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In memory of my teacher Vu Thi Hai (1968-1991)

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Abstract In this paper, we investigate several properties of the solution maps of variational inequalities with polynomial data. First, we prove some facts on the R_0-property, the local boundedness, and the upper semicontinuity of the solution maps. Second, we establish results on the solution existence and stability under the copositivity condition. Third, when the constraint set is semi-algebraic, we discuss the genericity of the R_0-property and the finite-valuedness of the solution maps.

Keywords Polynomial variational inequality · R_0-property · Upper semicontinuity · Stability · Genericity · Semi-algebraic set

Mathematics Subject Classification (2000) 90C33 · 90C31 · 14P10 · 47H05 · 54C60

1 Introduction

Polynomial complementarity problems and polynomial variational inequalities have been recently investigated by many authors, see, for example [8,11,14, 19] and the references therein. These problems are natural extensions of affine variational inequalities (see [15] and the references therein) and special cases of weakly homogeneous variational inequalities introduced by Gowda and Sossa [9].

In the recent past, several authors studied different properties of polynomial complementarity problems, tensor complementarity problems and tensor variational inequalities, which are subclasses of polynomial variational inequalities. In particular, the solvability, the global uniqueness, and the boundedness

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of the solution sets have been studied in [2,8,16,20,22]. Very recently, Hieu [10] discussed the upper semicontinuity, the stability, and the finite-valuedness of the solution maps of tensor complementarity problems. In [14], the authors investigated the genericity and the Hölder stability for semi-algebraic variational inequalities where the related maps are the sums of semi-algebraic maps and parametric vectors.

In this paper, we investigate several properties of the solution maps of variational inequalities with polynomial data. First, we introduce the R_0-property and show that it plays an important role in the investigation. Since polynomial maps are weakly homogeneous, the normalization argument (see, e.g. [1, 9,17]) can be applied. Several facts on the local boundedness and the upper semicontinuity of the solution maps are shown. Second, we obtain a result on the existence of solutions under the copositivity. We develop the results on solution stability of copositive linear complementarity problems and affine variational inequalities in [5,15] for copositive polynomial variational inequalities. Third, when the constraints are polynomial, techniques from semi-algebraic geometry (see, e.g. [4]) and differential geometry (see, e.g. [21]) can be used. Under some mild conditions of the constraint set, we prove the genericity of the R_0-property and the finite-valuedness of the solution maps.

The present paper is organized as follows: In the next section, we give a short introduction to variational inequalities, asymptotic cones, polynomial maps, and semi-algebraic sets. In Section 3, we investigate the R_0-property and the upper semicontinuity of the solution maps. In Section 4, we prove some facts on the solution existence and stability of copositive polynomial variational inequalities. In the last section, two results on genericity are shown.

2 Preliminaries

This section gives a short introduction to variational inequalities, asymptotic cones, polynomial maps, and semi-algebraic sets.

2.1 Variational inequalities and asymptotic cones

The usual scalar product of two vectors $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y\rangle$. Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector-valued map. The *variational inequality* defined by K and F is the problem:

$$\text{Find } x \in K \text{ such that } \langle F(x), y - x\rangle \geq 0,\ \forall y \in K.$$

The problem and the corresponding solution set are denoted by $\mathrm{VI}(K, F)$ and $\mathrm{SOL}(K, F)$, respectively. By the continuity of F and the closedness of K, it is not difficult to check that $\mathrm{SOL}(K, F)$ is closed.

If F is a polynomial map then $\mathrm{VI}(K, F)$ is called a *semi-polynomial variational inequality*. Furthermore, if K is defined by finitely many polynomial

equations and inequalities, then the problem is a *polynomial variational inequality.*

Theorem 2.1 (The Hartman-Stampacchia Theorem, [12, Chapter 1, Theorem 3.1]) *If K is compact, then the solution set* $\mathrm{SOL}(K, F)$ *is nonempty.*

Let us recall that a nonempty set $C \subset \mathbb{R}^m$ is called a *cone* if $\lambda > 0$ and $x \in C$ then $\lambda x \in C$. The cone C is bounded if and only if $C = \{0\}$. Note that C is a cone if and only if $\mathbb{R}^m \setminus C$ is a cone. We denote by $\operatorname{int} C$ and C^* the interior and the dual cone of C, respectively.

When (the closed convex set) K is a cone, the *complementarity problem* defined by K and F, denoted by $\mathrm{CP}(K, F)$, is to find a vector $x \in \mathbb{R}^n$ satisfying the following conditions:

$$x \in K,\ F(x) \in K^*,\ \langle F(x), x\rangle = 0. \tag{2.1}$$

In this setting, it is known that a vector x solves $\mathrm{CP}(K, F)$ if and only if x solves $\mathrm{VI}(K, F)$ [7, Proposition 1.1.3]. Therefore, the solution set of $\mathrm{CP}(K, F)$ is also denoted by $\mathrm{SOL}(K, F)$.

Remark 2.1 It is easy to see that a vector x solves $\mathrm{CP}(K, F)$ if and only if there is $\lambda \in \mathbb{R}^n$ such that

$$x \in K,\ F(x) - \lambda = 0,\ \langle \lambda, x\rangle = 0,\ \langle \lambda, y\rangle \geq 0\ \forall y \in K. \tag{2.2}$$

Remark 2.2 If K is a cone and F is homogeneous of degree $d > 0$, i.e. $F(tx) = t^d F(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$, then the solution set of $\mathrm{CP}(K, F)$ contains 0 and is a closed cone.

The *asymptotic cone* of K is defined by

$$K^\infty = \left\{ v \in \mathbb{R}^n : \exists t_k \to \infty, \exists x_k \in K \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = v \right\}.$$

By the convexity of K, K^∞ coincides with the recession cone of K which is defined by the set of vectors $v \in \mathbb{R}^n$ such that for some vector $x \in K$ the ray $\{x + tv : t \geq 0\}$ is contained in K [7, p.158]. So, one has $K = K + K^\infty$. Recall that the cone K^∞ is closed and convex; K is bounded if and only if $K^\infty = \{0\}$; if K is a cone then $K^\infty = K$.

Let P be a polynomial map, i.e. $P = (P_1, \ldots, P_n)$ where P_l is polynomial in n variables, $l = 1, \ldots, n$. The maximum of the numbers $\deg P_l$ is called the *degree* of the polynomial map P and one denotes $\deg P = d$. We denote $P^\infty = (P_1^\infty, \ldots, P_m^\infty)$, where P_l^∞ is the homogeneous component of degree d of P_l, $l = 1, \ldots, n$. Clearly, P^∞ is the *leading term* of the polynomial map P, i.e.

$$P^\infty(x) = \lim_{\lambda\to+\infty} \frac{P(\lambda x)}{\lambda^d},\ \forall x \in \mathbb{R}^n,$$

and the map P^∞ is homogeneous of degree d.

Throughout the paper, we assume that K is a nonempty, closed, and convex set, P is a polynomial map of degree d, where d is a positive integer, and let $\mathcal{S} := \mathrm{SOL}(K^\infty, P^\infty)$.

Let $\mathcal{P}_d$ be the linear space of all polynomial maps $Q = (Q_1, \ldots, Q_n)$ of degree at most d, m be the dimension of $\mathcal{P}_d$, and X be the vector consisting of all monomials degree at most d which is listed by the lexicographic ordering

$$X := (1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_1x_n, \ldots, x_1^d, x_1^{d-1}x_2, \ldots, x_n^d)^T. \quad (2.3)$$

For any polynomial map $Q \in \mathcal{P}_d$, there exists a unique matrix $A \in \mathbb{R}^{n\times m}$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}, \quad (2.4)$$

such that $Q(x) = AX$. The norm $\|Q\|$ of Q is defined by the Frobenius norm of the coefficients of matrix A. Let $\{Q^k\}$ be a convergent sequence in $\mathcal{P}_d$ with $Q^k \to Q$ and $\{x^k\}$ be a convergent sequence in $\mathbb{R}^n$ with $x^k \to \bar{x}$. Then $Q^k(x^k)$ also is convergent with $Q^k(x^k) \to Q(\bar{x})$.

Remark 2.3 Let $\{Q^k\}$ be a sequence in $\mathcal{P}_d$ with $Q^k \to P$. Assume that $Q^k(x) = A^kX$ and $P(x) = BX$. Clearly, one has $A^k \to B$. It follows that $(Q^k)^\infty \to P^\infty$.

The R_0-property of linear complementarity problems and affine variational inequalities has been investigated in [13,17] and [7, p.189]. We introduce a generalization of this property for semi-polynomial variational inequalities.

Definition 2.1 One says that the problem $\mathrm{VI}(K, P)$ has the R_0-*property* or (K, P) is an R_0-*pair* if $\mathrm{SOL}(K^\infty, P^\infty) = \{0\}$.

Remark 2.4 Let $Q \in \mathcal{P}_{d-1}$. Then $(P + Q)^\infty = P^\infty$. Furthermore, if (K, P) is an R_0-pair, then $(K, P + Q)$ also is an R_0-pair.

Remark 2.5 If K is compact, then $K^\infty = \{0\}$, and hence (K, P) is an R_0-pair.

Let $\mathcal{R}_0(K, d)$ be the set of all polynomial maps Q of degree d such that (K, Q) is an R_0-pair.

Remark 2.6 The set $\mathcal{R}_0(K, d)$ is a cone. Indeed, for each $t > 0$ and each polynomial map Q, one has $(tQ)^\infty = tQ^\infty$. This implies that

$$\mathrm{SOL}(K^\infty, (tQ)^\infty) = \mathrm{SOL}(K^\infty, tQ^\infty).$$

Moreover, it is easy to check that above sets coincide with $\mathrm{SOL}(K^\infty, Q^\infty)$, i.e.,

$$\mathrm{SOL}(K^\infty, (tQ)^\infty) = \mathrm{SOL}(K^\infty, tQ^\infty) = \mathrm{SOL}(K^\infty, Q^\infty).$$

Therefore, $\mathrm{SOL}(K^\infty, Q^\infty)$ is bounded iff $\mathrm{SOL}(K^\infty, (tQ)^\infty)$ is bounded, for any $t > 0$. This implies that $\mathcal{R}_0(K, d)$ is a cone in $\mathcal{P}_d$.

This paper mostly focuses on two solution maps SOL and SOL_P which are respectively defined by

$$\mathrm{SOL} : \mathcal{P}_d \rightrightarrows \mathbb{R}^n,\ Q \to \mathrm{SOL}(K, Q), \tag{2.5}$$

and

$$\mathrm{SOL}_P : \mathbb{R}^n \rightrightarrows \mathbb{R}^n,\ p \to \mathrm{SOL}(K, P + p). \tag{2.6}$$

Remark 2.7 By definition, it is easy to see that the map SOL is closed, i.e. the graph of SOL, which is defined by

$$\mathrm{gph}(\mathrm{SOL}) = \big\{(Q, x) \in \mathcal{P}_d \times \mathbb{R}^n : x \in \mathrm{SOL}(K, Q)\big\},$$

is closed in $\mathcal{P}_d \times \mathbb{R}^n$. Similarly, the map SOL_P also is closed.

2.2 Semi-algebraic sets and the LICQ

Recall that a set in $\mathbb{R}^n$ is *semi-algebraic*, if it is the union of finitely many subsets of the form

$$\big\{x \in \mathbb{R}^n \,:\, f_1(x) = \cdots = f_\ell(x) = 0,\ g_{\ell+1}(x) < 0, \ldots, g_m(x) < 0\big\},$$

where ℓ, m are natural numbers, and $f_1, \ldots, f_\ell, g_{\ell+1}, \ldots, g_m$ are polynomials with real coefficients. For further details on semi-algebraic geometry, we refer to [4].

Suppose that $S_1 \subset \mathbb{R}^m$ and $S_2 \subset \mathbb{R}^n$ are semi-algebraic sets. A vector-valued map $G : S_1 \to S_2$ is said to be semi-algebraic [4, Definition 2.2.5], if its graph is a semi-algebraic subset in $\mathbb{R}^m \times \mathbb{R}^n$.

Let $S \subset \mathbb{R}^m$ be a semi-algebraic set. Then there exists a decomposition of S into a disjoint union of semi-algebraic subsets [4, Theorem 2.3.6]

$$S = \bigcup_{i=1}^{s} S_i,$$

where each S_i is semi-algebraically homeomorphic to $(0,1)^{d_i}$, i.e., there is a map $h : S_i \to (0,1)^{d_i}$ such that h is semi-algebraic and homeomorphic. Let $(0,1)^0$ be a point and $(0,1)^{d_i} \subset \mathbb{R}^{d_i}$ be the set of points $x = (x_1, \ldots, x_{d_i})$ such that $x_j \in (0,1)$ for all $j = 1, \ldots, d_i$. The *dimension* of S is defined by

$$\dim S := \max\{d_1, \ldots, d_s\}.$$

The dimension is well-defined and does not depend on the decomposition of S.

If the semi-algebraic set S is nonempty and $\dim S = 0$, then S has finitely many points. If $\dim(\mathbb{R}^m \setminus S) < m$, then S is *generic* in $\mathbb{R}^m$ in the sense that S contains a countable intersection of dense and open sets in $\mathbb{R}^m$.

Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be manifolds. The tangent spaces of X at x and of Y at y are denoted by $T_x X$ and $T_y Y$, respectively. Consider the differentiable

map $\Phi : X \to Y$. A point $y \in \mathbb{R}^n$ is called a *regular value* of Φ if either the level set $\Phi^{-1}(y)$ is empty or the derivative map

$$D\Phi(x) : T_x X \to T_y Y$$

is surjective at every point $x \in \Phi^{-1}(y)$. So, y is a regular value of Φ if and only if rank $D\Phi(x) = n$ for all $x \in \Phi^{-1}(y)$.

Remark 2.8 Consider a differentiable semi-algebraic map $\Phi : X \to \mathbb{R}^n$, where $X \subset \mathbb{R}^n$. Assume that $y \in \mathbb{R}^n$ is a regular value of Φ and $\Phi^{-1}(y)$ is nonempty. Applying the regular level set theorem [21, Theorem 9.9], one has $\dim \Phi^{-1}(y) = 0$; this implies that $\Phi^{-1}(y)$ has finitely many points.

Remark 2.9 Let $\Phi : \mathbb{R}^m \times X \to \mathbb{R}^n$ be a differentiable semi-algebraic map, where $X \subset \mathbb{R}^n$. Assume that $y \in \mathbb{R}^n$ is a regular value of Φ. The Sard theorem with parameter [6, Theorem 2.4] says that there is a generic semi-algebraic set $\mathbb{S} \subset \mathbb{R}^m$ such that, for every $p \in \mathbb{S}$, y is a regular value of the map $\Phi_p : X \to \mathbb{R}^n$ with $x \mapsto \Phi(p, x)$.

Suppose that the constraint K is represented by finitely many convex polynomial functions $g_i(x), i \in I$, and finitely many affine functions $h_j(x), j \in J$, as follows:

$$K = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I,\ h_j(x) = 0, j \in J\}. \tag{2.7}$$

For each index set $\alpha \subset I = \{1, \ldots, s\}$, the *pseudo-face* K_α of K is defined by

$$K_\alpha = \{x \in \mathbb{R}^n : g_i(x) = 0, \forall i \in \alpha, g_i(x) < 0, \forall i \notin \alpha, h_j(x) = 0, \forall j \in J\}.$$

The number of pseudo-faces of K is finite and these pseudo-faces establish a disjoint decomposition of K. So, we obtain

$$\mathrm{SOL}(K, F) = \bigcup_{\alpha \subset I} [\mathrm{SOL}(K, F) \cap K_\alpha]. \tag{2.8}$$

For each $x \in K$, denote by $I(x)$ the active index set at x which is defined by $I(x) = \{i \in I : g_i(x) = 0\}$. One says that K satisfies the *linearly independent constraint qualification* (LICQ for short), if the gradient vectors

$$\{\nabla g_i(x), \nabla h_j(x), i \in I(x), j \in J\}$$

are linearly independent for all point $x \in K$. If the LICQ holds on K, then the Abadie constraint qualification (see, e.g. [7, p. 17]) also holds on K. Hence, the Karush-Kuhn-Tucker conditions can be applied (see, e.g. [7, Proposition 1.3.4]).

3 Upper semicontinuity of solution maps

This section focuses on upper semicontinuity of the solution maps SOL and SOL_P given by (2.5) and (2.6), respectively. A close relation between the upper semicontinuity and the R_0-property is shown.

3.1 Local boundedness

The following theorem describes a relation between the R_0-property and the boundedness of certain solution sets.

Proposition 3.1 *Consider the following statements:*

(a) (K, P) *is an* R_0*-pair;*
(b) $\mathrm{SOL}(K, P+Q)$ *is bounded, for every* $Q \in \mathcal{P}_{d-1}$*;*
(c) *For any bounded open set* $\mathcal{O} \subset \mathcal{P}_{d-1}$*, the following set is bounded:*

$$S_{\mathcal{O}} := \bigcup_{Q \in \mathcal{O}} \mathrm{SOL}(K, P+Q).$$

One has (a) $\Rightarrow$ (c) $\Rightarrow$ (b)*. Moreover, if* K *is a cone then the three statements are equivalent.*

Proof (a) $\Rightarrow$ (c) Assume that (K, P) is an R_0-pair and, on the contrary, there is a bounded open set $\mathcal{O}$ such that $S_{\mathcal{O}}$ is unbounded. There exists an unbounded sequence $\{x^k\} \subset K$ and a sequence $\{Q^k\} \subset \mathcal{O}$ such that $x^k \in \mathrm{SOL}(K, P+Q^k)$ for every k. By the unboundedness of $\{x^k\}$, without loss of generality, we can assume that $\|x^k\|^{-1}x^k \to \bar{x}$ with $\|\bar{x}\| = 1$. Because of the boundedness of $\{Q^k\}$, we can suppose that $Q^k \to \overline{Q}$ with $\deg \overline{Q} \leq d-1$.

By assumptions, one has

$$\left\langle (P+Q^k)(x^k), y - x^k \right\rangle \geq 0, \ \forall y \in K. \tag{3.1}$$

Let $u \in K$ be fixed. Then, for every $v \in K^\infty$, we have $u + \|x^k\|v \in K$ for any k. From (3.1), we deduce that

$$\left\langle (P+Q^k)(x^k), u + \|x^k\|v - x^k \right\rangle \geq 0.$$

Dividing this inequality by $\|x^k\|^{d+1}$ and letting $k \to +\infty$, we obtain

$$\left\langle (P+\overline{Q})^\infty(\bar{x}), v - \bar{x} \right\rangle = \langle P^\infty(\bar{x}), v - \bar{x} \rangle \geq 0,$$

and hence it follows that $\bar{x} \in \mathcal{S} = \{0\}$. As $\|\bar{x}\| = 1$, this is a contradiction; therefore, $S_{\mathcal{O}}$ must be bounded. The assertion (c) is proved.

(c) $\Rightarrow$ (b) Assume that (c) holds, but there is $\overline{Q} \in \mathcal{P}_{d-1}$ such that $\mathrm{SOL}(K, P+\overline{Q})$ is unbounded. Then there exists a bounded and open set $\mathcal{O}$ such that $\overline{Q} \in \mathcal{O}$. Clearly,

$$\mathrm{SOL}(K, P+\overline{Q}) \subset S_{\mathcal{O}}.$$

It is impossible, because $S_{\mathcal{O}}$ is bounded; hence (b) follows.

Assuming that K is a cone, we need only to prove that (b) $\Rightarrow$ (a). Consider $Q := P - P^\infty \in \mathcal{P}_{d-1}$. Then the assertion (b) implies that $\mathrm{SOL}(K, P^\infty)$ is a bounded cone; consequently, we get $\mathrm{SOL}(K, P^\infty) = \{0\}$. The proof is complete. □

The following lemma will be used in the proof of Theorem 3.1.

Lemma 3.1 *The cone $\mathcal{R}_0(K,d)$ is open in $\mathcal{P}_d$.*

Proof We need only to prove that $\mathcal{P}_d \setminus \mathcal{R}_0(K,d)$ is closed. Let $\{Q^k\}$ be a sequence in $\mathcal{P}_d \setminus \mathcal{R}_0(K,d)$ such that $Q^k \to Q$. Then $(Q^k)^\infty \to Q^\infty$ by Remark 2.3. Moreover, for each k, $\mathrm{SOL}(K^\infty,(Q^k)^\infty)$ is unbounded. Hence, there exists an unbounded sequence $\{x^k\}$ such that, for each k, $x^k \in \mathrm{SOL}(K^\infty,(Q^k)^\infty)$. Without loss of generality we can assume that $x^k \neq 0$ for all k and $\|x^k\|^{-1}x^k \to \bar{x}$ with $\|\bar{x}\| = 1$. By (2.2), for each $y \in K^\infty$, we get

$$\langle (Q^k)^\infty(x^k), x^k\rangle = 0,\ \langle (Q^k)^\infty(x^k), y\rangle \geq 0.$$

Dividing the above equation and inequality by, respectively, $\|x^k\|^{d_k+1}$ and $\|x^k\|^{d_k}$, where d_k is the degree of Q^k, and letting $k \to +\infty$, one gets

$$\langle Q^\infty(\bar{x}), \bar{x}\rangle = 0,\ \langle Q^\infty(\bar{x}), y\rangle \geq 0.$$

This leads to $\bar{x} \in \mathrm{SOL}(K^\infty, Q^\infty)$. As $\|\bar{x}\| = 1$, we have $\bar{x} \neq 0$. It follows that Q belongs to $\mathcal{P}_d \setminus \mathcal{R}_0(K,d)$. The proof is complete. □

Denote by $\mathcal{B}(0,\varepsilon)$ the open ball in $\mathcal{P}_d$ with center at 0 and radius ε. The closure of this ball is denoted by $\overline{\mathcal{B}}(0,\varepsilon)$.

The following theorem establishes a result on the local boundedness of the solution map SOL given in (2.5).

Theorem 3.1 *If (K,P) is an R_0-pair, then the map* SOL *is locally bounded at P, i.e. there exists $\varepsilon > 0$ such that the set*

$$O_\varepsilon := \bigcup_{Q \in \mathcal{B}(0,\varepsilon)} \mathrm{SOL}(K, P+Q)$$

is bounded. Consequently, $\mathrm{SOL}(K,P+Q)$ is bounded for every $Q \in \mathcal{B}(0,\varepsilon)$.

Proof According to Lemma 3.1, the cone $\mathcal{R}_0(K,d)$ is open in $\mathcal{P}_d$. Then there is some ε small enough such that

$$P + \overline{\mathcal{B}}(0,\varepsilon) \subset \mathcal{R}_0(K,d). \tag{3.2}$$

Suppose on the contrary, O_ε is unbounded. Then there exists an unbounded sequence $\{x^k\}$ and a sequence $\{Q^k\} \subset \mathcal{B}(0,\varepsilon)$ such that $x^k \in \mathrm{SOL}(K, P+Q^k)$, $x^k \neq 0$ for every k, and $\|x^k\|^{-1}x^k \to \bar{x}$ with $\|\bar{x}\| = 1$.

By the compactness of $\overline{\mathcal{B}}(0,\varepsilon)$, without loss of generality, we can assume that $Q^k \to Q$. Clearly, $P + Q^k \to P + Q$ and

$$P + Q \in P + \overline{\mathcal{B}}(0,\varepsilon). \tag{3.3}$$

By repeating the argument of the proof of Proposition 3.1, we can show that $\bar{x} \in \mathrm{SOL}\,(K^\infty, (P+Q)^\infty)$. From (3.2) and (3.3), $(K, P+Q)$ is an R_0-pair. This gives $\bar{x} = 0$ which contradicts $\|\bar{x}\| = 1$. Therefore, O_ε is bounded.

The proof is complete. □

3.2 Upper semicontinuity

A set-valued map $\Psi : X \rightrightarrows Y$ between two topological spaces X, Y is *upper semicontinuous* at $x \in X$ iff for any open set $V \subset Y$ such that $\Psi(x) \subset V$ there is a neighborhood U of x such that $\Psi(x') \subset V$ for all $x' \in U$. If Ψ upper semicontinuous at every $x \in X$ then one says that Ψ is upper semicontinuous on X. Recall that if Ψ is closed, i.e. its graph is a closed set in $X \times Y$, and locally bounded at x then Ψ is upper semicontinuous at x (see, e.g., [7, p.139]).

Proposition 3.2 *If (K, P) is an R_0-pair and $\mathrm{SOL}(K, P) \neq \emptyset$, then the map* SOL *is upper semicontinuous at P.*

Proof Assume that (K, P) is an R_0-pair and $\mathrm{SOL}(K, P) \neq \emptyset$. From Remark 2.7 and Theorem 3.1, the map SOL is closed and locally bounded at P. Hence, SOL is upper semicontinuous at P. □

Corollary 3.1 *If K is compact, then the solution map* SOL *is upper semicontinuous on $\mathcal{P}_d$.*

Proof Suppose that K is compact. Let $Q \in \mathcal{P}_d$. From Remark 2.5, $\mathrm{VI}(K, Q)$ has the R_0-property. Besides, Theorem 2.1 says that $\mathrm{SOL}(K, Q)$ is nonempty. According to Proposition 3.2, SOL is upper semicontinuous at Q. □

Corollary 3.2 *Assume that (K, P) is an R_0-pair and $p \in \mathbb{R}^n$. If the set $\mathrm{SOL}(K, P + p)$ is nonempty, then the solution map SOL_P is upper semicontinuous at p.*

Proof Let $p \in \mathbb{R}^n$ be given. Clearly, $(K, P + p)$ is an R_0-pair. According to Proposition 3.2, SOL is upper semicontinuous at $P + p$. Then for any open set V containing $\mathrm{SOL}(K, P + p)$, there exists an open ball $\mathcal{B}(0, \varepsilon)$ such that

$$V \supset \bigcup_{Q \in \mathcal{B}(0,\varepsilon)} \mathrm{SOL}(K, (P + p) + Q) \supset \bigcup_{\|q\| < \varepsilon} \mathrm{SOL}(K, (P + p) + q).$$

So, $U := \{p + q \in \mathbb{R}^n : \|q\| < \varepsilon\}$ is an open neighbourhood of p in $\mathbb{R}^n$ and $\mathrm{SOL}_P(U) \subset V$. It follows that SOL_P is upper semicontinuous at p. □

The following theorem gives a sufficient condition for the R_0-property. The proof is a modification of one in [15, Theorem 18.1].

Theorem 3.2 *Assume that K is a cone. If there exists $Q \in \mathcal{P}_{d-1}$ such that the following two conditions are satisfied:*

(a) $\mathrm{SOL}(K, P + Q)$ *is nonempty and bounded;*
(b) *The solution map* SOL *is upper semicontinuous at $P + Q$;*

then (K, P) is an R_0-pair.

Proof Since K is a cone, we have $K = K^\infty$. Suppose that there is $Q \in \mathcal{P}_{d-1}$ such that (a) and (b) hold, but (K, P) is not an R_0-pair. Let $0 \neq z \in \mathrm{SOL}(K, P^\infty)$. From (2.2), there exists $\lambda \in \mathbb{R}^n$ such that

$$P^\infty(z) - \lambda = 0,\ \langle \lambda, z\rangle = 0,\ \langle \lambda, y\rangle \geq 0\ \ \forall y \in K.$$

For each $t \in (0,1)$, we take $z_t := t^{-1}z$ and $\lambda_t := t^{-d}\lambda$. We prove the existence of $Q_t \in \mathcal{P}_d$, with $Q_t \to P + Q$ as $t \to 0$, satisfying

$$Q_t(z_t) - \lambda_t = 0,\ \langle \lambda_t, z_t\rangle = 0,\ \langle \lambda_t, y\rangle \geq 0\ \forall y \in K. \tag{3.4}$$

Suppose that

$$P = P^\infty + P^{d-1} + \cdots + P^1 + P^0$$

and

$$Q = Q^{d-1} + \cdots + Q^1 + Q^0,$$

where P^k, Q^k are homogeneous polynomial maps of degree k ($k = 1, \ldots, d-1$) and $P^0 \in \mathbb{R}^n, Q^0 \in \mathbb{R}^n$. The sum $P + Q$ can be written as

$$P + Q = P^\infty + [P^{d-1} + Q^{d-1}] + \cdots + [P^1 + Q^1] + [P^0 + Q^0]. \tag{3.5}$$

Because $z = (z_1, \ldots, z_n)$ is nonzero, there exists $l \in \{1, \ldots, n\}$ such that $z_l \neq 0$; hence $z_l^k \neq 0$ for $k = 1, \ldots, d$. Take $\overline{Q} \in \mathcal{P}_d$ with

$$\overline{Q}(x) = \overline{Q}^d(x) + \cdots + \overline{Q}^1(x),$$

where $\overline{Q}^k$ is a homogeneous polynomial map of degree k defined by

$$\overline{Q}^k(x) = \left(a_{k1}x_l^k, \ldots, a_{kn}x_l^k\right),\ a_{ki} = -\frac{P_i^{k-1}(z) + Q_i^{k-1}(z)}{z_l^k},\ i = 1, \ldots, n,$$

with P_i^{k-1}, Q_i^{k-1} are the $i-$th components of P^{k-1}, Q^{k-1}, respectively. It is easy to check that

$$P^{k-1}(z) + Q^{k-1}(z) + \overline{Q}^k(z) = 0. \tag{3.6}$$

Choosing $Q_t = (P + Q) + t\overline{Q}$, we now prove that the system (3.4) is valid. Indeed, the last one in (3.4) is obvious. The second one in (3.4) is obtained by

$$\langle \lambda_t, z_t\rangle = \langle t^{-d}\lambda, t^{-1}z\rangle = t^{-d-1}\langle \lambda, z\rangle = 0.$$

We now prove the first equality of (3.4). From (3.5), we get

$$\begin{aligned} Q_t(z_t) - \lambda_t &= [(P + Q) + t\overline{Q}](t^{-1}z) - t^{-d}\lambda \\ &= t^{-d}\left[P^\infty(z) - \lambda\right] + \textstyle\sum_{k=1}^{d} t^{-(k-1)}\left[P^{k-1}(z) + Q^{k-1}(z) + \overline{Q}^k(z)\right]. \end{aligned}$$

This and (3.6) imply that $Q_t(z_t) - \lambda_t = 0$. Hence, we get $z_t \in \mathrm{SOL}(K, Q_t)$. This holds for all $t \in (0,1)$.

Since $\mathrm{SOL}(K, P+Q)$ is bounded, there is a bounded open set V containing $\mathrm{SOL}(K, P+Q)$. By the upper semicontinuity of SOL at $P+Q$, there is $\varepsilon > 0$ such that $\mathrm{SOL}(K, P+Q+Q') \subset V$ for all $Q' \in \mathcal{P}_n$ and $\|Q' - (P+Q)\| < \varepsilon$. Taking t small enough such that $\|Q_t - (P+Q)\| < \varepsilon$, we have $\mathrm{SOL}(K, Q_t) \subset V$. So, $z_t \in V$ for every $t > 0$ sufficiently small. This is impossible, because V is bounded and z_t is unbounded as $t \to 0$.

The proof is complete. □

4 Solution existence and stability under copositivity condition

In this section, we will establish some results on solution existence and stability of semi-polynomial variational inequalities whose involved maps are copositive.

4.1 Solution existence

Let C be a nonempty and closed subset of $\mathbb{R}^n$. Note that $q \in \operatorname{int} C^*$ if and only if $\langle v, q\rangle > 0$ for all $v \in C$ and $v \neq 0$ (see, e.g., [15, Lemma 6.4]).

Recall that the map F is copositive on K if $\langle F(x), x\rangle \geq 0$ for all $x \in K$, and monotone on K if

$$\langle F(y) - F(x), y - x\rangle \geq 0, \tag{4.1}$$

for all $x, y \in K$. If the inequality in (4.1) is strict for all $y \neq x$, then F is strictly monotone on K. If $0 \in K$, $F(0) = 0$, and F is monotone on K, then F is copositive on K.

Theorem 6.2 in [9] gives a result on the solution existence under a copositivity condition along with $\mathrm{int}(K^*) \neq \emptyset$. We will now improve this result by weakening the interiority condition.

Theorem 4.1 *Assume that $0 \in K$ and P is copositive on K. If $p \in \mathrm{int}(\mathcal{S}^*)$, then $\mathrm{SOL}(K, P+p)$ is nonempty and bounded.*

Proof If K is compact then the assertion is obvious. Hence, we suppose that K be unbounded. Let $p \in \mathrm{int}(\mathcal{S}^*)$ be given. For each $k = 1, 2, \ldots$, we denote

$$K_k = \{x \in \mathbb{R}^n : x \in K, \|x\| \leq k\}.$$

Clearly, the set K_k is compact. Without loss of generality, we can assume that K_k is nonempty. According to Theorem 2.1, $\mathrm{VI}(K_k, P+p)$ has a solution denoted by x_k.

We will show that the sequence $\{x^k\}$ is bounded. Suppose on the contrary that $\{x^k\}$ is unbounded with $x^k \neq 0$, for all k, and $\|x^k\|^{-1}x^k \to \bar{x}$. Clearly, one has $\bar{x} \in K^\infty$ and $\|\bar{x}\| = 1$. For each k, it is true that

$$\left\langle P(x^k) + p, y - x^k\right\rangle \geq 0, \tag{4.2}$$

for all $y \in K_k$. By fixing $y \in K_1$ in (4.2), dividing this inequality by $\|x^k\|^{d+1}$ and letting $k \to +\infty$, we obtain $\langle P^\infty(\bar{x}), \bar{x}\rangle \leq 0$. Moreover, by the copositity of P, one has $\langle P(x^k), x^k\rangle \geq 0$. This leads to $\langle P^\infty(\bar{x}), \bar{x}\rangle \geq 0$. We thus get $\langle P^\infty(\bar{x}), \bar{x}\rangle = 0$.

Let $v \in K^\infty \setminus \{0\}$ be fixed. For each k, we set $y_k := \|x^k\|\|v\|^{-1}v$. Since $K = K + K^\infty$ and $0 \in K$, one has $y_k \in K$ for any k. It is easy to see that $\|y^k\| = \|x^k\| \leq k$, hence that $y_k \in K_k$. Now (4.2) becomes

$$\langle P(x^k) + p, \|x^k\|\|v\|^{-1}v - x^k\rangle \geq 0.$$

Dividing this inequality by $\|x^k\|^{d+1}$ and taking $k \to +\infty$, we obtain

$$\langle P^\infty(\bar{x}), v\rangle \geq \|v\| \langle P^\infty(\bar{x}), \bar{x}\rangle .$$

From what has already been proved, we have

$$\bar{x} \in K^\infty,\ \langle P^\infty(\bar{x}), \bar{x}\rangle = 0,\ \langle P^\infty(\bar{x}), v\rangle \geq 0\ \forall v \in K^\infty.$$

This means that $\bar{x} \in \mathcal{S}$.

Since P is copositive, letting $y = 0$ in (4.2), one has

$$-\langle p, x^k\rangle \geq \langle P(x^k), x^k\rangle \geq 0.$$

Dividing this inequality by $\|x^k\|$, as $k \to +\infty$, we get $\langle p, \bar{x}\rangle \leq 0$. This contradicts the assumption $p \in \text{int}(\mathcal{S}^*)$. Thus, the sequence $\{x^k\}$ must be bounded.

We can assume that $x^k \to \hat{x}$. We now prove that $\hat{x}$ solves $\text{VI}(K, P + p)$. Indeed, for any $y \in K$, from (4.2), taking $k \to +\infty$, one has

$$\langle P(\hat{x}) + p, y - \hat{x}\rangle \geq 0.$$

Hence, the nonemptiness of $\text{SOL}(K, P + p)$ is proved.

The boundedness of $\text{SOL}(K, P+p)$ is proved by assuming that there exists an unbounded sequence of solutions $\{x_k\} \subset \text{SOL}(K, P + p)$ with $x^k \neq 0$, for all k, and $\|x^k\|^{-1}x^k \to \bar{x}$ with $\|\bar{x}\| = 1$. Applying the normalization argument, we can show that $\bar{x} \in \mathcal{S}$ and $\langle p, \bar{x}\rangle \leq 0$. This contradicts the assumption $p \in \text{int}(\mathcal{S}^*)$.

The proof is complete. □

To illustrate Theorem 4.1, we consider the following example.

Example 4.1 Consider the polynomial variational inequalities $\text{VI}(K, P + p)$, where $K = \mathbb{R}^2_+$, $p = (p_1, p_2)^T \in \mathbb{R}^2$, and P is given by

$$P(x_1, x_2) = \begin{bmatrix}(x_1 - x_2)^2\\ (x_1 - x_2)^2\end{bmatrix}.$$

Clearly, one has $P^\infty = P$, $K^\infty = K$, and P is copositive on K. Since

$$\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 - x_2 = 0\},$$

one has

$$\text{int}(\mathcal{S}^*) = \{(p_1, p_2) \in \mathbb{R}^2 : p_1 + p_2 > 0\}.$$

From Theorem 4.1, $\mathrm{SOL}(K, P+p)$ is nonempty and bounded for any $p \in \mathrm{int}(\mathcal{S}^*)$. In fact, an easy computation shows that

$$\mathrm{SOL}_P(p_1,p_2) = \begin{cases} L_0 & \text{if } p_1=0,\ p_2=0, \\ L_{-p_1} & \text{if } p_1=p_2, p_2<0, \\ \{(0,0)\}\cup\{(0,\sqrt{-p_2})\} & \text{if } p_1>p_2, p_2<0, \\ \{(0,0)\}\cup\{(\sqrt{-p_1},0)\} & \text{if } p_1<0, p_1<p_2, \\ \{(0,0)\} & \text{if otherwise,} \end{cases}$$

where

$$L_{-p_1} = \{(x_1,x_2)\in\mathbb{R}^2_+ : x_1 - x_2 = \sqrt{-p_1}\}.$$

Clearly, $\mathrm{SOL}_P(p_1,p_2)$ is nonempty and bounded for any $p_1+p_2>0$.

4.2 Upper semicontinuity

We now give a sufficient condition for the upper semicontinuity of SOL_P at p under the copositivity.

Proposition 4.1 *Assume that $0 \in K$ and P is copositive on K. If $p \in \mathrm{int}(\mathcal{S}^*)$, then SOL_P is upper semicontinuous at p.*

Proof Theorem 4.1 says that $\mathrm{SOL}(K, P+p) \neq \emptyset$. Suppose that SOL_P is not upper semicontinuous at p. Then there exist a nonempty open set V containing $\mathrm{SOL}(K, P+p)$, a sequence $\{p^k\} \subset \mathbb{R}^n$, and a sequence $\{x^k\} \subset K$ such that $p^k \to p$ and

$$x^k \in \mathrm{SOL}(K, P+p^k) \setminus V, \tag{4.3}$$

for each k. By repeating the argument of the proof of Theorem 4.1, one can prove that the sequence $\{x^k\}$ is bounded. So, without loss of generality we can assume that $x^k \to \bar{x}$. It is easy to check that $\bar{x} \in \mathrm{SOL}(K, P+p)$, hence that $\bar{x} \in V$. Besides, since V is open, the relation (4.3) implies that $\bar{x} \notin V$. One obtains a contradiction. Therefore, SOL_P is upper semicontinuous at p. □

Corollary 4.1 *Assume that $0 \in K$ and P is copositive on K. If (K,P) is an R_0-pair, then the two following assertions hold:*

(a) $\mathrm{SOL}_P(p)$ *is nonempty and bounded for any $p \in \mathbb{R}^n$.*
(b) SOL_P *is upper semicontinuous on $\mathbb{R}^n$.*

Proof Since (K,P) is an R_0-pair, one has $\mathcal{S} = \{0\}$ and $\mathrm{int}(\mathcal{S}^*) = \mathbb{R}^n$. From Theorem 4.1, the assertion (a) follows. From Proposition 4.1, SOL_P is upper semicontinuous at p, for any $p \in \mathbb{R}^n$. □

4.3 Solution stability

In [15], the authors gave a result for the solution stability of copositive affine variational inequalities. In this section, we extend these results for copositive polynomial variational inequalities.

Let $p \in \mathbb{R}^n$ be given. If there exist $L > 0, c > 0$ and a neighborhood U_p of p such that

$$\mathrm{SOL}_P(q) \subset \mathrm{SOL}_P(p) + L\|q-p\|^c\,\mathbb{B}(0,1) \quad \forall q \in U_p,$$

where $\mathbb{B}(0,1)$ is the closed unit ball in $\mathbb{R}^n$, then one says that SOL_P is *locally upper-Hölder stable* at p.

When K is semi-algebraic, the following result says that the upper semicontinuity and the local upper-Hölder stability of SOL_P at p are equivalent.

Theorem 4.2 (see [14]) *Assume that K is semi-algebraic and $\mathrm{SOL}_P(p)$ is nonempty. Then the map SOL_P is upper semicontinuous at p iff it is locally upper-Hölder stable at p.*

Proposition 4.2 *Assume that K is semi-algebraic, $0 \in K$, and P is copositive on K. If $p \in \mathrm{int}(\mathcal{S}^*)$, then SOL_P is locally upper-Hölder stable at p.*

Proof Suppose that $p \in \mathrm{int}(\mathcal{S}^*)$. Proposition 4.1 says that SOL_P is upper semicontinuous at q. According to Theorem 4.2, SOL_P is locally upper-Hölder stable at p. □

Theorem 4.3 *Assume that K is semi-algebraic, $0 \in K$, and P is copositive on K. Let $p \in \mathrm{int}(\mathcal{S}^*)$ be given. Then there exist constants $\varepsilon > 0, L > 0$ and $c > 0$ with the following property: If $Q \in \mathcal{P}_d$, Q is copositive on K, and $q \in \mathbb{R}^n$ satisfy*

$$\max\{\|Q-P\|, \|q-p\|\} < \varepsilon,$$

then $\mathrm{SOL}(K, Q+q)$ is nonempty and bounded; and

$$\mathrm{SOL}(K,Q+q) \subset \mathrm{SOL}(K,P+p) + L(\|Q-P\| + \|q-p\|)^c\,\mathbb{B}(0,1). \quad (4.4)$$

Proof We first prove that there exists $\delta > 0$ such that if $Q \in \mathcal{P}_d$, where Q is copositive on K, and $p \in \mathbb{R}^n$ with

$$\max\{\|Q-P\|, \|q-p\|\} < \delta, \quad (4.5)$$

then $\mathrm{SOL}(K, Q+q)$ is nonempty and bounded. Suppose that the assertion is false. Then there is a sequence $\{(Q^k, q^k)\} \subset \mathcal{P}_d \times \mathbb{R}^n$ such that $(Q^k, q^k) \to (P, p)$ where, for each k, Q^k is copositive on K, $\mathrm{SOL}(K, Q^k + q^k)$ is empty or unbounded. Due to Theorem 4.1, one has

$$q^k \notin \mathrm{int}(\mathrm{SOL}((Q^k)^\infty, K^\infty)^*).$$

This means that there exists $x^k \in \mathrm{SOL}((Q^k)^\infty, K^\infty)$ satisfying $x^k \neq 0$ and $\langle x^k, q^k \rangle \leq 0$. We can assume that $\|x^k\|^{-1}x^k \to \bar{x} \in K^\infty$ with $\|\bar{x}\| = 1$.

Clearly, one has $\langle \bar{x}, p\rangle \leq 0$. If we prove that $\bar{x} \in \mathcal{S}$, then this contradicts the assumption $p \in \text{int}(\mathcal{S}^*)$ and hence (a) will be proved. Thus, we only need to show that $\bar{x} \in \mathcal{S}$. From (2.1), one has

$$\langle (Q^k)^\infty(x^k), x^k\rangle = 0,\ \langle (Q^k)^\infty(x^k), y\rangle \geq 0\ \forall y \in K^\infty. \tag{4.6}$$

It follows from Remark 2.3 that $(Q^k)^\infty \to P^\infty$. Let $y \in K^\infty$ be fixed; by dividing the equation and inequality in (4.6) by $\|x^k\|^{d+1}$ and $\|x^k\|^d$, respectively, and letting $k \to +\infty$, we obtain

$$\langle P^\infty(\bar{x}), \bar{x}\rangle = 0,\ \langle P^\infty(\bar{x}), y\rangle \geq 0.$$

As this holds for every $y \in K^\infty$, we get $\bar{x} \in \mathcal{S}$.

We now prove the inclusion (4.4). According to Proposition 4.2, there exist $L_0 > 0, c > 0$ and ε such that

$$\text{SOL}(K, P+q) \subset \text{SOL}(K, P+p) + L_0\|q-p\|^c\,\mathbb{B}(0,1) \tag{4.7}$$

for all q satisfying $\|q-p\| < \varepsilon$.

Suppose Q (copositive on K) and q (in $\mathbb{R}^n$) satisfy (4.5). For each $z_q \in \text{SOL}(K, Q+q) \neq \emptyset$, by setting

$$\widehat{q} := q + (Q-P)(z_q), \tag{4.8}$$

we have $P(z_q) + \widehat{q} = Q(z_q) + q$ and

$$\langle P(z_q) + \widehat{q}, y - z_q\rangle = \langle Q(z_q) + q, y - z_q\rangle \geq 0\ \forall y \in K.$$

This gives

$$z_q \in \text{SOL}(K, P+\widehat{q}). \tag{4.9}$$

Since $\text{SOL}(K, Q+q)$ is compact, there exists $\beta > 0$ such that

$$\|(Q-P)(z)\| \leq \beta\|Q-P\| \tag{4.10}$$

for all $z \in \text{SOL}(K, Q+q)$. From (4.8), (4.10), and (4.5), we get

$$\begin{aligned}\|\widehat{q}-p\| &\leq \|(Q-P)(z_q)\| + \|q-p\|\\ &\leq \beta\|Q-P\| + \|q-p\|\\ &\leq (1+\beta)\delta.\end{aligned}$$

Choosing δ small enough such that $(1+\beta)\delta < \varepsilon$, we have $\|\widehat{q}-p\| < \varepsilon$. From (4.7), (4.9) and (4.10), there exists $x \in \text{SOL}(K, P+p)$ such that

$$\begin{aligned}\|z_q - x\| &\leq L_0\|\widehat{q}-p\|^c\\ &\leq L_0\left(\|q-p\| + \beta\|Q-P\|\right)^c\\ &\leq L\left(\|q-p\| + \|Q-P\|\right)^c,\end{aligned}$$

where $L := \max\{L_0^c\beta, L_0^c\}$.

Since the inequality holds for any z_q in $\text{SOL}(K, Q+q)$, the inclusion (4.4) holds. □

4.4 The GUS-property

If $\mathrm{VI}(K, P+p)$ has a unique solution for every $p \in \mathbb{R}^n$, then $\mathrm{VI}(K, P)$ is said to have the globally uniquely solvable property (GUS-property). The following theorem develops Theorem 4.3 in [22] which concerning the GUS-property of tensor variational inequalities.

Proposition 4.3 *Assume that $0 \in K$ and P is strictly monotone on K. If (K, P) is an R_0-pair, then $\mathrm{VI}(K, P)$ has the GUS-property and SOL_P is single-valued and continuous on $\mathbb{R}^n$.*

Proof Suppose that (K, P) is an R_0-pair. By the monotonicity of P, $Q(x) := P(x) - P(0)$ is monotone on K. Because 0 belongs to K, the map Q is copositive on K. From Remark 2.4, (K, Q) is an R_0-pair. According to the assertion (a) in Corollary 4.1, the set

$$\mathrm{SOL}(K, Q+q) = \mathrm{SOL}(K, P - P(0) + q)$$

is nonempty for all $q \in \mathbb{R}^n$. This is equivalent to saying that $\mathrm{SOL}(K, P+p) \neq \emptyset$ for all $p \in \mathbb{R}^n$. Since P is strictly monotone, so is $P + p$. According to [7, Theorem 2.3.3], $\mathrm{VI}(K, P+p)$ has at most one solution. Thus, $\mathrm{VI}(K, P+p)$ has a unique solution for every $p \in \mathbb{R}^n$.

Because $\mathrm{VI}(K, P)$ has the R_0-property, Corollary 3.2 says that SOL_P is upper semicontinuous on $\mathbb{R}^n$. Hence, the map is single-valued and continuous on $\mathbb{R}^n$. The proof is complete. □

To illustrate Proposition 4.3, we consider the following example.

Example 4.2 Consider the polynomial variational inequality given by

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\},\ P(x) + p = \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},$$

where $(p_1, p_2) \in \mathbb{R}^2$. It is easy to check that $0 \in K$, P is strictly monotone on K, and (K, P) is an R_0-pair. According to Proposition 4.3, the problem has the GUS-property and SOL_P is single-valued and continuous on $\mathbb{R}^2$. In fact, an easy computation shows that

$$\mathrm{SOL}_P(p_1, p_2) = \begin{cases} \{(\sqrt[3]{-p_1}, \sqrt[3]{-p_2})\} & \text{if } p_1 < 0, \\ \{(0, \sqrt[3]{-p_2})\} & \text{if } p_1 \geq 0. \end{cases}$$

This map is single-valued and continuous on $\mathbb{R}^2$.

5 Genericity

In this section, we first prove the genericity of the R_0-property of polynomial variational inequalities under some mild conditions. Then, we show that the solution map SOL is finite-valued on a generic semi-algebraic set of the parametric space. Note that the finite-valuedness on a generic set of $\mathbb{R}^n$ of the map SOL_P was announced and proved in [14, Theorem 3.2].

5.1 Genericity of the R_0-property

Let $\mathcal{H}_d$ be the vector space spanned by homogeneous polynomial maps of degree d. The dimension of $\mathcal{H}_d$ is denoted by ρ. Clearly, $\mathbb{R}^{n\times\rho}$ and $\mathcal{H}_d$ are isomorphic. Let X_d be a vector whose components are monomials of degree d listed by lexicographic ordering

$$X_d = \left(x_1^d, x_1^{d-1}x_2, \ldots, x_n^d\right)^T.$$

For any homogeneous polynomial map $H \in \mathcal{H}_d$, there is a unique $B \in \mathbb{R}^{n\times\rho}$,

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\rho} \\ b_{21} & b_{22} & \cdots & b_{2\rho} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\rho} \end{bmatrix},$$

such that $H(x) = BX_d$, where

$$H_l(x) = b_{l1}x_1^d + b_{l2}x_1^{d-1}x_2 + \cdots + b_{l\rho}x_n^d \ (l = 1, \ldots, n).$$

Assume that K is an unbounded polyhedral convex cone, which is the intersection of finitely many half-spaces containing the origin, given by

$$K = \{x \in \mathbb{R}^n : Cx \leq 0\}, \tag{5.1}$$

where $C = (c_{ij}) \in \mathbb{R}^{s\times n}$. The following lemma shows that the solution map of homogeneous polynomial complementarity problems,

$$\Gamma_K : \mathbb{R}^{n\times\rho} \rightrightarrows \mathbb{R}^n,\ B \mapsto \Gamma_K(B) = \mathrm{SOL}(K, BX_d),$$

is constant on a generic semi-algebraic set of $\mathbb{R}^{n\times\rho}$ provided that the LICQ holds on K.

Lemma 5.1 *Assume that K is a polyhedral convex cone given by* (5.1) *and the* LICQ *holds on K. Then there exists a generic semi-algebraic set $\mathbb{S} \subset \mathbb{R}^{n\times\rho}$ such that $\Gamma_K(B) = \{0\}$ for any $B \in \mathbb{S}$.*

Proof Firstly, since the LICQ holds on K, applying [7, Proposition 1.3.4] for the $\mathrm{VI}(K, H)$, we have $x \in \mathrm{SOL}(K, H)$ if and only if there exists $\lambda \in \mathbb{R}^s$ such that

$$\begin{cases} H(x) + C^T\lambda = 0, \\ \lambda^T(Cx) = 0,\ \lambda \geq 0,\ Cx \leq 0. \end{cases} \tag{5.2}$$

Let $K_\alpha \neq \{0\}$ be a nonempty pseudo-face of K, given by

$$K_\alpha = \left\{x \in \mathbb{R}^n : C_i x = 0\ \forall i \in \alpha,\ C_i x < 0\ \forall i \in I \setminus \alpha\right\},$$

where C_i is the i-th row of C. Thus, X_d is nonzero on this pseudo-face. We consider the function

$$\Phi_\alpha : \mathbb{R}^{n\times\rho} \times K_\alpha \times \mathbb{R}_+^{|\alpha|} \to \mathbb{R}^{n+|\alpha|},$$

which is defined by

$$\Phi_\alpha(B, x, \lambda_\alpha) = \Big(BX_d + \sum_{i\in\alpha} \lambda_i C_i, C_\alpha x\Big),$$

where $C_\alpha x = (C_{i_1}x, \ldots, C_{i_{|\alpha|}}x), i_j \in \alpha$. Clearly, Φ_α is a smooth semi-algebraic function. The Jacobian matrix of Φ_α is given by

$$D\Phi_\alpha = \left[\begin{array}{c|c|c} D_B(BX_d) & * & C_\alpha^T \\ \hline 0_{|\alpha|\times\rho} & C_\alpha & 0_{|\alpha|\times|\alpha|} \end{array}\right],$$

where $0_{u\times v}$ is the zero $u\times v$–matrix. Here the $n\times\rho$–matrix $D_B(BX_d)$ is given by

$$D_B(BX_d) = \begin{bmatrix} X_d & 0_{1\times\rho} & \cdots & 0_{1\times\rho} \\ 0_{1\times\rho} & X_d & \cdots & 0_{1\times\rho} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{1\times\rho} & 0_{1\times\rho} & \cdots & X_d \end{bmatrix}.$$

Since X_d is nonzero on K_α, the rank of $D_B(BX_d)$ is n. By our assumptions, the rank of the matrix $D\Phi_\alpha$ is $n+|\alpha|$ for all $x \in K_\alpha$. Therefore, $0 \in \mathbb{R}^{n+|\alpha|}$ is a regular value of Φ_α. The Sard theorem with parameter [6, Theorem 2.4] says that there exists a generic semi-algebraic set $\mathbb{S}_\alpha \subset \mathbb{R}^{n\times\rho}$, such that if $B \in \mathbb{S}_\alpha$ then 0 is a regular value of the map

$$\Phi_{\alpha,B} : K_\alpha \times \mathbb{R}^{|\alpha|} \to \mathbb{R}^{n+|\alpha|},\ \Phi_{\alpha,B}(x, \lambda_\alpha) = \Phi_\alpha(B, x, \lambda_\alpha).$$

According to the regular level set theorem [21, Theorem 9.9], if $\Phi^{-1}_{\alpha,B}(0)$ is nonempty then it is a $0-$dimensional semi-algebraic set. It follows that $\Phi^{-1}_{\alpha,B}(0)$ is a finite set. Moreover, from (5.2), one has

$$\Gamma_K(B) \cap K_\alpha = \pi(\Phi^{-1}_{\alpha,B}(0)),$$

where π is the projection $\mathbb{R}^{n+|\alpha|} \to \mathbb{R}^n$ defined by $\pi(x, \lambda_\alpha) = x$. Therefore, $\Gamma_K(B) \cap K_\alpha$ is a finite set.

Since $0 \in \Gamma_K(B)$, $\Gamma_K(B) \cap K_\alpha = \{0\}$ if $K_\alpha = \{0\}$. By the finite decomposition (2.8),

$$\Gamma_K(B) = \bigcup_{\alpha\subset I} \Gamma_K(B) \cap K_\alpha$$

is a finite set. Taking $\mathbb{S} = \cap_{\alpha\subset I}\mathbb{S}_\alpha$, we see that $\Gamma_K(B)$ has finite points for any $B \in \mathbb{S}$. Recall that $\Gamma_K(B)$ is a closed cone which contains 0. Hence, $\Gamma_K(B) = \{0\}$ for all B in $\mathbb{S}$.

The proof is complete. □

Consider the isomorphism $\varphi : \mathbb{R}^{n\times m} \to \mathcal{P}_d$ defined by $\varphi(A) = AX$, where X, A are as in (2.3) and (2.4), respectively. If $\mathbb{S}$ is generic in $\mathbb{R}^{n\times m}$, then $\varphi(\mathbb{S})$ is generic in $\mathcal{P}_d$.

Theorem 5.1 *Suppose that the constraint K is given by* (2.7). *If the* LICQ *holds on K^∞, then the set $\mathcal{R}_0(K, d)$ is generic in $\mathcal{P}_d$.*

Proof For each $i \in I$, $j \in J$, one denotes $K_i = \{x \in \mathbb{R}^n : g_i(x) \leq 0\}$ and $K'_j = \{x \in \mathbb{R}^n : h_j(x) = 0\}$. Clearly, $K_i^\infty, K_j'^\infty$ are polyhedral convex cone (see [3, p.39]). Since

$$K = \big(\bigcap_{i\in I} K_i\big)\bigcap\big(\bigcap_{j\in J} K'_j\big),$$

it follows from [1, Proposition 2.1.9] that

$$K^\infty = \big(\bigcap_{i\in I} K_i^\infty\big)\bigcap\big(\bigcap_{j\in J} K_j'^\infty\big).$$

Thus, K^∞ is a nonempty polyhedral convex cone. We can assume that K^∞ is given by (5.1).

From the direct sum $\mathcal{P}_d = \mathcal{H}_d \oplus \mathcal{P}_{d-1}$ and Remark 2.4, we can see that

$$\mathcal{R}_0(K,d) = (\mathcal{R}_0(K,d) \cap \mathcal{H}_d) \oplus \mathcal{P}_{d-1}\,.$$

Since $\mathcal{H}_d$ and $\mathbb{R}^{n\times\rho}$ are isomorphic, according to Lemma 5.1, $\mathcal{R}_0(K,d) \cap \mathcal{H}_d$ is generic in $\mathcal{H}_d$. Hence, $\mathcal{R}_0(K,d)$ is also generic in $\mathcal{P}_d$.

The proof is complete. □

5.2 Genericity of the finite-valuedness

Recall that a set-valued map Ψ is *finite-valued* on X if the cardinality of $\Psi(x)$ is finite, i.e. $|\Psi(x)| < +\infty$ for every $x \in X$.

Theorem 5.2 *Assume that K is a semi-algebraic set given by* (2.7) *and the* LICQ *holds on K. Then the map* SOL *is finite-valued on a generic set of $\mathcal{P}_d$.*

Proof For $Q \in \mathcal{P}_d$, one has $Q(x) = AX$, where X and A are given by (2.3) and (2.4), respectively. For each nonempty pseudo-face K_α of K, we consider the function

$$\Phi_\alpha : \mathbb{R}^{n\times m} \times K_\alpha \times \mathbb{R}^{|\alpha|+|J|} \to \mathbb{R}^{n+|\alpha|+|J|},$$

which is defined by

$$\Phi_\alpha(A,x,\lambda_\alpha,\mu) = \Big(AX + \sum_{i\in\alpha}\lambda_i\nabla g_i(x) + \sum_{j\in J}\mu_i\nabla h_j(x), g_\alpha(x), h(x)\Big),$$

where $g_\alpha(x) = (g_{i_1}(x),\ldots,g_{i_{|\alpha|}}(x)), i_j \in \alpha$. An easy computation shows that the Jacobian matrix $D_A(AX)$ is the following $n \times m-$matrix

$$D_A(AX) = \begin{bmatrix} X & 0_{1\times m} & \cdots & 0_{1\times m} \\ 0_{1\times m} & X & \cdots & 0_{1\times m} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{1\times m} & 0_{1\times m} & \cdots & X \end{bmatrix}. \tag{5.3}$$

From (5.3), the rank of $D_A(AX)$ is n. Since K has the LICQ property, the rank of the Jacobian matrix $D\Phi_\alpha$ is $n + |\alpha| + |J|$ for all $x \in K_\alpha$.

Repeating the argument of the proof of Lemma 5.1 (by using the Sard theorem with parameter and the regular level set theorem), one can assert that there exists a semi-algebraic set $\mathbb{S}$ which is generic in $\mathbb{R}^{n\times m}$ such that $\mathrm{SOL}(K, AX)$ is a finite set for every $A \in \mathbb{S}$.

The assertion is proved. □

Remark 5.1 Suppose that the assumptions in Theorem 5.2 are satisfied. We can show that if the map SOL is lower semicontinous at P, then $\mathrm{SOL}(K, P)$ has finitely many points. Hence, if $\dim \mathrm{SOL}(K, P) \geq 1$, then SOL is not lower semicontinuous at P.

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