INTERIORS OF COMPLETELY POSITIVE CONES

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Abstract. A symmetric matrix A is completely positive (CP) if there exists an entrywise nonnegative matrix B such that $A = BB^T$. We characterize the interior of the CP cone. A semidefinite algorithm is proposed for checking interiors of the CP cone, and its properties are studied. A CP-decomposition of a matrix in Dickinson's form can be obtained if it is an interior of the CP cone. Some computational experiments are also presented.

1. Introduction

A real $n \times n$ symmetric matrix A is completely positive (CP) if there exist nonnegative vectors $b_1, \cdots, b_m \in \mathbb{R}^n$ such that

$$A = b_1 b_1^T + \cdots + b_m b_m^T, \tag{1.1}$$

where m is called the length of the decomposition (1.1). The smallest m in the above is called the CP-rank of A. If A is CP, we call (1.1) a CP-decomposition of A. Clearly, A is CP if and only if $A = BB^T$ for an entrywise nonnegative matrix B. Hence, a CP-matrix is not only positive semidefinite but also nonnegative entrywise.

Let $\mathcal{S}_n$ be the space of real $n \times n$ symmetric matrices. For a cone $\mathcal{C} \subseteq \mathcal{S}_n$, its dual cone is defined as

$$\mathcal{C}^* := \{G \in \mathcal{S}_n : \langle A, G \rangle \geqslant 0 \text{ for all } A \in \mathcal{C}\},$$

where $\langle A, G \rangle = \text{trace}(AG)$ is the trace inner product. Denote

$$\mathcal{C}_n = \{A \in \mathcal{S}_n : A = BB^T \text{ with } B \geqslant 0\}, \text{the completely positive cone,}$$

$$\mathcal{C}_n^* = \{G \in \mathcal{S}_n : x^T G x \geqslant 0 \text{ for all } x \geqslant 0\}, \text{the copositive cone.}$$

Both $\mathcal{C}_n$ and $\mathcal{C}_n^*$ are proper cones (i.e. closed, pointed, convex and full-dimensional). Moreover, they are dual to each other [17].

The completely positive cone and copositive cone have wide applications in mixed binary quadratic programming [6], standard quadratic optimization problems and general quadratic programming [4], etc. Some NP-hard problems can also be formulated as linear optimization problems over the CP cone and the copositive cone (cf. [8]). We refer to [3, 5, 14] for the work in the field.

The membership problems for the completely positive cone and the copositive cone are NP-hard (cf. [1, 13]). To compute a CP-decomposition of a CP-matrix is also hard. Dickinson & Dür [9] studied the CP-checking and CP-decomposition of some sparse matrices. Sponseldur & Dür [28] used polyhedral approximations to project a matrix to $\mathcal{C}_n$; a CP-decomposition of a matrix can be obtained if it

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is an interior of $\mathcal{C}_n$. In [30], a semidefinite algorithm is proposed for solving the CP-matrix completion problem, which includes the CP-checking as a special case; a CP-decomposition for a general CP-matrix can be found by the algorithm.

Denote $\mathrm{int}(\mathcal{C}_n)$ and $\mathrm{bd}(\mathcal{C}_n)$ the interior and the boundary of $\mathcal{C}_n$, respectively. Shaked-Monderer, Bomze, Jarre & Schachinger [27] showed that the maximal CP-rank of $n \times n$ CP-matrices is attained at a positive definite matrix on $\mathrm{bd}(\mathcal{C}_n)$. Denote $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x \geqslant 0\}$ and $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n \mid x > 0\}$. Dür & Still [15] characterized $\mathrm{int}(\mathcal{C}_n)$ as:

$$\begin{aligned} \mathrm{int}(\mathcal{C}_n) =& \{BB^T \mid B = (B_1, B_2) \text{ with } B_1 > 0 \text{ nonsingular},\ B_2 \geqslant 0\} \\ =& \left\{ \sum_{i=1}^m b_i b_i^T \,\middle|\, \begin{array}{c} m \geqslant n, b_i \in \mathbb{R}^n_+ \text{ for all } i, \\ \{b_1, \ldots, b_n\} \subseteq \mathbb{R}^n_{++}, \\ \mathrm{span}\{b_1, \ldots, b_n\} = \mathbb{R}^n \end{array} \right\}. \end{aligned} \tag{1.2}$$

Dickinson [12] further characterized $\mathrm{int}(\mathcal{C}_n)$ as:

$$\begin{aligned} \mathrm{int}(\mathcal{C}_n) =& \{BB^T \mid \mathrm{rank}(B) = n, B = (b_1, \tilde{B}), b_1 \in \mathbb{R}^n_{++}, \tilde{B} \geqslant 0\} \\ =& \left\{ \sum_{i=1}^m b_i b_i^T \,\middle|\, \begin{array}{c} b_1 \in \mathbb{R}^n_{++}, b_i \in \mathbb{R}^n_+ \text{ for } i = 2, \cdots, m, \\ \mathrm{span}\{b_1, \cdots, b_m\} = \mathbb{R}^n \end{array} \right\}. \end{aligned} \tag{1.3}$$

The above characterizations are very useful in checking interiors of $\mathcal{C}_n$.

How do we check whether a matrix is in the interior of C_n if it is not given in Dür & Still's form (1.2) or Dickinson's form (1.3)? Little is known for checking interiors or boundaries of $\mathcal{C}_n$. In this paper, we characterize interiors of $\mathcal{C}_n$ from the view of optimization. A semidefinite algorithm is proposed to check whether a symmetric matrix $A \notin \mathcal{C}_n$, or $A \in \mathrm{bd}(\mathcal{C}_n)$, or $A \in \mathrm{int}(\mathcal{C}_n)$. If $A \notin \mathcal{C}_n$, we can get a certificate. If $A \in \mathcal{C}_n$, we can get a CP-decomposition of A. Moreover, a CP-decomposition in Dickinson's form can also be obtained by a similar algorithm.

The paper is organized as follows. In Section 2, we give a new sufficient and necessary condition to characterize interiors of $\mathcal{C}_n$. In Section 3, we formulate the problem of checking the membership and interiors of $\mathcal{C}_n$ as the linear optimization with moments. In Section 4, we present a semidefinite algorithm for the problem. Its basic properties are also studied. Some computational results are reported in Section 5. Finally in Section 6, we discuss how to give a CP-decomposition of a matrix in Dickinson's form if it is an interior of $\mathcal{C}_n$.

2. A Characterization of interiors

In this section, we characterize interiors of $\mathcal{C}_n$ from the view of optimization.

Lemma 2.1. *Suppose $A \in \mathcal{S}_n$. Then $A \in int(\mathcal{C}_n)$ if and only if for some $C \in int(\mathcal{C}_n)$, there exists a $\lambda > 0$ such that $A - \lambda C \in \mathcal{C}_n$.*

Proof. Given $A \in \mathrm{int}(\mathcal{C}_n)$, then there exists a $\delta > 0$ such that for any $D \in \mathcal{S}_n$ with $\|A - D\| \leqslant \delta$, we have $D \in \mathcal{C}_n$. Choose an arbitrary $C \in \mathrm{int}(\mathcal{C}_n)$. Obviously, C is positive and nonsingular. Let $\lambda = \delta/\|C\|$. Then $\|A - (A - \lambda C)\| \leqslant \delta$, which implies that $A - \lambda C \in \mathcal{C}_n$.

Conversely, suppose $C \in \mathrm{int}(\mathcal{C}_n)$ and $A - \lambda C \in \mathcal{C}_n$, where $\lambda > 0$. By (1.2), there exist $B_1 > 0$ nonsingular and $B_2 \geqslant 0$ such that $C = (B_1, B_2)(B_1, B_2)^T$. Meanwhile, there exists a $B_3 \geqslant 0$ such that $A - \lambda C = B_3 B_3^T$. Hence,

$$A = \lambda C + B_3 B_3^T = (\sqrt{\lambda} B_1, \sqrt{\lambda} B_2, B_3)(\sqrt{\lambda} B_1, \sqrt{\lambda} B_2, B_3)^T.$$

So, by (1.2), $A \in \operatorname{int}(\mathcal{C}_n)$. □

Lemma 2.1 gives an equivalent condition for a matrix A to be an interior of $\mathcal{C}_n$. We wonder how to compute such a λ. This can be done by solving the linear optimization problem:

$$(P_1): \quad \begin{cases} f_1^* = & \max \quad \lambda \\ & \text{s.t.} \quad A - \lambda C \in \mathcal{C}_n \end{cases}$$

for some given $C \in \operatorname{int}(\mathcal{C}_n)$. A simple choice of C is $I_n + E_n$. Here, I_n denotes the $n \times n$ identity matrix and E_n the $n \times n$ matrix of all ones. By Lemma 2.1, if $f_1^* > 0$, then $A \in \operatorname{int}(\mathcal{C}_n)$; if $f_1^* = 0$, then $A \in \operatorname{bd}(\mathcal{C}_n)$; if $f_1^* < 0$, then $A \notin \mathcal{C}_n$.

Since $\mathcal{C}_n$ and $\mathcal{C}_n^*$ are dual to each other, we know

$$A \notin \mathcal{C}_n \iff \exists X \in \mathcal{C}_n^* \text{ such that } \langle A, X \rangle < 0. \tag{2.1}$$

Hence, $A \notin \mathcal{C}_n$ if and only if there exists $X \in \mathcal{C}_n^*$ satisfying

$$\langle A, X \rangle < 0, \quad \langle X, C \rangle = 1. \tag{2.2}$$

On the other hand, as shown in [2, 12],

$$A \in \operatorname{int}(\mathcal{C}_n) \iff \langle A, X \rangle > 0 \text{ for all } X \in \mathcal{C}_n^* \backslash \{0\}. \tag{2.3}$$

Hence, $A \in \mathcal{C}_n$ if and only if for all $X \in \mathcal{C}_n^*$ with $\langle X, C \rangle = 1$,

$$\langle A, X \rangle > 0. \tag{2.4}$$

Therefore, checking interiors of $\mathcal{C}_n$ is equivalent to solving the linear optimization problem over $\mathcal{C}_n^*$:

$$(D_1): \quad \begin{cases} g_1^* = & \min \quad \langle A, X \rangle \\ & \text{s.t.} \quad \langle X, C \rangle = 1 \\ & \qquad\;\; X \in \mathcal{C}_n^*. \end{cases}$$

By (2.2) and (2.4), if $g_1^* > 0$, then $A \in \operatorname{int}(\mathcal{C}_n)$; if $g_1^* = 0$, then $A \in \operatorname{bd}(\mathcal{C}_n)$; if $g_1^* < 0$, then $A \notin \mathcal{C}_n$.

In fact, the optimization problems (P_1) and (D_1) are dual to each other. Denote by $\operatorname{Feas}(P)$ the feasible region of an optimization problem (P). By the standard duality theory, we have $g_1^* \geqslant f_1^*$ for all $X \in \operatorname{Feas}(D_1)$ and $\lambda \in \operatorname{Feas}(P_1)$. This is referred to as weak duality. If there exists a $\lambda \in \operatorname{Feas}(P_1)$ such that $A - \lambda C \in \operatorname{int}(\mathcal{C}_n)$, we say that Slater's condition holds for (P_1) and λ is a strictly feasible point of (P_1). If there exists a $X \in \operatorname{Feas}(D_1) \cap \operatorname{int}(\mathcal{C}_n^*)$, we say that Slater's condition holds for (D_1) and X is a strictly feasible point of (D_1). Under Slater's conditions, strong duality holds (i.e. $g_1^* = f_1^*$).

The optimization problems (P_1) and (D_1) have the following properties.

Theorem 2.2. *Suppose $A \in \mathcal{S}_n$ and $C \in int(\mathcal{C}_n)$. Then the optimums of (P_1) and (D_1) are finite and equal, and the optimal solution sets of (P_1) and (D_1) are nonempty. Furthermore, if $f_1^* < 0$, then $A \notin \mathcal{C}_n$; if $f_1^* = 0$, then $A \in bd(\mathcal{C}_n)$; if $f_1^* > 0$, then $A \in int(\mathcal{C}_n)$.*

Proof. We first show that Slater's condition holds for (P_1). If $A = 0$, then all $\lambda < 0$ are strictly feasible points of (P_1). If $A \neq 0$, due to $C \in \operatorname{int}(\mathcal{C}_n)$, there exists a $\delta > 0$ such that $D \in \operatorname{int}(\mathcal{C}_n)$ for all $D \in \mathcal{S}_n$ with $\|C - D\| \leqslant \delta$. Let $\lambda \leqslant -\frac{\|A\|}{\delta}$. As $\|C - (C - \frac{1}{\lambda} \cdot A)\| \leqslant \delta$, we have $C - \frac{1}{\lambda} A \in \operatorname{int}(\mathcal{C}_n)$. So, $A - \lambda C \in \operatorname{int}(\mathcal{C}_n)$. That is, λ is a strictly feasible point of (P_1).

Choose an arbitrary $P \in \text{int}(\mathcal{C}_n^*)$. Since $C \in \text{int}(\mathcal{C}_n)$, we have $\langle P, C\rangle > 0$. Thus, $\langle P, C\rangle^{-1} P \in \text{Feas}(D_1) \cap \text{int}(\mathcal{C}_n^*)$. So, Slater's condition holds for (D_1).

It is obvious that the optimum of (P_1) is finite. Therefore, the optimums of (P_1) and (D_1) are finite and equal, and the optimal solution sets of (P_1) and (D_1) are both nonempty by the duality theory given in [11, Theorems 1.25 and 1.26].

By Lemma 2.1, we obtain the rest part of the theorem. □

Therefore, checking interiors of $\mathcal{C}_n$ is equivalent to solving (P_1) or (D_1). For all $A \in \mathcal{S}_n$ and $C \in \text{int}(\mathcal{C}_n)$, a maximizer λ^* of (P_1) always exists. This leads to an interesting result for $A - \lambda^* C$.

Proposition 2.3. *Suppose $A \in \mathcal{S}_n$, $C \in int(\mathcal{C}_n)$, and λ^* is a maximizer of (P_1). Then $A - \lambda^* C \in bd(\mathcal{C}_n)$.*

Proof. We prove by contradiction. Obviously, $A - \lambda^* C \in \mathcal{C}_n$. Suppose $A - \lambda^* C \in \text{int}(\mathcal{C}_n)$. Then, there exists a $\delta > 0$ such that $D \in \mathcal{C}_n$ for all $D \in \mathcal{S}_n$ with $\|A - \lambda^* C - D\| \leqslant \delta$. Hence, $A - (\lambda^* + \varepsilon)C \in \mathcal{C}_n$ for all $0 < \varepsilon \leqslant \delta / \|C\|$. Thus $\lambda^* + \varepsilon$ is a feasible point of (P_1), which contradicts that λ^* is the maximizer of (P_1). The proof is completed. □

3. A linear moment optimization approach

As shown above, checking interiors of $\mathcal{C}_n$ is equivalent to a linear optimization problem with $\mathcal{C}_n$. Generally, it is difficult to solve it directly. In this section, we formulate (P_1) as a linear optimization problem with the cone of moments. We begin with some basics about moments.

3.1. **Formulation as a moment problem.** A symmetric matrix can be identified by a vector consisting of its upper triangular entries. Let $\mathbb{N}$ be the set of nonnegative integers. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, denote $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Let

$$\mathcal{A} := \{\alpha \in \mathbb{N}^n : \alpha = e_i + e_j, j \geqslant i, i, j = 1, \cdots, n\}, \tag{3.1}$$

where e_i is the i-th unit vector. So, a matrix $A \in \mathcal{S}_n$ can be identified as a vector a as:

$$a = (a_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}^{\mathcal{A}}, \quad a_\alpha = A_{ij} \text{ if } \alpha = e_i + e_j,$$

where $\mathbb{R}^{\mathcal{A}}$ denotes the space of real vectors indexed by $\alpha \in \mathcal{A}$. We call a an $\mathcal{A}$-truncated moment sequence ($\mathcal{A}$-tms). Let

$$K = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 - 1 = 0, x_1 \geqslant 0, \cdots, x_n \geqslant 0\} \tag{3.2}$$

be the nonnegative part of the unit sphere. Every nonnegative vector is a multiple of a vector in K. So, by (1.1), $A \in \mathcal{C}_n$ if and only if there exist vectors $b_1, \cdots, b_m \in K$ and $\rho_1, \cdots, \rho_m > 0$ such that

$$A = \rho_1 b_1 b_1^T + \cdots + \rho_m b_m b_m^T. \tag{3.3}$$

The $\mathcal{A}$-truncated K-moment problem ($\mathcal{A}$-TKMP) studies whether or not a given $\mathcal{A}$-tms a admits a K-measure μ, i.e., a nonnegative Borel measure μ supported in K such that

$$a_\alpha = \int_K x^\alpha d\mu, \quad \forall\, \alpha \in \mathcal{A},$$

where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. A measure μ satisfying the above is called a K-representing measure for a. A measure is called finitely atomic if its support is a finite set, and is called m-atomic if its support consists of at most m distinct points. We refer to [23] for representing measures of truncated moments sequences.

Hence, by (3.3), a symmetric matrix A, with the identifying vector $a \in \mathbb{R}^{\mathcal{A}}$, is completely positive if and only if a admits an m-atomic K-measure, i.e.,

$$a = \rho_1[b_1]_{\mathcal{A}} + \cdots + \rho_m[b_m]_{\mathcal{A}}, \tag{3.4}$$

where each $b_i \in K$, $\rho_i > 0$, and

$$[b]_{\mathcal{A}} := (b^{\alpha})_{\alpha\in\mathcal{A}}.$$

In other words, checking CP is equivalent to an $\mathcal{A}$-TKMP with $\mathcal{A}$ and K given in (3.1) and (3.2) respectively.

Denote

$$\mathbb{R}[x]_{\mathcal{A}} := \operatorname{span}\{x^{\alpha} : \alpha \in \mathcal{A}\}.$$

We say $\mathbb{R}[x]_{\mathcal{A}}$ is K-full if there exists a polynomial $p \in \mathbb{R}[x]_{\mathcal{A}}$ such that $p|_K > 0$ (cf. [16]). An $\mathcal{A}$-tms $a \in \mathbb{R}^{\mathcal{A}}$ defines an $\mathcal{A}$-Riesz functional $\mathscr{L}_a$ acting on $\mathbb{R}[x]_{\mathcal{A}}$ as

$$\mathscr{L}_a\Big(\sum_{\alpha\in\mathcal{A}} p_{\alpha}x^{\alpha}\Big) := \sum_{\alpha\in\mathcal{A}} p_{\alpha}a_{\alpha}. \tag{3.5}$$

For convenience, we also denote $\langle p, a\rangle := \mathscr{L}_a(p)$. Let

$$\mathbb{N}^n_d := \{\alpha \in \mathbb{N}^n : |\alpha| \leqslant d\}$$

and

$$\mathbb{R}[x]_d := \operatorname{span}\{x^{\alpha} : \alpha \in \mathbb{N}^n_d\}.$$

For $s \in \mathbb{R}^{\mathbb{N}^n_{2k}}$ and $q \in \mathbb{R}[x]_{2k}$, the k-th localizing matrix of q generated by s is the symmetric matrix $L^{(k)}_q(s)$ satisfying

$$\mathscr{L}_s(qp^2) = vec(p)^T(L^{(k)}_q(s))vec(p), \quad \forall p \in \mathbb{R}[x]_{k-\lceil deg(q)/2\rceil}. \tag{3.6}$$

In the above, $vec(p)$ denotes the coefficient vector of p in the graded lexicographical ordering, and $\lceil t\rceil$ denotes the smallest integer that is not smaller than t. In particular, when $q = 1$, $L^{(k)}_1(s)$ is called a k-th order moment matrix and denoted as $M_k(s)$. We refer to [16, 18, 23] for more details about localizing and moment matrices.

Denote the polynomials:

$$h(x) := x_1^2 + \cdots + x_n^2 - 1, g_0(x) := 1, g_1(x) := x_1, \cdots, g_n(x) := x_n.$$

Note that K given in (3.2) is nonempty compact. We can also describe K equivalently as

$$K = \{x \in \mathbb{R}^n : \ h(x) = 0, g(x) \geqslant 0\}, \tag{3.7}$$

where $g(x) = (g_0(x), g_1(x), \cdots, g_n(x))$. As shown in [23], a necessary condition for $s \in \mathbb{R}^{\mathbb{N}^n_{2k}}$ to admit a K-measure is

$$L^{(k)}_h(s) = 0, \quad \text{and} \quad L^{(k)}_{g_j}(s) \succeq 0, \quad j = 0, 1, \cdots, n. \tag{3.8}$$

If, in addition to (3.8), s satisfies the rank condition

$$\operatorname{rank} M_{k-1}(s) = \operatorname{rank} M_k(s), \tag{3.9}$$

then s admits a unique K-measure, which is $\operatorname{rank} M_k(s)$-atomic (cf. Curto and Fialkow [7]). We say that s is flat if both (3.8) and (3.9) are satisfied.

Given two tms' $y \in \mathbb{R}^{\mathbb{N}^n_d}$ and $z \in \mathbb{R}^{\mathbb{N}^n_e}$, we say z is an extension of y, if $d \leqslant e$ and $y_{\alpha} = z_{\alpha}$ for all $\alpha \in \mathbb{N}^n_d$. We denote by $z|_{\mathcal{A}}$ the subvector of z, whose entries are indexed by $\alpha \in \mathcal{A}$. For convenience, we denote by $z|_d$ the subvector $z|_{\mathbb{N}_d^n}$. If z is flat and extends y, we say z is a flat extension of y. Note that an $\mathcal{A}$-tms $a \in \mathbb{R}^{\mathcal{A}}$ admits a K-measure if and only if it is extendable to a flat tms $z \in \mathbb{R}^{\mathbb{N}_{2k}^n}$ for some k (cf. [23]). By (3.4), a matrix A is CP if and only if its identifying vector a has a flat extension.

3.2. **Linear optimization with moments.** Let $\mathcal{A}$ and K be given as (3.1) and (3.7), respectively. Denote

$$\mathscr{R}_{\mathcal{A}}(K) = \{a \in \mathbb{R}^{\mathcal{A}} : meas(a, K) \neq \emptyset\},$$

where $meas(a, K)$ is the set of all K-measures admitted by a. By (3.4), $\mathscr{R}_{\mathcal{A}}(K)$ is the CP cone (cf. [24]).

Suppose $A \in \mathcal{S}_n$ and $C \in \mathrm{int}(\mathcal{C}_n)$. Let $a, c \in \mathscr{R}_{\mathcal{A}}(K)$ be the identifying vectors of A and C, respectively. Replacing $\mathcal{C}_n$ by $\mathscr{R}_{\mathcal{A}}(K)$, we formulate (P_1) as the linear optimization problem with the cone of moments:

$$(P_2): \quad \left\{ \begin{array}{lll} f_2^* = & \max & \lambda \\ & \text{s.t.} & a - \lambda c \in \mathscr{R}_{\mathcal{A}}(K). \end{array} \right.$$

Similar to Theorem 2.2, we have:

Proposition 3.1. *Suppose $A \in \mathcal{S}_n$ and $C \in int(\mathcal{C}_n)$. Then, the optimum f_2^* of (P_2) is finite. Furthermore, if $f_2^* < 0$, then $A \notin \mathcal{C}_n$; if $f_2^* = 0$, then $A \in bd(\mathcal{C}_n)$; if $f_2^* > 0$, then $A \in int(\mathcal{C}_n)$.*

Actually, we can further formulate (P_2) in the form with $\mathscr{R}_{\mathcal{A}}(K)$ and some linear constraints. Obviously, $c \neq 0$. Suppose $\{p_1, \cdots, p_{\bar{n}}\}$ is a basis of the orthogonal complement of span$\{c\}$, where $\bar{n} = \frac{n(n+1)}{2} - 1$. Let

$$p_0 = (c^T c)^{-1} c. \tag{3.10}$$

Then,

$$\langle p_0, c \rangle = 1, \quad \langle p_i, c \rangle = 0, i = 1, \cdots, \bar{n}. \tag{3.11}$$

Hence, $z = a - \lambda c$ for some λ if and only if

$$p_i^T z = p_i^T a, \quad i = 1, \cdots, \bar{n}. \tag{3.12}$$

Moreover,

$$\lambda = (c^T c)^{-1} c^T (a - z). \tag{3.13}$$

The vectors p_i can also be considered as polynomials in $\mathbb{R}[x]_{\mathcal{A}}$. Note that

$$\langle p_0, z \rangle = (c^T c)^{-1} c^T z = -\lambda + (c^T c)^{-1} c^T a. \tag{3.14}$$

By (3.12)-(3.14), we know (P_2) is equivalent to

$$(P_3): \quad \left\{ \begin{array}{lll} f_3^* = & \min & \langle p_0, z \rangle \\ & \text{s.t.} & \langle p_i, z \rangle = p_i^T a, \ i = 1, \cdots, \bar{n} \\ & & z \in \mathscr{R}_{\mathcal{A}}(K). \end{array} \right.$$

Proposition 3.2. *Suppose $A \in \mathcal{S}_n$ and $C \in int(\mathcal{C}_n)$. If z^* is a minimizer of (P_3), then*

$$\lambda^* = (c^T c)^{-1} c^T (a - z^*) \tag{3.15}$$

is a maximizer of (P_2), and vice versa.

4. A semidefinite algorithm for checking interiors

In this section, we present a semidefinite algorithm for checking the membership and interiors of $\mathcal{C}_n$. The cone $\mathscr{R}_{\mathcal{A}}(K)$ is typically difficult to describe. However, it has nice semidefinite relaxations.

Let h and g be as in (3.7). For each $k \in \mathbb{N}$, denote

$$\Gamma_k(h,g) = \left\{ y \in \mathbb{R}^{\mathbb{N}^n_{2k}} : L_h^{(k)}(y) = 0, L_{g_j}^{(k)}(y) \geq 0, j = 0, 1, \cdots, n \right\}. \tag{4.1}$$

By (3.8) and (3.9), we relax $\mathscr{R}_{\mathcal{A}}(K)$ by $\Gamma_k(h,g)$. Then the k-th order relaxation of (P_2) is

$$(P_2^k): \quad \begin{cases} f_2^k = \max\limits_{\lambda,y} & \lambda \\ \qquad\quad \text{s.t.} & a - \lambda c = y|_{\mathcal{A}},\ y \in \Gamma_k(h,g). \end{cases}$$

Since $\text{Feas}(P_2) \subseteq \text{Feas}(P_2^k)$, we have $f_2^k \geqslant f_2^*$ for all k. If $f_2^k < 0$, then, by Theorem 2.2, $A \notin \mathcal{C}_n$. Let $\lambda^{*,k}$ be the maximizer of (P_2^k). If $a(\lambda^{*,k}) := a - \lambda^{*,k} c \in \mathscr{R}_{\mathcal{A}}(K)$, then $f_2^* = f_2^k$ and $\lambda^{*,k}$ is the maximizer of (P_2), i.e., the relaxation (P_2^k) is tight for (P_2). If $f_2^k = 0$, then $A \in \text{bd}(\mathcal{C}_n)$; otherwise $A \in \text{int}(\mathcal{C}_n)$.

Based on the above, we propose a semidefinite algorithm for checking interiors of $\mathcal{C}_n$.

Algorithm 4.1. An algorithm for checking interiors of $\mathcal{C}_n$.
Input: $A \in \mathcal{S}_n$ and K as (3.2).
Output: An answer $A \notin \mathcal{C}_n$, or $A \in \text{bd}(\mathcal{C}_n)$ or $A \in \text{int}(\mathcal{C}_n)$, with a CP-decomposition.
Procedure:

Step 0: Let $k := 1$.

Step 1: Compute an optimal pair $(\lambda^{*,k}, y^{*,k})$ of (P_2^k).

Step 2: If $f_2^k < 0$, output $A \notin \mathcal{C}_n$ and stop. Otherwise, let $t := 1$.

Step 3: Let $v := y^{*,k}|_{2t}$. If the rank condition (3.9) is not satisfied, go to Step 6.

Step 4: If $f_2^k = 0$, output $A \in \text{bd}(\mathcal{C}_n)$ and stop. Otherwise, go to Step 5.

Step 5: Compute the finitely atomic measure μ admitted by v:

$$\mu = \rho_1 \delta(b_1) + \cdots + \rho_m \delta(b_m),$$

where $m = \text{rank}(M_t(v))$, $b_i \in K$, $\rho_i > 0$, and $\delta(b_i)$ is the Dirac measure supported on the points $b_i \in K$. Output $A \in \text{int}(\mathcal{C}_n)$ with a CP-decomposition of A (3.3) and stop.

Step 6: If $t < k$, set $t := t + 1$ and go to Step 3; otherwise, set $k := k + 1$ and go to Step 1.

Algorithm 4.1 gives a certificate for whether $A \notin \mathcal{C}_n$, or $A \in \text{bd}(\mathcal{C}_n)$, or $A \in \text{int}(\mathcal{C}_n)$. A CP-decomposition can also be obtained if $A \in \mathcal{C}_n$.

Remark 4.2. We use Henrion and Lasserre's method [19] to get a m-atomic K-measure for $y^{*,k}$. The CP-decomposition of the boundary point $A - \lambda^* C$ (see Proposition 2.3) is computed, with which the CP-decomposition of A can be further obtained if $A \in \mathcal{C}_n$ (i.e. $\lambda^* \geqslant 0$).

Remark 4.3. We apply Step 3 - Step 6 to check whether $a(\lambda^{*,k}) := a - \lambda^{*,k} c \in \mathscr{R}_{\mathcal{A}}(K)$ or not. It might be possible that $a(\lambda^{*,k})$ belongs to $\mathscr{R}_{\mathcal{A}}(K)$ while $y^{*,k}|_{2t}$ is not flat for all t. In such cases, we can apply Algorithms given in [23, 30] to check if $a(\lambda^{*,k}) \in \mathscr{R}_{\mathcal{A}}(K)$ or not. In computational experiments, the finite convergence always occurs.

Remark 4.4. In Step 1, we solve (P_2^k). By Proposition 3.2, we can instead solve the relaxation of (P_3):

$$(P_3^k): \quad \begin{cases} f_3^k = & \min \quad \langle p_0, z\rangle \\ & \text{s.t.} \quad \langle p_i, z\rangle = p_i^T a, \ i = 1, \cdots, \bar{n} \\ & \qquad\ \ z = y|_{\mathcal{A}}, \ y \in \Gamma_k(h, g). \end{cases}$$

Proposition 4.5. *Suppose $A \in \mathcal{S}_n$ and $C \in int(\mathcal{C}_n)$. If $z^{*,k}$ is a minimizer of (P_3^k), then*

$$\lambda^{*,k} = (c^T c)^{-1} c^T (a - z^{*,k}) \tag{4.2}$$

is a maximizer of (P_2^k), and vice versa.

Since K as in (3.7) is nonempty compact and $\mathcal{A}$ as in (3.1) is finite, $\mathbb{R}[x]_{\mathcal{A}}$ is K-full (cf. [30]). Note that (P_2) always has feasible points. Combining Nie's result [24, Theorem 4.3] with Proposition 3.2 and Proposition 4.5, we have the following asymptotic convergence of Algorithm 4.1.

Proposition 4.6. *Algorithm 4.1 has the following properties:*

(i) *For all k sufficiently large, (P_2^k) has a maximizing pair $(\lambda^{*,k}, y^{*,k})$.*
(ii) *The sequence $\{\lambda^{*,k}\}$ is monotonically decreasing and converges to the maximizer of (P_2). Furthermore, the sequence $\{\lambda^{*,k}\}$ is bounded, and each of its accumulation points is a maximizer of (P_2).*

The finite convergence also happens, under some general conditions in optimization [24].

5. Numerical experiments

In this section, we present numerical experiments for checking the membership and interiors of $\mathcal{C}_n$ by using Algorithm 4.1. A CP-decomposition of a matrix is also given if it is CP. We use softwares GloptiPoly 3 [20] and SeDuMi [29] to solve (P_3^k) in Step 1 of Algorithm 4.1. If $|\lambda^{*,k}| < 10^{-4}$, we regard that the matrix is on the boundary of $\mathcal{C}_n$.

Example 5.1. Consider the matrix A given as (cf. [3, Example 2.9]):

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 6 \end{pmatrix}. \tag{5.1}$$

We have $A \notin \mathcal{C}_5$ (cf. [3]). We apply Algorithm 4.1 to verify this fact. Choose $C = I_5 + E_5$. Then the identifying vector of C is

$$c = (2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 2, 1, 2)^T.$$

We can choose

$$p_i = -e_1 + e_{i+1}, \quad i \in T_1 = \{5, 9, 12, 14\},$$
$$p_i = -e_1 + 2e_{i+1}, \quad i \in \{1, \ldots, 14\} \backslash T_1$$

to be basis vectors of $\text{span}\{c\}^{\perp}$. Let

$$p_0 = (c^T c)^{-1} c = \frac{1}{30} \cdot c.$$

Since $\lambda^{*,k} = -0.3982 < 0$ at $k = 1$, we know $A \notin \mathcal{C}_5$.

Example 5.2. Consider the matrix A given as (similar to [3, Exercise 2.22]):

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}. \tag{5.2}$$

It is shown in [3] that nonnegative symmetric diagonally dominant matrices are completely positive. So, $A \in \mathcal{C}_7$. Since $A \nsucc 0$, we have $A \in \mathrm{bd}(\mathcal{C}_7)$. We now verify it by Algorithm 4.1. Choose $C = I_7 + E_7$. Then the identifying vector of C is

$$c = (2,1,1,1,1,1,1,2,1,1,1,1,1,2,1,1,1,1,2,1,1,1,2,1,1,2,1,2)^T.$$

We choose

$$p_i = -e_1 + e_{i+1}, \quad i \in T_2 = \{7, 13, 18, 22, 25, 27\},$$

$$p_i = -e_1 + 2e_{i+1}, \quad i \in \{1, \dots, 27\} \backslash T_2$$

to be basis vectors of $\mathrm{span}\{c\}^{\perp}$. Let

$$p_0 = (c^T c)^{-1} c = \frac{1}{49} \cdot c.$$

Algorithm 4.1 terminates at $k = 4$, with $|\lambda^{*,k}| = 2.0815e{-}008 < 10^{-4}$ and $y(\lambda^{*,k}) \in \mathscr{R}_{\mathcal{A}}(K)$. As $\lambda^{*,k} \approx 0$, we regard $A \in \mathrm{bd}(\mathcal{C}_7)$. We obtain the CP-decomposition $A = \sum_{i=1}^{7} \rho_i b_i b_i^T$, where the points and their weights are:

$$\begin{aligned}
\rho_1 &= 2.0000, \quad b_1 = (0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.7071, 0.7071)^T, \\
\rho_2 &= 2.0000, \quad b_2 = (0.0000, 0.0000, 0.0000, 0.0000, 0.7071, 0.7071, 0.0000)^T, \\
\rho_3 &= 2.0000, \quad b_3 = (0.7071, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.7071)^T, \\
\rho_4 &= 2.0000, \quad b_4 = (0.0000, 0.0000, 0.7071, 0.7071, 0.0000, 0.0000, 0.0000)^T, \\
\rho_5 &= 2.0000, \quad b_5 = (0.0000, 0.7071, 0.7071, 0.0000, 0.0000, 0.0000, 0.0000)^T, \\
\rho_6 &= 2.0000, \quad b_6 = (0.0000, 0.0000, 0.0000, 0.7071, 0.7071, 0.0000, 0.0000)^T, \\
\rho_7 &= 2.0000, \quad b_7 = (0.7071, 0.7071, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)^T.
\end{aligned}$$

In fact, we get the minimal CP-decomposition (cf. [27, Remark 3.1]).

Example 5.3. Consider the matrix A given as:

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 & 1 & 1 \\ 1 & 3 & 6 & 4 & 1 & 1 \\ 1 & 1 & 4 & 11 & 3 & 1 \\ 1 & 1 & 1 & 3 & 9 & 3 \\ 2 & 1 & 1 & 1 & 3 & 3 \end{bmatrix}. \tag{5.3}$$

Since A can be written as

$$A = \begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}^T + \begin{pmatrix}0\\1\\2\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\2\\0\\0\\0\end{pmatrix}^T + \begin{pmatrix}0\\0\\1\\3\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\3\\0\\0\end{pmatrix}^T$$
$$+ \begin{pmatrix}0\\0\\0\\1\\2\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\\2\\0\end{pmatrix}^T + \begin{pmatrix}0\\0\\0\\0\\2\\1\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\2\\1\end{pmatrix}^T + \begin{pmatrix}1\\0\\0\\0\\0\\1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\\0\\1\end{pmatrix}^T, \tag{5.4}$$

by Dickinson's result (1.3), $A \in \text{int}(\mathcal{C}_6)$. We now verify it by Algorithm 4.1. Choose $C = I_6 + E_6$. Then the identifying vector of C is

$$c = (2, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 2, 1, 2)^T.$$

Choose

$$p_i = -e_1 + e_{i+1}, \quad i \in T_1 = \{6, 11, 15, 18, 20\},$$
$$p_i = -e_1 + 2e_{i+1}, \quad i \in \{1, \dots, 20\}\backslash T_1$$

to be basis vectors of $\text{span}\{c\}^\perp$. Let

$$p_0 = (c^T c)^{-1} c = \frac{1}{39} \cdot c.$$

Algorithm 4.1 terminates at $k = 3$, with $\lambda^{*,k} = 0.0726 > 0$. So, $A \in \text{int}(\mathcal{C}_6)$. We obtain the CP-decomposition $A = 0.0726 \cdot (I_6 + E_6) + \sum_{i=1}^{7} \rho_i b_i b_i^T$, where

$$\begin{aligned}
\rho_1 &= 2.9447, & b_1 &= (0.1034, 0.0000, 0.1929, 0.9757, 0.0000, 0.0000)^T,\\
\rho_2 &= 5.5366, & b_2 &= (0.0945, 0.0561, 0.4340, 0.8734, 0.0000, 0.1918)^T,\\
\rho_3 &= 5.8588, & b_3 &= (0.0000, 0.0030, 0.0000, 0.6941, 0.7199, 0.0000)^T,\\
\rho_4 &= 3.0631, & b_4 &= (0.0986, 0.3668, 0.7263, 0.5729, 0.0000, 0.0000)^T,\\
\rho_5 &= 4.1047, & b_5 &= (0.0790, 0.5271, 0.8462, 0.0000, 0.0000, 0.0000)^T,\\
\rho_6 &= 2.8900, & b_6 &= (0.7372, 0.2209, 0.0000, 0.0000, 0.0000, 0.6386)^T,\\
\rho_7 &= 7.7308, & b_7 &= (0.1383, 0.1364, 0.1383, 0.0000, 0.8676, 0.4365)^T.
\end{aligned}$$

6. Dickinson's form

We present Algorithm 4.1 for checking the membership and interiors of $\mathcal{C}_n$. If $A \in \mathcal{C}_n$, Algorithm 4.1 can give a CP-decomposition of A. Actually, we can also design a similar algorithm to give a CP-decomposition of A in Dickinson's form if $A \in \text{int}(\mathcal{C}_n)$.

Lemma 6.1. *Suppose $A \in \mathcal{S}_n$. Then $A \in int(\mathcal{C}_n)$ if and only if $rank(A) = n$ and, for some $b_1 \in R^n_{++}$, there exists a $\lambda > 0$ such that $A - \lambda b_1 b_1^T \in \mathcal{C}_n$.*

The proof of Lemma 6.1 is similar to that of Lemma 2.1, so we omit here. Lemma 6.1 gives an equivalent characterization of the interior of $\mathcal{C}_n$. Therefore, we can also transform the problem of checking interiors of $\mathcal{C}_n$ to the following linear optimization problem:

$$(\bar{P}_1): \quad \begin{cases} \bar{f}_1^* = & \max \quad \lambda \\ & \text{s.t.} \quad A - \lambda b_1 b_1^T \in \mathcal{C}_n, \end{cases}$$

where $b_1 \in R^n_{++}$. A simple choice of b_1 is $\mathbf{1}_n$, the n dimensional vector of all ones. The difference between $(\bar{P}_1)$ and (P_1) is that we use $b_1 b_1^T \in \mathrm{bd}(\mathcal{C}_n)$ instead of $C \in \mathrm{int}(\mathcal{C}_n)$.

By repeating similar arguments as in Sections 2 and 3, we can get

(1) If $(\bar{P}_1)$ is infeasible, then $A \notin \mathcal{C}_n$.
(2) If $(\bar{P}_1)$ is feasible, we have:
 (i) If $\bar{f}_1^* < 0$, then $A \notin \mathcal{C}_n$.
 (ii) If $\bar{f}_1^* = 0$, then $A \in \mathrm{bd}(\mathcal{C}_n)$.
 (iii) If $\bar{f}_1^* > 0$ and $\mathrm{rank}(A) < n$, then $A \in \mathrm{bd}(\mathcal{C}_n)$.
 (iv) If $\bar{f}_1^* > 0$ and $\mathrm{rank}(A) = n$, then $A \in \mathrm{int}(\mathcal{C}_n)$.

We formulate $(\bar{P}_1)$ as the linear optimization problem:

$$(\bar{P}_2): \quad \begin{cases} \bar{f}_2^* = & \max \quad \lambda \\ & \text{s.t.} \quad a - \lambda \bar{b} \in \mathscr{R}_{\mathcal{A}}(K), \end{cases}$$

where a and $\bar{b}$ are the identifying vectors of A and $b_1 b_1^T$, respectively. The k-th order semidefinite relaxation of $(\bar{P}_2)$ is

$$(\bar{P}_2^k): \quad \begin{cases} \bar{f}_2^k = & \max\limits_{\lambda, y} \quad \lambda \\ & \text{s.t.} \quad a - \lambda \bar{b} = y|_{\mathcal{A}}, \ y \in \Gamma_k(h, g). \end{cases}$$

We present another algorithm for checking the membership and interiors of $\mathcal{C}_n$ as follows.

Algorithm 6.2.
Input: $A \in \mathcal{S}_n$ and K as (3.2).
Output: $A \notin \mathcal{C}_n$, or $A \in \mathrm{bd}(\mathcal{C}_n)$, or $A \in \mathrm{int}(\mathcal{C}_n)$ with a CP-decomposition in Dickinson's form (1.3).
Procedure:

Step 0: Let $k := 1$.

Step 1: Solve the relaxation $(\bar{P}_2^k)$. If $(\bar{P}_2^k)$ is infeasible, stop and output that $A \notin \mathcal{C}_n$; otherwise, compute an optimal pair $(\lambda^{*,k}, y^{*,k})$ of (P_2^k).

Step 2: If $\bar{f}_2^k < 0$, stop and output that $A \notin \mathcal{C}_n$; else let $t := 1$.

Step 3: Let $v := y^{*,k}|_{2t}$. If the rank condition (3.9) is not satisfied, go to Step 6.

Step 4: Compute the finitely atomic measure μ admitted by v:

$$\mu = \rho_2 \delta(b_2) + \cdots + \rho_m \delta(b_m),$$

where $m = \mathrm{rank}(M_t(v))$, $b_i \in K$, $\rho_i > 0$, and $\delta(b_i)$ is the Dirac measure supported on the point $b_i \in K$.

Step 5: If $\mathrm{rank}(A) < n$ or $f_2^k = 0$, output $A \in \mathrm{bd}(\mathcal{C}_n)$ with a CP-decomposition and stop. Otherwise, output $A \in \mathrm{int}(\mathcal{C}_n)$ with a CP-decomposition of A in Dickinson's form (1.3) and stop.

Step 6: If $t < k$, set $t := t + 1$ and go to Step 3; otherwise, set $k := k + 1$ and go to Step 1.

Algorithm 6.2 can check whether a matrix $A \in \mathcal{S}_n$ is CP or not. If it is CP, Algorithm 6.2 can further check whether $A \in \mathrm{bd}(\mathcal{C}_n)$ or $A \in \mathrm{int}(\mathcal{C}_n)$. If $A \in \mathrm{int}(\mathcal{C}_n)$, a CP-decomposition of A in Dickinson's form (1.3) can be given. The convergence results of Algorithm 6.2 are similar to those of Algorithm 4.1, so we omit here.

We test Algorithm 6.2 on some examples.

Example 6.3. Consider the matrix A given as (5.3). We now use Algorithm 6.2 to verify $A \in \mathrm{int}(\mathcal{C}_6)$. Let $b_1 = \mathbf{1}_6$. Then the identifying vector of $b_1 b_1^T$ is $\bar{b} = \mathbf{1}_{21}$.

$$\bar{b} = (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)^T.$$

Choose

$$p_i = -e_1 + e_{i+1}, \quad i \in \{1, \ldots, 20\}$$

to be basis vectors of $\mathrm{span}\{\bar{b}\}^{\perp}$, and let

$$p_0 = (\bar{b}^T \bar{b})^{-1} \bar{b} = \frac{1}{21} \cdot \bar{b}.$$

Algorithm 6.2 terminates at $k = 3$, with $\lambda^{*,k} = 1.0000 > 0$ and $y(\lambda^{*,k}) \in \mathscr{R}_{\mathcal{A}}(K)$. So, $A \in \mathrm{int}(\mathcal{C}_6)$. We obtain the CP-decomposition $A = \sum_{i=1}^{6} \rho_i b_i b_i^T$ in Dickinson's form, where

$$\begin{aligned}
\rho_1 &= 1.0000, & b_1 &= (1.0000, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000)^T,\\
\rho_2 &= 5.0000, & b_2 &= (0.0000, 0.4472, 0.8944, 0.0000, 0.0000, 0.0000)^T,\\
\rho_3 &= 10.0000, & b_3 &= (0.0000, 0.0000, 0.3162, 0.9487, 0.0000, 0.0000)^T,\\
\rho_4 &= 5.0000, & b_4 &= (0.0000, 0.0000, 0.0000, 0.0000, 0.8944, 0.4472)^T,\\
\rho_5 &= 2.0000, & b_5 &= (0.7071, 0.0000, 0.0000, 0.0000, 0.0000, 0.7071)^T,\\
\rho_6 &= 5.0000, & b_6 &= (0.0000, 0.0000, 0.0000, 0.4472, 0.8944, 0.0000)^T.
\end{aligned}$$

The computed decomposition above is the same as (5.4).

Example 6.4. Consider the matrix A given as (cf. [28]):

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 6 & 5 & 1 \\ 1 & 1 & 5 & 6 & 2 \\ 2 & 1 & 1 & 2 & 3 \end{pmatrix}. \tag{6.1}$$

Since A can be written as

$$\begin{aligned}
A = & \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}^T + \begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}^T + \begin{pmatrix}0\\0\\2\\2\\0\end{pmatrix}\begin{pmatrix}0\\0\\2\\2\\0\end{pmatrix}^T \\
& + \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}^T + \begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}^T,
\end{aligned} \tag{6.2}$$

by Dickinson's form (1.3), $A \in \mathrm{int}(\mathcal{C}_5)$. Moreover, the decomposition above is minimal (cf. [12]). We verify $A \in \mathrm{int}(\mathcal{C}_5)$ by Algorithm 6.2. Choose $b_1 = \mathbf{1}_5$. Then the identifying vector of $b_1 b_1^T$ is $\bar{b} = \mathbf{1}_{15}$. Choose

$$p_i = -e_1 + e_{i+1}, \quad i = 1, \ldots, 14$$

to be basis vectors of $\mathrm{span}\{\bar{b}\}^\perp$, and let

$$p_0 = (\bar{b}^T \bar{b})^{-1} \bar{b} = \frac{1}{15} \cdot \mathbf{1}_{15}.$$

Algorithm 6.2 terminates at $k = 3$, with $\lambda^{*,k} = 1.0000 > 0$ and $y(\lambda^{*,k}) \in \mathscr{R}_{\mathcal{A}}(K)$. So, $A \in \mathrm{int}(\mathcal{C}_5)$. We obtain the CP-decomposition $A = \sum_{i=1}^5 \rho_i b_i b_i^T$ in Dickinson's form, where

$$\begin{aligned}
\rho_1 &= 1.0000, \quad b_1 = (1.0000, 1.0000, 1.0000, 1.0000, 1.0000)^T, \\
\rho_2 &= 2.0000, \quad b_2 = (0.0000, 0.0000, 0.0000, 0.7071, 0.7071)^T, \\
\rho_3 &= 8.0000, \quad b_3 = (0.0000, 0.0000, 0.7071, 0.7071, 0.0000)^T, \\
\rho_4 &= 2.0000, \quad b_4 = (0.7071, 0.0000, 0.0000, 0.0000, 0.7071)^T, \\
\rho_5 &= 2.0000, \quad b_5 = (0.0000, 0.7071, 0.7071, 0.0000, 0.0000)^T.
\end{aligned} \tag{6.3}$$

The computed decomposition (6.3) is the same as (6.2). We get a minimum CP-decomposition.

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