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On an extension of Pólya's Positivstellensatz

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Abstract In this paper we provide a generalization of a Positivstellensatz by Pólya [Pólya in Naturforsch Ges Zürich 73:141–145 1928]. We show that if a homogeneous polynomial is positive over the intersection of the non-negative orthant and a given basic semialgebraic cone (excluding the origin), then there exists a "Pólya type" certificate for non-negativity. The proof of this result uses the original Positivstellensatz by Pólya, and a Positivstellensatz by Putinar and Vasilescu [Putinar and Vasilescu C R Acad Sci Ser I Math 328(7) 1999].

Keywords Positivstellensatz · Semialgebraic set · Non-negativity certificate · Polynomial optimization

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1 Introduction

A *Positivstellensatz* is a theorem that relates positivity of certain functions (typically polynomials) to algebraic representations of these functions [14]. In the real algebraic geometry people also use term *Nichtnegativestellensatz* for results about algebraic certificates for non-negative polynomials. Sometimes these two names are reserved only for theorems that are "if and only if", see e.g. Scheiderer [28]. In this paper we will use *Positivstellensatz* for results which provide algebraic certificates for positivity (or non-negativity) for positive polynomials.

The first theorem carrying this name is due to Stengle and Krivine. The so-called Krivine–Stengle Positivstellensatz has been initially attributed to Stengle [29], but later it became clear

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that the main ideas of this result were already published few years earlier by Krivine [11]. This theorem says that for a polynomial f (in this paper we consider only polynomials with real coefficients) that is positive over the set, defined by polynomial inequalities:

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n \mid f_i(\mathbf{x}) \geq 0, \quad i = 1, \ldots, m\} \tag{1}$$

we can find under some assumptions polynomials p, q such that $pf = 1 + q$ and p, q belong to so-called preordering generated by polynomials f_i, see e.g. [20,28]. This is a pure Positivstellensatz since it deals with positivity and is "if and only if". Note that we call (1) a *basic closed semialgebraic set*. If all of the polynomials f_i are homogeneous, then we shall refer to (1) as a *semialgebraic cone*.

In our opinion the most famous Positivstellensätze are due to Pólya [10,18], Schmüdgen [26] and Putinar [22]. Pólya proved (see Theorem 2.1 in Sect. 2) that if given real homogeneous polynomial f is positive on $\mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ then multiplying it with $(\sum_i x_i)^r$, where r is sufficiently large, gives polynomial with non-negative coefficients, i.e. a certificate that the original polynomial is nonnegative on $\mathbb{R}^n_+$. Note that there exist also a "positive" version of the Pólya's theorem stating that all coefficients of $(\sum_i x_i)^r f$ are positive for r sufficiently large [21].

Theorems of Schmüdgen [26] and Putinar [22] refined the Krivine–Stengle theorem by showing that (i) if the semialgebraic set (1) is compact then f belongs to the preordering generated by $\{f_i\}$ (Schmüdgen) and (ii) if the quadratic module generated by $\{f_i\}$ is Archimedean then f belongs to this module (Putinar). In both cases we do not have "if and only if", i.e. we only have certificates for non-negativity. For complexity issues related with Schmüdgen and Putinar Positivstellensätze see [16,27], while a comprehensive overview of this type of results can be found in [14,28].

The Pólya's Positivstellensatz therefore implies non-negativity certificate based on polynomials with non-negative coefficients while Schmüdgen's and Putinar's theorems guaranty non-negativity certificates that are based on polynomials which are sum-of-squares (SOS) (see definition in Sect. 1.1).

Reznick [25] (see Theorem 2.2) and Putinar and Vasilescu [23] (Theorem 2.3) proved two results which together with Pólya's theorem motivated our research. Reznick provided a SOS type non-negativity certificate for homogeneous polynomials of even degree which are positive on all non-zero points from $\mathbb{R}^n$ while Putinar and Vasilescu proved that there exists a SOS certificate for non-negativity of homogeneous polynomial of even degree if it is positive on the semialgebraic set (1) defined by $\{f_i\}$ of even degree.

In this paper, we prove a new Positivstellensatz yielding a non-negativity certificate for homogeneous polynomials which are positive on the non-zero points from the non-negative orthant intersected by semialgebraic set (1) where polynomials defining (1) are homogeneous. This certificate is based on polynomials with all their coefficients being non-negative, hence our result is in this sense a generalization of the Pólya's theorem.

When we try to use Positivstellensätze mentioned above to construct non-negativity certificates or compute the infimum of f over the semialgebraic set (1) we naturally meet the approximation hierarchies. Indeed, the Pólya nonnegativity certificate for fixed r can be found by solving a linear programming feasibility problem (see e.g. [5,17]). Since we do not know in advance for which r there will exist a certificate of this type we have to consider the hierarchy of linear programs, parameterized by r.

Similarly Reznick's and Putinar–Vasilescu's theorems imply that we can search for non-negativity certificates mentioned in these theorems by solving a hierarchy of semidefinite programming problems. Indeed, finding a certificate from Reznick theorem for fixed r amounts to solving one semidefinite programming feasibility problem with linear equations implied by demand $(\mathbf{x}^\mathsf{T}\mathbf{x})^r f(\mathbf{x})$ is SOS. If this problem is infeasible for given r, we continue with $r+1$. The theorem guaranties that we will stop in a finite time if the polynomial f satisfies the conditions of the theorem, but with increasing r the complexity of semidefinite programming problems increases very fast. Similarly Putinar–Vasilescu's theorem naturally implies a hierarchy of semidefinite programming feasibility problems for the non-negativity certificates of order r. Here the elements of this hierarchy are defined by putting uniform bound on the degree of summands in the SOS certificate. We suggest the reader to consider [12] for more details about these hierarchies, see also Remark 3.5.

Among others Pólya's Positivstellensatz implies also a linear programming approximation hierarchies for the copositive programming problems [5,17], while Reznick's and Putinar–Vasilescu's theorems imply semidefinite programming approximation hierarchies for the copositive programming problems. We suggest the reader to consider also [1,3,4,6,8,19] for other results about linear and semidefinite programming approximation hierarchies for the copositive programming problems.

Our new Positivstellensatz has strong potential to construct similar hierarchies for the linear and non-linear optimization problems over the semialgebraic sets, which are subset of the non-negative orthant. Some new results in this direction will be presented in the paper [7].

1.1 Notation

We let $\mathbf{e} \in \mathbb{R}^n$ denote the all-ones vector. We use $\mathbb{Z}_+ = \{0, 1, 2\ldots\}$ and $\mathbb{Z}_{++} = \{1, 2, 3\ldots\}$. For $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{m} \in \mathbb{Z}_+^n$ we let $\mathbf{x}^\mathbf{m} := \prod_{i=1}^n x_i^{m_i}$ (where $0^0 := 1$).

We let $\mathbb{R}[\mathbf{x}]$ denote the ring of multivariate polynomials on $\mathbb{R}^n$ with real coefficients in variables $\mathbf{x} := (x_1, \ldots, x_n)$. For a polynomial $f(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}_+^n} f_\mathbf{m}\mathbf{x}^\mathbf{m} \in \mathbb{R}[\mathbf{x}]$, we let $\deg(f)$ denote its degree, i.e. the highest degree of its terms: $\deg(f) = \max\{\mathbf{e}^\mathsf{T}\mathbf{m} \mid f_\mathbf{m} \neq 0\}$, and for $f(\mathbf{x}) = 0$ we define $\deg(f) := 0$. When $\deg(f)$ is even number we say that polynomial has even degree.

Polynomial is *homogeneous* if all of its terms have the same degree. Note that for a homogeneous polynomial $f \in \mathbb{R}[\mathbf{x}]$ of degree d, we have $f(\lambda\mathbf{x}) = \lambda^d f(\mathbf{x})$ for all $\lambda \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^n$.

For two polynomials $f, g \in \mathbb{R}[\mathbf{x}]$, we write $f = g$, or equivalently $f(\mathbf{x}) = g(\mathbf{x})$, if all the corresponding coefficients of these polynomials are equal.

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we define their Hadamard product $(\mathbf{a}\circ\mathbf{b}) \in \mathbb{R}^n$ such that for all i we have $(\mathbf{a}\circ\mathbf{b})_i = a_i b_i$. Note that we have $(\mathbf{x}\circ\mathbf{x})^\mathbf{m} = \mathbf{x}^{2\mathbf{m}} = (\mathbf{x}^\mathbf{m})^2 \in \mathbb{R}[\mathbf{x}]$ for all $\mathbf{m} \in \mathbb{Z}_+^n$. We call such terms *even terms*. The terms that are not even we call *odd terms*.

For a polynomial $f \in \mathbb{R}[\mathbf{x}]$ and a set $\mathcal{M} \subseteq \mathbb{R}$, we let $f^{-1}(\mathcal{M}) := \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \in \mathcal{M}\}$.

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ is defined to be *sum-of-squares* (SOS), if there exists $p \in \mathbb{Z}_{++}$ and polynomials $h_1, \ldots, h_p \in \mathbb{R}[\mathbf{x}]$ such that $f(\mathbf{x}) = \sum_{i=1}^p \left(h_i(\mathbf{x})\right)^2$. We now note that:

1. if $f, g \in \mathbb{R}[\mathbf{x}]$ are SOS then both $(f+g)$ and fg are SOS,
2. if $f \in \mathbb{R}[\mathbf{x}]$ is SOS then $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,
3. if $f \in \mathbb{R}[\mathbf{x}]$ such that $f(\mathbf{x}\circ\mathbf{x})$ is SOS then $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}_+^n$.

We also consider polynomials having only non-negative coefficients, for which we similarly have:

1. if all the coefficients of $f, g \in \mathbb{R}[\mathbf{x}]$ are non-negative then so are all the coefficients of both $(f+g)$ and fg,
2. if all the coefficients of $f \in \mathbb{R}[\mathbf{x}]$ are non-negative then $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}_+^n$,
3. if all the coefficients of $f \in \mathbb{R}[\mathbf{x}]$ are non-negative then $f(\mathbf{x}\circ\mathbf{x})$ is SOS.

1.2 Contribution

The main contributions of this paper are twofold, and are summarized below. In these, when we say that a homogeneous polynomial f is positive on a set $\mathcal{K}$, we mean that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{K} \setminus \{\mathbf{0}\}$. The origin is excluded as for all homogeneous polynomials f we have $f(\mathbf{0}) = \mathbf{0}$.

1. The central result of this paper is Theorem 2.4, where we prove a non-trivial generalization of the well-known Pólya's theorem [10,18]. Our new Positivstellensatz is closely related to Putinar and Vasilescu's Positivstellensatz [23,24]. The original Pólya's theorem states that there exists a certificate for non-negativity for all homogeneous polynomials which are positive on the non-negative orthant, while the Putinar–Vasilescu's theorem states that there exists an SOS certificate for non-negativity for all homogeneous polynomials of even degree that are positive on a semialgebraic cone defined by homogeneous polynomials of even degree. Our version states roughly the same, but we do not demand polynomials of even degree and we only consider semialgebraic cones that are subsets of the non-negative orthant.
2. If we allow the semialgebraic cone to be defined by infinitely many polynomial inequalities, then we can prove that a given polynomial is positive on this set if and only if it is positive on a semialgebraic cone which is defined by some finite subset of the polynomial inequalities.

To the best of our knowledge, neither of these results have previously been published.

2 Positivstellensätze

In this section we recall few well-known Positivstellensätze which motivated our result in Theorem 2.4 and are important basis to prove it. We first recall the well-known Pólya's Positivstellensatz:

Theorem 2.1 ([10, Section 2.24], [18]) *Let $f \in \mathbb{R}[\mathbf{x}]$ be a homogeneous polynomial on $\mathbb{R}^n$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$. Then for some $r \in \mathbb{Z}_+$, we have that all the coefficients of $(\mathbf{e}^\mathsf{T}\mathbf{x})^r f(\mathbf{x})$ are non-negative.*

Powers and Reznick [21] proved stronger result. If r is larger than a certain number which depends only on the degree of f and its minimum on the standard simplex then all coefficients of $(\mathbf{e}^\mathsf{T}\mathbf{x})^r f(\mathbf{x})$ are positive, hence the Póly's theorem is actually "if and only if".

Another well known Positivstellensatz is the following from Reznick, which provides a constructive solution to Hilbert's seventeenth problem for the case of positive definite forms:

Theorem 2.2 [25] *Let $f \in \mathbb{R}[\mathbf{x}]$ be a homogeneous polynomial of even degree on $\mathbb{R}^n$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Then for some $r \in \mathbb{Z}_+$, we have that $(\mathbf{x}^\mathsf{T}\mathbf{x})^r f(\mathbf{x})$ is SOS.*

Faybusovich [9, Theorem 1] provided an explicit bound for the exponent r in the theorem above. Putinar and Vasilescu extended this Positivstellensatz to give the following theorem.

Theorem 2.3 [23, Theorem 1] *Let $m \in \mathbb{Z}_{++}$ and $f_0, \ldots, f_m \in \mathbb{R}[\mathbf{x}]$ be homogeneous polynomials of even degree on $\mathbb{R}^n$ such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \bigcap_{i=1}^m f_i^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$ and $f_1(\mathbf{x}) = 1$. (Note that $f_1^{-1}(\mathbb{R}_+) = \mathbb{R}^n$.) Then for some $r \in \mathbb{Z}_+$, there exists homogeneous SOS polynomials $g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$ such that $(\mathbf{x}^\mathsf{T}\mathbf{x})^r f_0(\mathbf{x}) = \sum_{i=1}^m f_i(\mathbf{x}) g_i(\mathbf{x})$.*

We suggest the reader to consider also [24, Theorem 4.5] where this theorem was generalized for non-homogeneous polynomials of even degree.

In Sect. 3 we shall prove the following new Positivstellensatz, which can be seen as an extension of Pólya's Positivstellensatz and is closely related to Putinar–Vasilescu's Positivstellensatz:

Theorem 2.4 *Let be $m \in \mathbb{Z}_{++}$ and $f_0, \ldots, f_m \in \mathbb{R}[\mathbf{x}]$ be homogeneous polynomials on $\mathbb{R}^n$ such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n_+ \cap \bigcap_{i=1}^m f_i^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$ and $f_1(\mathbf{x}) = 1$. Then for some $r \in \mathbb{Z}_+$, there exist homogeneous polynomials $g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$ such that all of their coefficients are non-negative and $(\mathbf{e}^\mathsf{T}\mathbf{x})^r f_0(\mathbf{x}) = \sum_{i=1}^m f_i(\mathbf{x}) g_i(\mathbf{x})$.*

Example 2.5 Consider $n = 3$ and $m = 4$, with

$$\begin{aligned} f_0(\mathbf{x}) &= 3x_1 - 2x_2 - 2x_3, \\ f_1(\mathbf{x}) &= 1, \\ f_2(\mathbf{x}) &= x_1 - x_2, \\ f_3(\mathbf{x}) &= x_1 - x_3, \\ f_4(\mathbf{x}) &= x_1^2 - 4x_2x_3. \end{aligned}$$

For $r = 1$, a certificate for $f_0(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n_+ \cap \bigcap_{i=1}^4 f_i^{-1}(\mathbb{R}_+)$ is given by:

$$(\mathbf{e}^\mathsf{T}\mathbf{x}) f_0(\mathbf{x}) = (x_1 + 2x_2) f_2(\mathbf{x}) + (x_1 + 2x_3) f_3(\mathbf{x}) + f_4(\mathbf{x}).$$

We can say even more, this is a certificate for positivity of f_0 since there exists no $\mathbf{x} \geq 0$ such that $f_i(\mathbf{x}) = 0$ for $i = 2, 3, 4$.

In Sect. 4, the final section of this paper, we shall look at how Theorem 2.3 and 2.4 can be extended for infinitely many polynomials.

3 Proof of new Positivstellensatz

In this section we shall consider $m \in \mathbb{Z}_{++}$ and homogeneous polynomials $f_1, \ldots, f_m \in \mathbb{R}[\mathbf{x}]$ on $\mathbb{R}^n$, with $f_1(\mathbf{x}) = 1$, and we define the set

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \|\mathbf{x}\|_2 = 1,\ f_i(\mathbf{x}) \geq 0 \quad \text{for all } i = 1, \ldots, m\}. \tag{2}$$

Note that $f_1(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, so this is a redundant constraint in the description of $\mathcal{X}$, however it will simplify the notation later on. It should also be noted that $\mathcal{X}$ is a compact set, as it is a closed and bounded set.

We next consider a homogeneous polynomial f_0 such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$. Note that this is equivalent to having $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \operatorname{cone} \mathcal{X} \setminus \{0\}$, where

$$\begin{aligned} \operatorname{cone} \mathcal{X} &:= \{0\} \cup \{\lambda\mathbf{x} \mid \lambda > 0, \quad \mathbf{x} \in \mathcal{X}\} \\ &= \{\mathbf{x} \in \mathbb{R}^n_+ \mid f_i(\mathbf{x}) \geq 0 \quad \text{for all } i = 1, \ldots, m\}. \end{aligned}$$

The aim of this section is to find simple certificates, based on polynomials with nonnegative coefficients, to certify that $f_0(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \operatorname{cone} \mathcal{X}$.

We begin by considering how Theorem 2.3 can be extended for $\mathbf{x}$ being restricted to the non-negative orthant but with the polynomials not being restricted to have even degree. We will use the fact that if $\mathbf{x} \in \mathbb{R}^n_+$ then $\mathbf{x} = \mathbf{z} \circ \mathbf{z}$ for some $\mathbf{z} \in \mathbb{R}$.

Theorem 3.1 *Let f_0 to be a homogeneous polynomial such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$, where $\mathcal{X}$ is as given in (2). Then for some $r \in \mathbb{Z}_+$, there exists homogeneous polynomials $g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$ such that the polynomials $g_i(\mathbf{z} \circ \mathbf{z})$ (in the variable $\mathbf{z} = (z_1, \ldots, z_n)$) are SOS for all i and $(\mathbf{e}^\mathsf{T}\mathbf{x})^r f_0(\mathbf{x}) = \sum_{i=1}^m f_i(\mathbf{x}) g_i(\mathbf{x})$.*

Proof Since $\mathcal{X} \subset \mathbb{R}^n_+$ we can substitute every $\mathbf{x} \in \mathcal{X}$ with $\mathbf{z} \circ \mathbf{z}$ for $\mathbf{z} \in \mathbb{R}$. Polynomials $f_i(\mathbf{z} \circ \mathbf{z})$ in variable $\mathbf{z} = (z_1, \ldots, z_n)$ are satisfying assumptions of Theorem 2.3, hence this theorem implies that for some $r \in \mathbb{Z}_+$ there exist sets of homogeneous polynomials $\{h_j \mid j \in \mathcal{I}_i\} \subseteq \mathbb{R}[\mathbf{x}]$ for all $i = 1, \ldots, m$ such that

$$(\mathbf{e}^\mathsf{T}(\mathbf{z} \circ \mathbf{z}))^r f_0(\mathbf{z} \circ \mathbf{z}) = (\mathbf{z}^\mathsf{T}\mathbf{z})^r f_0(\mathbf{z} \circ \mathbf{z}) = \sum_{i=1}^m f_i(\mathbf{z} \circ \mathbf{z}) \sum_{j \in \mathcal{I}_i} \left(h_j(\mathbf{z})\right)^2.$$

We now note that for all j there exists a unique set of homogenous polynomials $\{h_{j,\mathbf{t}} \mid \mathbf{t} \in \{0,1\}^n\} \subseteq \mathbb{R}[\mathbf{x}]$ such that $h_j(\mathbf{z}) = \sum_{\mathbf{t} \in \{0,1\}^n} \mathbf{z}^\mathbf{t} h_{j,\mathbf{t}}(\mathbf{z} \circ \mathbf{z})$. We then get that

$$\begin{aligned}(\mathbf{e}^\mathsf{T}(\mathbf{z} \circ \mathbf{z}))^r f_0(\mathbf{z} \circ \mathbf{z}) &= \sum_i f_i(\mathbf{z} \circ \mathbf{z}) \sum_{j \in \mathcal{I}_i} \sum_{\mathbf{t} \in \{0,1\}^n} (\mathbf{z} \circ \mathbf{z})^\mathbf{t} \left(h_{j,\mathbf{t}}(\mathbf{z} \circ \mathbf{z})\right)^2 \\ &\quad + \sum_i f_i(\mathbf{z} \circ \mathbf{z}) \sum_{j \in \mathcal{I}_i} \sum_{\substack{\mathbf{s},\mathbf{t} \in \{0,1\}^n: \\ \mathbf{s} \neq \mathbf{t}}} \mathbf{z}^{\mathbf{s}+\mathbf{t}} h_{j,\mathbf{s}}(\mathbf{z} \circ \mathbf{z}) h_{j,\mathbf{t}}(\mathbf{z} \circ \mathbf{z}).\end{aligned}$$

We shall call terms of the form $\mathbf{z}^{2\mathbf{m}}$ for some $\mathbf{m} \in \mathbb{Z}^n_+$ even terms and all other terms odd terms. We note that all terms on the left hand side are even while on the right hand side we have only even terms in the first part and only odd terms in the second part. Therefore the second part on the right hand side must be identically zero, hence all its coefficients are equal zero. Therefore

$$(\mathbf{e}^\mathsf{T}(\mathbf{z} \circ \mathbf{z}))^r f_0(\mathbf{z} \circ \mathbf{z}) = \sum_i f_i(\mathbf{z} \circ \mathbf{z}) \sum_{\mathbf{t} \in \{0,1\}^n} (\mathbf{z} \circ \mathbf{z})^\mathbf{t} \sum_{j \in \mathcal{I}_i} \left(h_{j,\mathbf{t}}(\mathbf{z} \circ \mathbf{z})\right)^2.$$

The equality above states that the polynomials on the left hand side and on the right hand side are equal on the non-negative orthant. Since this set has an interior point these polynomials are equal (all corresponding coefficients are equal) on whole $\mathbb{R}^n$, i.e.

$$(\mathbf{e}^\mathsf{T}\mathbf{x})^r f_0(\mathbf{x}) = \sum_i f_i(\mathbf{x}) \sum_{\mathbf{t} \in \{0,1\}^n} \mathbf{x}^\mathbf{t} \sum_{j \in \mathcal{I}_i} \left(h_{j,\mathbf{t}}(\mathbf{x})\right)^2, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Now letting $g_i(\mathbf{x}) = \sum_{\mathbf{t} \in \{0,1\}^n} \mathbf{x}^\mathbf{t} \sum_{j \in \mathcal{I}_i} \left(h_{j,\mathbf{t}}(\mathbf{x})\right)^2$ for all i we get the required result.

□

Remark 3.2 In the previous theorem we have for all i either

$$g_i(\mathbf{x}) = 0 \quad \text{or} \quad \deg(g_i) + \deg(f_i) = r + \deg(f_0),$$

since all polynomials are homogeneous.

We next consider the following proposition and the corresponding corollary.

Proposition 3.3 *Let f_0 to be a homogeneous polynomial such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$, where $\mathcal{X}$ is as given in (2). Then for any homogeneous polynomial $h \in \mathbb{R}[\mathbf{x}]$ such that $\deg(h) = \deg(f_0)$, there exists $\varepsilon > 0$ such that $f_0(\mathbf{x}) - \varepsilon h(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$.*

Proof If $h(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{X}$ then the result trivially holds. From now on we assume that this is not the case and thus $M_h := \max_{\mathbf{x}}\{|h(\mathbf{x})| \mid \mathbf{x} \in \mathcal{X}\} > 0$. Moreover, by assumption $m_{f_0} := \min_{\mathbf{x}}\{f_0(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X}\} > 0$. Here we have used the fact that $\mathcal{X}$ is compact and that continuous function on a compact set attains their maximum and minimum. Therefore $\varepsilon = \frac{m_{f_0}}{2M_h} > 0$ and thus the polynomial $f_0 - \varepsilon h$ is positive on $\mathcal{X}$. □

Corollary 3.4 *Let f_0 to be a homogeneous polynomial such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$, where $\mathcal{X}$ is as given in (2). We let*

$$c_i = -\deg(f_i) + \max\{\deg(f_j) \mid j = 0, \ldots, m\} \qquad \text{for all } i = 0, \ldots, m,$$

and for all $\varepsilon \in \mathbb{R}$ we define the homogeneous polynomial

$$f_\varepsilon(\mathbf{x}) := (\mathbf{e}^\mathsf{T}\mathbf{x})^{c_0} f_0(\mathbf{x}) - \varepsilon \sum_{i=1}^{m} (\mathbf{e}^\mathsf{T}\mathbf{x})^{c_i} f_i(\mathbf{x}).$$

Then there exists an $\varepsilon > 0$ such that $f_\varepsilon(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$.

Proof Since $F(\mathbf{x}) = (\mathbf{e}^\mathsf{T}\mathbf{x})^{c_0} f_0(\mathbf{x})$ is homogeneous and positive on $\mathcal{X}$ and $\sum_{i=1}^m (\mathbf{e}^\mathsf{T}\mathbf{x})^{c_i} f_i(\mathbf{x})$ is homogeneous with degree equal to $\deg(F)$ we can apply Propsition 3.3. □

We can prove our main result now. It was formulated in Sect. 2 already and for convenience we repeat it here again. The proof relies on Pólya's Positivstellensatz from Theorem 2.1 and Putinar–Vasilescu's Theorem 2.3.

Theorem 2.4 *Let f_0 to be a homogeneous polynomial such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$, where $\mathcal{X}$ is as given in (2). Then for some $r \in \mathbb{Z}_+$, there exists homogeneous polynomials $g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$ such that all their coefficients are non-negative and $(\mathbf{e}^\mathsf{T}\mathbf{x})^r f_0(\mathbf{x}) = \sum_{i=1}^m f_i(\mathbf{x}) g_i(\mathbf{x})$.*

Proof The polynomial f_0 satisfies assumptions of Corollary 3.4, hence we can use the notation and results from this corollary to show that there exists $\varepsilon > 0$ such that $f_\varepsilon(\mathbf{x})$ is a homogeneous polynomial with $f_\varepsilon(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$. Theorem 3.1 applied to f_ε implies that for some $r_1 \in \mathbb{Z}_+$ there exist homogenous polynomials $h_1, \ldots, h_m \in \mathbb{R}[\mathbf{x}]$ such that $h_i(\mathbf{z} \circ \mathbf{z})$ is SOS for all i and $(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_1} f_\varepsilon(\mathbf{x}) = \sum_{i=1}^m f_i(\mathbf{x}) h_i(\mathbf{x})$. Therefore

$$(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_1+c_0} f_0(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{x}) \left(\varepsilon(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_1+c_i} + h_i(\mathbf{x})\right).$$

Furthermore, by Remark 3.2, without loss of generality, for all $i = 1, \ldots, m$, either $h_i(\mathbf{x}) = 0$ or

$$\deg(f_i) + \deg(h_i) = r_1 + \deg(f_\varepsilon) = r_1 + \deg(f_i) + c_i.$$

In other words, for all $i = 1, \ldots, m$, either $h_i(\mathbf{x}) = 0$ or $\deg(h_i) = r_1 + c_i$. From this it can be seen that $\left(\varepsilon(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_1+c_i} + h_i(\mathbf{x})\right)$ is homogeneous and positive for all $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, and thus, by Theorem 2.1, for some $r_2 \in \mathbb{Z}_+$ there exist homogeneous polynomials $g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$, with all their coefficients being non-negative, such that

$$(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_2} \left(\varepsilon(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_1+c_i} + h_i(\mathbf{x})\right) = g_i(\mathbf{x}) \qquad \text{for all } i = 1, \ldots, m.$$

Now letting $r = (r_1 + r_2 + c_0) \in \mathbb{Z}_+$, we get the required result. □

Remark 3.5 We briefly recall that searching for polynomials with nonnegative coefficients (as are implied by Theorems 2.1 and 2.4) for fixed r can be done by linear programming (LP), whilst searching for SOS polynomials from Theorems 2.2 and 2.3 can be done by semidefinite programming (SDP). In practice we find such non-negativity certificates numerically by wide range of efficient methods implemented in the state-of-the-art software like Mosek [15] or CPLEX [2], see also the comprehensive list of available LP and SDP software together with list of benchmarks [13].

Our main result from Theorem 2.4 was also the key tool in [7] to find new LP and SDP based lower bounds for the polynomial optimization problems where we want to minimize a homogeneous polynomial f_0 over the basic closed semialgebraic set (1) interested by constraint $g(x) = 1$ with $\deg(f_0) = \deg(g)$. The main idea of approach in [7] to find largest ε such that $f_0 - \varepsilon g$ has a non-negativity certificate from Theorem 2.4.

4 Infinite number of polynomials

We finish this paper by noting that the Positivstellensätze given in Theorem 2.3 and 2.4 can be extended for infinitely many polynomials using the following theorem.

Theorem 4.1 *Consider a set of homogeneous polynomials* $\{f_0\} \cup \{f_i \mid i \in \mathcal{I}\} \subseteq \mathbb{R}[\mathbf{x}]$ *with infinite cardinality. Then* $f_0(\mathbf{x}) > 0$ *for all* $\mathbf{x} \in \bigcap_{i\in\mathcal{I}} f_i^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$ *if and only if there exists a subset* $\mathcal{J} \subseteq \mathcal{I}$ *of finite cardinality such that* $f_0(\mathbf{x}) > 0$ *for all* $\mathbf{x} \in \bigcap_{i\in\mathcal{J}} f_i^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$.

Proof The "if" part is trivial. We proceed with the "only if" part.

We begin by letting $\mathcal{Z}_\emptyset = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_2 = 1\}$, which is a compact set, and we note that for any $\mathcal{J} \subseteq \mathcal{I}$ we have $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \bigcap_{i\in\mathcal{J}} f_i^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$ if and only if $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{Z}_\emptyset \cap \bigcap_{i\in\mathcal{J}} f_i^{-1}(\mathbb{R}_+)$.

Without loss of generality, we assume that for all $i \in \mathcal{I}$ we have $\deg(f_i) \geq 1$ and

$$1 \geq \max_{\mathbf{x}\in\mathbb{R}^n}\{\|\nabla f_i(\mathbf{x})\|_2 \mid \|\mathbf{x}\|_2 \leq 1\}.$$

Considering the mean value theorem for this implies that for all $\mathbf{x}, \mathbf{y} \in \mathcal{Z}_\emptyset$ and all $i \in \mathcal{I}$, there exists $\theta \in [0, 1]$ and $\mathbf{z} = \theta\mathbf{x} + (1-\theta)\mathbf{y}$ (note that $\|\mathbf{z}\| \leq 1$) such that

$$\begin{aligned} f_i(\mathbf{x}) - f_i(\mathbf{y}) &= (\mathbf{x}-\mathbf{y})^{\mathsf{T}}\nabla f_i(\mathbf{z}) \\ &\leq \|\mathbf{x}-\mathbf{y}\|_2 \, \|\nabla f_i(\mathbf{z})\|_2 \\ &\leq \|\mathbf{x}-\mathbf{y}\|_2 \max_{\mathbf{x}\in\mathbb{R}^n}\{\|\nabla f_i(\mathbf{x})\|_2 \mid \|\mathbf{x}\|_2 \leq 1\} \\ &\leq \|\mathbf{x}-\mathbf{y}\|_2. \end{aligned} \tag{3}$$

By the same line of reasoning we obtain also that for all $\mathbf{x} \in \mathcal{Z}_\emptyset$ and all $i \in \mathcal{I}$ we have

$$|f_i(\mathbf{x})| \leq 1. \tag{4}$$

For all $j \in \mathcal{I}$ and $\mathcal{J} \subseteq \mathcal{I}$, we define the compact sets

$$\begin{aligned} \mathcal{Y} &= \mathcal{Z}_\emptyset \cap f_0^{-1}(-\mathbb{R}_+), \\ \mathcal{Z}_j &= \mathcal{Z}_\emptyset \cap f_j^{-1}(\mathbb{R}_+), \\ \mathcal{Z}_\mathcal{J} &= \mathcal{Z}_\emptyset \cap \bigcap_{i\in\mathcal{J}} f_i^{-1}(\mathbb{R}_+) = \bigcap_{i\in\mathcal{J}} \mathcal{Z}_i. \end{aligned}$$

We have that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{Z}_\emptyset \cap \bigcap_{i\in\mathcal{J}} f_i^{-1}(\mathbb{R}_+)$ if and only if $\mathcal{Y} \cap \mathcal{Z}_\mathcal{J} = \emptyset$. In particular we have $\mathcal{Y} \cap \mathcal{Z}_\mathcal{I} = \emptyset$.

Next we define the following function from $\mathcal{Z}_\emptyset$ to $\mathbb{R}$

$$\xi(\mathbf{x}) := \sup\{-f_i(\mathbf{x}) \mid i \in \mathcal{I}\} \qquad \text{for all } \mathbf{x} \in \mathcal{Z}_\emptyset.$$

From (3) and (4) we get that this is a continuous function from $\mathcal{Z}_\emptyset$ to $[-1, 1]$. Furthermore, as $\mathcal{Y} \cap \mathcal{Z}_\mathcal{I} = \emptyset$, we have that $\xi(\mathbf{x}) \in (0, 1]$ for all $\mathbf{x} \in \mathcal{Y}$. We now set $\varepsilon = \min_\mathbf{x}\{\xi(\mathbf{x}) \mid \mathbf{x} \in \mathcal{Y}\}$. As $\mathcal{Y}$ is compact, ξ is a continuous function and $\xi(\mathbf{x}) \in (0, 1]$ for all $\mathbf{x} \in \mathcal{Y}$, we get that $\varepsilon \in (0, 1]$.

For any $\mathbf{x} \in \mathcal{Y}$, there exists an $i \in \mathcal{I}$ such that $-f_i(\mathbf{x}) \geq \frac{2}{3}\xi(\mathbf{x}) \geq \frac{2}{3}\varepsilon > 0$. Now, for all $\mathbf{y} \in \mathcal{Z}_\emptyset$ such that $\|\mathbf{x} - \mathbf{y}\|_2 \leq \frac{1}{3}\varepsilon$, we have $f_i(\mathbf{y}) \leq f_i(\mathbf{x}) + \|\mathbf{x} - \mathbf{y}\|_2 \leq -\frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon < 0$ and thus $\mathbf{y} \notin \mathcal{Z}_i$. Keeping this in mind, we consider Algorithm 1.

Algorithm 1 Finding a set $\mathcal{J} \subseteq \mathcal{I}$ of finite cardinality such that $\mathcal{Y} \cap \mathcal{Z}_\mathcal{J} = \emptyset$.

1: Let $\mathcal{J} \leftarrow \emptyset$.
2: **while** $\exists \mathbf{z} \in \mathcal{Y} \cap \mathcal{Z}_\mathcal{J}$ **do**
3: Let $i \in \mathcal{I}$ such that $f_i(\mathbf{z}) \leq -\frac{2}{3}\varepsilon$ and let $\mathcal{J} \leftarrow \mathcal{J} \cup \{i\}$.
4: **end while**
5: **print** $\mathcal{J}$.

We see that no $\mathbf{z}$ in this algorithm can be within a distance of $\frac{1}{3}\varepsilon$ of a previous $\mathbf{z}$. Therefore, as $\mathcal{Y}$ is a bounded subset of $\mathbb{R}^n$, the algorithm finishes within a finite number of iterations. The resultant $\mathcal{J}$ then conforms to the requirements in the theorem.

Note that Algorithm 1 is purely there to aid the proof and is not meant for use in practice. This is due to the fact that simply checking whether $\mathcal{Y} \cap \mathcal{Z}_\mathcal{J} = \emptyset$ is in general an NP-hard problem.

We will now consider a couple of examples connected to Theorem 4.1.

Example 4.2 Let us consider

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2_+ \mid (x_1 - ax_2)(x_1 - (a+1)x_2) \geq 0 \ \forall a \in \mathbb{Z}\} = (\mathbb{R}_+ \times \{0\}) \ \cup \ \text{cone}(\mathbb{Z} \times \{1\})$$

and $f_0(\mathbf{x}) = 4x_1^3 - x_1x_2^2$. Obviously we have $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X} \setminus \{\mathbf{0}\}$. We can write $f_0(\mathbf{x}) = 4(x_1 + x_2)x_1(x_1 - x_2) + 3x_1x_2^2$. Therefore $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^2_+ \setminus \{\mathbf{0}\}$ such that $x_1(x_1 - x_2) \geq 0$, hence we can keep only one single constraint (the one that corresponds to $a = 0$) out of countable many.

Example 4.3 Let us consider $\{f_0\} \cup \{h_\nu \mid \nu \in \mathbb{R}\} \subseteq \mathbb{R}[x, y, z]$ such that

$$f_0(x, y, z) = 4x^2 + 4y^2 - 9z^2,$$
$$h_\nu(x, y, z) = \big(x - z\sin(\nu\pi/6)\big)^2 + \big(x - z\cos^3(\nu\pi/6)\big)^2 - 2z^2\cos(\nu\pi/6).$$

Note that $f_0(x, y, 0) > 0$ for all $(x, y) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. From this, and the fact that we are dealing with homogeneous polynomials of degree two, in this example we shall equivalently consider $\mathcal{Z}_\emptyset = \{(x, y, z) \in \mathbb{R}^3 \mid z = 1\}$. This makes the visualizations somewhat simpler.

We then have the following for $\mathcal{I} \subseteq \mathbb{R}$:

$$\mathcal{Y} = \{(x, y, 1) \mid f_0(x, y, 1) \leq 0\}, \quad \mathcal{Z}_\mathcal{I} = \{(x, y, 1) \mid h_\nu(x, y, 1) \geq 0 \text{ for all } \nu \in \mathcal{I}\}.$$

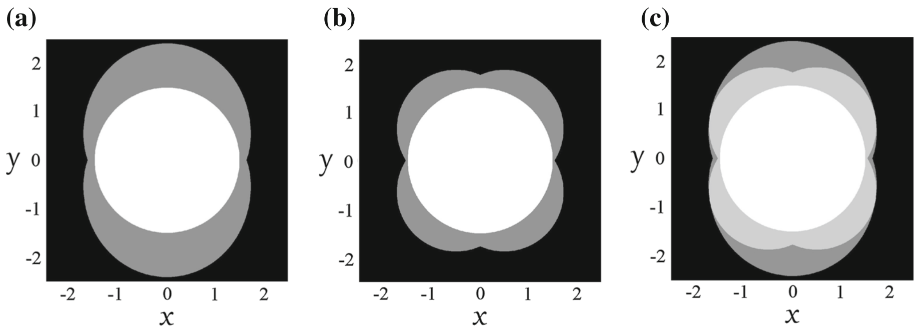

Fig. 1 Representation of Example 4.3. **a** The *inner white circle* represents $\mathcal{Y}$. The *black area* (and outwards to infinity) represents $\mathcal{Z}_{\mathbb{R}}$. **b** The *inner white circle* represents $\mathcal{Y}$. The *black area* (and outwards to infinity) represents $\mathcal{Z}_{\mathcal{J}}$. **c** A combination of representations for $\mathcal{Z}_{\mathbb{R}}$, $\mathcal{Z}_{\mathcal{J}}$ and $\mathcal{Y}$

Note that $\mathcal{Z}_{\mathbb{R}}$ is built from uncountably many constraints and we visualize the sets $\mathcal{Y}$ and $\mathcal{Z}_{\mathbb{R}}$ in Fig. 1a. We have $\mathcal{Y} \cap \mathcal{Z}_{\mathbb{R}} = \emptyset$ and thus $f_0(x, y, z) > 0$ for all $(x, y, z) \in \bigcap_{\nu \in \mathbb{R}} h_\nu^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$.

If we now consider the set $\mathcal{J} = \{1, 5, 7, 11\}$ of cardinality four, then visualizing $\mathcal{Y}$ and $\mathcal{Z}_{\mathcal{J}}$ in Fig. 1b, we have $\mathcal{Y} \cap \mathcal{Z}_{\mathcal{J}} = \emptyset$ and thus $f_0(x, y, z) > 0$ for all $(x, y, z) \in \bigcap_{\nu \in \mathcal{J}} h_\nu^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$.

5 Conclusions

In this paper we proved a generalization of a well-known Positivstellensatz from Pólya [10, 18]. The proof of this used the original Positivstellensatz from Pólya, and a Positivstellensatz from Putinar and Vasilescu [23,24]. We showed that for homogeneous polynomials which are positive on the semialgebraic cones defined by homogeneous polynomials and intersected by the non-negative orthant, there exists a Pólya type certificate which can be numerically found by solving an instance of a linear programming problem. We also showed that this can further be extended for infinitely many polynomials in the definition of the semialgebraic cone. An application of these theorems in constructing linear programming based hierarchies for polynomial optimization problems will be presented in the forthcoming paper [7].

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