How to project onto extended second order cones

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Abstract

The extended second order cones were introduced by S. Z. Németh and G. Zhang in [S. Z. Németh and G. Zhang. *Extended Lorentz cones and variational inequalities on cylinders.* J. Optim. Theory Appl., 168(3):756-768, 2016] for solving mixed complementarity problems and variational inequalities on cylinders. R. Sznajder in [R. Sznajder. *The Lyapunov rank of extended second order cones.* Journal of Global Optimization, 66(3):585-593, 2016] determined the automorphism groups and the Lyapunov or bilinearity ranks of these cones. S. Z. Németh and G. Zhang in [S.Z. Németh and G. Zhang. *Positive operators of Extended Lorentz cones.* arXiv:1608.07455v2,2016] found both necessary conditions and sufficient conditions for a linear operator to be a positive operator of an extended second order cone. In this note we give formulas for projecting onto the extended second order cones. In the most general case the formula depends on a piecewise linear equation for one real variable which is solved by using numerical methods.

Keywords: Semi-smooth equation, extended second order cone, metric projection, piecewise linear Newton method

1 Introduction

The Lorentz cone is an important object in theoretical physics. In recent times it has been rebranded as second order cone and used for various application in optimization. Some robust optimization, plant location and investment portfolio manangement problems were formulated as as a second order cone program [1]. Another good survey paper with a wide range of applications of second order cone programming is [14]. More recent connections of second order cone programming and second order cone complementarity problem with physics, mechanics, economics, game theory, robotics, optimization and neural networks were considered in [4, 7, 12, 13, 15, 16, 22, 28, 29]. The importance of the second order cone is nowadays notorious not only in theoretical physics, but in optimization as well.

Thus far, there is no closed-form expression for metric (orthogonal) projection onto a general closed convex cone. A nice property of the second order cone is that it admits an explicit representation of the projection mapping onto it (see [6, Proposition 3.3]). The original motivation for extending the second order cone was inspired by using iterative methods for solving complementarity problems and variational inequalities [20,21]. These iterative methods are based on the property that the projection onto the closed convex set defining the problem is isotone with respect to the order defined out by a cone. Usually this is

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a very restrictive condition. However, cylinders and in particular cylinders with cone base admit isotone projections onto them with respect to the extended second order cones. Therefore, variational inequalities on cylinders and mixed complementarity problems can be solved by using such iterative techniques based on monotone convergence [19, 20].

Later it turned out that many of these cones could be even more useful because the bilinearity rank (or Lyapunov rank) [9, 10, 23, 24, 26] of them is higher than the dimension of the underlying space and therefore they have good numerical properties. More specifically, for $p > 1$ this is true whenever $q^2-3q+2 > 2p$ [25], where p, q are from the definition of the extended second order cone (see Definition 1). Such cones are "numerically good" cones when solving complementarity problems defined on them. The extended second order cones are also irreducible [25]. But to be really usable from optimization point of view we need easy ways of projecting onto them. In this paper we show that projecting onto an extended second order cone it is "almost possible" by using closed-form expressions. We present a set of formulas for projecting onto an extended second order cone which is subject to solving a piecewise linear equation with one real variable only. The method of finding these expressions is based on the special form of the complementarity set of the extended second order cone and Moreau's decomposition theorem [18] for projecting onto cones. The latter problem of projecting onto the extended second order cone is a particular conic optimization problem with respect to this cone. Although, the problem of projecting the point $(x, u) \in \mathbb{R}^p \times \mathbb{R}^q$ into the extended second order cone L (see Definition 1) can be transformed into the second order conic optimization problem

$$\text{Minimize}\left\{\|y-x\|^2+\|v-u\|^2\ :\ (y,v)\in\mathbb{R}^p\times\mathbb{R}^q,\ \ell_i(y,v)\in\mathcal{L},\quad i=1,\dots,p\right\},$$

where $\mathcal{L} = \{(t, u) \in \mathbb{R}\times\mathbb{R}^q : t \geq \|u\|\}$ is the second order cone in $\mathbb{R}^{q+1} \equiv \mathbb{R}\times\mathbb{R}^q$ and $\ell_i : \mathbb{R}^p\times\mathbb{R}^q \to \mathbb{R}\times\mathbb{R}^q$ are the linear mappings defined by $\ell_i(y, v) = (y_i, v)$, the complexity of our method is much simpler than solving the reformulated problem, because apart from closed-form expressions, it contains only one piecewise linear equation. By considering such a reformulation one would lose the useful special structure of the cone, which is the cornerstone for the simplicity of our method.

Certainly, the explicit representation of the projection mapping onto the second order cone (see [6, Proposition 3.3]) should not be handled as a conic optimization problem and the need to solve a simple piecewise linear equation for $p > 1$ makes our method just slightly more complex. The above observation about why one shouldn't reformulate the projection onto the extended second order cone into a second order conic optimization problem, together with the irreducibility of the second order cone, clearly shows that this cone "deserves a closer look".

The structure of the paper is as follows: In Section 2 we fix the notation and the terminology used throughout the paper. In Section 3 we present the formulas for projecting onto the extended second order cone. In Section 4 we solve the piecewise linear equation involved in these formulas by using the semi-smooth Newton's method and a method based on Picard's iteration. Finally, we make some remarks in the last section.

2 Preliminaries

Let ℓ, m, p, q be positive integers such that $m = p+q$. We identify the the vectors of $\mathbb{R}^\ell$ with $\ell\times 1$ matrices with real entries. The scalar product in $\mathbb{R}^\ell$ is defined by the mapping

$$\mathbb{R}^\ell\times\mathbb{R}^\ell \ni (x, y) \mapsto \langle x, y\rangle := x^\top y \in \mathbb{R}$$

and the corresponding norm by

$$\mathbb{R}^\ell \ni x \mapsto \|x\| := \sqrt{\langle x, x\rangle} \in \mathbb{R}.$$

For $x, y \in \mathbb{R}^\ell$ denote $x \perp y$ if $\langle x, y\rangle = 0$. We identify the elements of $\mathbb{R}^p \times \mathbb{R}^q$ with the elements of $\mathbb{R}^m$ through the correspondence

$$\mathbb{R}^p \times \mathbb{R}^q \ni (x, y) \mapsto (x^\top, y^\top)^\top.$$

Through this identification the scalar product in $\mathbb{R}^p \times \mathbb{R}^q$ is defined by

$$\langle (x, y), (u, v)\rangle := \langle (x^\top, y^\top)^\top, (u^\top, v^\top)^\top\rangle = \langle x, u\rangle + \langle y, v\rangle.$$

A closed set $K \subset \mathbb{R}^\ell$ with nonempty interior is called a *proper cone* if $K + K \subset K$, $K \cap (-K) = \{0\}$ and $\lambda K \subset K$, for any λ positive real number. The *dual cone* of a proper cone $K \subset \mathbb{R}^\ell$ is a proper cone defined by

$$K^* := \{x \in \mathbb{R}^\ell \ : \ \langle x, y\rangle \geq 0, \ \forall y \in K\}.$$

A proper cone $K \subset \mathbb{R}^\ell$ is called subdual if $K \subset K^*$, superdual if $K^* \subset K$ and self-dual if $K^* = K$. If $K, D \subset \mathbb{R}^\ell$ are proper cones such that $D = K^*$, then $D^* = K$ and the cones K, D are called *mutually dual*.

For a proper cone $K \in \mathbb{R}^\ell$ denote

$$C(K) := \{(x, y) \in K \times K^* \ : \ x \perp y\}$$

the *complementarity set* of K.

Let $C \in \mathbb{R}^\ell$ be a closed convex set. The projection mapping $P_C \colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ onto C is the mapping defined by

$$P_C(x) := \operatorname{argmin}\{\|x - y\| : y \in C\}.$$

We recall here Moreau's decomposition Theorem [18] (stated here for proper cones only):

Theorem 1. *Let $K \subset \mathbb{R}^\ell$ be a proper cone, K^* its dual cone and $z \in \mathbb{R}^\ell$. Then, the following two statements are equivalent:*

(i) $z = x - y$ and $(x, y) \in C(K)$,

(ii) $x = P_K(z)$ and $y = P_{K^*}(-z)$.

Theorem 1 implies

$$z = P_K(z) - P_{K^*}(-z),$$

with $P_K(z) \perp P_{K^*}(-z)$.

For $z \in \mathbb{R}^\ell$ we denote $z = (z_1, \dots, z_\ell)^\top$. Let $\geq$ denote the component-wise order in $\mathbb{R}^\ell$, that is, the order defined by $\mathbb{R}^\ell \ni x \geq y \in \mathbb{R}^\ell$ if and only if $x_i \geq y_i$ for $i = 1, \dots, \ell$. Denote by 0 the vector in $\mathbb{R}^\ell$ or a scalar zero (it will not lead to any confusion), by e the vector of ones in $\mathbb{R}^\ell$ and by $\mathbb{R}^\ell_+ = \{x \in \mathbb{R}^\ell \ : \ x \geq 0\}$ the nonnegative orthant. The proper cone $\mathbb{R}^\ell_+$ is self-dual. For a real number $\alpha \in \mathbb{R}$ denote $\alpha^+ := \max(\alpha, 0)$ and $\alpha^- := \max(-\alpha, 0)$. For a vector $z \in \mathbb{R}^\ell$ denote $z^+ := (z_1^+, \dots, z_\ell^+)$, $z^- := (z_1^-, \dots, z_\ell^-)$, $|z| := (|z_1|, \dots, |z_\ell|)$, $\operatorname{sgn}(z) := (\operatorname{sgn}(z_1), \dots, \operatorname{sgn}(z_\ell))$ and $\operatorname{diag}(z)$ the $\ell \times \ell$ diagonal matrix with entries $\operatorname{diag}(z)_{ij} := \delta_{ij} z_i$, where $i, j \in \{1, \dots, \ell\}$. It is known that $z^+ = P_{\mathbb{R}^\ell_+}(z)$ and $z^- = P_{\mathbb{R}^\ell_+}(-z)$.

We recall from [20] the following definition of a pair of mutually dual extended second order cones L, M:

Definition 1.

$$L := \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : x \geq \|u\| e\},$$
$$M := \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : \langle x, e\rangle \geq \|u\|, x \geq 0\}.$$

where $\geq$ denotes the component-wise order.

It is known that both L and M are proper cones, L is subdual M is superdual and if $p = 1$, then both cones reduce to the second order cone. The cones L and M are polyhedral if and only if $q = 1$. If we allow $q = 0$ as well, then the cones L and M reduce to the nonnegative orthant. More properties of the extended second order cones can be found in [19, 20, 25].

3 Projection formulas for extended second order cones

In this section we give formulas for projecting onto the pair of mutually dual extended second order cones. Before presenting our main theorem, we need some preliminary results for these cones. Let p, q be positive integers.

Proposition 1. *Let $x, y \in \mathbb{R}^p$ and $u, v \in \mathbb{R}^q \setminus \{0\}$. We have that $(x, u, y, v) := ((x, u), (y, v)) \in C(L)$ if and only if there exists a $\lambda > 0$ such that $v = -\lambda u$, $\langle y, e\rangle = \|v\|$ and $(x - \|u\|e, y) \in C(\mathbb{R}^p_+)$.*

Proof. Suppose first that there exists $\lambda > 0$ such that $v = -\lambda u$, $\langle y, e\rangle = \|v\|$ and $(x - \|u\|e, y) \in C(\mathbb{R}^p_+)$. Hence, $(x, u) \in L$ and $(y, v) \in M$. Moreover,

$$\langle (x, u), (y, v)\rangle = \langle x, y\rangle + \langle u, v\rangle = \|u\|\langle e, y\rangle - \lambda\|u\|^2 = \|u\|\|v\| - \lambda\|u\|^2 = 0.$$

Thus, $(x, u, y, v) \in C(L)$. Conversely, suppose that $(x, u, y, v) \in C(L)$. Then, $(x, u) \in L$, $(y, v) \in M$ and

$$0 = \langle (x, u), (y, v)\rangle = \langle x, y\rangle + \langle u, v\rangle \geq \langle \|u\|e, y\rangle + \langle u, v\rangle \geq \|u\|\|v\| + \langle u, v\rangle \geq 0.$$

Hence, there exists $\lambda > 0$ such that $v = -\lambda u$, $\langle e, y\rangle = \|v\|$ and $\langle x - \|u\|e, y\rangle = 0$. It follows that $(x - \|u\|e, y) \in C(\mathbb{R}^p_+)$. □ □

Before presenting the main result of this section we introduce a piecewise linear function and establish some important properties of it. This function will play an important role in the sequel, namely, the formulas for the projection will depend on its single positive zero. The piecewise linear function $\psi\colon [0, +\infty) \to \mathbb{R}$ is defined by

$$\psi(\lambda) := -\lambda\|w\| + \left\langle e, [(\lambda + 1)z - \|w\|e]^-\right\rangle. \tag{1}$$

For stating the next proposition we need to define the following diagonal matrix, which we will see is related to the subdifferential $\partial\psi$ of ψ:

$$N(\lambda) := \operatorname{diag}\left(-\operatorname{sgn}\left([(\lambda + 1)z - \|w\|e]^-\right)\right), \qquad \lambda \in [0, +\infty). \tag{2}$$

Proposition 2. *The function ψ is convex. Moreover, if*

$$z^+ \not\geq \|w\|e, \qquad \langle z^-, e\rangle < \|w\|,$$

then we have:

1. *$-\|w\| + \langle e, N(\lambda)z\rangle \in \partial\psi(\lambda)$ and $-\|w\| + \langle e, N(\lambda)z\rangle < 0$, for all $\lambda \geq 0$;*
2. *ψ has a unique zero $\lambda_* > 0$.*

Proof. We first note that the function ψ can be equivalently given by

$$\psi(\lambda) := -\lambda\|w\| + \sum_{i=1}^{p} \psi_i(\lambda), \qquad \psi_i(\lambda) := [(\lambda + 1)z_i - \|w\|]^-, \qquad \lambda \geq 0. \tag{3}$$

Since the sum and the maximum of two convex functions is convex, it follows that the function $\psi_i(\lambda) = \max\{-(\lambda + 1)z_i + \|w\|,\ 0\}$ is convex for all $i = 1, \ldots p$. Hence, the result of the first part follows.

1. The definitions of ψ and ψ_i in (3) imply that $\partial\psi(\lambda) = -\|w\| + \sum_{i=1}^{p} \partial\psi_i(\lambda)$. Moreover, considering that $\psi_i(\lambda) = \max\{-(\lambda+1)z_i + \|w\|,\ 0\}$, we have $-\mathrm{sgn}\left([(\lambda+1)z_i - \|w\|]^-\right) z_i \in \partial\psi_i(\lambda)$, for all $i = 1, \dots p$. Therefore, using (2), the inclusion follows. To prove the inequality, note that (2) implies that the entries of $N(\lambda)$ are equal to 0 or -1, for all $\lambda \geq 0$. Thus, from the assumption $\langle z^-, e\rangle < \|w\|$ we have $-\|w\| + \langle e, N(\lambda)z\rangle < 0$, for all $\lambda \geq 0$.

2. First, we show that (1) has a positive zero. Note that $z \not\geq \|w\|e$, otherwise it would follow that $z^+ = z \geq \|w\|e$, which contradicts our assumptions. Then, there exists $i_0 \in \{1, \dots, p\}$ such that $z_{i_0} < \|w\|$. Hence, from (3) we have $\psi(0) > \|w\| - z_{i_0} > 0$. If $\lambda > 0$ is sufficiently large, then $\mathrm{sgn}[(\lambda+1)z_i - \|w\|] = \mathrm{sgn} z_i$ and consequently $[(\lambda+1)z_i - \|w\|]^- \leq (\lambda+1)z_i^- + \|w\|$. By using the last inequality, (3) and the assumption $\langle z^-, e\rangle < \|w\|$, we conclude that for $\lambda > 0$ sufficiently large, it is true that

$$\psi(\lambda) \leq -\lambda\|w\| + \left\langle e, (\lambda+1)z^- + \|w\|e\right\rangle = \left[-\|w\| + \left\langle z^-, e\right\rangle\right]\lambda + \|w\| + \left\langle e, z^-\right\rangle < 0. \quad (4)$$

Since ψ is continuous, there is a $\lambda_* > 0$ such that $\psi(\lambda_*) = 0$. By contradiction we assume that ψ has two positive zeroes $\bar{\lambda}$ and $\hat{\lambda}$. Let $0 < \hat{\lambda} < \bar{\lambda}$. Since ψ is convex and $-\|w\| + \langle e, N(\lambda)z\rangle \in \partial\psi(\lambda)$, we have $\psi(\hat{\lambda}) \geq \psi(\bar{\lambda}) + [-\|w\| + \left\langle e, N(\bar{\lambda})z\right\rangle][\hat{\lambda} - \bar{\lambda}]$. Due to $\psi(\hat{\lambda}) = \psi(\bar{\lambda}) = 0$ and considering that $0 < \hat{\lambda} < \bar{\lambda}$, the last inequity implies that $-\|w\| + \langle e, N(\lambda)z\rangle \geq 0$, which contradicts the second part of item 1. Therefore, ψ has a unique positive zero. □

□

Now we ready to state and prove the main result of the paper.

Theorem 2. *Let $(z, w) \in \mathbb{R}^p \times \mathbb{R}^q$. Then, we have*

1. *If $z^+ \geq \|w\|e$, then $P_L(z, w) = (z^+, w)$ and $P_M(-z, -w) = (z^-, 0)$.*
2. *If $\langle z^-, e\rangle \geq \|w\|$, then $P_L(z, w) = (z^+, 0)$ and $P_M(-z, -w) = (z^-, -w)$.*
3. *If $z^+ \not\geq \|w\|e$ and $\langle z^-, e\rangle < \|w\|$, then the piecewise linear equation*

$$\lambda\|w\| = \left\langle e, [(\lambda+1)z - \|w\|e]^-\right\rangle. \quad (5)$$

has a unique positive solution $\lambda > 0$,

$$P_L(z, w) = \left(\left[z - \frac{1}{\lambda+1}\|w\|e\right]^+ + \frac{1}{\lambda+1}\|w\|e,\ \frac{1}{\lambda+1}w\right) \quad (6)$$

and

$$P_M(-z, -w) = \left(\left[z - \frac{1}{\lambda+1}\|w\|e\right]^-,\ -\frac{\lambda}{\lambda+1}w\right) \quad (7)$$

Proof. We will use Moreau's decomposition theorem for L for proving all three items. In this case this theorem states that, $P_L(z, w) = (x, u)$ and $P_M(-z, -w) = (y, v)$ if and only if $(z, w) = (x, u) - (y, v)$ and $(x, u, y, v) \in C(L)$.

1. This is exactly the case when $v = 0$.

   Indeed, $v = 0$ implies $P_L(z, w) = (x, u)$ and $P_M(-z, -w) = (y, 0)$. Hence, $z = x - y$, $w = u$, $x \geq \|u\|e$, $y \geq 0$ and $\langle x, y\rangle = 0$. By using Moreau's decomposition theorem for $\mathbb{R}^p_+$, we have that $z = x - y$, $x \geq 0$, $y \geq 0$ and $\langle x, y\rangle = 0$ implies $x = z^+$ and $y = z^-$. Since, $w = u$ and $x \geq \|u\|e$, we get $z^+ \geq \|w\|e$.

   Conversely, suppose that $z^+ \geq \|w\|e$. Then $(z^+, w, z^-, 0) \in C(L)$. Hence, by Moreau's decomposition Theorem for L, we get $P_L(z, w) = (z^+, w)$ and $P_M(-z, -w) = (z^-, 0)$. Thus, $v = 0$.

2. This is exactly the case when $u = 0$.

   Indeed, $u = 0$ implies $P_L(z, w) = (x, 0)$ and $P_M(-z, -w) = (y, v)$. Hence, $z = x - y$, $w = -v$, $x \geq 0$, $\langle y, e\rangle \geq \|v\|$, $y \geq 0$ and $\langle x, y\rangle = 0$. By using Moreau's decomposition theorem for $\mathbb{R}^p_+$, we have that $z = x - y$, $x \geq 0$, $y \geq 0$ and $\langle x, y\rangle = 0$ implies $x = z^+$ and $y = z^-$. Since $w = -v$ and $\langle y, e\rangle \geq \|v\|$, we get $\langle z^-, e\rangle \geq \|w\|$.

   Conversely, suppose that $\langle z^-, e\rangle \geq \|w\|$. Then, it is easy to check that $(z^+, 0, z^-, -w) \in C(L)$. Then, by Moreau's decomposition Theorem for L, we get $P_L(z, w) = (z^+, 0)$ and $P_M(-z, -w) = (z^-, -w)$. Thus, $u = 0$.

3. This is exactly the case when $u \neq 0$ and $v \neq 0$.

   From Proposition 1 it follows that $(z, w) = (x, u) - (y, v)$ and $(x, u, y, v) \in C(L)$ is equivalent to $z = x - y$, $w = u - v$ and the existence of a $\lambda > 0$ such that $v = -\lambda u$, $\langle y, e\rangle = \|v\|$ and $(x - \|u\|e, y) \in C(\mathbb{R}^p_+)$. On the other hand, by Moreau's decomposition theorem for $\mathbb{R}^p_+$, $(x - \|u\|e, y) \in C(\mathbb{R}^p_+)$ is equivalent to $x - \|u\|e = [x - \|u\|e - y]^+$ and $y = [x - \|u\|e - y]^-$. Hence,

$$P_L(z, w) = \left(x, \frac{1}{\lambda + 1} w\right) \tag{8}$$

   and

$$P_M(-z, -w) = \left(y, -\frac{\lambda}{\lambda + 1} w\right) \tag{9}$$

   if and only if $z = x - y$ and $\lambda > 0$ is such that

$$\langle y, e\rangle = \frac{\lambda}{\lambda + 1}\|w\|, \tag{10}$$

$$x = \left[z - \frac{1}{\lambda + 1}\|w\|e\right]^+ + \frac{1}{1 + \lambda}\|w\|e \tag{11}$$

   and

$$y = \left[z - \frac{1}{\lambda + 1}\|w\|e\right]^- . \tag{12}$$

   From equations (8) and (11) follows equation (6) and from equations (9) and (12) follows equation (7), where $\lambda > 0$ is given by equation (5), which is a combination of equations (10) and (12). The uniqueness of $\lambda > 0$ which satisfies (5) follows from the uniqueness of $P_L(z, w)$ and $P_M(z, w)$. □

□

The next remark will recover the well known formulas for projecting onto the second order cone (see for example [6, Proposition 3.3]).

Remark 1. *Let* $(z, w) \in \mathbb{R} \times \mathbb{R}^q$ *and* L *be the second order cone. Then, letting* $u := [z - \|w\|]^+$ *and* $v := [z + \|w\|]^+$ *we conclude that Theorem 2 implies that*

$$P_L(z, w) = \begin{cases} \frac{1}{2}\left(u + v,\, [v - u]\,\dfrac{w}{\|w\|}\right), & w \neq 0, \\ \\ (z^+,\, 0)\,, & w = 0. \end{cases} \tag{13}$$

Indeed, for $p = 1$*, the conditions in item 3 in Theorem* (2) *hold if and only if* $0 \leq |z| < \|w\|$ *and equation* (5) *becomes* $\lambda\|w\| = [(\lambda + 1)z - \|w\|]^-$*, which obviously can have only nonnegative solutions, because the right hand side of the equation is nonnegative. Moreover,* $\lambda = 0$ *cannot be a solution because that would imply* $|z| - \|w\| \geq z - \|w\| > 0$*. Hence, the conditions in item 3 hold if and only if* (5) *becomes* $\lambda\|w\| = (\|w\| - (\lambda + 1)z)$*. This latter equation has the unique positive solution*

$$\lambda = \frac{\|w\| - z}{\|w\| + z}. \tag{14}$$

By using equation (6) *and* (14)*, it is just a matter of algebraic manipulations to check that* (13) *holds for this case. The cases described by items 1 and 2 can be similarly checked.*

4 Numerical methods for projecting

In this section we present three well known numerical methods to find the unique zero of the piecewise linear equation (5), in order to project onto the extended second order cones. We note that $(z, w) \in \mathbb{R}^p \times \mathbb{R}^q$ satisfies the two conditions in item 3 of Theorem 2 if and only if

$$\exists\ i_0 \in \{1, \ldots, p\}; \quad 0 \leq z_{i_0}^+ < \|w\|, \qquad 0 \leq \sum_{i=1}^{p} z_i^- < \|w\|. \tag{15}$$

Throughout this section we will assume that $(z, w) \in \mathbb{R}^p \times \mathbb{R}^q$ *satisfies* (15).

4.1 Semi-smooth Newton method

In order to study (5), we consider the piecewise linear function ψ defined by (1). It follows from Proposition 2 that ψ is convex and its unique zero, namely $\lambda_* > 0$, is the solution of (5). The *semi-smooth Newton method* for finding the zero of ψ, with a starting point $\lambda_0 \in (0, +\infty)$, it is formally defined by

$$\psi(\lambda_k) + s_k\,(\lambda_{k+1} - \lambda_k) = 0, \qquad s_k \in \partial\psi(\lambda_k), \qquad k = 0, 1, \ldots, \tag{16}$$

where s_k is any subgradient in $\partial\psi(\lambda_k)$. Let $N(\lambda)$ be defined by equation (2). Item 1 of Proposition 2 implies that $-\|w\| + \langle e, N(\lambda)z\rangle \in \partial\psi(\lambda)$. Since $N(\lambda)[(\lambda + 1)z - \|w\|e] = [(\lambda + 1)z - \|w\|e]^-$, by setting $s_k = -\|w\| + \langle e, N_k z\rangle$ with

$$N_k := N(\lambda_k), \tag{17}$$

equation (16) implies

$$-\lambda_k\|w\| + \langle e, N_k\,[(\lambda_k + 1)z - \|w\|e]\rangle + [-\|w\| + \langle e, N_k z\rangle]\,[\lambda_{k+1} - \lambda_k] = 0.$$

After simplification, we get

$$[-\|w\| + \langle e, N_k z\rangle]\,\lambda_{k+1} = -\,\langle e, N_k\,[z - \|w\|e]\rangle\,, \qquad k = 0, 1, \ldots, \tag{18}$$

which formally defines the *semi-smooth Newton sequence* $\{\lambda_k\}$ for solving (5).

Remark 2. *For $p = 1$, the conditions in* (15) *hold if and only if $0 \le |z| < \|w\|$. Thus, if $z \le 0$, then $N_k \equiv -1$ and $\lambda_{k+1} = [\|w\| - z]/[\|w\| + z]$ for all $k = 0, 1, \ldots$. Now, if $z > 0$ then letting $0 < \lambda_0 < [\|w\| - z]/z$, we have $N_0 \equiv -1$ and $\lambda_1 = [\|w\| - z]/[\|w\| + z]$. Therefore, from Remark 1, we conclude that the semi-smooth Newton sequence* (18) *solves equation* (5) *for $p = 1$ with only one iteration.*

The proof of the next proposition is based on ideas similar to some arguments in [2].

Proposition 3. *For any $\lambda_0 > 0$ the sequence $\{\lambda_k\}$ defined in* (18) *is well defined and converges after at most 2^p steps to the unique solution $\lambda_* > 0$ of* (5).

Proof. Proposition 2 implies that ψ is convex and $-\|w\| + \langle e, N(\lambda)z\rangle \in \partial\psi(\lambda)$. Thus, we have

$$\psi(\mu) - \psi(\lambda) - [-\|w\| + \langle e, N(\lambda)z\rangle](\mu - \lambda) \ge 0, \qquad \mu, \lambda \in [0, +\infty). \tag{19}$$

On the other hand, it follows from (16) and (17) that the sequence $\{\lambda_k\}$ is equivalently defined as follows

$$\psi(\lambda_k) + [-\|w\| + \langle e, N_k z\rangle]\,(\lambda_{k+1} - \lambda_k) = 0, \qquad k = 0, 1, \ldots. \tag{20}$$

By combining the above equality with the definition in (17) and the equality in (19), we can conclude that

$$\psi(\lambda_{k+1}) \ge \psi(\lambda_k) + [-\|w\| + \langle e, N_k z\rangle](\lambda_{k+1} - \lambda_k) = 0, \qquad k = 0, 1, \ldots. \tag{21}$$

By letting $\mu = \lambda^*$ and $\lambda = \lambda_k$ in inequality (19) and by using again the definition in (17), we obtain that

$$0 = \psi(\lambda_*) \ge \psi(\lambda_k) + [-\|w\| + \langle e, N_k z\rangle](\lambda_* - \lambda_k), \qquad k = 0, 1, \ldots. \tag{22}$$

Proposition 2 implies that $-\|w\| + \langle e, N_k z\rangle < 0$, for all $k = 0, 1, \ldots$. Then, by dividing both sides of (22) by $-\|w\| + \langle e, N_k z\rangle$ and by using (20), after some algebras we obtain

$$\lambda_{k+1} = \lambda_k - [-\|w\| + \langle e, N_k z\rangle]^{-1}\psi(\lambda_k) \le \lambda_*, \qquad k = 0, 1, \ldots. \tag{23}$$

On the other hand, $\psi(\lambda_k) \ge 0$, for all $k = 0, 1, \ldots$. Thus, after dividing both sides of the equality in (21) by $\|w\| - \langle e, N_k z\rangle$ and some algebraic manipulations, we conclude

$$0 < \lambda_k \le \lambda_k - [-\|w\| + \langle e, N_k z\rangle]^{-1}\psi(\lambda_k) = \lambda_{k+1}, \qquad k = 0, 1, \ldots. \tag{24}$$

Hence, by combining (23) with (24), we conclude that $0 < \lambda_k \le \lambda_{k+1} \le \lambda_*$, for all $k = 0, 1, \ldots$. Hence, $\{\lambda_k\}$ converges to some $\bar{\lambda} > 0$. By using again (20) and that the entries of N_k are equal to 0 or -1, we have

$$\begin{aligned} |\psi(\bar{\lambda})| = \lim_{k\to\infty} |\psi(\lambda_k)| &= \lim_{k\to\infty} |\,[-\|w\| + \langle e, N_k z\rangle]\,(\lambda_{k+1} - \lambda_k)\,| \\ &\le [\|w\| + \langle e, |z|\rangle] \lim_{k\to\infty} |\lambda_{k+1} - \lambda_k| = 0. \end{aligned}$$

Hence, $\{\lambda_k\}$ converges to $\bar{\lambda} = \lambda_*$ the unique zero of ψ, which is the solution of (5).

Finally, we establish the finite termination of the sequence $\{\lambda_k\}$ at λ_*, the unique solution of (5). Since the entries of $N(\lambda)$ are equal to 0 or -1, $N(\lambda)$ has at most 2^p different possible configurations. Then, there exist $j, \ell \in \mathbb{N}$ with $1 \le j < 2^p$ and $1 \le \ell < 2^p$ such that $N(\lambda_j) = N(\lambda_{j+\ell})$. Hence, from (18) we have

$$\begin{aligned} \lambda_{j+1} &= -\left[-\|w\| + \langle e, N_j z\rangle\right]^{-1} \langle e, N_j\left[z - \|w\|e\right]\rangle \\ &= -\left[-\|w\| + \langle e, N_{j+\ell} z\rangle\right]^{-1} \langle e, N_{j+\ell}\left[z - \|w\|e\right]\rangle = \lambda_{j+\ell+1}. \end{aligned}$$

By applying this argument inductively, $\lambda_{j+1} = \lambda_{j+\ell+1}$, $\lambda_{j+2} = \lambda_{j+\ell+2}$, …, $\lambda_{j+\ell} = \lambda_{j+2\ell}$, $\lambda_{j+\ell+1} = \lambda_{j+2\ell+1} = \lambda_{j+1}$. Thus, by using (24) and the last equality, we conclude that

$$\lambda_{j+1} \le \lambda_{j+2} \le \cdots \le \lambda_{j+\ell+1} \le \lambda_{j+1}.$$

Hence, $\lambda_{j+1} = \lambda_{j+2}$ and in view of (20) we conclude that $\psi(\lambda_{j+1}) = 0$ and λ_{j+1} is the solution of (5), i.e., $\lambda_{j+1} = \lambda_*$. □ □

The next proposition shows that under a further restriction on the point which is projected the convergence of the semi-smooth Newton sequence is linear.

Proposition 4. *Assume that* $0 < \alpha < 1$ *and* $\langle e, |z|\rangle < \alpha(1+\alpha)^{-1}\|w\|$*. Then, for any* $\lambda_0 > 0$ *, the sequence* $\{\lambda_k\}$ *in* (18) *is well defined and converges linearly to the unique solution* λ_* *of* (5)*:*

$$|\lambda_* - \lambda_{k+1}| \leq \alpha|\lambda_* - \lambda_k|, \qquad k = 0, 1, \ldots. \tag{25}$$

Proof. Proposition 2 and (17) imply $-\|w\| + \langle e, N_k z\rangle < 0$ for all $k = 0, 1, \ldots$, which implies that the sequence $\{\lambda_k\}$ is well defined. Proposition 2 also implies that (5) has a zero $\lambda_* \in (0, +\infty)$. Hence, by using (17), (18) and the definition of ψ, after some algebra we obtain that

$$\begin{aligned}\lambda_* - \lambda_{k+1} = [-\|w\| + \langle e, N_k z\rangle]^{-1} \big[&\lambda_*\|w\| - \langle e, [(\lambda_* + 1)z - \|w\|e]^-\rangle \\ &- \lambda_k\|w\| + \langle e, [(\lambda_k + 1)z - \|w\|e]^-\rangle + [-\|w\| + \langle e, N_k z\rangle]\, [\lambda_* - \lambda_k]\big],\end{aligned}$$

for all $k = 0, 1, \ldots$. On the other hand, since $N(\lambda)[(\lambda + 1)z - \|w\|e] = [(\lambda + 1)z - \|w\|e]^-$, after some calculations we have

$$\begin{aligned}&\lambda_*\|w\| - \langle e, [(\lambda_* + 1)z - \|w\|e]^-\rangle - \\ &\qquad \lambda_k\|w\| + \langle e, [(\lambda_k + 1)z - \|w\|e]^-\rangle + [-\|w\| + \langle e, N_k z\rangle]\, [\lambda_* - \lambda_k] = \\ &\qquad\qquad - \langle e, N_*[(\lambda_* + 1)z - \|w\|e]\rangle + \langle e, N_k[(\lambda_k + 1)z - \|w\|e]\rangle + \langle e, N_k z\rangle\, [\lambda_* - \lambda_k]\,,\end{aligned}$$

for all $k = 0, 1, \ldots$, where $N_* := N(\lambda_*)$. By combining the above two equalities, we obtain that

$$\begin{aligned}\lambda_* - \lambda_{k+1} = [-\|w\| + \langle e, N_k z\rangle]^{-1} \big[&- \langle e, N_*[(\lambda_* + 1)z - \|w\|e]\rangle + \\ &\langle e, N_k[(\lambda_k + 1)z - \|w\|e]\rangle + \langle e, N_k z\rangle\, [\lambda_* - \lambda_k]\big].\end{aligned}$$

Define the auxiliary piecewise linear convex function $\zeta(\lambda) := \langle e, N(\lambda)[(\lambda + 1)z - \|w\|e]\rangle$. Thus, except possibly at p points, ζ is differentiable and there holds

$$\zeta(\lambda_*) = \zeta(\lambda_k) + \int_0^1 \langle e, N(\lambda_k + t(\lambda_* - \lambda_k))z\rangle\, [\lambda_* - \lambda_k] dt,$$

due to $\langle e, N(\lambda)z\rangle \in \partial\zeta(\lambda)$; see [11, Remark 4.2.5, pag. 26]. Hence, by simple combination of the two latter equalities, we have

$$\begin{aligned}&\lambda_* - \lambda_{k+1} = \\ &\qquad - [-\|w\| + \langle e, N_k z\rangle]^{-1} \int_0^1 \langle e, [N(\lambda_k + t(\lambda_* - \lambda_k)) - N_k]\, z\rangle\, dt[\lambda_* - \lambda_k],\end{aligned}$$

for all $k = 0, 1, \ldots$. Since (2) implies that the entries of the matrix N are equal to 0 or -1, we obtain

$$|\langle e, [N(\lambda_k + t(\lambda_* - \lambda_k)) - N_k]\, z\rangle| \leq \sum_{j=1}^{p} |z_j| = \langle e, |z|\rangle\,.$$

Thus, combining above equality with last inequality, we obtain that

$$|\lambda_* - \lambda_{k+1}| \leq |\|w\| - \langle e, N_k z\rangle|^{-1} \langle e, |z|\rangle\, |\lambda_* - \lambda_k|, \qquad k = 0, 1, \ldots.$$

Therefore, as we are under the assumption $\langle e, |z|\rangle < \alpha(1 + \alpha)^{-1}\|w\|$, we have $\langle e, |z|\rangle / [\|w\| - \langle e, N_k z\rangle] < \alpha < 1$, (25) holds and the sequence $\{\lambda_k\}$ converges to λ_*, which concludes the proof. □ □

4.2 Picard's method

In this section we present a method based on Picard's iteration for solving equation (5) under a further restriction on the point which is projected. The statement of the result is as follows:

Proposition 5. *If* $\langle e, |z| \rangle < \|w\|$, *then for all* $\lambda_0 > 0$ *the sequence given by the iteration*

$$\lambda_{k+1} = \frac{1}{\|w\|} \left\langle e, [(\lambda_k + 1)z - \|w\|e]^- \right\rangle, \qquad k = 1, \dots, \tag{26}$$

converges to the unique solution of the semi-smooth equation (5).

Proof. It is sufficient to prove that $\varphi \colon [0, +\infty) \to \mathbb{R}$ defined by

$$\varphi(\lambda) = \frac{1}{\|w\|} \left\langle e, [(\lambda + 1)z - \|w\|e]^- \right\rangle$$

is a contraction. Indeed, the definition of φ implies

$$\begin{aligned} |\varphi(\lambda) - \varphi(\mu)| &\leq \frac{1}{\|w\|} \sum_{i=1} \left| [(\lambda + 1)z_i - \|w\|]^- - [(\mu + 1)z_i - \|w\|]^- \right| \\ &\leq \frac{1}{\|w\|} \sum_{i=1} |z_i(\lambda - \mu)| = \frac{\langle e, |z| \rangle}{\|w\|} |\lambda - \mu|, \qquad \lambda, \mu \in [0, +\infty). \end{aligned}$$

Since we are under the assumption $\langle e, |z| \rangle < \|w\|$, the last inequality implies that φ is a contraction and the result follows. $\square$ $\square$

Final remarks

The *extended second order cones* (ESOCs) are likely the most natural extensions of the second order cones. Also, the complementarity problems defined on them often have nice computational properties as remarked in the introduction. Finally, we found almost closed-form formulas for projecting onto them. The formulas depend only on a piecewise linear equation for a real parameter. Not so much of the ESOCs is known, nevertheless, we stipulate that they will become an important class of cones in optimization.

For a given point in the ambient space the projection can be obtained easily in at most 2^p steps, by assigning signs to the components of the second vector in the scalar product on the right hand side of the piecewise linear equation (5), solving for λ, and if there is a solution, then checking that the solution corresponds to the a priori assumed signs. However, this method is computationally unviable for larger p. Therefore, we developed numerical methods for solving (5) based on the semismooth Newton method and Picards iterations. Although the semismooth Newton method always converges in at most 2^p steps, it needs some restriction on the point which is projected to prove that is globally linearly convergent. A similar type of restriction is needed for Picard's method to prove that it is globally convergent.

The complexity of our projection method is considerably lower than the complexity of solving the reformulation of the projection problem into a second order conic optimization problem. It is expected that there are other conic optimization problems with respect to the extended second order cone which are easier to solve than transforming them into second order conic optimization problems. We plan to solve conic optimization and complementarity problems on the extended second order cone (similarly to the second order cone in [3]) and to find practical examples which can be modeled by such problems. Early studies of Lianghai Xiao (PhD student of the second author) suggest that the extended second order cones could be useful for portfolio selection, see [17, 27] and signal processing problems, see [5, 8].

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