ON STABLE UNIQUENESS IN LINEAR SEMI-INFINITE OPTIMIZATION [1]

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Abstract. This paper is intended to provide conditions for the stability of the strong uniqueness of the optimal solution of a given linear semi-infinite optimization (LSIO) problem, in the sense of the maintaining of the strong uniqueness property under sufficiently small perturbations of all the data. We consider LSIO problems such that the family of gradients of all the constraints is unbounded, extending earlier results of Nürnberger for continuous LSIO problems, and of Helbig and Todorov for LSIO problems with bounded set of gradients. To do this we characterize the absolutely (affinely) stable problems, i.e., those LSIO problems whose feasible set (its affine hull, respectively) remains constant under sufficiently small perturbations.

Key words. Linear semi-infinite optimization, stable strong uniqueness, extended Nürnberger condition, affine stability, absolute stability.

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1 Introduction

This paper deals with linear semi-infinite optimization (LSIO) problems in $\mathbb{R}^n$ of the form

$$\pi : \text{ Inf } c'x \text{ s.t. } a_t'x \geq b_t, t \in T, \tag{1}$$

where T is a fixed arbitrary (possible infinite) set, $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$ are functions, and $x \in \mathbb{R}^n$. It is focussed on the identification of those LSIO problems which have a strongly unique (or just unique) optimal solution and this desirable property is *stable* in the sense that it is preserved by sufficiently small perturbations of the cost vector c and the functions a and b. For short,

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we will say "(strong) uniqueness" to mean "uniqueness of the (strong) unique optimal solution" of an LSIO problem.

The problem (1) will also be denoted by $\pi = (a, b, c)$; and by $\Pi = (\mathbb{R}^n \times \mathbb{R})^T \times \mathbb{R}^n$ we will denote the linear space of all these kind of problems (the n decision variables and the index set T are fixed), i.e., the result of arbitrary perturbations of the *nominal problem* π preserving n and T. In the next section we shall introduce a topology on Π, which will be called *space of parameters*.

The pioneering work on uniqueness in linear programming (LP), due to Dantzig (1963, [5]), was completed by Mangasarian (1979, [18]), who provided characterizations not involving optimal basis. Uniqueness and strong uniqueness are no longer equivalent in LSIO, where strong uniqueness plays a crucial role in numerical analysis (together with some regularity conditions it implies superlinear convergence of multiple exchange methods, see [16] - [15]) as well as in sensitivity analysis (it characterizes those problems for which the optimal value is a linear function of the costs on some neighborhood of c, as shown in [7]). The literature about uniqueness and strong uniqueness in LSIO up to 1995 was surveyed in [9]. One of the characterizations of the uniqueness in LP included in [18] consists of maintaining the uniqueness under arbitrary but sufficiently small perturbations of the cost vector. The first works on stable (strong) uniqueness are due to Nürnberger ([19] - [20]) and Strauss ([21]); Nürnberger considered the case of continuous LSIO problems and continuous perturbations of the whole triple (a, b, c) (π is a *continuous* LSIO problem if T is a compact Hausdorff space and the coefficient functions a and b are continuous); while Strauss allowed only perturbations of the RHS function b. In this last setting, Cánovas et al. (2007, [3]) have recently shown that Nürnberger condition ([19, Condition (2) in Thm. 1.4]) turns out to be equivalent to the metric regularity of the inverse of the optimal set mapping. In 1998, Helbig and Todorov ([14]) characterized the stable (strong) uniqueness for LSIO problems such that the set of the LHS coefficients of the constraints is bounded by means of a suitable Nürnberger-type condition. Under the same assumption on the boundedness of the LHS coefficients, Goberna, López and Todorov (2003, [12]) proved that the set of problems with strongly unique optimal solution contains an open and dense subset of the set of solvable problems. This is a generic result and in view of it we can say that most (in a topological sense) solvable problems satisfying this boundedness condition have a strongly unique optimal solution.

In this paper we analyze the stable (strong) uniqueness of problems which are not upper bounded in the sense of [14], but satisfy a suitable property, e.g., positive distance from the origin to the set of LHS coefficients, or absolutely stability in the sense that the feasible set remains constant under sufficiently small perturbations. In the first case we adapt to our purpose the idea (already used in 1965 by Charnes, Cooper and Kortanek ([4]) in the context of duality in LSIO) of improving the properties of a given program by means of suitable reformulations: dividing each nontrivial constraint $a_t'x \geq b_t$ by $\|((a_t, b_t))\|$ (called canonical normalization of π), aggregating redundant constraints until the set of coefficients of the constraints is compact (regularization), and eliminating variables in order to get an equivalent problem with full dimensional feasible set (dimension reduction). Here, we use dimension reduction and scaling (not necessarily canonical normalization), not only for the nominal problem π (as in [4]) but for the problems in some neighborhood of π. In doing this we show a very interesting characterization of the lower semicontinuity of the feasible set mapping at π : the affine hull of the feasible set remains constant on some neighborhood of π, that is π is *affinely stable*. Finally, it is important to remark that the strong uniqueness property is a geometric property essentially related to the shape of the feasible set F and the relative position of the gradient vector c, while the stability of the strong uniqueness is related to the representation of the set F.

In Section 2 we define a topology on Π and state some necessary definitions; Section 3 gives the definition of the extended Nürnberger condition, and recalls some known results about it. In Section 4 we characterize the affine stability and apply it to reduce the dimension of π. These two sections contain Nürnberger-type necessary conditions for stable strong uniqueness based on Theorem 4.1 in [14]. Section 5 gives Nürnberger-type sufficient conditions when $\pi \notin \Pi_{UB}$, by applying the extended Nürnberger condition to suitable reformulations of π. Finally, Section 6 characterizes the absolutely stable problems and the subclass of stably strongly unique problems; we also provide a generic result about the absolutely stable problems which are uniquely solvable.

2 Preliminaries

We represent by F the feasible set and by F^* the optimal set of π. The problem π is *consistent* if $F \neq \emptyset$ and *solvable* if $F^* \neq \emptyset$.

$\widehat{x} \in F$ is a *strong Slater* element for π if there exists $\varepsilon > 0$ such that $a_t'\widehat{x} \geq b_t + \varepsilon$ for all $t \in T$. In this case π is said to satisfy the *strong Slater condition* (SSC). In the particular case that π is a *continuous* LSIO problem the strong Slater elements are the Slater points.

π *has a unique solution* if there exists some $\overline{x} \in F$ such that $F^* = \{\overline{x}\}$, this optimal solution being *strongly unique* if there exists $\alpha > 0$ such that

$$c'x \geq c'\overline{x} + \alpha \|x - \overline{x}\| \quad \text{for all} \quad x \in F,$$

where $\|\cdot\|$ stands for the Chebyshev norm. (We will always consider $\mathbb{R}^n$ equipped with the Chebyshev norm).

We say that π is *LHS-positively lower bounded* if $\inf \{\|a_t\|, t \in T\} > 0$ (i.e., $\{a_t, t \in T\}$ does not intersect some neighborhood of the null vector 0_n). In the same fashion, π is *LHS-upper bounded* if $\sup \{\|a_t\|, t \in T\} < +\infty$ (with an analogous interpretation, just replacing the origin by the point at infinity). Moreover, we say that π is *absolutely stable* if its feasible set is nonempty and remains constant under arbitrary but sufficiently small perturbations of all the data (a, b, c). In Example 1 of [8], it is shown, for any infinite index set T, that π admits a reformulation in Π (with the same objective function, and feasible set assumed to be bounded) which is absolutely stable.

We consider the following subsets of parameters:

$\Pi_C = \{\pi \in \Pi : \pi \text{ is consistent}\}$,

$\Pi_{SS} = \{\pi \in \Pi : \pi \text{ satisfies the strong Slater condition}\}$,

$\Pi_U = \{\pi \in \Pi : \pi \text{ has a unique solution}\}$,

$\Pi_{SU} = \{\pi \in \Pi : \pi \text{ has a strongly unique solution}\}$,

$\Pi_{PLB} = \{\pi \in \Pi : \pi \text{ is LHS-positively lower bounded}\}$,

$\Pi_{UB} = \{\pi \in \Pi : \pi \text{ is LHS-upper bounded}\}$,

and

$\Pi_{AS} = \{\pi \in \Pi : \pi \text{ is absolutely stable}\}$.

Obviously, $\Pi_{SU} \subset \Pi_U \subset \Pi_C$ and $\Pi_{SS} \subset \Pi_C$. On the other hand, it is well-known that $\pi \in \Pi_{SS}$ if and only if π is stable with respect to consistency ([10, Thm. 6.1]). Thus, $\Pi_{AS} \subset \Pi_{SS}$.

Notice that, if $\pi \in \Pi_{UB} \cap \Pi_{SS}$, then F contains interior points, i.e., F is full dimensional. In fact, if $\rho := \sup \{\|a_t\|, t \in T\}$ and $a_t' \overline{x} \geq b_t + \varepsilon$ for all $t \in T$, then the ball centered at $\overline{x}$ with radius $\frac{\varepsilon}{n\rho}$ is contained in F by the Cauchy-Schwartz inequality.

For the next definitions we will make use of the following notation for a given set X :
◦ If X is a nonempty subset of $\mathbb{R}^n$, $\operatorname{aff} X$, $\operatorname{conv} X$, $\operatorname{cone} X$ and $\dim X$ denote its affine hull, its convex hull, its convex conical hull containing 0_n, and the dimension of $\operatorname{aff} X$, respectively.
◦ If X is a convex cone in $\mathbb{R}^n$, its positive polar is denoted by X^0 and its lineality space by $\operatorname{lin} X$.
◦ If X is convex in $\mathbb{R}^n$ and $\overline{x} \in X$, we denote by $D(X; \overline{x})$ the (convex) cone

of feasible directions of X at $\overline{x}$.
$\circ$ If X is a nonempty subset of either $\mathbb{R}^n$ or Π, $\operatorname{int} X$, $\operatorname{cl} X$ and $\operatorname{bd} X$ represent its interior, closure, and boundary, respectively.

The *data set* of the constraint system of π is

$$D := \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\},$$

whereas the *characteristic cone* of π is

$$K := \operatorname{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}.$$

Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The nonhomogeneous Farkas lemma establishes that $a'x \geq b$ is a consequence of the consistent system $\{a_t'x \geq b_t, t \in T\}$ if and only if $(a, b) \in \operatorname{cl} K$.

Given $\overline{x} \in F$, the constraint $a_t'x \geq b_t$ is *active at* $\overline{x}$ if $a_t'\overline{x} = b_t$. We denote by $T(\overline{x})$ the set of active indexes at $\overline{x}$, $T(\overline{x}) = \{t \in T : a_t'\overline{x} = b_t\}$. The *cone of active constraints* at $\overline{x}$ is $\operatorname{cone}\{a_t, t \in T(\overline{x})\}$. The KKT condition $c \in \operatorname{cone}\{a_t, t \in T(\overline{x})\}$ is sufficient for $\overline{x} \in F^*$ and it is also necessary if some constraint qualification holds (e.g., π is a continuous LSIO problem satisfying the Slater condition). The condition $c \in \operatorname{int} \operatorname{cone}\{a_t, t \in T(\overline{x})\}$ is sufficient for $\overline{x} \in F$ to be a strongly unique optimal solution of π, and it is also necessary under some constraint qualification (see [10, Theorems 7.1 and 10.6]). We say that $a'x \geq b$ is an *implicit active constraint* at $\overline{x} \in F$ if $a'\overline{x} = b$ and $(a, b) \in \operatorname{cl} D$, in which case $a'x \geq b$ for all $x \in F$. Each implicit active constraint $a'x \geq b$ is characterized just by the vector a; in this fashion the set

$$A(\overline{x}) := \left\{ a \in \mathbb{R}^n : \begin{pmatrix} a \\ a'\overline{x} \end{pmatrix} \in \operatorname{cl} D \right\}$$

will be called the *set of implicit active constraints* at $\overline{x}$. Obviously, $\{a_t, t \in T(\overline{x})\} \subset A(\overline{x})$ and the equality holds if D is closed (e.g., if π is continuous). Moreover, if $\pi \in \Pi_{UB}$, then $A(\overline{x})$ is bounded for all $\overline{x} \in F$.

From the topological side, we consider Π equipped with the pseudometric of the uniform convergence, i.e., given two parameters $\pi_1 = (a^1, b^1, c^1)$ and $\pi_2 = (a^2, b^2, c^2)$,

$$d(\pi_1, \pi_2) := \max \left\{ \left\| c^1 - c^2 \right\|, \sup_{t \in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \right\};$$

we use the same symbol to mark a perturbation of the nominal problem π and its associated objects.

It is immediate to prove that Π_{UB}, Π_{PLB} and Π_{AS} are open cones, that Π_{UB} is closed and that Π_{PLB} is dense in Π. In fact, given $\varepsilon > 0$, replacing in π each constraint $a_t' x \geq b_t$ such that $\|a_t\| < \frac{2\varepsilon}{3}$ by another one $\left(a_t^1\right)' x \geq b_t$ such that $\frac{\varepsilon}{6} < \|a_t^1\| < \frac{\varepsilon}{3}$, maintaining b and c, we get a perturbed problem $\pi_1 \in \Pi_{PLB}$ such that $d\left(\pi_1, \pi\right) < \varepsilon$. The only relationship between these three sets is that $\Pi_{UB} \cap \Pi_{AS} = \emptyset$ (this will be a consequence of Theorem 6.1).

To analyze the stability of the LSIO problems which have a strongly unique (or just unique) solution is to determine the topological interiors of Π_{SU} (Π_U, respectively). Since $\operatorname{int} \Pi_{SU} \subset \operatorname{int} \Pi_U \subset \Pi_{SS}$ only the elements of Π_{SS} are relevant in this paper.

3 Extended Nürnberger condition and known results

According to [19], given a continuous LSIO problem π satisfying the Slater condition, strong uniqueness holds in a neighborhood of π (in the topological subspace of Π formed by the continuous problems) if and only if there exists $\overline{x} \in F$ that satisfies:

(a) there exists $\left\{\overline{d}_1, ..., \overline{d}_n\right\} \subset \left\{a_t, t \in T\left(\overline{x}\right)\right\}$ such that $c \in \operatorname{cone}\left\{\overline{d}_1, ..., \overline{d}_n\right\}$,

and

(b) for any set $\left\{d_1, ..., d_n\right\} \subset \left\{a_t, t \in T\left(\overline{x}\right)\right\}$ such that $c \in \operatorname{cone}\left\{d_1, ..., d_n\right\}$, all the subsets of cardinality n of $\left\{c, d_1, ..., d_n\right\}$ are linearly independent.

Condition (a) means that $\overline{x} \in F^*$ (by the optimality theorem and Carathéodory's theorem) and both conditions together imply that $c \neq 0_n$. A simple algebraic argument shows that (b) can be replaced in the above *Nürnberger condition* by the following geometric one:

(b') If $\left\{d_1, ..., d_n\right\} \subset \left\{a_t, t \in T\left(\overline{x}\right)\right\}$ and $c \in \operatorname{cone}\left\{d_1, ..., d_n\right\}$, then $c \in \operatorname{int} \operatorname{cone}\left\{d_1, ..., d_n\right\}$.

Nürnberger theorem can be extended from continuous to general LSIO problems by replacing the set of active constraints $\left\{a_t, t \in T\left(\overline{x}\right)\right\}$ by the enlarged set of implicit active constraints at $\overline{x}$, $A\left(\overline{x}\right)$, and strengthening the Slater condition by considering the strong Slater condition (SSC).

Definition 3.1 $\pi \in \Pi$ *satisfies the Extended Nürnberger condition at* $\overline{x} \in F$ *if there exists* $\left\{\overline{d}_1, ..., \overline{d}_n\right\} \subset A\left(\overline{x}\right)$ *such that* $c \in \operatorname{cone}\left\{\overline{d}_1, ..., \overline{d}_n\right\}$ *and* $c \in \operatorname{int} \operatorname{cone}\left\{d_1, ..., d_n\right\}$ *for any set* $\left\{d_1, ..., d_n\right\} \subset A\left(\overline{x}\right)$ *such that* $c \in \operatorname{cone}\left\{d_1, ..., d_n\right\}$. *We say that* $\pi \in \Pi$ *satisfies the Extended Nürnberger condition (ENC for short) if* π *satisfies this condition at some* $\overline{x} \in F$.

Observe that

$$c \in \operatorname{cone}\left\{\overline{d}_1, ..., \overline{d}_n\right\} \subset \operatorname{cone} A\left(\overline{x}\right)$$

implies that $\overline{x} \in F^*$ by Corollary 5.5 in [11]. This means that only the points $\overline{x} \in F^*$ (contained in $\operatorname{bd} F$ if $c \neq 0_n$) have to be considered when checking ENC.

The next key result is proved in Theorem 4.1 of [14].

Theorem 3.2 *Given $\pi \in \Pi_{UB} \cap \Pi_{SS}$, the following statements are equivalent:*
(i) $\pi \in \operatorname{int} \Pi_{SU}$.
(ii) $\pi \in \operatorname{int} \Pi_U$.
(iii) π satisfies ENC.

Observe that (i)-(iii) fail if $c = 0_n$ because, in this case, $\pi \in \Pi_{UB} \cap \Pi_{SS}$ entails that $F^* = F$ is an infinite set, i.e., $\pi \notin \Pi_U$. The next two examples show that the assumption $\pi \in \Pi_{UB}$ in Theorem 3.2 is not superfluous.

Example 3.3 *Here we present a problem $\pi \in \Pi_{SS} \diagdown \Pi_{UB}$ that does not satisfy ENC even though $\pi \in \operatorname{int} \Pi_{SU}$, showing that the implications $(i) \Longrightarrow (iii)$ and $(ii) \Longrightarrow (iii)$ do not hold. Let $n = 3$, $T = \mathbb{Z}$, and consider the problem*

$$\begin{aligned} \pi : \operatorname{Inf}\ & x_2 \\ s.t.\ & x_1 + x_2 + kx_3 \geq 0, \quad k \geq 0, \\ & -x_1 + x_2 + kx_3 \geq 0, \quad k < 0. \end{aligned}$$

Its feasible set is $F = \{x \in \mathbb{R}^3 : \pm x_1 + x_2 \geq 0, x_3 = 0\}$, and its optimal set is $F^* = \{0_3\}$. We have $\pi \notin \Pi_{UB}$ and $\pi \in \Pi_{SS} \cap \Pi_{SU}$. The data set D is discrete (and so it is closed), thus

$$A\left(0_3\right) = \left\{a \in \mathbb{R}^n : \begin{pmatrix} a \\ 0 \end{pmatrix} \in D\right\} = \left\{a_t, t \in T\right\}.$$

Since $c = (0,1,0) \in \operatorname{bd} \operatorname{cone}\{a_{-1}, a_0, a_1\}$, ENC fails at 0_3. Now, for any π_1 such that the $d(\pi_1, \pi)$ is small enough, $\dim F_1 = \dim F = 2$ (because $\pi, \pi_1 \in \Pi_{SS}$). The nonhomogeneous Farkas lemma implies that $x_3 \geq 0$ and $-x_3 \geq 0$ are consequences of the constraints of π_1. Therefore, $\operatorname{aff} F_1 = \{x \in \mathbb{R}^3 : x_3 = 0\}$ (see also Theorem 4.1 below). Then there exists $\mu > 0$ such that $d(\pi_1, \pi) < \mu$ implies that F_1 can be expressed as the set of points of the form $(x_1, x_2, 0)$ such that $x_2 \geq \sup\left\{\alpha_r^1 x_1 + \beta_r^1, r \in \mathbb{N}; \gamma_r^1 x_1 + \delta_r^1, r \in \mathbb{N}\right\}$, for some $\alpha_r^1 < -\frac{1}{2}$ and $\gamma_r^1 > \frac{1}{2}$ for all $r \in \mathbb{N}$, so that $\pi_1 \in \Pi_{SU}$. We conclude that $\pi \in \operatorname{int} \Pi_{SU}$.

Example 3.4 *This example shows that both implications $(iii) \Longrightarrow (i)$ and $(iii) \Longrightarrow (ii)$ in the previous theorem are not valid in general when $\pi \in$*

$\Pi_{SS} \diagdown \Pi_{UB}$. *Consider the problem in* $\mathbb{R}^2$,

$$\pi : \operatorname{Inf} x_2 \ \ s.t. \ \ x_1 + x_2 \geq 0, -x_1 + x_2 \geq 0, \ \ and \ kx_2 \geq -1, k \in \mathbb{N},$$

with $F^* = \{0_2\}$ *and* $A(0_2) = \{(1,1), (-1,1)\}$. *Then,* $\pi \in \Pi_{SS} \diagdown \Pi_{UB}$ *satisfies ENC and it is not (strongly) unique stable. Indeed, for each* $\delta > 0$, *the problem*

$$\pi_1 : \operatorname{Inf} x_2 \ \ s.t. \ \ x_1 + x_2 \geq -\delta, -x_1 + x_2 \geq -\delta, \ and \ kx_2 \geq -1, k \in \mathbb{N},$$

is such that $d(\pi, \pi_1) = \delta$ *and* $F_1^* = \{(x_1, 0) : |x_1| \leq \delta\}$.

The proof of $(ii) \implies (iii)$ in Theorem 3.2 given in [14] remains valid by replacing the condition $\pi \in \Pi_{UB}$ by the weaker one that the set of implicit active constraints at its optimal solution $\overline{x}$, $A(\overline{x})$, is bounded. So we can state the following Nürnberger-type necessary condition for stable (strong) uniqueness:

Theorem 3.5 *If* $\pi \in \operatorname{int} \Pi_U$, *with* $F^* = \{\overline{x}\}$, *and* $A(\overline{x})$ *is bounded, then* π *satisfies ENC.*

4 Dimension reduction and Nürnberger-type necessary conditions

As shown in the next section, the full dimension of the feasible set is a desirable property in order to check the stable strong uniqueness of π. Here we show that, given an LSIO problem π satisfying SSC and such that $0 < \dim F < n$, it is possible to reformulate π in a lower dimensional space in such a way that the equivalent problem (in a sense to be made precise later) possesses a full dimensional feasible set. In doing that we need a new characterization of the stable consistency of π.

Theorem 4.1 $\pi \in \Pi_C$ *satisfies the strong Slater condition if and only if there exists* $\varepsilon > 0$ *such that* $\operatorname{aff} F_1 = \operatorname{aff} F$ *for all* $\pi_1 \in \Pi$ *with* $d(\pi_1, \pi) < \varepsilon$.

Proof. Let $\pi \in \Pi_C$. Given $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, by the Farkas Lemma, $a'x = b$ for all $x \in F$ if and only if $\pm(a,b) \in \operatorname{cl} K$ if and only if $(a,b) \in \operatorname{lin} \operatorname{cl} K$. Accordingly,

$$\operatorname{aff} F = \{x \in \mathbb{R}^n : a'x = b \text{ for all } (a,b) \in \operatorname{lin} \operatorname{cl} K\}. \tag{2}$$

Now we assume that $\pi \in \Pi_{SS}$. By the argument of *(ii)* $\Rightarrow$ *(vii)* in [10, Theorem 6.1], there exists $\varepsilon > 0$ such that

$$\operatorname{lin} \operatorname{cl} K_1 = \operatorname{lin} \operatorname{cl} K \tag{3}$$

for all $\pi_1 \in \Pi$ such that $d(\pi_1, \pi) < \varepsilon$. Combining (2) and (3) we get the conclusion.

Conversely, if $\operatorname{aff} F_1 = \operatorname{aff} F$ for all $\pi_1 \in \Pi$ such that $d(\pi_1, \pi) < \varepsilon$, then $\pi \in \operatorname{int} \Pi_C = \Pi_{SS}$. $\square$

Remark 4.2 *Observe that* $\pi \in \Pi_{SS}$ *and* $\dim F = 0$ *implies that* $\dim F_1 = 0$ *for any* π_1 *in some neighborhood of* π, *and hence* $\pi \in \operatorname{int} \Pi_{SU}$.

A reduced problem. Assume that π satisfies SSC and that $\dim F = d$, $1 \leq d \leq n-1$. Let $\varepsilon > 0$ be such that $\operatorname{aff} F_1 = \operatorname{aff} F$ if $d(\pi_1, \pi) < \varepsilon$. Since the translation of the feasible sets preserves all the properties of the corresponding problems, we can assume without loss of generality that $V := \operatorname{aff} F \subset \mathbb{R}^n$ is a linear subspace of dimension d. Let $\widetilde{\pi}_1 \in \Pi_C$ be the problem obtained by replacing in π_1 the cost vector c^1 by its orthogonal projection on V, say $\widetilde{c}^1$. Obviously, $\widetilde{F}_1^* = F_1^*$. Without loss of generality we can assume that $x_{d+1}, ..., x_n$ are linear combinations of $x_1, ..., x_d$ for all $x = (x_1, ..., x_n) \in V$. Let $(x_{d+1}, ..., x_n)' = R(x_1, ..., x_d)'$, where R is a $(n-d) \times d$ matrix. The equation $x = M(x_1, ..., x_d)'$, with $M = \begin{bmatrix} I_d \\ R \end{bmatrix}$, where I_d represents the $d \times d$ unit matrix, provides the full dimensional problem equivalent to π_1, say θ_1, called *reduced problem* of π_1:

$$\theta_1 : \ \operatorname{Inf} \left(e^1\right)' (x_1, ..., x_d) \ \text{ s.t. } \ \left(d_t^1\right)' (x_1, ..., x_d) \geq b_t, t \in T,$$

where $e^1 = M' \widetilde{c}^1$ and $d_t^1 = M' a_t^1$ for all $t \in T$. We denote by Θ the parameter space corresponding to index set T and d variables.

It is easy to prove that $\pi \in \Pi_{SU}$ ($\pi \in \Pi_U$) if and only if $\theta \in \Theta_{SU}$ ($\theta \in \Theta_U$, respectively) and that $\widehat{x}$ is a strong Slater element for π if and only if $(\widehat{x}_1, ..., \widehat{x}_d)$ is a strong Slater element for θ. Moreover,

$$d(\theta, \theta_1) \leq \left(\sqrt{n} \|M\| + 1\right) d(\pi, \pi_1). \tag{4}$$

It is also obvious that $\pi \in \Pi_{UB}$ implies that $\theta \in \Theta_{UB}$ because

$$\sup \{\|d_t\|, t \in T\} \leq \sqrt{n} \|M\| \sup \{\|a_t\|, t \in T\}.$$

However, the LHS-positive lower boundedness of π is not inherited by θ, e.g., for $n = 2$, if the constraint system is

$$\frac{1}{k} x_1 + k x_2 \geq -1, \qquad \frac{1}{k} x_1 - k x_2 \geq -1, \qquad k \in \mathbb{N},$$

for which $F = \{(x_1, 0) : x_1 \geq -1\}$, then $\pi \in \Pi_{PLB} \cap \Pi_{SS}$ but its reduced problem $\theta \notin \Theta_{PLB}$. Observe also that $\pi \notin \Pi_{UB}$ whereas $\theta \in \Theta_{UB}$.

Proposition 4.3 *Let $\pi \in \Pi_{SS}$ be such that $1 \leq \dim F \leq n-1$ and let $\theta \in \Theta$ be its reduced problem. Then, $\pi \in \operatorname{int}\Pi_{SU}$ if and only if $\theta \in \operatorname{int}\Theta_{SU}$.*

Proof. Let $\eta > 0$ be such that $\operatorname{aff} F_1 = \operatorname{aff} F$ for all $\pi_1 \in \Pi$ with $d(\pi_1, \pi) < \eta$. Suppose that ε is such that $0 < \varepsilon < \eta$ and $\pi_1 \in \Pi_{SU}$ for any $\pi_1 \in \Pi$ with $d(\pi_1, \pi) < \varepsilon$. For any $\theta_1 \in \Theta$ such that $d(\theta_1, \theta) < \frac{\varepsilon}{2}$ there exists $\pi_1 \in \Pi$ such that θ_1 is the reduced problem of π_1 and $d(\pi_1, \pi) < \varepsilon$. Since $\pi_1 \in \Pi_{SU}$, $\theta_1 \in \Theta_{SU}$. Now assume that $\theta \in \operatorname{int}\Theta_{SU}$ and let $0 < \varepsilon < \eta$ be such that $\theta_1 \in \Theta_{SU}$ if $d(\theta_1, \theta) < \varepsilon$. According to (4), if $\pi_1 \in \Pi$ satisfies $d(\pi_1, \pi) < \frac{\varepsilon}{\sqrt{n}\|M\|+1}$, then $d(\theta_1, \theta) < \varepsilon$, and this yields $\pi_1 \in \Pi_{SU}$. $\square$

Let us revisit Example 3.3, where $\dim F < n = 3$. Here $\operatorname{aff} F = \{x \in \mathbb{R}^3 : x_3 = 0\}$ and the full dimensional equivalent problem is

$$\theta : \ \operatorname{Inf}\ x_2 \quad \text{s.t.} \ -x_1 + x_2 \geq 0, t \in T_1 \text{ and } x_1 + x_2 \geq 0, t \in T_2,$$

where $T_1 = \mathbb{Z}_- \cup \{t_2\}$ and $T_2 = \mathbb{Z}_+ \cup \{t_1\}$. Since $\theta \in \Theta_{UB} \cap \Theta_{SS}$ and satisfies ENC at 0_2, we conclude that $\pi \in \operatorname{int}\Pi_{SU}$.

Proposition 4.4 *Let $\pi \in \Pi_{SS}$ be such that $1 \leq \dim F \leq n-1$ and let $\theta \in \Theta$ be its reduced problem. If $\pi \in \operatorname{int}\Pi_{SU}$ and the set of implicit active constraints at the optimal solution of θ is bounded, then θ satisfies ENC.*

Proof. It is an immediate consequence of Theorem 3.2, applied to θ, and of Proposition 4.3. $\square$

5 Upper bounding scaling and Nürnberger-type sufficient conditions

In order to get sufficient conditions for $\pi \in \operatorname{int}\Pi_{SU}$ when $\pi \notin \Pi_{UB}$ we appeal to suitable scalings of the constraint systems. We say that a mapping $\varphi : \Pi \to \Pi_{UB}$ is an *upper bounding* (UB in short) *scaling* when $\varphi(\pi_1)$ has the same objective function and the same feasible set as π_1, so that the optimal set of $\varphi(\pi_1)$ also coincides with F_1^* for all $\pi_1 \in \Pi$. Thus $\varphi(\pi)$ belongs to Π_{SU} (Π_U, Π_C) if and only if π belongs to Π_{SU} (Π_U, Π_C, respectively). Examples of UB-scaling mappings are the canonical normalization (divide each nontrivial constraint $a_t'x \geq b_t$ by $\|((a_t, b_t))\|$) referred to in Section 1, and the mappings ν and ω which associate with $\pi_1 = (a^1, b^1, c^1) \in \Pi$ the problems

$$\nu(\pi_1) = \pi_1^\nu : \ \operatorname{Inf}\ c_1'x \ \text{ s.t. } \left(a_t^{1\,\nu}\right)' x \geq b_t^{1\,\nu}, t \in T,$$

with

$$\left(a_t^{1\,\nu}, b_t^{1\,\nu}\right) = \begin{cases} \left(\frac{a_t^1}{\|a_t^1\|}, \frac{b_t^1}{\|a_t^1\|}\right), & \text{if } a_t^1 \neq 0_n, \\ \left(a_t^1, b_t^1\right), & \text{otherwise,} \end{cases}$$

and

$$\omega\left(\pi_1\right) = \pi_1^{\omega} : \ \text{Inf} \ \ c_1' x \ \ \text{s.t.} \ \left(a_t^{1\,\omega}\right)' x \geq b_t^{1\,\omega}, t \in T,$$

with

$$\left(a_t^{1\,\omega}, b_t^{1\,\omega}\right) = \begin{cases} \left(\frac{a_t^1}{\|a_t^1\|}, \frac{b_t^1}{\|a_t^1\|}\right), & \text{if } \|a_t^1\| > 1, \\ \left(a_t^1, b_t^1\right), & \text{otherwise,} \end{cases}$$

respectively (we use the same symbol to mark the image of a problem π_1 by a given UB-scaling mapping and its associated objects). These two UB-scalings are useful due to the properties shown in the next result.

Proposition 5.1 *(i)* ν *is continuous at any* $\pi \in \Pi_{PLB}$. *Moreover,* $\pi^{\nu} \in \Pi_{SS}$ *if* $\dim F = n$ *and no* a_t *is null.*
(ii) ω *is continuous. Moreover,* $\pi^{\omega} \in \Pi_{SS}$ *if* $\pi \in \Pi_{SS}$ *and* $\dim F = n$.

Proof. (i) Let $\pi \in \Pi_{PLB}$ and $\rho > 0$ be such that $\|a_t\| > \rho$ for all $t \in T$. Consider any sequence $\{\pi_r\} \subset \Pi$ such that $\pi_r \to \pi \in \Pi_{PLB}$. Given $0 < \varepsilon < \rho/2$, there exists $r_0 \in \mathbb{N}$ such that $d\left(\pi_r, \pi\right) < \varepsilon$ for all $r \geq r_0$. Then, for each $t \in T$ and $r \geq r_0$, we have $a_t^r \neq 0_n$ and

$$\begin{aligned} \left\| \frac{a_t}{\|a_t\|} - \frac{a_t^r}{\|a_t^r\|} \right\| &= \frac{\left\| \|a_t^r\| \, a_t - \|a_t^r\| \, a_t^r + \|a_t^r\| \, a_t^r - \|a_t\| \, a_t^r \right\|}{\|a_t\| \, \|a_t^r\|} \\ &\leq \frac{\|a_t^r\| \, \|a_t - a_t^r\| + \left| \|a_t^r\| - \|a_t\| \right| \ \|a_t^r\|}{\|a_t\| \, \|a_t^r\|} \\ &\leq \frac{\|a_t - a_t^r\| + \left| \|a_t^r\| - \|a_t\| \right|}{\|a_t\|} < \frac{2\varepsilon}{\rho}, \end{aligned}$$

so that

$$\left\| \frac{a_t}{\|a_t\|} - \frac{a_t^r}{\|a_t^r\|} \right\| < \frac{2\varepsilon}{\rho}. \tag{5}$$

Analogously,

$$\left| \frac{b_t}{\|a_t\|} - \frac{b_t^r}{\|a_t^r\|} \right| < \frac{2\varepsilon}{\rho}. \tag{6}$$

From (5) and (6) we conclude that $d\left(\pi_r^{\nu}, \pi^{\nu}\right) \leq \frac{2\varepsilon}{\rho}$ whenever $r \geq r_0$. Hence, ν is continuous at π.

Now, assume that $\pi \in \Pi$, $\dim F = n$, and $a_t \neq 0_n$ for all $t \in T$. The feasible set of π^ν is $F_\nu = F$. Since $\pi^\nu \in \Pi_{PLB} \cap \Pi_{UB}$, $\dim F_\nu = n$ if and only if π^ν satisfies SSC ([11, Proposition 2.1]). Thus π^ν satisfies SSC.

(ii) Next we prove that ω is continuous at an arbitrary $\pi \in \Pi$.

Consider $\{\pi_r\} \subset \Pi$ such that $\pi_r \to \pi$. Let $\varepsilon > 0$ and take $r_0 \in \mathbb{N}$ such that $d(\pi_r, \pi) < \varepsilon$ for all $r \geq r_0$. Let $t \in T$ and take $r \geq r_0$. Four cases can arise:

(a) $\|a_t\|, \|a_t^r\| \leq 1$. Since $(a_t^{r\,\omega}, b_t^{r\,\omega}) = (a_t^r, b_t^r)$ and $(a_t^\omega, b_t^\omega) = (a_t, b_t)$, we have

$$\left\| \begin{pmatrix} a_t^{r\,\omega} \\ b_t^{r\,\omega} \end{pmatrix} - \begin{pmatrix} a_t^\omega \\ b_t^\omega \end{pmatrix} \right\| \leq d(\pi_r, \pi) < \varepsilon.$$

(b) $\|a_t\|, \|a_t^r\| > 1$. Then $(a_t^{r\,\omega}, b_t^{r\,\omega}) = (a_t^{r\,\nu}, b_t^{r\,\nu})$ and $(a_t^\omega, b_t^\omega) = (a_t^\nu, b_t^\nu)$, and we get the aimed conclusion from (5) and (6).

(c) $\|a_t\| \leq 1 < \|a_t^r\|$. On one hand,

$$\|a_t^r\| = \|a_t + a_t^r - a_t\| \leq \|a_t\| + \|a_t^r - a_t\| < 1 + \varepsilon.$$

Thus, $0 \leq \|a_t^r\| - 1 < \varepsilon$, which gives

$$\begin{aligned} \left\| \frac{a_t^r}{\|a_t^r\|} - a_t \right\| &= \frac{\|a_t^r - \|a_t^r\| a_t\|}{\|a_t^r\|} \leq \|a_t^r - \|a_t^r\| a_t\| \\ &= \|a_t^r - a_t + (1 - \|a_t^r\|) a_t\| \\ &\leq \|a_t^r - a_t\| + (\|a_t^r\| - 1) \|a_t\| \\ &< 2\varepsilon. \end{aligned}$$

An analogous argument shows that $\left| \frac{b_t^r}{\|b_t^r\|} - b_t \right| < 2\varepsilon$. Hence $d(\pi_r^\omega, \pi^\omega) \leq 2\varepsilon$.

(d) $\|a_t^r\| \leq 1 < \|a_t\|$. The proof is similar to the previous one.

Finally, assume that $\pi \in \Pi_{SS}$ and $\dim F = n$. Then $\pi^\nu \in \Pi_{SS}$ by *(ii)*. Let $\widehat{x}, \overline{x} \in \mathbb{R}^n$, $\alpha > 0$, and $\beta > 0$ be such that $a_t' \widehat{x} \geq b_t + \alpha$ and $(a_t^\nu)' \overline{x} \geq b_t^\nu + \beta$ for all $t \in T$. Two cases are possible for any $t \in T$:

If $\|a_t\| \leq 1$, $(a_t^\omega, b_t^\omega) = (a_t, b_t)$, so that $(a_t^\omega)' \widehat{x} \geq b_t^\omega + \alpha$.

If $\|a_t\| > 1$, $(a_t^\omega, b_t^\omega) = (a_t^\nu, b_t^\nu)$, so that $(a_t^\omega)' \overline{x} \geq b_t^\omega + \beta$.

Thus, $(a_t^\omega)' \left(\frac{\widehat{x} + \overline{x}}{2} \right) \geq b_t^\omega + \frac{1}{2} \min\{\alpha, \beta\}$.

We conclude that $\pi^{\omega} \in \Pi_{SS}$. $\square$

Observe that for any $\bar{\pi} \in \Pi_{UB} \cap \Pi_{PLB}$ with bounded $\{b_t\}_{t \in T}$, the sequence $\pi_r := r^{-1}\bar{\pi}$, $r = 1, 2, ..$, converges to the null parameter whereas $\pi_r^{\nu} = \bar{\pi}^{\nu}$ for all $r \in \mathbb{N}$. Thus ν is not continuous on the whole space Π. On the other hand, $\pi \in \Pi_{SS}$ (the general assumption in any test of stable strong uniqueness) does not guarantee that $\pi^{\nu}, \pi^{\omega} \in \Pi_{SS}$.

Example 5.2 *Consider an LSIO problem π of a single variable with constraint system $\{kx \geq -1, k \in \mathbb{N}; -kx \geq -1, k \in \mathbb{N}\}$, whose solution set is $\{0\}$. Obviously, $\pi \in \Pi_{SS}$. Now, the constraint system of π^{ν}, and of π^{ω}, is given by $\left\{x \geq \frac{1}{k}, k \in \mathbb{N}; -x \geq -\frac{1}{k}, k \in \mathbb{N}\right\}$, whose unique solution, 0, is not a strong Slater element. Thus $\pi^{\nu}, \pi^{\omega} \notin \Pi_{SS}$.*

Theorem 5.3 *Let $\pi \in \Pi_{SS}$ and let φ be a UB-scaling mapping continuous at π such that $\varphi(\pi) \in \Pi_{SS}$. If $\varphi(\pi)$ satisfies ENC, then $\pi \in \operatorname{int} \Pi_{SU}$.*

Proof. Since $\varphi(\pi) \in \Pi_{UB} \cap \Pi_{SS}$ and $\varphi(\pi)$ satisfies ENC, then $\varphi(\pi) \in \operatorname{int} \Pi_{SU}$.

Assume now that $\pi \notin \operatorname{int} \Pi_{SU}$. Let $\{\pi_r\} \subset \Pi \diagdown \Pi_{SU}$ be such that $\pi_r \to \pi$. Since $\pi \in \Pi_{PLB} \cap \Pi_{SS}$ and this set is open, $\pi_r \in \Pi_{PLB} \cap \Pi_{SS}$ for r large enough. By the continuity assumption, $\varphi(\pi_r) \to \varphi(\pi) \in \operatorname{int} \Pi_{SU}$. This implies that $\varphi(\pi_r) \in \Pi_{SU}$ for r large enough, i.e., $\pi_r \in \Pi_{SU}$ for r large enough, in contradiction with $\{\pi_r\} \subset \Pi \diagdown \Pi_{SU}$. $\square$

Example 5.4 *Example 3.4 shows that the EN property is not inherited by $\varphi(\pi)$, even in the case that φ is continuous at π and the feasible set has full dimension. In fact,*

$$\pi^{\nu} : \operatorname{Inf} x_2 \quad s.t. \ \frac{x_1 + x_2}{\sqrt{2}} \geq 0, \frac{-x_1 + x_2}{\sqrt{2}} \geq 0, \text{ and } x_2 \geq -1/k, k \in \mathbb{N},$$

is in $\Pi_{UB} \cap \Pi_{SS}$ and does not satisfy ENC.

The next example shows that the converse statement of Theorem 5.3 is not true, at least for $\varphi = \nu$ and $\varphi = \omega$.

Example 5.5 *Consider the problem*

$$\begin{aligned} \pi : \operatorname{Inf} \quad & x_1 + x_2 \\ s.t. \quad & kx_1 \geq -1, k \in \mathbb{N}, \\ & kx_2 \geq -1, k \in \mathbb{N}, \\ & kx_1 + kx_2 \geq -1, k \in \mathbb{N}, \\ & -kx_1 - kx_2 \geq -k - 1, k \in \mathbb{N}. \end{aligned}$$

We have $F = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$, $F^* = \{0_2\}$, $\pi \in \Pi_{PLB} \cap \Pi_{SS}$, *and* $\pi \in \operatorname{int}\Pi_{SU}$ *(the last statement will be justified after Theorem 6.2). Nevertheless, the sets of implicit active constraints at* 0_2 *relative to* π *and* $\pi^\nu = \pi^\omega$ *are* $\emptyset$ *and* $\{(1,1),(1,0),(0,1)\}$, *respectively. Thus neither* π *nor* $\pi^\nu = \pi^\omega$ *satisfy ENC.*

Corollary 5.6 *Let* $\pi \in \Pi_{PLB} \cap \Pi_{SS}$ *be such that* $\dim F = n$. *If* π^ν *satisfies ENC, then* $\pi \in \operatorname{int}\Pi_{SU}$.

Corollary 5.7 *Let* $\pi \in \Pi_{SS}$ *be such that* $\dim F = n$. *If* π^ω *satisfies ENC, then* $\pi \in \operatorname{int}\Pi_{SU}$.

Corollary 5.8 *Let* $\pi \in \Pi_{SS}$ *be such that* $1 \leq \dim F \leq n-1$ *and let* $\theta \in \Theta$ *be its reduced problem. If* φ *is a UB-scaling mapping continuous at* θ *such that* $\varphi(\theta) \in \Theta_{SS}$ *and* $\varphi(\theta)$ *satisfies ENC, then* $\pi \in \operatorname{int}\Pi_{SU}$.

6 Absolute stability

In this section we need to consider the *feasible set mapping* $\mathcal{F} : \Pi \rightrightarrows \mathbb{R}^n$, which is the multivalued function that associates with each $\pi_1 \in \Pi$ its feasible set F_1. The following concepts are due to Bouligand and Kuratowski (see [1, Section 1.4], or [2]).

$\mathcal{F}$ is *lower semicontinuous* at $\pi \in \Pi$ (lsc, in brief) if, for each open set $W \subset \mathbb{R}^n$ such that $W \cap \mathcal{F}(\pi) \neq \emptyset$, there exists an open set $V \subset \Pi$, containing π, such that $W \cap \mathcal{F}(\pi_1) \neq \emptyset$ for each $\pi_1 \in V$. Obviously, $\mathcal{F}$ is lsc at $\pi \notin \operatorname{dom}\mathcal{F}$ and $\pi \in \operatorname{int}\operatorname{dom}\mathcal{F}$ if $\mathcal{F}$ is lsc at $\pi \in \operatorname{dom}\mathcal{F}$.

It is well-known that the feasible set mapping $\mathcal{F}$ is lower semicontinuous at π if and only if $\pi \in \Pi_{SS}$ ([10, Thm. 6.1]), which is also equivalent to π being stable with respect to consistency.

$\mathcal{F}$ is *upper semicontinuous* at $\pi \in \Pi$ (usc, in brief) if, for each open set $W \subset \mathbb{R}^n$ such that $\mathcal{F}(\pi) \subset W$, there exists an open set $V \subset \Pi$, containing π, such that $\mathcal{F}(\pi_1) \subset W$ for each $\pi_1 \in V$. If $\mathcal{F}$ is usc at $\pi \notin \operatorname{dom}\mathcal{F}$, then $\pi \in \operatorname{int}(\Pi \diagdown \operatorname{dom}\mathcal{F})$.

$\mathcal{F}$ is *closed* at $\pi \in \operatorname{dom}\mathcal{F}$ if for all sequences $\{\pi_r\} \subset \Pi$ and $\{z_r\} \subset \mathbb{R}^n$ satisfying $z_r \in \mathcal{F}(\pi_r)$ for all $r \in \mathbb{N}$, $\pi_r \to \pi$, and $z_r \to z$, one has $z \in \mathcal{F}(\pi)$. If $\mathcal{F}$ is usc at $\pi \in \operatorname{dom}\mathcal{F}$ and $\mathcal{F}(\pi)$ is closed, then $\mathcal{F}$ is closed at π. Conversely, if $\mathcal{F}$ is closed and *locally bounded* at $\pi \in \operatorname{dom}\mathcal{F}$ (i.e., if there exists a neighborhood of π, say V, and a bounded set in $\mathbb{R}^n$ containing $\mathcal{F}(\pi_1)$ for every $\pi_1 \in V$), then $\mathcal{F}$ is usc at π.

$\mathcal{F}$ is lsc (usc, closed) if it is lsc (usc, closed) at π for all $\pi \in \Pi$.

Finally, recall that π is *absolutely stable* (in the feasible set sense) if there exists $\delta > 0$ such that $\mathcal{F}(\pi_1) = F$ for all $\pi_1 \in \Pi$ with $d(\pi_1, \pi) < \delta$. This is the strongest conceivable form of stability of the feasible set mapping $\mathcal{F} : \Pi \rightrightarrows \mathbb{R}^n$.

The following result gives necessary and sufficient conditions for π being *absolutely stable* when the feasible set F is bounded.

Theorem 6.1 *Let $\pi \in \Pi_C$. If π is absolutely stable then there exists a positive scalar ε such that:*
(i) $a_t' x \geq b_t + \varepsilon$ for all $t \in T$ and for all $x \in F$, and
(ii) for each $x \notin F$ there exists some $t \in T$ such that $a_t' x < b_t - \varepsilon$.
Conversely, (i) and (ii) are sufficient conditions for π being absolutely stable if F is bounded.
Moreover, π absolutely stable implies that $\pi \in \Pi_{SS} \diagdown \Pi_{UB}$, and that F is bounded if, in addition, $\pi \notin \Pi_{PLB}$ and $a_t \neq 0_n$ for all $t \in T$.

Proof. Assume that π is absolutely stable and let $\delta > 0$ be such that $F_1 = F$ if $d(\pi_1, \pi) < \delta$. Put $\varepsilon = \delta/2$ and consider the problems π_1 and π_2 with $a_t^1 = a_t, b_t^1 = b_t + \varepsilon$, and $a_t^2 = a_t, b_t^2 = b_t - \varepsilon$, for $t \in T$, respectively. Then $F_1 = F_2 = F \neq \emptyset$. Hence, by taking any $x \in F = F_1$, *(i)* is satisfied. Moreover, if $x \notin F$ then $x \notin F_2$, which yields *(ii)*.
Suppose now that F is bounded and that there exists $\varepsilon > 0$ satisfying *(i)* and *(ii)*. $\mathcal{F}$ is upper semicontinuous at π by the boundedness of F ([10, Corollary 6.2.1]). Let $\rho, \gamma > 0$ be such that F_1 is contained in the open ball centered at 0_n with radius ρ, $B(0_n; \rho)$, if $d(\pi_1, \pi) < \gamma$.
Put

$$\delta = \frac{1}{2} \min \left\{ \varepsilon (n+1)^{-1/2} (\rho + 1)^{-1}, \gamma \right\} > 0.$$

If $x \in B(0_n; \rho)$ and π_1 satisfies $d(\pi_1, \pi) < \delta$, we have

$$\left\| \begin{pmatrix} x \\ -1 \end{pmatrix} \right\| \leq \rho + 1$$

and so, by the Cauchy-Schwartz inequality,

$$\left| \left[\begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right]' \begin{pmatrix} x \\ -1 \end{pmatrix} \right| \leq \delta (n+1)^{1/2} (\rho + 1) < \varepsilon. \tag{7}$$

If $x \in F$, given any $t \in T$, $\binom{a_t}{b_t}' \binom{x}{-1} \geq \varepsilon$ by assumption *(i)*; hence (7) yields $\binom{a_t^1}{b_t^1}' \binom{x}{-1} > 0$, so that $x \in F_1$. Thus $F \subset F_1$.
In the case $x \notin F$, take $t \in T$ such that $a_t' x < b_t - \varepsilon$, i.e., $\binom{a_t}{b_t}' \binom{x}{-1} < -\varepsilon$.

Reasoning as before we get $\binom{a_t^1}{b_t^1}'\binom{x}{-1} < 0$, so that $x \notin F_1$. Hence $F_1 \subset F$. Therefore $F_1 = F$.
Now, if π is absolutely stable, then $\pi \in \operatorname{int}\Pi_C = \Pi_{SS}$. Moreover, there exists $\delta > 0$ such that $F_1 = F_2$ for all $\pi_1, \pi_2 \in \Pi$ with $d(\pi_1, \pi) < \delta$ and $d(\pi_2, \pi) < \delta$. Then, according to Proposition 4.1 in [13], the set $\{\|a_t\|, t \in T\}$ cannot be bounded. In particular the index set T is infinite.
Suppose, additionally, that $\pi \notin \Pi_{PLB}$ and $a_t \neq 0_n$ for all $t \in T$. Let $\delta > 0$ be such that $F_1 = F$ if $d(\pi_1, \pi) \leq \delta$. Let $t_1, s_1, ..., t_n, s_n$ be non repeated elements of T such that $\|a_{t_i}\| < \frac{\delta}{2}$ and $\|a_{s_i}\| < \frac{\delta}{2}$, $i = 1, ..., n$. Let π_1 be the problem obtained by replacing $a_{t_i}'x \geq b_{t_i}$ by $\frac{\delta}{2}x_i \geq b_{t_i}$ and $a_{s_i}'x \geq b_{s_i}$ by $-\frac{\delta}{2}x_i \geq b_{s_i}$, $i = 1, ..., n$. Since $d(\pi_1, \pi) \leq \delta$, any $x \in F = F_1$ satisfies $\frac{2}{\delta}b_{t_i} \leq x_i \leq -\frac{2}{\delta}b_{s_i}$, $i = 1, ..., n$, i.e., F is contained in a box. $\square$

Observe that condition *(i)* in Theorem 6.1 can be seen as a uniform strong Slater condition. The problem π in $\mathbb{R}$ with the unique constraint $0x \geq -1$ satisfies conditions *(i)* and *(ii)*, but it is not absolutely stable; its feasible set F is unbounded. Notice that the feasible set mapping $\mathcal{F}$ is upper semicontinuous at π because $n = 1$, so the condition "F is bounded" can not be substituted by "$\mathcal{F}$ is upper semicontinuous at π".

The characterization of stable strong uniqueness for absolutely stable problems is quite simple. The next result can be seen as the LSIO counterpart of the Mangasarian characterization of uniqueness in LP in terms of perturbations of the cost vector.

Theorem 6.2 *Let $\pi \in \Pi_{AS}$. Then $\pi \in \operatorname{int}\Pi_{SU}$ if and only if $\pi \in \Pi_{SU}$.*

Proof. Let $\delta > 0$ be such that $F_1 = F$ if $d(\pi_1, \pi) < \delta$. Assume that $\pi \in \Pi_{SU}$ and let $\overline{x} \in F$ be the strongly unique optimal solution of π. This happens if and only if $c \in \operatorname{int} D(F; \overline{x})^0$ ([10, Theorem 10.5]). Thus, there exists $\rho > 0$ such that $B(c; \rho) \subset D(F; \overline{x})^0$. Let $\varepsilon = \min\{\delta, \rho\}$. Then $d(\pi_1, \pi) < \varepsilon$ implies $c^1 \in \operatorname{int} D(F; \overline{x})^0 = \operatorname{int} D(F_1; \overline{x})^0$, so that $\pi_1 \in \Pi_{SU}$. $\square$

Consider again Example 5.5, where $\pi \in \Pi_{SU}$ and

$$F = \operatorname{conv}\{(0,0), (1,0), (0,1)\} = \left\{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\right\}. \tag{8}$$

By Theorem 6.2, $\pi \in \operatorname{int}\Pi_{SU}$ provided $\pi \in \Pi_{AS}$. Since $(0,0), (1,0)$, and $(0,1)$ satisfy $a_t'x \geq b_t + 1$ for all $t \in T$, condition *(i)* in Theorem 6.1 hods with $\varepsilon = 1$. Moreover, if $x \notin F$ at least one of the three constraints in (8) fails. Assume, e.g., that $x_1 + x_2 > 1$. In this case there exists $r \in \mathbb{N}$ such that $r(x_1 + x_2 - 1) > 2$, and so condition *(ii)* also holds with $\varepsilon = 1$ (the other two cases, $x_1 < 0$ and $x_2 < 0$, are similar). Thus $\pi \in \Pi_{AS}$, which implies that $\pi \in \operatorname{int}\Pi_{SU}$.

Now consider that $\pi \in \Pi_{AS}$. Observe that, according to Corollary 5.5 in [11], if there exists $\overline{x} \in F$ such that $c \in \operatorname{int} \operatorname{cone} A(\overline{x})$, then $\overline{x}$ is a strongly unique solution of π and so $\pi \in \operatorname{int} \Pi_{SU}$. On the other hand, if $n = 2$, $c = (0,1)$, and $F = \operatorname{cl} \operatorname{conv} \left\{ \left(\pm \frac{1}{r}, \frac{1}{r^2} \right), r \in \mathbb{N} \right\}$, we have $\pi \in \Pi_U \diagdown (\operatorname{int} \Pi_U)$. Thus, it is not possible to replace Π_{SU} by Π_U in Theorem 6.2. Finally we give a generic result.

Theorem 6.3 *Let $\pi \in \Pi_{AS}$ and let W be a neighborhood of π where $\mathcal{F}$ is constant. Then there exists a dense and G_δ subset of W formed by problems which have at most one optimal solution.*

Proof. Suppose that F is the feasible set of π. For each $c \in \mathbb{R}^n$, consider the parametric problem $P(c)$: $\operatorname{Inf} c'x$ subject to $x \in F$. The property will follows by showing that there exists a dense and G_δ subset A of $\mathbb{R}^n$ such that the optimal set of $P(c)$, $F^*(c)$, satisfies $|F^*(c)| \leq 1$ for all $c \in A$. Given $m \in \mathbb{N}$, let $F_m := F \cap \operatorname{cl} B(0_n; m)$, which is nonempty for m large enough, say $m \geq m_0$. Consider the problem $P_m(c)$: $\operatorname{Inf} c'x$ subject to $x \in F_m$ and its optimal set mapping $\mathcal{F}_m^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. This mapping $\mathcal{F}_m^*$ is uniformly bounded and closed, so it is usc and Fort's theorem ([6], [17]) implies that it is also lsc on some dense and G_δ subset A_m of $\mathbb{R}^n$. The set $A := \cap_{m \geq m_0} A_m$ is dense and G_δ because $\mathbb{R}^n$ is a Baire space. Now we will prove that $|F^*(c)| \leq 1$ for any $c \in A$. Suppose on the contrary that there exist $c \in A$, and x^*, $y^* \in F^*(c)$, $x^* \neq y^*$. Taking $m > \max\{\|x^*\|, \|y^*\|\}$, it follows that $x^*, y^* \in F_m^*(c)$.
Consider $\varepsilon > 0$ small enough so that $(y^* - x^*)'(z - x^*) > 0$ for all $z \in B(y^*; \varepsilon)$. Put $c^k := c + \frac{1}{k}(y^* - x^*)$ and observe that, for any $z \in F \cap B(y^*; \varepsilon)$,

$$\left(c^k\right)'(z - x^*) = c'(z - x^*) + \frac{1}{k}(y^* - x^*)'(z - x^*) > 0.$$

Hence $\left(c^k\right)' z > \left(c^k\right)' x^*$ and so $z \notin \mathcal{F}_m^*\left(c^k\right)$, a contradiction with $\mathcal{F}_m^*$ being lsc at c, because $\mathcal{F}_m^*(c) \cap B(y^*; \varepsilon) \neq \emptyset$. □

The final example shows that we cannot replace "unique" by "strongly unique" in Theorem 6.3. It also shows that $(\operatorname{int} \Pi_{SU}) \cap \Pi_{SS} \subsetneqq (\operatorname{int} \Pi_U) \cap \Pi_{SS}$, so that $\operatorname{int} \Pi_{SU} \subsetneqq \operatorname{int} \Pi_U$.

Example 6.4 *Let $n = 2$, and $T = \mathbb{N} \times [0, 2\pi]$. If c is any non null vector in $\mathbb{R}^2$, $a_{(m,\alpha)} = -(m \cos \alpha, m \sin \alpha)$ and $b_{(m,\alpha)} = -m - 1$ for all $(m, \alpha) \in T$. Obviously, $\pi \in \Pi_{PLB} \cap \Pi_{SS}$. Since $\left\| a_{(m,\alpha)} \right\| = m$, $a_{(m,\alpha)}^\nu = -(\cos \alpha, \sin \alpha)$ and $b_{(m,\alpha)}^\nu = -1 - \frac{1}{m}$, we have*

$$\begin{aligned} F = F_\nu &= \bigcap_{m=1}^{\infty} \left\{ x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq \left(1 + \tfrac{1}{m}\right)^2 \right\} \\ &= \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} \end{aligned}$$

and $\{\pi, \pi^\nu\} \subset \Pi_U \diagdown \Pi_{SU}$*. Since* $A^\nu(\overline{x}) = \{-\overline{x}\}$ *for all* $\overline{x} \in \operatorname{bd} F$, π^ν *does not satisfy ENC. Nonetheless we will show that* π *is absolutely stable. In fact, given* $x \in F$ *and* $t = (m, \alpha) \in T$, *we have*

$$a_t' x - b_t = 1 + m\left[1 - \left((\cos\alpha)\, x_1 + (\sin\alpha)\, x_2\right)\right] \geq 1,$$

and given $x \notin F$, *there exists* $\alpha \in [0, 2\pi]$ *such that* $(\cos\alpha)\, x_1 + (\sin\alpha)\, x_2 > 1$. *Then there exists* $m \in \mathbb{N}$ *such that* $m\left[(\cos\alpha)\, x_1 + (\sin\alpha)\, x_2 - 1\right] > 2$, *i.e.,*

$$(-m\cos\alpha)\, x_1 + (-m\sin\alpha)\, x_2 < -(m+1) - 1.$$

Hence conditions (i) and (ii) in Theorem 6.1 hold for $\varepsilon = 1$. *On the other hand, since* $c^1 \neq 0_n$ *in some neighborhood of* π *we conclude that* $\pi \in \operatorname{int} \Pi_U$.

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