y} \{c^T x : A_1 x + A_2 y = b', x \geq 0, y \geq 0\}. \tag{9}$$

Then, to study the equivalence between the integer programming problem (1) and the weighted linear programming problem (3), we turn to derive conditions under which problems (8) and (9) have the same optimal solutions. In the sequel, we adapt the concept of s-goodness [11] to problem (8), such that problems (8) and (9) have the same unique optimal solution. Through this optimal solution, we can obtain the optimal solution of problem (1).

3 Nonnegative partial s-goodness

Since problem (8) is NP-hard, we are interested in establishing some conditions, under which both of problems (8) and (9) have the same unique optimal solution. Firstly, we give the following nonnegative partial s-goodness definition of matrix A' and weighted vector c, where $A' = (A_1, A_2)$.

3.1 Definition of nonnegative partial s-goodness

Definition 1 Let A' be a $(m+n) \times (m+2n)$ matrix and s be an integer, $0 \le s \le n$. $0 < c_i \le 1$, $i = 1, 2, ..., n$. We say that (A', c) is nonnegative partially s-good with respect to the columns of A_1, if for any pair of vectors $w^1 \ge 0 \in R^n$, $w^2 \ge 0 \in R^{m+n}$ with that $w^1 \in R^n$ has at most s nonzero elements, $(w^1, w^2)^T$ is the unique optimal solution to the optimization problem

$$\min_{x,y} \left\{ c^T x : A_1 x + A_2 y = A_1 w^1 + A_2 w^2, x \ge 0, y \ge 0 \right\}. \tag{10}$$

For the convenient of description, we say (A', c) is nonnegative partially s-good to refer that (A', c) is nonnegative partially s-good with respect to the columns of A_1. Moreover, without loss of generality, for $s \in \{0, 1, 2, \cdots, n\}$, we say $w^1 \in R^n$ with $\|w^1\|_0 \le s$ to mean that the nonzero entries of w^1 is no more than s. Meanwhile, in this paper, a vector is said to be s-sparse when this vector contains at most s nonzero components.

It is obvious that, if the partially weighted linear problem (9) has multiple optimal solutions, then (A', c) is not nonnegative partially s-good. However, we want to recover the optimal solution of problem (8) from problem (9), in which c is not fixed. So we adjust the coefficient c in problem (9), such that one of the optimal solutions is the unique optimal solution of problem (9).

By Definition 1, taking a step closer to our goal, we obtain the following results, which characterize the consistency of solutions to the problems (8) and (9).

Theorem 5 *For any optimal solution $(w^1, w^2)^T$ of problem* (8)*, where w^1 is an s-sparse vector, if (A', c) is nonnegative partially s-good, then $(w^1, w^2)^T$ is the unique optimal solution to the partially weighted linear programming problem* (9)*.*

Proof For any optimal solution $(w^1, w^2)^T$ of problem (8), where w^1 is an s-sparse vector, it holds that $A_1 x + A_2 y = b' = A_1 w^1 + A_2 w^2$ and $w^1 \ge 0$, $w^2 \ge 0$. That is, $(w^1, w^2)^T$ is a feasible solution of problem (9). If (A', c) is nonnegative partially s-good, then according to Definition 1, $(w^1, w^2)^T$ with $\|w^1\|_0 \le s$ is a unique optimal solution of problem (9). $\square$

Under the nonnegative partial s-goodness condition, Theorem 5 shows that an optimal solution of problem (8) is a unique optimal solution of problem (9). In the above theorem, there is no requirement for the uniqueness of the solution of the l_0-norm minimization problem (8), while only the uniqueness of the partially weighted linear problem (9) is required.

Next, we give a stronger result that both problems (8) and (9) have a unique optimal solution.

Theorem 6 *Given an integer $0 \le s \le n$, and let $(w^1, w^2)^T$ with $\|w^1\|_0 \le s$ be a feasible solution to problem* (9)*. Suppose (A', c) is nonnegative partially s-good, then $(w^1, w^2)^T$ is both the unique optimal solution to the partially weighted linear problem* (9) *and the l_0-norm minimization problem* (8)*.*

Proof Suppose (A', c) is nonnegative partially s-good, and $(w^1, w^2)^T$ with $||w^1||_0 \leq s$ is a feasible solution to problem (9). Then by Definition 1, $(w^1, w^2)^T$ is the unique optimal solution to problem (9).

Next, we prove that $(w^1, w^2)^T$ is also the unique optimal solution to the l_0−norm minimization problem (8). Suppose $(x^1, y^1)^T$ is another solution to problem (8), and $||x^1||_0 \leq ||w^1||_0 \leq s$. Then $A_1 x^1 + A_2 y^1 = b' = A_1 w^1 + A_2 w^2$ and $x^1 \geq 0$, $y^1 \geq 0$. By Definition 1, $(x^1, y^1)^T$ is also a unique optimal solution to problem (9), and then we can get $(x^1, y^1)^T = (w^1, w^2)^T$. Hence $(w^1, w^2)^T$ is the unique optimal solution of the l_0−norm minimization problem (8). $\qquad\square$

According to Section 2.2 and Theorems 5 and 6, we can immediately get that an optimal solution of the integer programming problem (1) can be recovered from problem (9), as in the following corollary.

Corollary 2 *Suppose* (A', c) *is nonnegative partially s-good, let* $(w^1, w^2)^T$ *with* $||w^1||_0$ $\leq s$ *be an optimal solution to the partially weighted linear programming problem* (9). *Then* $\lceil w^1 \rceil$ *is an optimal solution of integer programming problem* (1).

It seems not easy to completely characterize the nonnegative partial s-goodness of the constraint matrix A' and the coefficient c of the objective function (3). In the next subsection, we utilize two quantities to characterize nonnegative partial s-goodness.

3.2 Two quantities of nonnegative partial s-goodness

In this section, we introduce two quantities: $\gamma_{s,K}(A', c, \beta)$ and $\hat{\gamma}_{s,K}(A', c, \beta)$, where $K := \{1, 2, \cdots, n\}$ is the index set of x, i.e., the index set of the columns of matrix A_1. In particular, for a vector $\theta \in R^{m+n}$, let $||\cdot||_*$ be the dual norm of $||\cdot||$ specified by $||\theta||_* = \max_d \{d^T \theta : ||d|| \leq 1\}$. In this paper, we consider the dual norm of $||\cdot||_1$.

Definition 2 Let $A' \in R^{(m+n)\times(m+2n)}$, s is an integer and $0 \leq s \leq n$, $0 < c_i \leq 1$, $i = 1, 2, \ldots, n$, $\beta \in [0, \infty]$. We define $\gamma_{s,K}(A', c, \beta)$, $\hat{\gamma}_{s,K}(A', c, \beta)$ as follows:

(1) $\gamma_{s,K}(A', c, \beta)$ is the infimum of $\gamma > 0$ such that for every pair of vectors $z^1 \in R^n$, $z^2 \in R^{m+n}$, where $z^1 \in R^n$ has s nonzero entries, each is equal to 1, there exists a vector $\theta \in R^{m+n}$ such that

$$||\theta||_* \leq \beta, \quad (A_1^T \theta)_i \begin{cases} = c_i z_i^1, & if \ z_i^1 = 1; \\ \in [-\gamma, \gamma], & if \ z_i^1 = 0, \end{cases} \quad and \ (A_2^T \theta)_i \begin{cases} = 0, & if \ z_i^2 \neq 0; \\ \leq 0, & if \ z_i^2 = 0. \end{cases} \quad (11)$$

If for some pair of vectors $z^1 \in R^n$, $z^2 \in R^{m+n}$ as above, there does no θ with $||\theta||_* \leq \beta$, such that $A_1^T \theta$ coincides with $c \circ z^1$ on the support set of z^1, and $A_2^T \theta$ coincides with 0 on the support set of z^2, then we let $\gamma_{s,K}(A', c, \beta) = +\infty$.

(2) $\hat{\gamma}_{s,K}(A', c, \beta)$ is the infimum of $\gamma > 0$ such that for every pair of vectors $z^1 \in R^n$, $z^2 \in R^{m+n}$, where $z^1 \in R^n$ has s nonzero entries, each is equal to 1, there exists a vector $\hat{\theta} \in R^{m+n}$ such that

$$||\hat{\theta}||_* \leq \beta, \quad ||(A_1^T \hat{\theta}) - c \circ z^1||_\infty \leq \gamma \ and \ (A_2^T \hat{\theta})_i \begin{cases} = 0, & if \ z_i^2 \neq 0; \\ \leq 0, & if \ z_i^2 = 0, \end{cases} \quad (12)$$

here $c \circ z^1$ means entry-wise product of the two vectors.

Furthermore, when $\beta = \infty$, we write $\gamma_{s,K}(A',c)$, $\hat{\gamma}_{s,K}(A',c)$ instead of $\gamma_{s,K}(A',c,\infty)$ and $\hat{\gamma}_{s,K}(A',c,\infty)$, respectively.

Remark 1 Obviously, the set of values of the γ is closed. Thus, if $\gamma_{s,K}(A',c,\beta) < +\infty$, then for every pair of vectors $z^1 \in R^n, z^2 \in R^{m+n}$, where $z^1 \in R^n$ has s nonzero entries, each is equal to 1, there exists a vector $\theta \in R^{m+n}$ such that

$$\|\theta\|_* \le \beta, \ (A_1^T\theta)_i \begin{cases} = c_i z_i^1, & \text{if } z_i^1 = 1; \\ \in [-\gamma_{s,K}(A',c,\beta), \gamma_{s,K}(A',c,\beta)], & \text{if } z_i^1 = 0, \end{cases}$$

$$(A_2^T\theta)_i \begin{cases} = 0, & \text{if } z_i^2 \ne 0; \\ \le 0, & \text{if } z_i^2 = 0. \end{cases} \tag{13}$$

Similarly, for every pair of vectors $z^1 \in R^n$, $z^2 \in R^{m+n}$, where $z^1 \in R^n$ has s nonzero entries, each is equal to 1, there exists a vector $\theta \in R^{m+n}$ such that

$$\|\hat{\theta}\|_* \le \beta, \ \|(A_1^T\hat{\theta}) - c \circ z^1\|_\infty \le \hat{\gamma}_{s,K}(A',c,\beta) \text{ and } (A_2^T\hat{\theta})_i \begin{cases} = 0, & \text{if } z_i^2 \ne 0; \\ \le 0, & \text{if } z_i^2 = 0. \end{cases} \tag{14}$$

Before characterizing nonnegative partial s-goodness of (A',c) more specifically, we need to give some basic properties of $\gamma_{s,K}(A',c,\beta)$ and $\hat{\gamma}_{s,K}(A',c,\beta)$, such as convexity and monotonicity. Since nonnegative partial s-goodness of (A',c) requires $\gamma_{s,K}(A',c) < \infty$, we assume it holds without loss of generality in the sequel.

Lemma 1 $\gamma_{s,K}(A',c,\beta)$ *and* $\hat{\gamma}_{s,K}(A',c,\beta)$ *are convex nonincreasing function of* $\beta \in [0,+\infty]$.

Proof Here, we only need to prove that the $\gamma_{s,K}(A',c,\beta)$ is a convex nonincreasing function with respect to $\beta \in [0,+\infty]$. The property with respect to $\hat{\gamma}_{s,K}(A',c,\beta)$ can be proved similarly.

Firstly, for the given A',c and s, we demonstrate that the $\gamma_{s,K}(A',c,\beta)$ is a nonincreasing function of β. For any $\beta_2 \ge \beta_1$, according to the definition of $\gamma_{s,K}(A',c,\beta)$ and Remark 1, for every pair of vectors $z^1 \in R^n, z^2 \in R^{m+n}$, where $z^1 \in R^n$ has s nonzero entries, each is equal to 1, there exists a vector $\theta \in R^{m+n}$ such that

$$\|\theta\|_* \le \beta_1 \ , \ (A_1^T\theta)_i \begin{cases} = c_i z_i^1, & \text{if } z_i^1 = 1; \\ \in [-\gamma_{s,K}(A',c,\beta_1), \gamma_{s,K}(A',c,\beta_1)], & \text{if } z_i^1 = 0, \end{cases}$$

$$(A_2^T\theta)_i \begin{cases} = 0, & \text{if } z_i^2 \ne 0; \\ \le 0, & \text{if } z_i^2 = 0. \end{cases}$$

Since $\beta_2 \ge \beta_1$, the θ in the above equation also satisfies that

$$\|\theta\|_* \le \beta_2 \ , \ (A_1^T\theta)_i \begin{cases} = c_i z_i^1, & \text{if } z_i^1 = 1; \\ \in [-\gamma_{s,K}(A',c,\beta_1), \gamma_{s,K}(A',c,\beta_1)], & \text{if } z_i^1 = 0, \end{cases}$$

$$(A_2^T\theta)_i \begin{cases} = 0, & \text{if } z_i^2 \ne 0; \\ \le 0, & \text{if } z_i^2 = 0. \end{cases}$$

Hence by the definition of $\gamma_{s,K}(A',c,\beta_2)$, $\gamma_{s,K}(A',c,\beta_1) \geq \gamma_{s,K}(A',c,\beta_2)$.

Next, we prove that $\gamma_{s,K}(A',c,\beta)$ is a convex function of β. That is to say, for any $\beta_1, \beta_2 \in [0,+\infty]$, for any $\alpha \in [0,1]$, we need to prove that

$$\gamma_{s,K}(A',c,\alpha\beta_1 + (1-\alpha)\beta_2) \leq \alpha\gamma_{s,K}(A',c,\beta_1) + (1-\alpha)\gamma_{s,K}(A',c,\beta_2). \qquad (15)$$

Note that, the above inequality (15) obviously holds if one of β_1 and β_2 is $+\infty$. Therefore, we only need to verify that for $\beta_1, \beta_2 \in [0,+\infty)$, the inequality (15) still holds. By the definition of $\gamma_{s,K}(A',c,\beta)$, it is easy to know that for every pair of vectors $z^1 \in R^n, z^2 \in R^{m+n}$, where $z^1 \in R^n$ has s nonzero entries, each is equal to 1, there exists a vector $\theta_\ell \in R^{m+n}$, $\ell \in \{1,2\}$ such that

$$\|\theta_\ell\|_* \leq \beta_\ell \,,\, (A_1^T\theta_\ell)_i \begin{cases} = c_i z_i^1, & if\ z_i^1 = 1; \\ \in [-\gamma_{s,K}(A',c,\beta_\ell), \gamma_{s,K}(A',c,\beta_\ell)], & if\ z_i^1 = 0, \end{cases}$$

$$(A_2^T\theta_\ell)_i \begin{cases} = 0, & if\ z_i^2 \neq 0; \\ \leq 0, & if\ z_i^2 = 0. \end{cases}$$

Clearly, for any $\alpha \in [0,1]$, we can easily get

$$\|\alpha\theta_1 + (1-\alpha)\theta_2\|_* \leq \alpha\beta_1 + (1-\alpha)\beta_2.$$

Moreover,

$$[A_1^T(\alpha\theta_1 + (1-\alpha)\theta_2)]_i \begin{cases} = c_i z_i^1, & if\ z_i^1 = 1; \\ \in [-k\rho, \rho], & if\ z_i^1 = 0, \end{cases}$$

$$[A_2^T(\alpha\theta_1 + (1-\alpha)\theta_2)]_i \begin{cases} = 0, & if\ z_i^2 \neq 0; \\ \leq 0, & if\ z_i^2 = 0, \end{cases}$$

where $\rho = \alpha\gamma_{s,K}(A',c,\beta_1) + (1-\alpha)\gamma_{s,K}(A',c,\beta_2)$. Hence, by the definition of $\gamma_{s,K}(\cdot)$, it holds that

$$\gamma_{s,K}(A',c,\alpha\beta_1 + (1-\alpha)\beta_2) \leq \alpha\gamma_{s,K}(A',c,\beta_1) + (1-\alpha)\gamma_{s,K}(A',c,\beta_2). \qquad \square$$

By Definition 2 and Lemma 1, the set of values of the γ is closed and has a infimum $\gamma_{s,K}(A',c,\beta)$. Namely, for the given A',c and s, if β is large enough, we can set $\gamma_{s,K}(A',c,\beta) = \gamma_{s,K}(A',c)$. In the same way, for the given A',c and s, if β is large enough, we can set $\hat{\gamma}_{s,K}(A',c,\beta) = \hat{\gamma}_{s,K}(A',c)$.

From the definitions of $\gamma_{s,K}(A',c,\beta)$ and $\hat{\gamma}_{s,K}(A',c,\beta)$, it is obvious that s is another important parameter of γ. Next, we give a property of $\gamma_{s,K}(A',c,\beta)$ and $\hat{\gamma}_{s,K}(A',c,\beta)$ with respect to s.

Lemma 2 $\gamma_{s,K}(A',c,\beta)$ and $\hat{\gamma}_{s,K}(A',c,\beta)$ *are monotonically nondecreasing functions of the parameter s.*

Proof Firstly, we prove that $\gamma_{s,K}(A',c,\beta)$ is a monotonically nondecreasing function of the parameter s. Let $\gamma_{s,K}(A',c,\beta) < \infty$. According to the definition of $\gamma_{s,K}(A',c,\beta)$

and Remark 1, for every pair of vectors $z^1 \in R^n$, $z^2 \in R^{m+n}$, where $z^1 \in R^n$ has s nonzero entries, each is equal to 1, there exists a vector $\theta \in R^{m+n}$ such that

$$\|\theta\|_* \leq \beta \ , \ \left(A_1^T \theta\right)_i \begin{cases} = c_i z_i^1, & if \ z_i^1 = 1; \\ \in [-\gamma_{s,K}\left(A',c,\beta\right), \gamma_{s,K}\left(A',c,\beta\right)], & if \ z_i^1 = 0, \end{cases}$$

$$\left(A_2^T \theta\right)_i \begin{cases} = 0, & if \ z_i^2 \neq 0; \\ \leq 0, & if \ z_i^2 = 0. \end{cases} \tag{16}$$

Then, let $t_s(z^1, z^2)$ be the minimal value of the optimization problem

$$\begin{aligned}
\min_{\theta} \ & \|(A_1^T \theta)\|_\infty \\
s.t. \ & \|\theta\|_* \leq \beta \\
& (A_1^T \theta)_i = c_i z_i^1, \ i \in I_1 \\
& (A_2^T \theta)_i = 0, \ i \in I_2 \\
& (A_2^T \theta)_i \leq 0, \ i \in \bar{I}_2,
\end{aligned} \tag{17}$$

where $I_1 = \{i : z_i^1 = 1\}$, $I_2 = \{i : z_i^2 \neq 0\}$ and $\bar{I}_2 = \{i : z_i^2 = 0\}$. Obviously, $t_s(z^1, z^2) \leq \gamma_{s,K}(A', c, \beta)$.

Let $s' = s - 1 < s$. For every pair of vectors $z^{1\prime} \in R^n$, $z^2 \in R^{m+n}$, where $z^{1\prime} \in R^n$ has $s - 1$ nonzero entries, each is equal to 1, we can construct a pair of vectors $z^1 \in R^n$, $z^2 \in R^{m+n}$, where z^1 is obtained from $z^{1\prime}$ by changing one entry with value 0 to 1. According to (17), it is obvious that

$$t_{s'}(z^{1\prime}, z^2) \leq t_s(z^1, z^2) \leq \gamma_{s,K}(A', c, \beta).$$

Hence, $\gamma_{s',K}(A', c, \beta) \leq \gamma_{s,K}(A', c, \beta)$.

Using the way similar to the above proof, we can also show that $\hat{\gamma}_{s',K}(A', c, \beta) \leq \hat{\gamma}_{s,K}(A', c, \beta)$, i.e., $\hat{\gamma}_{s,K}(A', c, \beta)$ is a monotonically nondecreasing function of s. $\quad\square$

Remark 2 According to Lemma 2, $\gamma_{s,K}(A', c, \beta)$ and $\hat{\gamma}_{s,K}(A', c, \beta)$ are nondecreasing functions of s, hence for all $s' \leq s$, Remark 1 holds. That means, in Remark 1 for every pair of vectors $z^1 \in R^n$, $z^2 \in R^{m+n}$ with $\|z^1\|_0 \leq s$, there exists θ with $\|\theta\|_* \leq \beta$ such that Equations (11) and (12) hold.

3.3 Sufficient condition and necessary condition of nonnegative partial s-goodness

In this subsection, via $\gamma_{s,K}(A', c)$ we propose a sufficient condition and a necessary condition for nonnegative partial s-goodness of the constraint matrix A' and the coefficient c of the objective function.

Theorem 7 *Given $A' \in R^{(m+n)\times(m+2n)}$, s is an integer and $0 \leq s \leq n$, $0 < c \leq 1$, we can obtain:*

(a) if (A', c) is nonnegative partially s-good, then $\gamma_{s,K}(A', c) \leq \max\limits_{0 < i \leq n} c_i$;

(b) if $\gamma_{s,K}(A', c) < \min\limits_{0 < i \leq n} c_i$, then (A', c) is nonnegative partially s-good.

Proof (a) Suppose (A', c) is nonnegative partially s-good. For any given $w = (w^1, w^2)^T \geq 0 \in R^n \times R^{m+n}$ with $\|w^1\|_0 \leq s$, let $I_1 = \{i : w_i^1 > 0\}$, $\bar{I}_1 = \{i : w_i^1 = 0\}$, $I_2 = \{i : w_i^2 > 0\}$ and $\bar{I}_2 = \{i : w_i^2 = 0\}$.

By Definition 1, w is the unique optimal solution to problem (10). According to the optimality condition, there exists $\theta \in R^{m+n}$ such that $f_\theta(x, y) = \sum_{i=1}^{n} c_i|x_i| - \theta^T(A_1 x + A_2 y - A_1 w^1 - A_2 w^2)$ attains its minimum value at $(x, y)^T = (w^1, w^2)^T$, i.e., $0 \in \partial f_\theta(w^1, w^2)$. This implies that

$$(A_1^T \theta)_i \begin{cases} = c_i, & i \in I_1; \\ \in [-\max_{0 < i \leq n} c_i, \max_{0 < i \leq n} c_i], & i \in \bar{I}_1, \end{cases} \text{ and } (A_2^T \theta)_i \begin{cases} = 0, & i \in I_2; \\ \leq 0, & i \in \bar{I}_2. \end{cases}$$

Since $w \geq 0$, hence, for the optimization problem

$$\min_{\theta, \gamma} \left\{ \gamma : (A_1^T \theta)_i \begin{cases} = c_i, & i \in I_1; \\ \in [-\gamma, \gamma], & i \in \bar{I}_1, \end{cases} \text{ and } (A_2^T \theta)_i \begin{cases} = 0, & i \in I_2; \\ \leq 0, & i \in \bar{I}_2, \end{cases} \right\}$$

the optimal value $\gamma \leq \max_{0 < i \leq n} c_i$. By Definition 2, $\gamma_{s,K}(A', c)$ is the infimum of γ, thus $\gamma_{s,K}(A', c) \leq \max_{0 < i \leq n} c_i$.

(b) Suppose $\gamma_{s,K}(A', c) < \min_{0 < i \leq n} c_i$, next we prove (A', c) is nonnegative partially s-good. That is, for a vector $w = (w^1, w^2)^T \geq 0$ with $A_1 w^1 + A_2 w^2 = b'$ and $\|w^1\|_0 \leq s$, we need to prove that w is the unique optimal solution to problem (10).

First, we consider the special case that $(w^1, w^2)^T = (0, w^2)^T$. Obviously, $(x, y)^T = (0, w^2)^T$ is the unique optimal solution to problem (10). The reason is that, when $w^1 = 0$, then $A_2 y = A_2 w^2$, and since $A_2 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ is injective, it is easy to see that $y = w^2$ is unique.

Now, suppose $\|w^1\|_0 = s'$, $0 \leq s' \leq s$, and its support set is I_1. Meanwhile, let I_2 be the support index set of w^2. According to Lemma 2, we have $\gamma := \gamma_{s',K}(A', c) \leq \gamma_{s,K}(A', c)$. Since $\gamma_{s,K}(A', c) < \min_{0 < i \leq n} c_i$, we get that $\gamma < \min_{0 < i \leq n} c_i$. Moreover, by the definition of $\gamma_{s,K}(\cdot)$, there exists $\theta \in R^{m+n}$ such that

$$\|\theta\|_* \leq \beta, \ (A_1^T \theta)_i \begin{cases} = c_i \text{sign}(w_i^1), & i \in I_1; \\ \in [-\gamma, \gamma], & i \in \bar{I}_1, \end{cases} \ (A_2^T \theta)_i \begin{cases} = 0, & i \in I_2; \\ \leq 0, & i \in \bar{I}_2, \end{cases} \tag{18}$$

where $I_1 = \{i : w_i^1 > 0\}$, $\bar{I}_1 = \{i : w_i^1 = 0\}$, $I_2 = \{i : w_i^2 > 0\}$ and $\bar{I}_2 = \{i : w_i^2 = 0\}$. Furthermore by (a), there exists $\theta \in R^{m+n}$ satisfying Eq. (18) which is the optimal Lagrange multiplier of problem (10). Then for any feasible solution $(x, y)^T$ of problem

(10), it holds that

$$
\begin{aligned}
f(x,y) &= c^T x - \theta^T (A_1 x + A_2 y - A_1 w^1 - A_2 w^2) \\
&= c^T x - (A_1^T \theta)^T (x - w^1) - (A_2^T \theta)^T (y - w^2) \\
&= \sum_{i \in I_1} c_i w_i^1 + \sum_{i \in \bar{I}_1} (c_i - (A_1^T \theta)_i) x_i - \sum_{i \in \bar{I}_2} (A_2^T \theta)_i y_i \\
&\geq \sum_{i \in I_1} c_i w_i^1 + \sum_{i \in \bar{I}_1} (c_i - (A_1^T \theta)_i) x_i \\
&\geq c^T w^1.
\end{aligned}
\tag{19}
$$

According to (19), it is obvious that the minimum value of the Lagrange function can be attained at $x = w^1$. Further, since (x,y) and (w^1, w^2) have the relationship $A_1 x + A_2 y = A_1 w^1 + A_2 w^2$, $A_1 x = A_1 w^1$ and $A_2 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ is an injective matrix, it is easy to show that $A_2 y = A_2 w^2$. Therefore, $(x,y)^T = (w^1, w^2)^T$ is an optimal solution of problem (10).

Next, it is necessary to prove that this optimal solution is unique. Suppose $(\tilde{x}, \tilde{y})^T$ is another optimal solution of problem (10). That is,

$$
f(\tilde{x}, \tilde{y}) - f(w^1, w^2) = \sum_{i \in \bar{I}_1} (c_i - (A_1^T \theta)_i) \tilde{x}_i - \sum_{i \in \bar{I}_2} (A_2^T \theta)_i \tilde{y}_i = 0.
$$

By the assumption that $\gamma < \min_{0 < i \leq n} c_i$, and from Eq. (18), $|A_1^T \theta|_i < \min_{0 < i \leq n} c_i$ for all $i \in \bar{I}_1$, which means that $\tilde{x}_i = 0$ and $\sum_{i \in \bar{I}_2} (A_2^T \theta)_i \tilde{y}_i = 0$. Therefore, $\tilde{x}_i = w_i^1 = 0$ for all $i \in \bar{I}_1$. Hence, $\|\tilde{x} - w^1\|_0 \leq s$.

Further, for the above vector $\tilde{x}_i - w_i^1$, define

$$
h(\tilde{x} - w^1, \tilde{y} - w^2) := \sum_{i=1}^{n} c_i |(\tilde{x} - w^1)_i| - \tilde{\theta}^T (A_1(\tilde{x} - w^1) + A_2(\tilde{y} - w^2)).
$$

Similar to the proof of part (a) in Theorem 7, there exists $\tilde{\theta} \in R^{m+n}$ such that

$$
(A_1^T \tilde{\theta})_i \begin{cases} = c_i sign((\tilde{x} - w^1)_i), & if \ (\tilde{x} - w^1)_i \neq 0; \\ \in [-\max_{0 < i \leq n} c_i, \max_{0 < i \leq n} c_i], & if \ (\tilde{x} - w^1)_i = 0, \end{cases}
$$

and

$$
(A_2^T \tilde{\theta})_i \begin{cases} = 0, & if \ (\tilde{y} - w^2)_i \neq 0; \\ \leq 0, & if \ (\tilde{y} - w^2)_i = 0. \end{cases}
$$

Therefore, for the $\tilde{\theta}$ in the function $h(\tilde{x} - w^1, \tilde{y} - w^2)$, we have

$$
\begin{aligned}
0 &= (A_1^T \tilde{\theta})^T (\tilde{x} - w^1) + (A_2^T \tilde{\theta})^T (\tilde{y} - w^2) \\
&= \sum_{i \in I_1} (A_1^T \tilde{\theta})_i^T (\tilde{x}_i - w_i^1) + \sum_{i \in I_2} (A_2^T \tilde{\theta})_i^T (\tilde{y}_i - w_i^2),
\end{aligned}
$$

and then we can get $\tilde{x}_i = w_i^1$ for all $i \in I_1$. This combined with the fact $A_1 \tilde{x} + A_2 \tilde{y} = A_1 w^1 + A_2 w^2$ can lead to $A_2 \tilde{y} = A_2 w^2$. Further, since $A_2 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ is an injective matrix, we have $\tilde{y} = w^2$ and then $(\tilde{x}, \tilde{y})^T = (w^1, w^2)^T$. $\qquad \square$

Below, we show the relationship between $\gamma_{s,K}(\cdot)$ and $\hat{\gamma}_{s,K}(\cdot)$.

Proposition 1 *For arbitrary $\beta \in [0, \infty]$, if $\hat{\gamma} := \hat{\gamma}_{s,K}(A', c, \beta) < \frac{1}{2} \min\limits_{0 < i \leq n} c_i$, then*

$$\gamma_{s,K}\left(A', c, \frac{\min\limits_{0 < i \leq n} c_i}{\min\limits_{0 < i \leq n} c_i - \hat{\gamma}} \beta\right) \leq \frac{\min\limits_{0 < i \leq n} c_i}{\min\limits_{0 < i \leq n} c_i - \hat{\gamma}} \hat{\gamma} < \min\limits_{0 < i \leq n} c_i. \tag{20}$$

Proof Suppose $\hat{\gamma} := \hat{\gamma}_{s,K}(A', c, \beta) < \frac{1}{2} \min\limits_{0 < i \leq n} c_i$. Now let I_1 be an s-element subset of $\{1, 2, \ldots, n\}$, $\bar{I}_1 := \{1, 2, \ldots, n\} \setminus I_1$, and let I_2 be a subset of $\{n+1, n+2, \ldots, m+2n\}$, $\bar{I}_2 := \{n+1, n+2, \ldots, m+2n\} \setminus I_2$.

For the I_1, $\bar{I}_1$ and I_2, $\bar{I}_2$, we define a closed convex set Π_{I_1} in R^n as

$$\Pi_{I_1} = \left\{ \tau' \in R^n : \exists \theta \in R^{m+n}, \|\theta\|_* \leq \beta, (A_1^T \theta)_i \begin{cases} = c_i \tau_i', & i \in I_1; \\ \in [-\hat{\gamma}, \hat{\gamma}], & i \in \bar{I}_1, \end{cases} (A_2^T \theta)_i \begin{cases} = 0, & i \in I_2; \\ \leq 0, & i \in \bar{I}_2 \end{cases} \right\}.$$

Similar to the proof of Proposition 2.1 in [11], we claim that Π_{I_1} contains the $\|\cdot\|_\infty$-ball B centered at the origin and with the radius $\frac{\min\limits_{0 < i \leq n} c_i - \hat{\gamma}}{\min\limits_{0 < i \leq n} c_i}$. The proof is as follows.

Define a subspaces $L_{I_1} := \{\tau' \in R^n : \tau_i' = 0, i \in \bar{I}_1\}$ and let $L_{I_1}^\perp := \{\tau' \in R^n : \tau_i' = 0, i \in I_1\}$ be the orthogonal complement of L_{I_1}. Let P be the projection of Π_{I_1} onto L_{I_1} and P' be the projection of Π_{I_1} onto $L_{I_1}^\perp$. Note that Π_{I_1} is the direct sum of P and P'. Thus, P is a closed convex set. Obviously, L_{I_1} can be naturally identified with R^s.

Hence, the claim in the above can be more precisely described as that the image $\bar{P} \subset R^s$ of P contains the $\|\cdot\|_\infty$-ball B_s centered at the origin and with the radius $\frac{\min\limits_{0 < i \leq n} c_i - \hat{\gamma}}{\min\limits_{0 < i \leq n} c_i}$ in R^s.

Next, we prove that $B_s \subseteq \bar{P}$. Contradictorily, suppose that $\bar{P}$ does not contain B_s. Since P is a closed convex set, $\bar{P}$ is also a closed convex set. According to the separating hyperplane theorem, for $v \in B_s \setminus \bar{P}$ and $\bar{v} \in \bar{P}$, there exists $u \in R^s$ with $\|u\|_1 = 1$, such that

$$u^T v > \max_{\bar{v} \in \bar{P}} u^T \bar{v}. \tag{21}$$

In the following, we prove that there does not exist u such that the inequality (21) holds.

First, define $\bar{z}^1 \in R^n$ with $\|\bar{z}^1\|_0 = s$ and $\bar{z}^2 \in R^{m+n}$ as

$$\bar{z}^1 = \begin{cases} 1, & i \in I_1; \\ 0, & i \in \bar{I}_1, \end{cases} \quad \bar{z}^2 \begin{cases} \neq 0, & i \in I_2; \\ = 0, & i \in \bar{I}_2. \end{cases}$$

By the definition of $\hat{\gamma}_{s,K}(A',c,\beta)$, for the given $\bar{z}^1 \in R^n$, $\bar{z}^2 \in R^{m+n}$, there exists $\bar{\theta} \in R^{m+n}$ such that

$$\|\bar{\theta}\|_* \leq \beta, \ \|(A_1^T\bar{\theta}) - c^T \circ \bar{z}^1\|_\infty \leq \hat{\gamma}, \ (A_2^T\theta)_i \begin{cases} = 0, & \text{if } \bar{z}_i^2 \neq 0; \\ \leq 0, & \text{if } \bar{z}_i^2 = 0. \end{cases}$$

Then for the u with $\|u\|_1 = 1$ in (21), combining the above inequality with the definitions of Π_{I_1} and $\bar{P}$, there exists a vector $v' \in \bar{P}$ such that

$$|c_i v_i' - c_i sign(u_i)| \leq \hat{\gamma}, \ i \in I_1.$$

Thus,

$$sign(u_i) - \frac{\hat{\gamma}}{\min_{0<i\leq n} c_i} \leq v_i' \leq sign(u_i) + \frac{\hat{\gamma}}{\min_{0<i\leq n} c_i}, \ i \in I_1.$$

Since $\hat{\gamma} < \frac{1}{2} \min_{0<i\leq n} c_i$, the above inequalities imply that the sign of v_i' is same as u_i, for all i. Moreover, according to the definition of $\hat{\gamma}_{s,K}(A',c,\beta)$, we can get

$$1 - \frac{\hat{\gamma}}{\min_{0<i\leq n} c_i} \leq v_i' \leq 1 + \frac{\hat{\gamma}}{\min_{0<i\leq n} c_i}, \ i \in I_1.$$

Hence, $v' > 0$, $u > 0$, and

$$v_i' \geq \frac{\min_{0<i\leq n} c_i - \hat{\gamma}}{\min_{0<i\leq n} c_i}, i \in I_1.$$

So

$$u^T v' \geq \sum_{i=1}^s |u_i| \frac{\min_{0<i\leq n} c_i - \hat{\gamma}}{\min_{0<i\leq n} c_i} = \frac{\min_{0<i\leq n} c_i - \hat{\gamma}}{\min_{0<i\leq n} c_i}.$$

Further, for the above given $v \in B_s$ and $\|u\|_1 = 1$, we have

$$\frac{\min_{0<i\leq n} c_i - \hat{\gamma}}{\min_{0<i\leq n} c_i} \geq \|v\|_\infty = \|u\|_1\|v\|_\infty \geq u^T v > u^T v' \geq \frac{\min_{0<i\leq n} c_i - \hat{\gamma}}{\min_{0<i\leq n} c_i},$$

which is a contradiction. So the claim holds.

Through the above proof, we can conclude that, for any $z = (z^1, z^2)^T \in R^n \times R^{m+n}$ with $z_i^1 = 1$ for $i \in I_1$, and $z_i^1 = 0$ else, there exists $\tau' \in \Pi_{I_1}$ such that

$$\tau_i' = (\frac{\min_{0<i\leq n} c_i - \hat{\gamma}}{\min_{0<i\leq n} c_i})z_i^1, \ i \in I_1.$$

Further, by the definition of Π_{I_1}, there exists $\hat{\theta} \in R^{m+n}$ with $\|\hat{\theta}\|_* \leq \frac{\min\limits_{0<i\leq n} c_i}{\min\limits_{0<i\leq n} c_i - \hat{\gamma}}\beta$ such that

$$(A_1^T\hat{\theta})_i \begin{cases} = \frac{\min\limits_{0<i\leq n} c_i}{\min\limits_{0<i\leq n} c_i - \hat{\gamma}} c_i \tau_i' = c_i z_i^1, & i \in I_1; \\ \in [-\frac{\min\limits_{0<i\leq n} c_i}{\min\limits_{0<i\leq n} c_i - \hat{\gamma}}\hat{\gamma}, \frac{\min\limits_{0<i\leq n} c_i}{\min\limits_{0<i\leq n} c_i - \hat{\gamma}}\hat{\gamma}], & i \in \bar{I}_1, \end{cases}$$

and

$$(A_2^T\hat{\theta})_i \begin{cases} = 0, & i \in I_2; \\ \leq 0, & i \in \bar{I}_2. \end{cases}$$

So by the definition of $\gamma_{s,K}(A',c,\beta)$, and since $\hat{\gamma} < \frac{1}{2}\min\limits_{0<i\leq n} c_i$, we can obtain

$$\gamma_{s,K}(A',c,\frac{\min\limits_{0<i\leq n} c_i}{\min\limits_{0<i\leq n} c_i - \hat{\gamma}}\beta) \leq \frac{\min\limits_{0<i\leq n} c_i}{\min\limits_{0<i\leq n} c_i - \hat{\gamma}}\hat{\gamma} < \min\limits_{0<i\leq n} c_i. \qquad \square$$

Based on Proposition 2, Theorem 7 can be equivalently written as:

Theorem 8 *Given $A' \in R^{(m+n)\times(m+2n)}$, s is an integer and $0 \leq s \leq n$, $0 < c \leq 1$, if $\hat{\gamma}_{s,K}(A',c) < \frac{1}{2}\min\limits_{0<i\leq n} c_i$, then (A',c) is nonnegative partially s-good.*

According to Theorem 7, to show that (A',c) is nonnegative partially s-good, we need to compare the magnitude of $\gamma_{s,K}(A',c)$ with $\min\limits_{0<i\leq n} c_i$. Now, due to Theorem 8, and we know that $\hat{\gamma}_{s,K}(A',c,\beta)$ is weaker than $\gamma_{s,K}(A',c,\beta)$, hence we focus on $\hat{\gamma}(\cdot)$, which has a specific representation presented in the next subsection.

3.4 Specific representation of $\hat{\gamma}_{s,K}(\cdot)$

$\hat{\gamma}_{s,K}(A',c,\beta)$ is given in Definition 2, which is essentially obtained from the optimality condition of problem (10). In this subsection, we give a specific representation of $\hat{\gamma}_{s,K}(A',c,\beta)$ in more detail.

Theorem 9 *Consider the polytope*

$$P_s = \{\tau \in R_+^{m+2n} : \tau = (\tau^1, \tau^2)^T, \tau^1 \in R^n, \tau^2 \in R^{m+n}, \|\tau^1\|_1 \leq s, \|\tau^1\|_\infty \leq 1\},$$

we have

$$\hat{\gamma}_{s,K}(A',c,\beta) = \max_{\tau,x}\{\sum_{i=1}^n \tau_i^1 c_i x_i - \beta\|A_1 x\|_1 : \tau \in P_s, \|x\|_1 \leq 1, x \geq 0\}. \qquad (22)$$

Particularly,

$$\hat{\gamma}_{s,K}(A',c) = \max_{\tau,x}\{\sum_{i=1}^n \tau_i^1 c_i x_i : \tau \in P_s, \|x\|_1 \leq 1, A_1 x = 0, x \geq 0\}. \qquad (23)$$

Proof For any vector $y \geq 0$, let $I_2(y)$ be its support set. According to Definition 2, define

$$B_\beta(y) = \{\theta \in R^{m+n} : \|\theta\|_* \leq \beta, \ (A_2^T\theta)_i = 0 \text{ for } i \in I_2(y), \ (A_2^T\theta)_i \leq 0 \text{ otherwise}\},$$

$$B = \{v \in R^n : \|v\|_\infty \leq 1\}.$$

Then, $\hat{\gamma}_{s,K}(A',c,\beta)$ is the smallest γ, such that the set $C_{1,\gamma,\beta} := A_1^T B_\beta(y) + \gamma B \subseteq R^n$ is closed, convex, and contains all vectors with s nonzero elements, which are selected from $c_i, i = 1, 2, \cdots, n$. This statement is equivalent to that $C_{1,\gamma,\beta}$ contains the convex hull of the vectors.

Let $\hat{c} = (c^T, 0, \cdots, 0)^T \in R^{m+2n}$. Then $C_{1,\gamma,\beta}$ contains the projection of $\hat{c} \circ P_s$ onto the R^n space. Thus, γ satisfies the relationship of $\hat{c} \circ P_s \subseteq C_{1,\gamma,\beta} \times R^{m+n}$, if and only if for any pair of $(x,y)^T \geq 0$ with $x \in R^n$ and $y \in R^{m+n}$,

$$
\begin{aligned}
\max_{\tau \in P_s} \sum_{i=1}^n \tau_i^1 c_i x_i &\leq \max_{\eta \in C_{1,\gamma,\beta}} \{\sum_{i=1}^n \eta_i x_i\} \\
&= \max_{\theta,v} \{<x, A_1^T\theta> + \gamma <x,v> : \ \theta \in B_\beta(y), \ \|v\|_\infty \leq 1\} \\
&\leq \max_{\theta,v} \{<x, A_1^T\theta> + \gamma <x,v> : \ \|\theta\|_* \leq \beta, \ \|v\|_\infty \leq 1\} \\
&= \max_{\theta,v} \{<A_1 x, \theta> + \gamma <x,v> : \ \|\theta\|_* \leq \beta, \ \|v\|_\infty \leq 1\} \\
&= \beta \|A_1 x\|_1 + \gamma \|x\|_1.
\end{aligned}
\tag{24}
$$

That is, $\hat{c} \circ P_s \subseteq C_{1,\gamma,\beta} \times R^{m+n}$ if and only if γ satisfies that

$$\max_{\tau \in P_s} \{\sum_{i=1}^n \tau_i^1 c_i x_i - \beta \|A_1 x\|_1 : \ x \geq 0\} \leq \gamma \|x\|_1,$$

namely,

$$\max_{\tau,x} \{\sum_{i=1}^n \tau_i^1 c_i x_i - \beta \|A_1 x\|_1 : \tau \in P_s, \|x\|_1 \leq 1, \ x \geq 0\} \leq \gamma.$$

Therefore Eq. (22) holds, since $\hat{\gamma}_{s,K}(A',c,\beta)$ is the smallest γ. Finally, Eq. (23) holds due to that Eq. (22) can be regarded as a penalty problem of Eq. (23). $\qquad\square$

For $c \circ x \geq 0 \in R^n$, we define the sum of the s largest entries of $c \circ x$ as

$$\|c \circ x\|_{s,K,1} := \max_{\tau \in P_s} \sum_{i=1}^n \tau_i^1 c_i x_i.$$

Then by taking Theorems 8 and 9 into consideration, we can obtain the following result:

Corollary 3 *Given a matrix A_1, $\hat{\gamma}_{s,K}(A',c)$ is the least upper bound on $\|c \circ x\|_{s,K,1} := \max_{\tau \in P_s} \sum_{i=1}^n \tau_i^1 c_i x_i$ over $x \geq 0$, $x \in Ker(A_1)$ and $\|x\|_1 \leq 1$. As a result, if the maximum of $\|c \circ x\|_{s,K,1}$ over $x \in Ker(A_1)$ and $\|x\|_1 \leq 1$ is less than $\frac{1}{2} \min_{0 < i \leq n} c_i$, then (A',c) is nonnegative partially s-good.*

Equations (22) and (23) provide the specific forms of $\hat{\gamma}_{s,K}(\cdot)$. Thus, we can judge the nonnegative partial s-goodness of (A',c) according to Theorem 8, as long as the Eqs. (22) and (23) can be calculated. However, in Eqs. (22) and (23), the calculation of $\hat{\gamma}_{s,K}(\cdot)$ is complicated, and sometimes it is not easy to directly calculate the specific value of $\hat{\gamma}_{s,K}(A',c,\beta)$. In order to make up for this shortcoming, in what follows, we give effective upper bounding of $\hat{\gamma}_{s,K}(A',c,\beta)$ to estimate the value of $\hat{\gamma}_{s,K}(A',c,\beta)$.

4 Efficient bounding of $\hat{\gamma}_{s,K}(\cdot)$

From the previous sections, we have shown that $\hat{\gamma}_{s,K}(A',c,\beta)$ plays an important role in distinguishing whether (A',c) is non-negative partially s-good. However, it still not easy to determine the exact value of $\hat{\gamma}_{s,K}(A',c,\beta)$ according to Eqs. (23) and (22). In this section, we introduce efficiently computable upper bound on the value of $\hat{\gamma}_{s,K}(A',c,\beta)$.

Since $\hat{\gamma}_{s,K}(A',c,\beta) \geq \hat{\gamma}_{s,K}(A',c)$ for any $\beta > 0$, we will use Eq. (23) to calculate an upper bound of $\hat{\gamma}_{s,K}(A',c)$. The difficulty of calculation is in the process of the linear constraint $A_1 x = 0$, which will be handled via Lagrange relaxation.

Since $\hat{\gamma}_{s,K}(A',c) > 0$, hence, we only consider the case where the elements in x are not all 0. For any matrix $Q = [q_1, \cdots, q_n] \in R^{(m+n) \times n}$ with $A_2^T Q \leq 0$, we have

$$Q^T A_1 x = \sum_{\ell=1}^{m+n} \sum_{j=1}^{n} q_{\ell,i} a_{\ell,j} x_j, \ i = 1, 2, \ldots, n$$
$$= 0,$$

where $a_{\ell,j}$ is the element of matrix A_1 in the ℓ-th row and j-th column, $q_{\ell,i}$ is the element of matrix Q in the ℓ-th row and i-th column.

Let $C \in R^{n \times n}$ be the diagonal matrix $diag(c_1, c_2, \cdots, c_n)$. For the above Q and by Eq. (23), we can get

$$\hat{\gamma}_{s,K}(A',c)$$

$$= \max_{\tau,x} \left\{ \sum_{i=1}^{n} \tau_i^1 c_i x_i : \ \tau \in P_s, \ \|x\|_1 \leq 1, A_1 x = 0, \ x \geq 0 \right\}$$

$$\leq \max_{\substack{\tau \in P_s \\ x \geq 0}} \left\{ \sum_{i=1}^{n} \tau_i^1 \left(c_i x_i - \sum_{\ell=1}^{m+n} \sum_{j=1}^{n} q_{\ell,i} a_{\ell,j} u_j \right) : \ \|x\|_1 \leq 1, Q^T A_1 x = 0, A_2^T Q \leq 0 \right\}$$

$$= \max_{\substack{\tau \in P_s \\ x \geq 0}} \left\{ \tau^{1T} (Cx - Q^T A_1 x) : \ \|x\|_1 \leq 1, \ Q^T A_1 x = 0, A_2^T Q \leq 0 \right\}$$

$$\leq \max_{\substack{\tau \in P_s \\ x \geq 0}} \left\{ \tau^{1T} (Cx - Q^T A_1 x) : \ \|x\|_1 \leq 1, A_2^T Q \leq 0 \right\}$$

$$= \max_{\substack{\tau \in P_s \\ x \geq 0}} \left\{ \tau^{1T} (C - Q^T A_1) x : \ \|x\|_1 \leq 1, A_2^T Q \leq 0 \right\}.$$

Hence we can solve the problem

$$\max_{\tau,x} \left\{ \tau^{1T} (C - Q^T A_1) x : \ x \geq 0, \|x\|_1 \leq 1, \tau \in P_s, A_2^T Q \leq 0 \right\} \tag{25}$$

to obtain an upper bound of $\hat{\gamma}_{s,K}(A',c)$, which is linear in x.

In the above problem, the feasible region for x is the convex hull of just n points e_i, $i = 1,2,\cdots,n$, where e_i is the n-dimensional vector with the i-th component being 1, and the remaining components being 0. Therefore, the above problem can be rewritten as

$$
\begin{aligned}
\max_{\tau,x} &\left\{ \tau^{1T}(C - Q^T A_1)x : \ x \geq 0, \|x\|_1 \leq 1, \tau \in P_s, A_2^T Q \leq 0 \right\} \\
&\leq \max_{\tau,0<j\leq n} \left\{ |\tau^{1T}(C - Q^T A_1)e_j| : \ \tau \in P_s, A_2^T Q \leq 0 \right\} \\
&= \max_{0<j\leq n} \left\{ \max_{\tau \in P_s} |\tau^{1T}(C - Q^T A_1)e_j| : \ A_2^T Q \leq 0 \right\} \\
&= \max_{0<j\leq n} \left\{ \|(C - Q^T A_1)e_j\|_{s,K,1} : \ A_2^T Q \leq 0 \right\}.
\end{aligned}
\tag{26}
$$

Define $g_{A_1,C,s,K}(Q)$ as $\displaystyle\max_{0<j\leq n} \|(C - Q^T A_1)e_j\|_{s,K,1}$, and let

$$
\eta_{s,K}(A_1,C,\infty) := \begin{cases} \min_Q \ g_{A_1,C,s,K}(Q) \\ \text{s.t.} \ \ A_2^T Q \leq 0. \end{cases}
$$

Then

$$
\hat{\gamma}_{s,K}(A',c) \leq \eta_{s,K}(A_1,C,\infty).
$$

Since $g_{A_1,C,s,K}(Q)$ is easy to compute, so $\eta_{s,K}(A_1,C,\infty)$ is easy to compute.

Further, from (24) we have

$$
\begin{aligned}
\max_{\tau \in P_s} \sum_{i=1}^n \tau_i^1 c_i x_i &\leq \max_{\theta,v} \{ <A_1 x, \theta> + \gamma <x,v> : \|\theta\|_* \leq \beta, \|v\|_\infty \leq 1 \} \\
&= \max_{\theta} \{ <A_1 x, \theta> : \|\theta\|_* \leq \beta \} + \gamma \|x\|_1 \}.
\end{aligned}
$$

Thus, γ satisfies that

$$
\max_{\substack{\tau \in P_s \\ x \geq 0}} \left\{ \sum_{i=1}^n \tau_i^1 c_i x_i - \max_{\theta} <A_1 x, \theta> : \|\theta\|_* \leq \beta, \|x\|_1 \leq 1 \right\} \leq \gamma.
$$

Note that $\hat{\gamma}_{s,K}(A',c,\beta)$ is the infimum of γ, hence,

$$
\hat{\gamma}_{s,K}(A',c,\beta) = \max_{\tau \in P_s, x \geq 0} \left\{ \sum_{i=1}^n \tau_i^1 c_i x_i - \max_{\theta} <A_1 x, \theta> : \|\theta\|_* \leq \beta, \|x\|_1 \leq 1 \right\}.
$$

For any matrix $Q = [q_1, \cdots, q_n] \in R^{(m+n) \times n}$ with $A_2^T Q \leq 0$, $\|q_i\|_* \leq \beta$ for all i, and $Q^T A_1 x = 0$, we can get

$$\hat{\gamma}_{s,K}(A', c, \beta)$$

$$= \max_{\substack{\tau \in P_s \\ x \geq 0 \\ \|x\|_1 \leq 1}} \left\{ \sum_{i=1}^{n} \tau_i^1 c_i x_i - \max_{\theta} < A_1 x, \theta > : \|\theta\|_* \leq \beta \right\}$$

$$\leq \max_{\substack{\tau \in P_s \\ x \geq 0 \\ \|x\|_1 \leq 1}} \left\{ \sum_{i=1}^{n} \tau_i^1 c_i x_i - < A_1 x, q_i > : \|q_i\|_* \leq \beta, Q^T A_1 x = 0, A_2^T Q \leq 0 \right\}$$

$$= \max_{\substack{\tau \in P_s \\ x \geq 0 \\ \|x\|_1 \leq 1}} \left\{ \sum_{i=1}^{n} \tau_i^1 (c_i x_i - \sum_{\ell=1}^{m+n} \sum_{j=1}^{n} q_{\ell,i} a_{\ell,j} x_j) : \|q_i\|_* \leq \beta, Q^T A_1 x = 0, A_2^T Q \leq 0 \right\}$$

$$= \max_{\substack{\tau \in P_s \\ x \geq 0 \\ \|x\|_1 \leq 1}} \left\{ \tau^{1T}(Cx - Q^T A_1 x) : \|q_i\|_* \leq \beta, Q^T A_1 x = 0, A_2^T Q \leq 0 \right\}$$

$$\leq \max_{\substack{\tau \in P_s \\ x \geq 0 \\ \|x\|_1 \leq 1}} \left\{ \tau^{1T}(C - Q^T A_1)x : \|q_i\|_* \leq \beta, A_2^T Q \leq 0 \right\},$$

where q_i, $i = 1, \ldots, n$, is the i-th column of matrix Q, a_ℓ, $\ell = 1, \ldots, n$, is the ℓ-th column of matrix A_1.

Similar to (26), we can solve the following problem

$$\max_{\substack{\tau \in P_s \\ x \geq 0}} \left\{ \tau^{1T}(C - Q^T A_1)x : \|x\|_1 \leq 1, \|q_i\|_* \leq \beta, A_2^T Q \leq 0 \right\}.$$

Moreover, it is easy to change the above upper bound of $\hat{\gamma}_{s,K}(A', c, \beta)$ to $\eta_{s,K}(A_1, C, \beta)$, which is defined as

$$\min_Q \max_{0 < j \leq n} \|(C - Q^T A_1)e_j\|_{s,K,1}$$
$$\text{s.t. } \|q_i\|_* \leq \beta, \ 0 < i \leq n, \tag{27}$$
$$A_2^T Q \leq 0,$$

where q_ℓ is the ℓ-th column of matrix Q.

Obviously, problem (27) is a convex programming and is solvable. Similar to the properties of $\hat{\gamma}_{s,K}(A', c, \beta)$, $\eta_{s,K}(A_1, C, \beta)$ is a nondecreasing function of s, and is a nonincreasing function of β. Thus we can get an upper bound on $\hat{\gamma}_{s,K}(A', c, \beta)$, by calculating the least upper bound of $g_{A_1, C, s, K}(Q)$ with respect to Q, i.e., Eq. (27).

In addition, according to the definition of $\| \cdot \|_{s,K,1}$, given positive integers s and t, we have $\| \cdot \|_{st,K,1} \leq t \| \cdot \|_{s,K,1}$, and

$$\eta_{s,K}(A_1, C, \beta) \leq s \eta_{1,K}(A_1, C, \beta). \tag{28}$$

So, the upper bound $\eta_{s,K}(A_1, C, \beta)$ of $\hat{\gamma}_{s,K}(A', c, \beta)$ can be replaced by $s \eta_{1,K}(A', C, \beta)$. This property allows us to reduce the calculation of $\eta_{s,K}(A_1, C)$ to $\eta_{1,K}(A_1, C)$, which greatly reduces the amount of calculation.

Let $\mathbf{Q} = \{Q : \|q_i\|_* \leq \beta, i = 1, \ldots, n, A_2^T Q \leq 0\}$. According to the definition of $\eta_{s,K}(A_1, C, \beta)$, we have

$$\eta_{1,K}(A_1, C, \beta) = \min_{Q \in \mathbf{Q}} \max_{0 < j \leq n} \|(C - Q^T A_1) e_j\|_\infty$$

$$= \min_{Q \in \mathbf{Q}} \max_{0 < j \leq n} \left\| \begin{pmatrix} |c_1 - q_1^T a_1|, & |-q_1^T a_2|, & \ldots, & |-q_1^T a_n| \\ |-q_2^T a_1|, & |c_2 - q_2^T a_2|, & \ldots, & |-q_2^T a_n| \\ & & \ddots & \\ |-q_n^T a_1|, & |-q_n^T a_2|, & \ldots, & |c_n - q_n^T a_n| \end{pmatrix} e_j \right\|_\infty$$

$$= \min_{Q \in \mathbf{Q}} \max_{0 < j \leq n} \|(C - A_1^T Q) e_j\|_\infty$$

$$= \min_{Q \in \mathbf{Q}} \max_{0 < j \leq n} \left\| \begin{pmatrix} |c_1 - a_1^T q_1|, & |-a_1^T q_2|, & \ldots, & |-a_1^T q_n| \\ |-a_2^T q_1|, & |c_2 - a_2^T q_2|, & \ldots, & |-a_1 q_n^T| \\ & & \ddots & \\ |-a_n^T q_1|, & |-a_n^T q_2|, & \ldots, & |c_n - a_n^T q_n| \end{pmatrix} e_j \right\|_\infty$$

$$= \min_{Q \in \mathbf{Q}} \max_{0 < j \leq n} \left\| \begin{pmatrix} 0 \\ 0 \\ \vdots \\ c_j \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} a_1^T q_j \\ a_2^T q_j \\ \vdots \\ a_j^T q_j \\ \vdots \\ a_n^T q_j \end{pmatrix} \right\|_\infty$$

$$= \min_{Q \in \mathbf{Q}} \max_{0 < j \leq n} \|C_j - A_1^T q_j\|_\infty,$$

which is equivalent to solve n convex optimization problems of dimension n:

$$\eta^j = \min_{q_j \in \mathbf{Q}} \|C_j - A_1^T q_j\|_\infty. \tag{29}$$

Obviously, here $\eta_{1,K}(A', c, \beta) = \max_{0 < j \leq n} \eta^j$.

In Theorem 8, the sufficient condition for nonnegative partial s-goodness is $\hat{\gamma}_{s,K}(A', c) < \frac{1}{2} \min_{0 < i \leq n} c_i$. Note that, $\hat{\gamma}_{s,K}(A', c)$ takes the value of $\hat{\gamma}_{s,K}(A', c, \beta)$ for a large enough β. Similarly, $\eta_{s,K}(A_1, C)$ takes the value of $\eta_{s,K}(A_1, C, \beta)$ for the above large enough β.

Given A', c and s, suppose $\hat{\gamma}_{s,K}(A', c) < \frac{1}{2} \min_{0 < i \leq n} c_i$, such that for every pair of vectors $z^1 \in R^n$ and $z^2 \in R^{m+n}$, where $z^1 \in R^n$ has s nonzero entries, each is equal to 1, there exists a vector $\theta \in R^{m+n}$ such that

$$\|(A_1^T \theta) - c \circ z^1\|_\infty \leq \hat{\gamma}_{s,K}(A', c), \text{ and } (A_2^T \theta)_i \begin{cases} = 0, & \text{if } z_i^2 \neq 0; \\ \leq 0, & \text{if } z_i^2 = 0. \end{cases}$$

Next, we need to show that under the assumptions the solution of θ is bounded, i.e., there exists $\bar{\beta}$ such that $\|\theta\|_* < \bar{\beta}$.

Proposition 2 *Suppose $\{u : \|u\|_1 \leq \rho, u \in R^{m+n}\} \subseteq \mathscr{R}(A_1)$, where $\mathscr{R}(A_1) = \{A_1 d : \|d\|_1 \leq 1, d \in R^n\}$. For every $s \leq n$, if $\beta \geq \bar{\beta} = \frac{1}{\rho}(\max_{0 < i \leq n} c_i + \frac{1}{2} \min_{0 < i \leq n} c_i)$ and $\hat{\gamma}_{s,K}(A', c) < \frac{1}{2} \min_{0 < i \leq n} c_i$, then $\hat{\gamma}_{s,K}(A', c) = \hat{\gamma}_{s,K}(A', c, \beta)$.*

Proof According to the definition of $\hat{\gamma}_{s,K}(A',c,\beta)$, we have $\|A_1^T\theta\|_\infty < \max\limits_{0<i\leq n} c_i + \frac{1}{2}\min\limits_{0<i\leq n} c_i$. Hence,

$$\max_{0<i\leq n} c_i + \frac{1}{2}\min_{0<i\leq n} c_i > \|A_1^T\theta\|_\infty$$
$$= \max_d\{d^T A_1^T\theta : \|d\|_1 \leq 1, d \in R^n\}$$
$$= \max_u\{u^T\theta : u = A_1 d, \|d\|_1 \leq 1, d \in R^n\}$$
$$\geq \max_u\{u^T\theta : \|u\|_1 \leq \rho\}$$
$$= \rho\|\theta\|_*.$$

Then

$$\|\theta\|_* < \frac{1}{\rho}(\max_{0<i\leq n} c_i + \frac{1}{2}\min_{0<i\leq n} c_i),$$

and we define

$$\bar{\beta} = \frac{1}{\rho}(\max_{0<i\leq n} c_i + \frac{1}{2}\min_{0<i\leq n} c_i).$$

The proposition is proven. $\qquad\square$

According to Proposition 2, if we could find a lower bound $\bar{\beta}$ of β such that $\hat{\gamma}_{s,K}(A',c,\bar{\beta}) < \frac{1}{2}\min\limits_{0<i\leq n} c_i$, then $\hat{\gamma}_{s,K}(A',c,\beta) \leq \hat{\gamma}_{s,K}(A',c,\bar{\beta}) < \frac{1}{2}\min\limits_{0<i\leq n} c_i$ for all $\beta \geq \bar{\beta}$, since $\hat{\gamma}_{s,K}(A',c,\beta)$ is a nonincreasing function of β. The same applies to $\eta_{s,K}(A_1,C,\beta)$.

5 Heuristic algorithm and examples

In this section, we give three examples to illustrate the proposed non-negative partial *s*-goodness condition for the equivalence between the integer programming problem (1) and the weighted linear programming problem (9). To this aim, according to Theorem 8, first we should verify that (A',c) is nonnegative partially *s*-good. Then, according to Definition 1, the partially weighted linear programming problem (9) has the unique optimal solution. Further, according to Theorem 5 or Theorem 6, the optimal solution of problem (9) is also the optimal solution of the l_0-norm minimization problem (8). Meanwhile, according to Theorem 3, it is also an optimal solution of problem (1).

The main idea of verifying the nonnegative partial *s*-goodness of (A',c) is as follows. Given the $\bar{\beta}$ in Proposition 2, for $\beta = \bar{\beta}$ and an arbitrary *s*, combining Theorem 8 and the definition of $\eta_{s,K}(A_1,C,\beta)$, it is obvious that $\eta_{s,K}(A_1,C,\bar{\beta}) < \frac{1}{2}\min\limits_{0<i\leq n} c_i$ is a sufficient condition for (A',c) to be nonnegative partially *s*-good.

Let $s^*(A',c)$ be a lower bound on the largest *s* such that $\eta_{s,K}(A_1,C,\bar{\beta}) < \frac{1}{2}\min\limits_{0<i\leq n} c_i$, which will be obtained from Eqs. (28) and (29). In other words, given $s^*(A',c)$ and $\beta \geq \bar{\beta}$, the value of $\eta_{s^*(A',c),K}(A_1,C,\beta)$ must be less than $\frac{1}{2}\min\limits_{0<i\leq n} c_i$.

According to Theorems 5 and 6, the partially weighted linear programming problem (9) must have a unique optimal solution. However in some cases, the optimal

solution of problem (9) may not be unique, and we need to adjust c, such that problem (9) has a unique optimal solution.

To better understand the role of the weight c, let us divide a non-negative optimal solution $(x^*, y^*)^T$ with $\|x^*\|_0 \leq s$ of problem (9) into the following three cases:

Case 1 : problem (9) has the unique optimal solution $(x^*, y^*)^T$ with $\|x^*\|_0 \leq s$;

Case 2 : problem (9) has multiple optimal solutions $(x^*, y^*)^T$ with $\|x^*\|_0 \leq s$, and the vectors x^* have the same sparsity;

Case 3 : problem (9) has multiple optimal solutions $(x^*, y^*)^T$ with $\|x^*\|_0 \leq s$, and the vectors x^* have different sparsity.

Clearly, by Definition 1, it is natural that (A', c) is nonnegative partially s-good in *Case* 1. Example 1 in this section will illustrate this case.

Case 2 shows that there are multiple optimal solutions of problems (8) and (9). At this point, we may adjust c to a suitable value, such that one of the optimal solutions of problem (9) is a unique optimal solution. Then we go to verify that (A', c) is nonnegative partially s-good. Example 2 in this section will illustrate this case.

Note that in *Case* 2, all optimal solutions of problem (9) are optimal solutions of problem (8).

Case 3 is a very special case, in which problem (9) has multiple optimal solutions $(x^*, y^*)^T$, and the vectors x^* have different sparsity. Firstly, suppose that $\eta_{s,K}(A_1, C, \bar{\beta}) < \frac{1}{2} \min_{0 < i \leq n} c_i$, we can obtain $s^*(A', c)$. Next, we choose one of the optimal solutions of problem (9), which satisfies $\|x^*\|_0 \leq s^*(A', c)$. If such solution cannot be found, then we should find another $s^*(A', c)$; otherwise, we may adjust c to a suitable value, such that this solution is unique, and then we continue to verify that (A', c) is nonnegative partially s-good. Example 3 in this section will illustrate this case.

The above idea of verifying the nonnegative partial s-goodness of (A', c) can be organized in the following steps. Initially, let $c_i = 1$, $i = 1, 2, \ldots, n$ and let $\beta = \bar{\beta} = \frac{1}{\rho}\left(\max_{0 < i \leq n} c_i + \frac{1}{2} \min_{0 < i \leq n} c_i\right)$ according to Proposition 2.

Step 1: According to Eq. (29), we calculate the value of $\eta_{1,K}(A_1, C, \bar{\beta})$. If $\eta_{1,K}(A_1, C, \bar{\beta}) < \frac{1}{2} \min_{0 < i \leq n} c_i$, then go to *Step* 2; otherwise (this will happen in *Cases* 2 or 3), we should use *Step* 5 to update c, such that $\eta_{1,K}(A_1, C, \bar{\beta}) < \frac{1}{2} \min_{0 < i \leq n} c_i$, and go to *Step* 2.

Step 2: Since s is not known at present, we suppose $\eta_{s,K}(A_1, C, \bar{\beta}) < \frac{1}{2} \min_{0 < i \leq n} c_i$. Then according to (28),

$$s^*(A', c) = \lfloor \frac{\frac{1}{2} \min_{0 < i \leq n} c_i}{\eta_{1,K}(A_1, C, \bar{\beta})} \rfloor.$$

Step 3: Consider an optimal solution (x^*, y^*) of problem (9). If the solution of problem (9) is unique and $\|x^*\|_0 = s$, then we compare s with $s^*(A', c)$. If $s = s^*(A', c)$, then we verify whether $s^*(A', c)\eta_{1,K}(A_1, C, \bar{\beta}) < \frac{1}{2} \min_{0 < i \leq n} c_i$. When it holds, go to *Step* 4.

Otherwise, such as *Case* 3, not all solutions satisfy $s\eta_{1,K}(A_1, C, \bar{\beta}) < \frac{1}{2} \min_{0 < i \leq n} c_i$. So we choose a solution with $\|x^*\|_0 = s^*(A', c)$, and use *Step* 5 to update c, such that this optimal solution of problem (9) is the unique optimal solution. Next, we

verify that whether $s^*(A',c)\eta_{1,K}(A_1,C,\bar{\beta}) < \frac{1}{2}\min_{0<i\leq n} c_i$. When it holds, go to *Step* 4; otherwise, update c again.

Step 4: According Theorem 8, it implies that (A',c) is nonnegative partially s-good. Stop the algorithm.

Step 5: Update c as follows: for the maximum component x_i^* in *Step* 3, select c_i such that $0 < c_i \leq \bar{\beta}$. For the minimum component x_j^* in *Step* 3, select c_j from $(\bar{\beta}, \frac{3}{2}\bar{\beta})$. The other components in c are randomly selected from $[c_i, c_j]$, i.e., c_i and c_j are the minimum and maximum components of c respectively. Let $\bar{\beta} = \frac{1}{\rho}(\max_{0<i\leq n} c_i + \frac{1}{2}\min_{0<i\leq n} c_i)$.

Below are three examples we provide. Example 1 does not comply with *TUM*, Example 2 does not comply with *TDI*, and Example 3 is neither *TUM* nor *TDI* compliant.

Example 1 Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $c = 1$, given $\bar{\beta} = 0.563$, according to Eq. (29), we can calculate the value of $\eta_{1,K}(A_1,C,\bar{\beta})$ is 0.2188. Suppose $\eta_{s,K}(A_1,C,\bar{\beta}) < \frac{1}{2}\min_{0<i\leq n} c_i$, then according to (28), $s^*(A',c) = \lfloor \frac{0.5}{0.2188} \rfloor = 2$. For $c = 1$, problem (9) has an optimal solution $(x^1,y^1)^T = \left((0,\frac{1}{2},\frac{1}{2}), (0,0,0,1,\frac{1}{2},\frac{1}{2})\right)^T$ and $\|x^1\|_0 \leq 2$. According to (28), $\eta_{s,K}(A_1,C,\bar{\beta}) \leq 2\eta_{1,K}(A_1,C,\bar{\beta}) = 0.4376 < \frac{1}{2}$. Then, by Theorem 8, (A',c) is nonnegative partially s-good. Hence, according to Definition 1, $(x^1,y^1)^T$ is the unique optimal solution of problem (9). Next, according to Theorem 6, $(x^1,y^1)^T$ is an optimal solution of problem (8). So by Theorem 3, $(0,1,1)^T$ is an optimal solution of problems (1) and (4) respectively.

It must be remarked that, sometimes problem (9) has multiple nonnegative optimal solutions when $c = 1$, and the vectors x have the same sparsity. This situation is contradict to Definition 1. So the coefficient c must be adjusted, such that problem (9) has a unique optimal solution. The following Example 2 shows that this can be achieved.

Example 2 Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}.$$

For $c = 1$, given $\bar{\beta} = 0.5$, according to Eq. (29), we can calculate the value of $\eta_{1,K}(A_1,C,\bar{\beta})$ is 0.5. For $c = 1$, problem (9) has two optimal solutions, which are $(x^1,y^1)^T = \left((1,\frac{1}{2},0), (1,0,0,0,\frac{1}{2},1)\right)^T$ and $(x^2,y^2)^T = \left((\frac{3}{4},\frac{3}{4},0), (\frac{3}{4},0,\frac{1}{4},\frac{1}{4},\frac{1}{4},1)\right)^T$, $c^T x^1 = c^T x^2$, and both vectors x^1 and x^2 are 2-sparse.

Next, let $c = (0.5, 0.7, 0.8)$. Given $\bar{\beta} = 0.7$, we have $\eta_{1,K}(A_1, C, \bar{\beta}) = 0.1$. Suppose $\eta_{s,K}(A_1, C, \bar{\beta}) < \frac{1}{2} \min_{0 < i \leq n} c_i$, then according to (28), $s^*(A', c) = \lfloor \frac{0.25}{0.125} \rfloor = 2$. For $c = (0.5, 0.7, 0.8)$, problem (9) has an optimal solution $(x^1, y^1)^T = ((1, \frac{1}{2}, 0), (1, 0, 0, 0, \frac{1}{2}, 1))^T$, and $\|x^*\|_0 \leq 2$. According to (28), $\eta_{s,K}(A_1, C, \bar{\beta}) \leq 2\eta_{1,K}(A_1, C) = 0.2 < \frac{1}{2} \min_{0 < i \leq n} c_i = 0.25$. Then, by Theorem 8, (A', c) is nonnegative partially s-good. Hence, according to Definition 1, $(x^1, y^1)^T$ is the unique optimal solution of problem (9). Furthermore, according to Theorem 5, $(x^1, y^1)^T$ is an optimal solution of problem (8). Since $\|x^1\|_0 = \|x^2\|_0$, it is natural that $(x^2, y^2)^T$ is also an optimal solution to problem (8). So by Theorem 3, $(1, 1, 0)$ is an optimal solution of problems (1) and (4) respectively.

Different from Example 2, another situation to be aware of is that, problem (9) has multiple nonnegative optimal solutions, and the vectors x have different sparsities, when $c = 1$. This situation is also contradict to Definition 1. So the coefficient c must be adjusted too, such that problem (9) has a unique optimal solution. The following Example 3 shows that this can be achieved.

Example 3 Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}.$$

For $c = 1$, given $\bar{\beta} = 0.375$, according to Eq. (29), we can calculate the value of $\eta_{1,K}(A_1, C, \bar{\beta})$ is 0.2917. Suppose $\eta_{s,K}(A_1, C, \bar{\beta}) < \frac{1}{2} \min_{0 < i \leq n} c_i$, then according to (28), $s^*(A', c) = \lfloor \frac{0.5}{0.2917} \rfloor = 1$. For $c = 1$, problem (9) has two optimal solutions, they are: $(x^1, y^1)^T = ((0, \frac{1}{6}, \frac{1}{3}), (\frac{1}{3}, 0, 0, 1, \frac{5}{6}, \frac{2}{3}))^T$ with $\|x^1\|_0 = 2$, and $(x^2, y^2)^T = ((0, 0, \frac{1}{2}), (0, 0, \frac{1}{6}, 1, 1, \frac{1}{2}))^T$ with $\|x^2\|_0 = 1$, $c^T x^1 = c^T x^2$. Clearly, $\|x^1\|_0 > s^*(A', c)$, $\|x^2\|_0 = s^*(A', c)$.

Next, let $c = (0.5, 0.35, 0.3)$. Given $\bar{\beta} = 0.7$, we have $\eta_{1,K}(A_1, C, \bar{\beta}) = 0.1$. Suppose $\eta_{s,K}(A_1, C, \bar{\beta}) < \frac{1}{2} \min_{0 < i \leq n} c_i$, then according to (28), $s^*(A', c) = \lfloor \frac{0.15}{0.1} \rfloor = 1$. For $c = (0.5, 0.7, 0.8)$, problem (9) has an optimal solution $(x^2, y^2)^T = ((0, 0, \frac{1}{2}), (0, 0, \frac{1}{6}, 1, 1, \frac{1}{2}))^T$ with $\|x^2\|_0 = 1$. According to (28), $\eta_{s,K}(A_1, C, \bar{\beta}) \leq \eta_{1,K}(A_1, C, \bar{\beta}) = 0.1 < \frac{1}{2} \min_{0 < i \leq n} c_i = 0.15$. Hence, by Theorem 8, (A', c) is nonnegative partially s-good, and according to Definition 1, $(x^2, y^2)^T$ is the unique optimal solution of problem (9). Furthermore, according to Theorem 6, $(x^2, y^2)^T$ is an optimal solution of problem (8). So by Theorem 3, $(0, 0, 1)^T$ is an optimal solution of problems (1) and (4) respectively.

6 Conclusion

In this paper, we studied the equivalence of a 0-1 linear program to a weighted linear programming problem. Firstly, we prove the equivalence between the integer pro-

gramming problem and a sparse minimization problem. Next, we define the nonnegative partial s-goodness of the constraint matrix and the weight vector in the objective function of the weighted linear programming problem. Utilizing two quantities $\gamma_{s,K}(\cdot)$ and $\hat{\gamma}_{s,K}(\cdot)$ of the nonnegative partial s-goodness, we propose a necessary and a sufficient condition for the constraint matrix and weighted vector to be nonnegative partial s-good. Since it is difficult to calculate the two quantities, we further provide an efficiently computable upper bound of $\hat{\gamma}_{s,K}(A', c, \beta)$, such that the above sufficient condition is verifiable. It is worthy of mentioning that the objective coefficient c of the weighted linear programming problem is not fixed. When the weighted linear programming problem has multiple optimal solutions, we may adjust c so that the weighted linear programming problem has only a unique optimal solution. At the end, we provide three examples to illustrate the theory in this article.

References

1. Bandeira, A.S., Scheinberg, K., Vicente, L.N.: On partial sparse recovery. arXiv preprint arXiv:1304.2809 (2013)
2. Billionnet, A.: Solving a cut problem in bipartite graphs by linear programming: Application to a forest management problem. Applied Mathematical Modelling **34**(4), 1042–1050 (2010)
3. Chandrasekaran, R., Kabadi, S., Sridhar, R.: Integer solution for linear complementarity problem. Mathematics of Operations Research **23**(2), 390–402 (1998)
4. Chen, J., Kou, L., Cui, X.: An approximation algorithm for the minimum vertex cover problem. Procedia Engineering **137**, 180–185 (2016)
5. Dakin, R.: A tree-search algorithm for mixed integer programming problems. Computer Journal **8**, 250–255 (1965)
6. Danna, E., Rothberg, E., Pape, C.L.: Exploring relaxations induced neighborhoods to improve mip solutions. Mathematical Programming **102**(A), 71–90 (2005)
7. Dubey, D., Neogy, S.: Total dual integrality and integral solutions of the linear complementarity problem. Linear Algebra and its Applications **557**, 359–374 (2018)
8. Edmonds, J., Giles, R.: A min-max relation for submodular functions on graphs. Annals of Discrete Mathematics **1**, 185–204 (1977)
9. Gomory, R.E.: Outline of an algorithm for integer solutions to linear programs. Bulletin of the American Mathematical Society **64**(5), 275–278 (1958)
10. Hoffman, A.J.: Total unimodularity and combinatorial theorems. Linear Algebra and its Applications **13**(1), 103–108 (1976)
11. Juditsky, A., Nemirovski, A.: On verifiable sufficient conditions for sparse signal recovery via l_1-minimization. Mathematical Programming **127**, 57–88 (2011)
12. Kong, L., Xiu, N., Liu, G.: Partial s-goodness for partially sparse signal recovery. Numerical Algebra, Control & Optimization **4**(1), 25–38 (2014)
13. Lodi, A.: Mixed Integer Programming Computation. Springer, Berlin, Heidelberg (2010)
14. Zhang, W., Nicholson, C.: Objective scaling ensemble approach for integer linear programming. Journal of Heuristics **26**, 1–19 (2020)
15. Zhao, Y.: Equivalence and strong equivalence between sparsest and least l_1-norm nonnegative solutions to linear systems and their applications. Journal of the Operations Research Society of China **2**(2), 171–193 (2014)
16. Zhao, Y.: Sparse Optimization Theory And Methods. CRC Press. Taylor & Francis Group (2018)
17. Zhao, Y., Li, D.: Reweighted l_1-minimization for sparse solutions to underdetermined linear systems. SIAM Journal on Optimization **22**, 1065–1088 (2012)
18. Zhao, Y., Luo, Z.: Constructing new weighted l_1-algorithms for the sparsest points of polyhedral sets. Mathematics of Operations Research **42**(1), 57–76 (2017)