Journal of Global Optimization (2022) 84:783–805
https://doi.org/10.1007/s10898-022-01155-x

# Simultaneous distributed-boundary optimal control problems driven by nonlinear complementarity systems

**Jinxia Cen[1] · Tahar Haddad[2] · Van Thien Nguyen[3] · Shengda Zeng[1,4,5]**

Received: 6 December 2021 / Accepted: 19 March 2022 / Published online: 8 April 2022


## Abstract

The primary goal of this paper is to study a nonlinear complementarity system (NCS, for short) with a nonlinear and nonhomogeneous partial differential operator and mixed boundary conditions, and a simultaneous distributed-boundary optimal control problem governed by (NCS), respectively. First, we formulate the weak formulation of (NCS) to a mixed variational inequality with double obstacle constraints (MVI, for short), and prove the existence and uniqueness of solution to (MVI). Then, a power penalty method is applied to (NCS) for introducing an approximating mixed variational inequality without constraints (AMVI, for short). After that, a convergence result that the unique solution of (MVI) can be approached by the unique solution of (AMVI) when a penalty parameter tends to infinity, is established. Moreover, we explore the solvability of the simultaneous distributed-boundary optimal control problem described by (MVI), and consider a family of approximating optimal control problems driven by (AMVI). Finally, we provide a result on asymptotic behavior of optimal controls, system states and minimal values to approximating optimal control problems.

✉ Shengda Zeng
zengshengda@163.com

Jinxia Cen
jinxiacen@163.com

Tahar Haddad
haddadtr2000@yahoo.fr

Van Thien Nguyen
Thiennv15@fe.edu.vn

1 Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, People's Republic of China

2 Laboratoire LMPEA, Faculté des Sciences Exactes et Informatique, Université Mohammed Seddik Benyahia, Jijel, B.P. 98, Jijel 18000, Algérie

3 Departement of Mathematics, FPT University, Education Zone, Hoa Lac High Tech Park, Km 29 Thang Long Highway, Thach That Ward, Hanoi, Vietnam

4 Department of Mathematics, Nanjing University, Nanjing, Jiangsu 210093, People's Republic of China

5 Faculty of Mathematics and Computer Science, Jagiellonian University in Krakow, ul. Lojasiewicza 6, 30348 Krakow, Poland

**Keywords** Simultaneous optimal control problem · Mixed boundary condition · Complementarity problem · Nonhomogeneous partial differential operator · Asymptotic behavior

**Mathematics Subject Classification** 49J20 · 47J20 · 49J40 · 90C33 · 35J25

## 1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N > 2$, be a bounded domain such that its boundary $\Gamma := \partial\Omega$ is Lipschitz continuous and $\Gamma$ is divided into three disjoint measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ with $\text{meas}(\Gamma_1) > 0$, i.e., $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_i \cap \Gamma_j = \emptyset$ for $i, j = 1, \ldots, 3$ with $i \neq j$. It should be pointed out that in our setting the parts $\Gamma_2$ and $\Gamma_3$ can be empty, i.e., $\Gamma_1$ could be the whole boundary $\Gamma_1 = \Gamma$. Let $1 < p < +\infty$ and let $\nu$ be the outward unit normal at the boundary $\Gamma$. Given functions $a \colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$, $f \colon \Omega \to \mathbb{R}$, $b \colon \Gamma_2 \to \mathbb{R}$ and $\phi \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, this paper is devoted to investigate the following nonlinear complementarity system involving a nonlinear and nonhomogeneous partial differential operator and mixed boundary conditions:

**Problem 1** *Find a pair of functions $u \colon \Omega \to \mathbb{R}$ and $\mu \colon \Omega \to \mathbb{R}$ such that*

$$Z(x) = -\operatorname{div} a(x, \nabla u) + g(x, u) - f(x) + \mu(x) \quad \text{in } \Omega, \tag{1.1}$$

$$\mu(x) \geq 0,\ u(x) - u^*(x) \leq 0,\ \mu(x)\left(u(x) - u^*(x)\right) = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$Z(x) \geq 0,\ u_*(x) - u(x) \leq 0,\ Z(x)\left(u_*(x) - u(x)\right) = 0 \quad \text{in } \Omega, \tag{1.3}$$

$$u = 0 \quad \text{on } \Gamma_1, \tag{1.4}$$

$$-\frac{\partial u}{\partial \nu_a} = b(x) \quad \text{on } \Gamma_2, \tag{1.5}$$

$$-\frac{\partial u}{\partial \nu_a} \in \partial\phi(x, u(x)) \quad \text{on } \Gamma_3, \tag{1.6}$$

*where $\partial\phi(x, u(x))$ is the convex subdifferential operator of the convex function $s \mapsto \phi(x, s)$, and*

$$\frac{\partial u}{\partial \nu_a} := (a(x, \nabla u), \nu)_{\mathbb{R}^N}.$$

The distinguishing feature of a complementarity problem (CP, for short) is the set of complementarity conditions. Each of these conditions requires that the product of two or more nonnegative/nonpositive quantities should be zero (here, each quantity is either a decision variable, or a function of the decision variables). Because, on the one hand, complementarity conditions can model various complicated natural phenomena and engineering problems, for example, mechanics problems with Signorini contact conditions and semipermeablity problems; on the other hand, complementarity conditions are exactly formulated to the optimality conditions for numerous linear/nonlinear optimization problems involving equalities and inequalities constraints. Therefore, it has attracted more and more scholars' attention to the development of both theoretical and applications aspects of problems involving complementarity conditions. Amongst the results we mention: Bai-Migórski-Zeng [1] introduced a new fuzzy vector complementarity problem and a generalized fuzzy vector variational inequality, respectively, and then they applied the Knaster-Kuratowski-Mazurkiewicz principle to prove the existence of solutions to the fuzzy vector complementarity problem and fuzzy vector variational inequality. Ramadurai-Ukkusuri-Zhao-Pang [35] formulated the dynamic equilibrium conditions for a single bottleneck model with heterogeneous commuters as a linear complementarity problem, and presented theoretical proofs for solution existence and uniqueness, as well as numerical results to the single bottleneck model. By introducing a new concept of upper $Z$-mapping, which generalizes the well-known concept of the single-valued $Z$-mapping and involves the diagonal multivalued mappings, Konnov [13] established several existence and uniqueness results to a generalized mixed complementarity problem (MCP, for short) with box constraints and multivalued cost mapping, and proposed an extension of the Jacobi algorithm to (MCP). Peng-Liu [33] made use of the efficiency of the filter technique to present a new derivative-free algorithm for a class of nonlinear complementarity problems, and obtained a global convergence result under the monotonicity assumption. For more details concerning complementarity problems and complementarity conditions, the reader is welcome to consult Zeng-Yao [40], Huang-Fang [11], Huang-Yang-Chan [12], Schaible-Yao [36], Tang-Huang [37], Konnov [14], Wang-Zhang [39], Liu-Migórski-Nguyen-Zeng [17], and Zhou-Wang-Yang [47].

Apart from their obvious importance in the theory of partial differential equations, obstacle problems have a natural theoretical interest in stochastic control. Additionally, they can be found in physics, biology, and mathematical finance. One of the most well-known financial challenges is establishing the arbitrage-free price of American-style options. Concerning the mathematical analysis of obstacle problems, we refer to the recent contribution of Zeng-Rădulescu-Winkert [46] considered a complicated elliptic differential inclusion problem with a double phase differential operator, an implicit obstacle condition, a nonlinear convection and multivalued mixed boundary value conditions, they employed the Kakutani-Ky Fan fixed point theorem for multivalued operators along with the theory of nonsmooth analysis and variational methods for pseudomonotone operators to examine the nonemptiness and compactness of solution set to the double phase implicit obstacle problem. Via using Kluge's fixed point theorem for the set-valued selection map and the Minty approach, Migórski-Khan-Zeng [24] developed a general regularization framework to provide an existence result of an identification inverse problem in a complicated mixed elliptic boundary value problem with $p$-Laplace operator, an implicit obstacle, and a nonmonotone multivalued boundary condition. Feng-Han-Huang [9] utilized a fully non-conforming virtual element method to a fourth-order obstacle problem for the Kirchhoff-Love plate, and under certain solution regularity assumptions, they also derived the optimal order error estimates in the discrete energy norm. For further results concerning problems with obstacle effect, we refer to the works of Liu-Yang-Zeng [16], Wang-Han-Cheng [38], Mermri-Han [21], Zeng-Bai-Gasiński-Winkert [42, 43], Peng [34], Liu-Motreanu-Zeng [20], Migórski-Khan-Zeng [23], and Gasiński-Migórski-Ochal [10].

Distributed and boundary optimal control of partial differential equations is an expanding and vibrant branch of applied mathematics and modern control theory that has found numerous applications. Because distributed and boundary optimal control problem can be a useful and power mathematical tool to solve numerous comprehensive practical problems arising in chemical reactions, signal process, and communications and transportation. Therefore, the research of distributed optimal control problems has attracted more and more researchers' attention. For the last more than fifty years, the development of both theoretical and applications aspects of problems involving distributed and boundary optimal control grew rapidly. We mention some works on the topic of distributed and boundary optimal control problems. Under the $\Gamma$-convergence of objective functionals, the parabolic $G$-convergence of operators in the state equations, and the Kuratowski convergence of control constraint sets, Migórski [22] established a convergence result for an optimal control problem govern by a nonlinear parabolic equation. By utilizing the notation of $PG$ and $G$ convergences as well as

Filippov-Gronwall principle, Papageorgiou-Rădulescu-Repovš [32] considered a nonlinear optimal control problem governed by a nonlinear evolution inclusion and depending on a parameter $\lambda$, and established the Hadamard well-posedness and continuity properties of the optimal multifunction. By exploring the relative compactness, continuity, and convergence in the Kuratowski sense of solution mapping for a new kind of differential variational-hemivariational inequalities, Zeng-Migórski-Liu [44] proved existence of a solution to the optimal control problem driven by a differential variational-hemivariational inequality in infinite Banach spaces, and studied sensitivity of a perturbed problem with multiparameters corresponding to the differential variational-hemivariational inequality. For more details concerning the research of distributed and boundary optimal control problems, the reader is welcome to consult Denkowski-Migórski [6], Papageorgiou-Rădulescu-Repovš [30, 31], Boukrouche-Tarzia [2–4], Migórski-Ochal [25], Zeng-Migórski-Khan [45], Li-Liu [15], Liu-Li-Motreanu [18], and Papageorgiou [27–29].

Indeed, it should be mentioned that if $\Gamma_2 = \Gamma_3 = \emptyset$ (i.e., $\Gamma_1 = \Gamma$) and $a$ is independent of the variable $x$, then Problem 1 becomes the complementarity problem with Dirichlet boundary condition which has been studied recently by Duan-Wang-Zhou [8]. More precisely, when $a$ satisfies the hypotheses H($a'$) (which requires that $a$ is strongly monotone, see Remark 15) and inequality $2 \leq N < p$ is satisfied, Duan-Wang-Zhou [8] obtained an existence theorem and a convergence result (see Theorems 4.1 and 4.2 of Duan-Wang-Zhou [8]). However, the strong monotonicity of $a$ and inequality $2 \leq N < p$ lead to inapplicability in a lot of problems. Naturally, a question arises that how to drop these strict assumptions. Actually, this is one of the motivation of this paper. The main contribution of this paper is fourfold. The first aim of this paper is to prove the existence and uniqueness of weak solution to Problem 1, i.e., the unique solvability of Problem 2 (see in Sect. 2), in which our method is based on an existence result for mixed variational inequalities (see Theorem 3.2 of Liu-Migórski-Zeng [19]). Under general assumptions ($a$ is not strongly monotone with respect to the second variable and $1 < p < +\infty$ is satisfied), the second goal is to employ a power penalty method to introduce a family of approximating problems without constrains (see Problem 9) and to establish a convergence result that the unique solution of Problem 2 can be approached by the approximating mixed variational inequality, Problem 9, when a penalty parameter tends to infinity. More particularly, our result, Corollary 14, extends Theorems 4.1 and 4.2 of Duan-Wang-Zhou [8]. Whereas, the third contribution of this paper is to investigate a nonlinear simultaneous distributed-boundary optimal control problem governed by Problem 2 and to deliver an existence theorem to the simultaneous distributed-boundary optimal control problem. Moreover, we introduce an optimal control problem driven by the approximating problem, Problem 9, rather than Problem 2. Spontaneously, it requires us to answer directly the challenging question whether we can establish the asymptotic behavior of optimal controls, system states and minimal values for the optimal control problem described by Problem 9, when the penalty parameter tends to infinity. Therefore, our last intention is to examine the significant result on asymptotic behavior of optimal controls, system states and minimal values for optimal control problem driven by Problem 9.

The rest of the paper is organized as follows. Section 2 is devoted to recall some useful and important preliminaries, and to derive the weak variational formulation of Problem 1, which is exactly formulated by a mixed variational inequality with double obstacle constraints, namely, Problem 2. In Sect. 3, we prove the unique solvability of Problem 2, and employ a power penalty method to introduce a family of approximating problems without constraints (see Problem 9). In the meanwhile, a significant convergence result that the unique solution of Problem 9 converges strongly to the unique solution of Problem 2 when penalty parameter $\lambda$ tends to infinity. Finally, in Sect. 4, we introduce two nonlinear simultaneous distributed-boundary optimal control problems governed by Problem 2 and Problem 9, respectively, prove the existence of solutions to these optimal control problems, and explore the asymptotic behavior of the optimal controls, system states and minimal values to optimal control problem driven by Problem 9.

## 2 Mathematical prerequisites

The section is devoted to review some basic notation, definitions and the necessary preliminary material, which will be used in next sections. More details can be found, for instance, in [7, 26, 41].

Let $(Y, \|\cdot\|_Y)$ be a Banach space and $Y^*$ stand for the dual space to $Y$. We denote by $\langle\cdot,\cdot\rangle$ the duality bracket for the pair of $Y^*$ and $Y$. Everywhere below, the symbols $\xrightarrow{w}$ and $\to$ represent the weak and strong convergences, respectively. We say that a mapping $F\colon Y\to Y^*$ is

(i) Monotone, if

$$\langle Fu - Fv, u - v\rangle \geq 0 \ \text{ for all } \ u, v \in Y.$$

(ii) Strictly monotone, if

$$\langle Fu - Fv, u - v\rangle > 0 \ \text{ for all } \ u, v \in Y \text{ with } u \neq v.$$

(iii) Of type $(S)_+$ (or $F$ satisfies $(S_+)$–property), if for any sequence $\{u_n\}\subset Y$ with $u_n \xrightarrow{w} u$ in $Y$ as $n\to\infty$ for some $u\in Y$ and $\limsup\limits_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, then the sequence $\{u_n\}$ converges strongly to $u$ in $Y$.

(iv) Coercive, if

$$\lim_{\|v\|_Y\to\infty}\frac{\langle Fv, v\rangle}{\|v\|_Y} = +\infty.$$

Recall that a function $\varphi\colon Y\to\overline{\mathbb{R}} := \mathbb{R}\cup\{+\infty\}$ is called to be proper, convex and lower semicontinuous, if the following conditions are satisfied:

- $D(\varphi) := \{u\in Y \mid \varphi(u) < +\infty\} \neq \emptyset$;
- for any $u, v\in Y$ and $t\in(0,1)$, it holds $\varphi(tu + (1-t)v) \leq t\varphi(u) + (1-t)\varphi(v)$;
- $\liminf_{n\to\infty}\varphi(u_n) \geq \varphi(u)$ where the sequence $\{u_n\}_{n\in\mathbb{N}}\subset Y$ is such that $u_n\to u$ as $n\to\infty$ for some $u\in Y$.

Suppose that the map $\varphi$ is convex. An element $x^*\in Y^*$ is called a subgradient of $\varphi$ at $u\in Y$, if

$$\langle x^*, v - u\rangle \leq \varphi(v) - \varphi(u) \quad \text{for all } v\in Y. \tag{2.1}$$

The set of all elements $x^*\in Y^*$ which satisfy (2.1) is called the convex subdifferential of $\varphi$ at $u$ and is denoted by $\partial\varphi(u)$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ and let $1\leq r<\infty$. For any subset $D$ of $\overline{\Omega}$, in what follows, we denote by $L^r(D) := L^r(D;\mathbb{R})$ the usual Lebesgue spaces endowed with the norm $\|\cdot\|_{r,D}$, that is,

$$\|u\|_{r,D} := \left(\int_D |u|^r\,dx\right)^{\frac{1}{r}} \quad \text{for all } u\in L^r(D).$$

We set $L^r(D)_+ := \{u \in L^r(D) \mid u(x) \geq 0 \text{ for a.e. } x \in D\}$. Moreover, $W^{1,r}(\Omega)$ stands for the Sobolev space endowed with the norm $\|\cdot\|_{1,r,\Omega}$, namely,

$$\|u\|_{1,r,\Omega} := \|u\|_{r,\Omega} + \|\nabla u\|_{r,\Omega} \quad \text{for all } u \in W^{1,r}(\Omega).$$

For any $1 < r < \infty$ we denote by $r'$ the conjugate exponent of $r$, that is, $\frac{1}{r} + \frac{1}{r'} = 1$. In the sequel, we denote by $r^*$ and $r_*$ the critical exponents to $r$ in the domain and on the boundary, respectively, given by

$$r^* = \begin{cases} \frac{Nr}{N-r} & \text{if } r < N, \\ +\infty & \text{if } r \geq N, \end{cases} \quad \text{and} \quad r_* = \begin{cases} \frac{(N-1)r}{N-r} & \text{if } r < N, \\ +\infty & \text{if } r \geq N, \end{cases} \tag{2.2}$$

respectively.

Since Problem 1 contains mixed boundary value conditions, we are now in a position to introduce a closed subspace $V$ of $W^{1,p}(\Omega)$ defined by

$$V := \{u \in W^{1,p}(\Omega) \mid u = 0 \text{ on} \Gamma_1\},$$

which is also a reflexive Banach space. For the sake of convenience, in what follows, we denote by $\|\cdot\|_V$ the norm of $V$, that is, $\|u\|_V = \|u\|_{1,p,\Omega}$ for all $u \in V$, and by $V^*$ we denote the dual space of $V$. Besides, we consider a subset $K$ of $V$ defined by

$$K := \{u \in V \mid u_*(x) \leq u(x) \leq u^*(x) \quad \text{in} \quad \Omega\}, \tag{2.3}$$

where $u_*, u^* \colon \Omega \to \mathbb{R}$ are two obstacles given in Problem 1.

We end the section to deliver the weak variational formulation of Problem 1. Assume that $(u, \mu)$ are smooth functions such that (1.1)–(1.6) are satisfied. Let $v \in K$ be arbitrary. We multiply (1.1) by $v - u$ and use Green's formula to obtain

$$\begin{aligned} \int_\Omega Z(x)(v(x) - u(x))\, dx = & \int_\Omega a(x, \nabla u) \cdot \nabla(v - u)\, dx - \int_\Gamma \frac{\partial u}{\partial \nu_a}(v(x) - u(x))\, d\Gamma \\ & + \int_\Omega [g(x, u) - f(x) + \mu(x)]\,(v(x) - u(x))\, dx. \end{aligned} \tag{2.4}$$

Recall that $v \in K$ (i.e., $u_* \leq v \leq u^*$ for a.e. $x \in \Omega$), it follows from (1.2) that

$$\begin{aligned} \int_\Omega \mu(x)(v(x) - u(x))\, dx = & \int_\Omega \mu(x)(u^*(x) - u(x))\, dx \\ & + \int_\Omega \mu(x)(v(x) - u^*(x))\, dx \leq 0. \end{aligned} \tag{2.5}$$

Using (1.3), it yields

$$\begin{aligned} \int_\Omega Z(x)(v(x) - u(x))\, dx = & \int_\Omega Z(x)(u_*(x) - u(x))\, dx \\ & + \int_\Omega Z(x)(v(x) - u_*(x))\, dx \geq 0. \end{aligned} \tag{2.6}$$

Noting that

$$\begin{aligned} \int_\Gamma \frac{\partial u}{\partial \nu_a}(v(x) - u(x))\, d\Gamma = & \int_{\Gamma_1} \frac{\partial u}{\partial \nu_a}(v(x) - u(x))\, d\Gamma + \int_{\Gamma_2} \frac{\partial u}{\partial \nu_a}(v(x) - u(x))\, d\Gamma \\ & + \int_{\Gamma_3} \frac{\partial u}{\partial \nu_a}(v(x) - u(x))\, d\Gamma, \end{aligned}$$

utilizing boundary conditions (1.4)–(1.6) derives

$$\begin{aligned}
&-\int_{\Gamma} \frac{\partial u}{\partial \nu_a}(v(x)-u(x))\,d\Gamma \\
&\quad\le \int_{\Gamma_2} b(x)(v(x)-u(x))\,d\Gamma + \int_{\Gamma_3} \phi(x,v(x))\,d\Gamma \\
&\quad-\int_{\Gamma_3} \phi(x,u(x))\,d\Gamma, \qquad (2.7)
\end{aligned}$$

where we have used the definition of convex subgradient. Inserting (2.5)–(2.7) into (2.4), we have

$$\begin{aligned}
&\int_{\Omega} a(x,\nabla u)\cdot\nabla(v-u)\,dx + \int_{\Omega} g(x,u(x))(v(x)-u(x))\,dx \\
&+\int_{\Gamma_3} \phi(x,v(x))\,d\Gamma - \int_{\Gamma_3} \phi(x,u(x))\,d\Gamma \\
&\ge \int_{\Omega} f(x)(v(x)-u(x))\,dx - \int_{\Gamma_2} b(x)(v(x)-u(x))\,d\Gamma.
\end{aligned}$$

We are now in a position to deliver the variational formulation of complementarity system, Problem 1, as follows, which is exactly a mixed variational inequality with double obstacle effect:

**Problem 2** *Find function $u \in K$ such that*

$$\begin{aligned}
&\int_{\Omega} a(x,\nabla u)\cdot\nabla(v-u)\,dx + \int_{\Omega} g(x,u(x))(v(x)-u(x))\,dx \\
&+\int_{\Gamma_3} \phi(x,v(x))\,d\Gamma - \int_{\Gamma_3} \phi(x,u(x))\,d\Gamma \\
&\ge \int_{\Omega} f(x)(v(x)-u(x))\,dx - \int_{\Gamma_2} b(x)(v(x)-u(x))\,d\Gamma \qquad (2.8)
\end{aligned}$$

*for all $v \in K$.*

## 3 Existence and convergence to nonlinear complementarity problems

In this section, we are going to prove the existence and uniqueness of weak solution to Problem 1, i.e., the unique solvability of Problem 2, and apply a power penalty approximation technique to introduce a family of approximating mixed variational inequalities without constraints (i.e., without double obstacles effect). Also, we shall establish a strong convergence result which reveals that the unique solution of Problem 2 can be approached by the approximating mixed variational inequalities when the penalty parameter $\lambda$ tends to infinity.

To this end, we first impose the following assumptions on the data of Problem 2.

Assume that $\vartheta \in C^1(0,\infty)$ is any function such that

$$0 < a_1 \le \frac{\vartheta'(t)t}{\vartheta(t)} \le a_2 \ \text{ and } \ a_3 t^{p-1} \le \vartheta(t) \le a_4(t^{q-1}+t^{p-1}) \ \text{ for all } \ t>0 \qquad (3.1)$$

with some constants $a_1, a_2, a_3, a_4 > 0$ and $1 < q < p < +\infty$. In what follows, we assume that the nonlinear and nonhomogeneous operator $a\colon \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies the following conditions:

<u>$H(a)$</u>: $a\colon \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}^N$ is such that $a(x,\xi)=a_0(x,|\xi|)\xi$ with $a_0\in C(\overline{\Omega}\times[0,+\infty))$ for all $\xi\in\mathbb{R}^N$ and with $a_0(x,t)>0$ for all $x\in\overline{\Omega}$ and for all $t>0$ and

(i) $a_0 \in C^1(\overline{\Omega}\times(0,+\infty))$, $t\mapsto a_0(x,t)t$ is strictly increasing on $(0,+\infty)$ with $\lim_{t\to 0^+} a_0(x,t)t\to 0$ for all $x\in\overline{\Omega}$, and

$$\lim_{t\to 0^+}\frac{a_0'(x,t)t}{a_0(x,t)}=c>-1$$

for all $x\in\overline{\Omega}$;

(ii) There exists a constant $a_5>0$ such that

$$|\nabla_\xi a(x,\xi)|\le a_5\frac{\vartheta(|\xi|)}{|\xi|}\quad\text{for all } x\in\overline{\Omega} \text{ and for all } \xi\in\mathbb{R}^N\backslash\{0\};$$

(iii) For all $x\in\overline{\Omega}$, for every $\xi\in\mathbb{R}^N\backslash\{0\}$ and for all $y\in\mathbb{R}^N$, the following inequality holds

$$\nabla_\xi a(x,\xi)y\cdot y\ge\frac{\vartheta(|\xi|)}{|\xi|}|y|^2.$$

<u>$H(g)$</u>: The function $g\colon\Omega\times\mathbb{R}\to\mathbb{R}$ is such that

(i) For all $s\in\mathbb{R}$, the function $x\mapsto g(x,s)$ is measurable in $\Omega$;
(ii) For a.e. $x\in\Omega$, the function $s\mapsto g(x,s)$ is continuous and nondecreasing such that

$$|g(x,s)|\le\alpha_g(x)+\beta_g|s|^{\frac{p}{q_1'}}$$

for all $s\in\mathbb{R}$ and a.e. $x\in\Omega$, where $\beta_g>0$, $\alpha_g\in L^{q_1'}(\Omega)$ and $1<q_1<p^*$ with $p^*$ being the critical exponent of $p$ in the domain (see (2.2) with $r=p$);
(iii) There exist $a_g>0$ and $b_g\in L^1(\Omega)$ such that

$$g(x,s)s\ge a_g|s|^\varsigma+b_g(x),$$

for all $s\in\mathbb{R}$ and for a.e. $x\in\Omega$, where $p\le\varsigma<p^*$.

<u>$H(\phi)$</u>: $\phi\colon\Gamma_3\times\mathbb{R}\to\mathbb{R}$ is such that

(i) For all $s\in\mathbb{R}$, the function $x\mapsto\phi(x,s)$ is measurable on $\Gamma_3$;
(ii) For a.e. $x\in\Gamma_3$, $s\mapsto\phi(x,s)$ is convex and lower semicontinuous;
(iii) For each $u\in L^{p_*}(\Gamma_3)$ the function $x\mapsto\phi(x,u(x))$ belongs to $L^1(\Gamma_3)$, where $p_*$ is the critical exponent of $p$ on the boundary (see (2.2) with $r=p$).

<u>$H(0)$</u>: $f\in L^{p'}(\Omega)$, $b\in L^{p'}(\Gamma_2)$ and $u_*,u^*\in L^1(\Omega)$ are such that

$$u_*(x)\le 0\le u^*(x)\text{ for a.e. } x\in\Omega.$$

***Remark 3*** It follows from hypothesis $H(g)$(ii) that the function $s\mapsto g(x,s)$ is monotone, i.e.,

$$(g(x,s_1)-g(x,s_2))(s_1-s_2)\ge 0\quad\text{for all } s_1,s_2\in\mathbb{R}\text{ and a.e. } x\in\Omega.$$

Let $p\le\varsigma<p^*$ be such that $\varsigma\le p-\frac{p}{q_1}+1$ for some $p<q_1<p^*$ and $\kappa\in L^{\varsigma'}(\Omega)$. Then, the function

$$g(x,s)=|s|^{\varsigma-2}s+\kappa(x)\quad\text{for all } s\in\mathbb{R}\text{ and a.e. } x\in\Omega$$

satisfies $H(g)$.

Let $1 < \delta \le p_*$ and $\omega \in L^\infty_+(\Gamma_3)$. Then, the function $\phi\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ reads assumptions $H(\phi)$

$$\phi(x,s) := \begin{cases} \omega(x)|s| & \text{if } |s| \le 1, \\ \omega(x)|s|^\delta & \text{if } |s| > 1, \end{cases} \quad \text{for a.e.} x \in \Gamma_3.$$

In fact, there are a plenty of functions which satisfy hypotheses $H(a)$. Here, we give below several particular interesting examples of the function $a$, see Zeng-Liu-Migórski [41] for more details.

***Example 4*** The following maps satisfy properties stated in hypothesis $H(a)$, in which we drop the dependence on $x$ just for simplicity.

(i) If $a(\xi) = |\xi|^{p-2}\xi$ for all $\xi \in \mathbb{R}^N$ with $1 < p < \infty$, then operator $u(x) \mapsto \operatorname{div}(a(\nabla u(x)))$ is the well-known $p$-Laplacian differential operator, i.e.,

$$\operatorname{div}(a(\nabla u(x))) = \Delta_p u(x) := \operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x))$$

for all $u \in W^{1,p}(\Omega)$.

(ii) If $a(\xi) = |\xi|^{p-2}\xi + |\xi|^{q-2}\xi$ for all $\xi \in \mathbb{R}^N$ with $1 < q < p < \infty$, then operator $u(x) \mapsto \operatorname{div}(a(\nabla u(x)))$ is the $(p,q)$-Laplacian differential operator, namely,

$$\operatorname{div}(a(\nabla u(x))) = \Delta_p u(x) + \Delta_q u(x)$$

for all $u \in W^{1,p}(\Omega)$.

(iii) If $a(\xi) = (1 + |\xi|^2)^{\frac{p-2}{2}}\xi$ for all $\xi \in \mathbb{R}^N$ with $1 < p < \infty$, then operator $u(x) \mapsto \operatorname{div}(a(\nabla u(x)))$ is the generalized $p$-mean curvature differential operator, that is,

$$\operatorname{div}(a(\nabla u(x))) = \operatorname{div}\left((1 + |\nabla u(x)|)^{\frac{p-2}{2}}\nabla u(x)\right)$$

for all $u \in W^{1,p}(\Omega)$.

(iv) If $a(\xi) = |\xi|^{p-2}\left(1 + \frac{1}{1+|\xi|^p}\right)\xi$ for all $\xi \in \mathbb{R}^N$ with $1 < p < \infty$, then operator $u(x) \mapsto \operatorname{div}(a(\nabla u(x)))$ corresponds the following differential operator, thus,

$$\operatorname{div}(a(\nabla u(x))) = \Delta_p u(x) + \operatorname{div}\left(\frac{|\nabla u(x)|^{p-2}\nabla u(x)}{1 + |\nabla u(x)|^p}\right)$$

for all $u \in W^{1,p}(\Omega)$.

(v) If $a(\xi) = \left(|\xi|^{p-2} + \ln(1 + |\xi|^2)\right)\xi$ for all $\xi \in \mathbb{R}^N$ with $1 < p < \infty$, then the operator $u(x) \mapsto \operatorname{div}(a(\nabla u(x)))$ corresponds the following differential operator, i.e.,

$$\operatorname{div}(a(\nabla u(x))) = \Delta_p u(x) + \operatorname{div}\left(\ln(1 + |\nabla u(x)|^2)\nabla u(x)\right)$$

for all $u \in W^{1,p}(\Omega)$.

We now recall the following crucial properties of the operator $a$, whose proof can be found in [41, Lemma 3].

**Lemma 5** *If hypotheses $H(a)$ are satisfied, then we have*

(i) $a \in C(\overline{\Omega}\times\mathbb{R}^N, \mathbb{R}^N) \cap C^1\left(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N\right)$ *and for all $x \in \overline{\Omega}$ the map $\xi \mapsto a(x,\xi)$ is continuous, strictly monotone, and so maximal monotone as well;*

(ii) *There exists a constant $a_6 > 0$ such that*

$$|a(x,\xi)| \le a_6(1 + |\xi|^{p-1}) \textit{ for all } \xi \in \mathbb{R}^N \textit{ and } x \in \overline{\Omega};$$

(iii) *It holds*

$$a(x,\xi)\cdot\xi \geq \frac{a_3}{p-1}|\xi|^p \text{ for all } x\in\overline{\Omega} \text{ and for all } \quad \xi\in\mathbb{R}^N,$$

*where $a_3$ is given in (3.1).*

Furthermore, let us consider the nonlinear operator $A\colon V\to V^*$ defined by

$$\langle Au, v\rangle := \int_\Omega a(x,\nabla u)\cdot\nabla v\,dx \tag{3.2}$$

for all $u, v\in V$.

**Lemma 6** *Assume that hypotheses $H(a)$ are fulfilled. Then, $A$ defined in (3.2) is continuous, and bounded (i.e., $A$ maps bounded subsets in $V$ to bounded subsets in $V^*$), strictly monotone (hence maximal monotone too) and of type $(S)_+$, i.e.,*

$$\text{if } u_n \xrightarrow{w} u \text{ in } V \text{ and } \limsup_{n\to\infty}\langle Au_n, u_n-u\rangle\leq 0,$$

*entail $u_n\to u$ in $V$.*

The existence and uniqueness of solution to Problem 2 is stated by the following theorem.

**Theorem 7** *Assume that $H(a)$, $H(g)$, $H(\phi)$ and $H(0)$ are satisfied. Then, Problem 2 has a unique solution.*

***Proof*** Let us consider the functions $G\colon V\to\mathbb{R}$ and $\varphi\colon V\to\mathbb{R}$ defined by

$$\langle G(u), v\rangle := \int_\Omega g(x,u(x))v(x)\,dx, \tag{3.3}$$

$$\varphi(u) := \int_{\Gamma_3}\phi(x,u(x))\,d\Gamma \tag{3.4}$$

for all $v, u\in V$. Using the notation above, Problem 2 can be rewritten equivalently to the following mixed variational inequality of first kind: find $u\in K$ such that

$$\langle Au+G(u)-f+b, v-u\rangle+\varphi(v)-\varphi(u)\geq 0 \tag{3.5}$$

for all $v\in K$.

By the definition of $\varphi$ and hypotheses H($\phi$), we can use a standard way to prove that $\varphi$ is a convex and l.s.c. function. From Lemma 6 and hypothesis H($g$)(ii), we infer that $A+G\colon V\to V^*$ is continuous and strictly monotone. Using condition H($g$)(ii), one has

$$\begin{aligned}
\|g(\cdot,u)\|_{q_1',\Omega}^{q_1'} &= \int_\Omega |g(x,u(x))|^{q_1'}\,dx\\
&\leq \int_\Omega \left(\alpha_g(x)+\beta_g|u(x)|^{\frac{p}{q_1'}}\right)^{q_1'}\,dx\\
&\leq 2^{q_1'-1}\int_\Omega \alpha_g(x)^{q_1'}+\beta_g^{q_1'}|u(x)|^p\,dx\\
&= 2^{q_1'-1}\left(\|\alpha_g\|_{q_1',\Omega}^{q_1'}+\beta_g^{q_1'}\|u\|_{p,\Omega}^p\right),
\end{aligned}$$

where we have applied the elementary inequality $(c+d)^t\leq 2^{t-1}(c^t+d^t)$ for all $c, d\geq 0$ and $t\geq 1$. The latter combined with Lemma 6 implies that $A+G\colon V\to V^*$ is a bounded mapping. Moreover, we assert that $A + G: V \to V^*$ is coercive. For any $u \in V$, owing to $0 \in K$, we employing Lemma 5(iii) and hypothesis H($g$)(iii) to find

$$\begin{aligned}\langle Au + Gu, u\rangle &\geq \frac{a_3}{p-1}\|\nabla u\|_{p,\Omega}^p + \int_\Omega a_g|u(x)|^\varsigma + b_g(x)\,dx\\ &\geq \begin{cases}\frac{a_3}{p-1}\|\nabla u\|_{p,\Omega}^p + a_g\|u\|_{p,\Omega}^p - \|b_g\|_{1,\Omega} & \text{if } \varsigma = p,\\ \frac{a_3}{p-1}\|\nabla u\|_{p,\Omega}^p + \|u\|_{p,\Omega}^p - \|b_g\|_{1,\Omega} - m_0 & \text{if } \varsigma > p,\end{cases}\end{aligned}$$

for some $m_0 > 0$, where we have used Young inequality for the case $\varsigma > p$. Hence, we have

$$\lim_{u\in K,\|u\|_V\to+\infty}\frac{\langle Au + Gu, u\rangle}{\|u\|_V} = +\infty,$$

i.e., $A + G: V \to V^*$ is coercive.

Therefore, all conditions of Theorem 3.2 of Liu-Migórski-Zeng [19] are verified. Utilizing this theorem, we conclude that Problem 2 has at least one solution. Remembering that $A+G$ is strictly monotone, we, however, can use a standard procedure to prove the uniqueness of Problem 2. □

In what follows, let $\delta \geq 1$, $r = 1 + \frac{1}{\delta}$ and $s = \delta + 1$, and $\varepsilon > 0$ be fixed. This means that $\frac{1}{r} + \frac{1}{s} = 1$. Also, let us introduce the functions $\omega: \Omega \times \mathbb{R} \to \mathbb{R}$ and $\omega_0: \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned}&\omega(x,t) := |(t - u_*(x))_-|^{\frac{p-1}{p}} + (t - u^*(x))_+^{\frac{p-1}{p}} \quad \text{and}\\ &\omega_0(x,t) := (t - u_*(x))_- + (t - u^*(x))_+\end{aligned}$$

for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$, where the symbols $(\cdot)_+$ and $(\cdot)_-$ are defined by

$$(t)_+ := \max\{t, 0\} \text{ and } (t)_- := \min\{t, 0\} \text{ for all } t \in \mathbb{R}.$$

In the meantime, we consider the function $\theta_\varepsilon: L^1(\Omega) \to L^{p'}(\Omega)$ given by

$$\theta_\varepsilon(u) = (|\omega(\cdot, u(\cdot))| + \varepsilon)^{\frac{1}{\delta}}\chi(\omega_0(\cdot, u(\cdot))) - \varepsilon^{\frac{1}{\delta}} \tag{3.6}$$

for all $u \in L^1(\Omega)$, here the function $\chi: \mathbb{R} \to \mathbb{R}$ is the sign function, namely,

$$\chi(t) := \begin{cases}1 & \text{if } t \geq 0,\\ -1 & \text{otherwise.}\end{cases}$$

The following lemma turns out that for each $\varepsilon > 0$ the function $\theta_\varepsilon$ is well-defined, continuous and monotone.

**Lemma 8** *Let $u_*, u^* \in L^1(\Omega)$ be such that $u_*(x) \leq 0 \leq u^*(x)$ for a.e. $x \in \Omega$, and $\varepsilon$ be arbitrary fixed. Then, $\theta_\varepsilon$ given in (3.6) is well-defined, continuous and monotone, and satisfies the following equality*

$$Ker(\theta_\varepsilon) = K, \tag{3.7}$$

*that is, $\theta_\varepsilon(u) = 0$ if and only if $u \in K$, where $Ker(\theta_\varepsilon)$ stands for the kernel of $\theta_\varepsilon$.*

***Proof*** For any $u \in L^1(\Omega)$, we have

$$\begin{aligned}\|\theta_\varepsilon(u)\|_{p',\Omega}^{p'} &= \int_\Omega \left( (|\omega(\cdot, u(\cdot))| + \varepsilon)^{\frac{1}{\delta}} \chi(\omega(\cdot, u(\cdot))) - \varepsilon^{\frac{1}{\delta}} \right)^{p'} dx \\ &\leq 2^{p'-1} \int_\Omega (|\omega(\cdot, u(\cdot))| + \varepsilon)^{\frac{p'}{\delta}} + \varepsilon^{\frac{p'}{\delta}} dx \\ &\leq 2^{p'-1} \varepsilon^{\frac{p'}{\delta}} |\Omega| + 2^{p'-1+\frac{p'-1}{\delta}} \int_\Omega \left( |\omega(\cdot, u(\cdot))|^{p'} + \varepsilon^{p'} \right)^{\frac{1}{\delta}} dx \\ &\leq 2^{p'-1} \varepsilon^{\frac{p'}{\delta}} |\Omega| + 2^{p'-1+\frac{p'-1}{\delta}} \\ &\quad \int_\Omega \left[ 2^{p'-1} \left( |u(x) - u_*(x)| + |u(x) - u^*(x)| \right) + \varepsilon^{p'} \right]^{\frac{1}{\delta}} dx \\ &< +\infty,\end{aligned}$$

where the last inequality is obtained via using Young inequality for the case $\delta > 1$ and $|\Omega|$ stands for the Lebesgue measure of $\Omega$. Therefore, $\theta_\varepsilon$ is well-defined and bounded.

For any $u, v \in L^1(\Omega)$, without any loss of generality, we may assume that

$$u(x) \leq v(x) \ \text{ for a.e. } x \in \Omega.$$

Since the conclusion is still valid for the domain $\{x \in \Omega \mid u(x) > v(x)\}$. Noting that

$$\Omega := \Omega_1(u) \cup \Omega_2(u) \cup \Omega_3(u) \ \text{ with } \Omega_i(u) \cap \Omega_j(u) = \emptyset \text{ for } i, j = 1, 2, 3 \text{ with } i \neq j,$$

where $\Omega_1(u)$, $\Omega_2(u)$ and $\Omega_3(u)$ are defined by

$$\begin{aligned}\Omega_1(u) &:= \{x \in \Omega \mid u(x) > u^*(x)\}, \\ \Omega_2(u) &:= \{x \in \Omega \mid u_*(x) \leq u(x) \leq u^*(x)\}, \\ \Omega_3(u) &:= \{x \in \Omega \mid u_*(x) > u(x)\},\end{aligned}$$

respectively. If $x \in \Omega_1(u)$, then we have $\omega(x, u(x)) = (u(x) - u^*(x))^{\frac{p-1}{p}} \leq (v(x) - u^*(x))^{\frac{p-1}{p}} = \omega(x, v(x))$, $\chi(\omega_0(x, u(x))) = \chi(\omega_0(x, v(x))) = 1$ and

$$\begin{aligned}&\left( (|\omega(\cdot, u(x))| + \varepsilon)^{\frac{1}{\delta}} \chi(\omega_0(x, u(x))) - (|\omega(\cdot, v(x))| + \varepsilon)^{\frac{1}{\delta}} \chi(\omega_0(x, v(x))) \right) (u(x) - v(x)) \\ &= \left( \left( (u(x) - u^*(x))^{\frac{p-1}{p}} + \varepsilon \right)^{\frac{1}{\delta}} - \left( (v(x) - u^*(x))^{\frac{p-1}{p}} + \varepsilon \right)^{\frac{1}{\delta}} \right) (u(x) - v(x)) \\ &\geq 0\end{aligned}$$

for a.e. $x \in \Omega_1(u)$. When $x \in \Omega_2(u)$, then we have $\omega_0(x, u(x)) = 0$ and

$$\begin{aligned}&\left( (|\omega(\cdot, u(x))| + \varepsilon)^{\frac{1}{\delta}} \chi(\omega_0(x, u(x))) - (|\omega(\cdot, v(x))| + \varepsilon)^{\frac{1}{\delta}} \chi(\omega_0(x, v(x))) \right) (u(x) - v(x)) \\ &= \begin{cases} \left( \varepsilon^{\frac{1}{\delta}} - \varepsilon^{\frac{1}{\delta}} \right) (u(x) - v(x)) = 0 & \text{if } v(x) \leq u^*(x) \\ \left( \varepsilon^{\frac{1}{\delta}} - \left( (v(x) - u^*(x))^{\frac{p-1}{p}} + \varepsilon \right)^{\frac{1}{\delta}} \right) (u(x) - v(x)) & \text{if } v(x) > u^*(x) \end{cases} \\ &\geq 0\end{aligned}$$

for a.e. $x \in \Omega_2(u)$. While $x \in \Omega_3(u)$, then we have $\omega(x, u(x)) = (u^*(x) - u(x))^{\frac{p-1}{p}}$, $\chi(\omega_0(x, u(x))) = -1$ and

$$
\begin{aligned}
&\Big((|\omega(\cdot, u(x))| + \varepsilon)^{\frac{1}{\delta}}\,\chi(\omega_0(x, u(x))) - (|\omega(\cdot, v(x))| + \varepsilon)^{\frac{1}{\delta}}\,\chi(\omega_0(x, v(x)))\Big) \\
&= \begin{cases}
\left(-\left((u_*(x) - u(x))^{\frac{p-1}{p}} + \varepsilon\right)^{\frac{1}{\delta}} + \left((u_*(x) - v(x))^{\frac{p-1}{p}} + \varepsilon\right)^{\frac{1}{\delta}}\right) & \text{if } v(x) < u_*(x), \\
-\left((u_*(x) - u(x))^{\frac{p-1}{p}} + \varepsilon\right)^{\frac{1}{\delta}} - \varepsilon^{\frac{1}{\delta}} & \text{if } u_*(x) \le v(x) \le u^*(x), \\
\left(-\left((u_*(x) - u(x))^{\frac{p-1}{p}} + \varepsilon\right)^{\frac{1}{\delta}} - \left((v(x) - u^*(x))^{\frac{p-1}{p}} + \varepsilon\right)^{\frac{1}{\delta}}\right) & \text{if } v(x) > u^*(x).
\end{cases}
\end{aligned}
$$

Hence,

$$
\Big((|\omega(\cdot, u(x))| + \varepsilon)^{\frac{1}{\delta}}\,\chi(\omega_0(x, u(x))) - (|\omega(\cdot, v(x))| + \varepsilon)^{\frac{1}{\delta}}\,\chi(\omega_0(x, v(x)))\Big)\,(u(x) - v(x)) \ge 0
$$

for a.e. $x \in \Omega$. From the analysis above, it yields

$$
\int_\Omega (\theta_\varepsilon(u) - \theta_\varepsilon(v))\,(u(x) - v(x))\,dx \ge 0,
$$

i.e., $\theta_\varepsilon$ is monotone. Whereas the continuity of $\theta_\varepsilon$ is a direct consequence of the continuity of $(\cdot)_+$ and $|(\cdot)_-|$.

Let $u \in K$ be arbitrary. Then, we have $\omega(x, u(x)) = 0$ and $\chi(\omega_0(x, u(x))) = 1$ for a.e. $x \in \Omega$. Hence, $\theta_\varepsilon(u) = 0$. Conversely, if $\theta_\varepsilon(u) = 0$, then $(|\omega(x, u(x))| + \varepsilon)^{\frac{1}{\delta}}\,\chi(\omega_0(x, u(x))) = \varepsilon^{\frac{1}{\delta}}$ for a.e. $x \in \Omega$. This means that $\chi(\omega_0(x, u(x))) = 1$ and $\omega(x, u(x)) = 0$ for a.e. $x \in \Omega$. So, we have $u_*(x) \le u(x) \le u^*(x)$ for all a.e. $x \in \Omega$, thus, $u \in K$. □

For any $\lambda > 0$, we introduce the following approximating problem involving the power penalty function $\theta_\varepsilon$ corresponding to Problem 2.

**Problem 9** *Find function $u_\lambda \in V$ such that*

$$
\begin{aligned}
&\int_\Omega a(x, \nabla u_\lambda)\cdot\nabla(v - u_\lambda)\,dx + \int_\Omega (g(x, u_\lambda(x)) + \lambda\theta_\varepsilon(u_\lambda(x)))\,(v(x) - u_\lambda(x))\,dx \\
&\quad + \int_{\Gamma_3} \phi(x, v(x))\,d\Gamma - \int_{\Gamma_3} \phi(x, u_\lambda(x))\,d\Gamma \\
&\quad \ge \int_\Omega f(x)(v(x) - u_\lambda(x))\,dx - \int_{\Gamma_2} b(x)(v(x) - u_\lambda(x))\,d\Gamma
\end{aligned}
$$

*for all $v \in V$.*

We deliver the second result of this section by the following theorem which shows that Problem 9 has unique solution $u_\lambda$ and $u_\lambda$ converges strongly to the unique solution $u_0$ of Problem 2 in $V$ as $\lambda \to \infty$.

**Theorem 10** *Assume that $H(a)$, $H(g)$, $H(\phi)$ and $H(0)$ are satisfied. Then, we have*

(i) *For each $\lambda > 0$, Problem 9 has a unique solution $u_\lambda \in V$.*
(ii) *Let sequence $\{\lambda_n\} \subset (0, +\infty)$ be such that $\lambda_n \to \infty$ as $n \to \infty$. Then the unique solution of Problem 9 corresponding to $\lambda_n$ converges strongly to the unique solution $u_0$ of Problem 2 in $V$ as $n \to \infty$.*

***Proof*** (i) The desired conclusion can be obtained directly by using Lemma 8 and Theorem 7.
(ii) Let sequence $\{\lambda_n\} \subset (0, +\infty)$ be such that $\lambda_n \to \infty$ as $n \to \infty$ and let $u_n \in V$ be the unique solution of Problem 9 corresponding to $\lambda_n$, i.e., for every $n \in \mathbb{N}$, we have

$$\begin{aligned}&\int_\Omega a(x, \nabla u_n) \cdot \nabla(v - u_n)\, dx + \int_\Omega (g(x, u_n(x)) + \lambda_n \theta_\varepsilon(u_n(x)))\,(v(x) - u_n(x))\, dx\\&\quad + \int_{\Gamma_3} \phi(x, v(x))\, d\Gamma - \int_{\Gamma_3} \phi(x, u_n(x))\, d\Gamma\\&\quad \geq \int_\Omega f(x)(v(x) - u_n(x))\, dx - \int_{\Gamma_2} b(x)(v(x) - u_n(x))\, d\Gamma \end{aligned}\tag{3.8}$$

for all $v \in V$.

We claim that solution sequence $\{u_n\}$ is bounded in $V$. Taking $v = 0$ into (3.8), we have

$$\begin{aligned}&\int_\Omega a(x, \nabla u_n) \cdot \nabla u_n\, dx + \int_\Omega g(x, u_n(x)) u_n(x)\, dx\\&\quad \leq \lambda_n \int_\Omega \theta_\varepsilon(u_n(x))(-u_n(x))\, dx + \int_{\Gamma_3} \phi(x, 0)\, d\Gamma - \int_{\Gamma_3} \phi(x, u_n(x))\, d\Gamma\\&\quad + \int_\Omega f(x) u_n(x)\, dx - \int_{\Gamma_2} b(x) u_n(x)\, d\Gamma. \end{aligned}\tag{3.9}$$

Recall that $\theta_\varepsilon$ is monotone, $\theta_\varepsilon(u) = 0$ for all $u \in K$ and $0 \in K$, so, one has

$$\int_\Omega \theta_\varepsilon(u_n(x))(-u_n(x))\, dx = \int_\Omega (\theta_\varepsilon(u_n(x)) - \theta_\varepsilon(0))\,(-u_n(x))\, dx \leq 0. \tag{3.10}$$

Since $\varphi$ defined in (3.4) is convex and l.s.c., so, from Brézis [5, Proposition 1.10], there are two constants $\alpha_\varphi, \beta_\varphi \geq 0$ such that

$$\varphi(v) \geq -\alpha_\varphi \|v\|_V - \beta_\varphi \text{ for all } v \in V. \tag{3.11}$$

Besides, by virtue of hypothesis H($g$)(iii) and Lemma 5(iii), it yields

$$\begin{aligned}\langle Au_n + Gu_n, u_n\rangle &\geq \int_\Omega a(x, \nabla u_n) \cdot \nabla u_n + g(x, u_n(x)) u_n(x)\, dx\\&\geq \frac{a_3}{p-1} \|\nabla u_n\|_{p,\Omega}^p + \int_\Omega a_g |u_n(x)|^\varsigma + b_g(x)\, dx\\&\geq \begin{cases} \frac{a_3}{p-1} \|\nabla u_n\|_{p,\Omega}^p + a_g \|u_n\|_{p,\Omega}^p - \|b_g\|_{1,\Omega} & \text{if } \varsigma = p,\\ \frac{a_3}{p-1} \|\nabla u_n\|_{p,\Omega}^p + \|u_n\|_{p,\Omega}^p - \|b_g\|_{1,\Omega} - m_1 & \text{if } \varsigma > p, \end{cases}\end{aligned}\tag{3.12}$$

for some $m_1 > 0$, where we have used the Young inequality for the case $\varsigma > p$. Inserting (3.10), (3.11) and (3.12) into (3.9), we can use Hölder inequality to get

$$\begin{aligned}&\|f\|_{p',\Omega} \|u_n\|_{p,\Omega} + \|b\|_{p',\Gamma_2} \|u_n\|_{p,\Gamma} + \int_{\Gamma_3} \phi(x, 0)\, d\Gamma + \beta_\varphi\\&\quad \geq \begin{cases} \frac{a_3}{p-1} \|\nabla u_n\|_{p,\Omega}^p + a_g \|u_n\|_{p,\Omega}^p - \|b_g\|_{1,\Omega} - \alpha_\varphi \|u_n\|_V & \text{if } \varsigma = p,\\ \frac{a_3}{p-1} \|\nabla u_n\|_{p,\Omega}^p + \|u_n\|_{p,\Omega}^p - \|b_g\|_{1,\Omega} - m_1 - \alpha_\varphi \|u_n\|_V & \text{if } \varsigma > p. \end{cases}\end{aligned}\tag{3.13}$$

This implies directly that solution sequence $\{u_n\}$ is bounded in $V$. Passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } Vasn \to \infty \tag{3.14}$$

for some $u \in V$.

We are going to show that $u \in K$. From (3.8), we have

$$\begin{aligned}
&\int_\Omega \theta_\varepsilon(u_n(x))(u_n(x) - v(x))\,dx \\
&\le \frac{1}{\lambda_n}\Bigg[\int_\Omega a(x, \nabla u_n)\cdot\nabla(v - u_n)\,dx + \int_\Omega g(x, u_n(x))(v(x) - u_n(x))\,dx \\
&\quad + \int_{\Gamma_3} \phi(x, v(x))\,d\Gamma - \int_{\Gamma_3} \phi(x, u_n(x))\,d\Gamma - \int_\Omega f(x)(v(x) - u_n(x))\,dx \\
&\quad + \int_{\Gamma_2} b(x)(v(x) - u_n(x))\,d\Gamma\Bigg] \\
&\le \frac{1}{\lambda_n} m_2(v),
\end{aligned}$$

for some $m_2(v) > 0$ which relies on $v$ but independent of $n$ (due to the boundedness of $\{u_n\}$). Letting $n \to \infty$ to the inequality above and applying Lebesgue dominated convergence theorem, it gives

$$\int_\Omega \theta_\varepsilon(u(x))(u(x) - v(x))\,dx = \lim_{n\to\infty} \int_\Omega \theta_\varepsilon(u_n(x))(u_n(x) - v(x))\,dx \le 0,$$

where we have used the compactness of the embedding from $V$ into $L^p(\Omega)$. The arbitrariness of $v \in V$ deduces that $\theta_\varepsilon(u) = 0$. This combined with Lemma 8 implies that $u \in K$.

Moreover, we shall prove that $u_n$ converges strongly to $u$ in $V$. Inserting $v = u$ into (3.8) and using the fact that $\theta_\varepsilon(u) = 0$, it yields

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n)\cdot\nabla(u_n - u)\,dx \\
&\le \int_\Omega \left(g(x, u_n(x)) + \lambda_n \theta_\varepsilon(u_n(x))\right)(u(x) - u_n(x))\,dx \\
&\quad + \int_{\Gamma_3} \phi(x, u(x))\,d\Gamma - \int_{\Gamma_3} \phi(x, u_n(x))\,d\Gamma \\
&\quad - \int_\Omega f(x)(u(x) - u_n(x))\,dx + \int_{\Gamma_2} b(x)(u(x) - u_n(x))\,d\Gamma \\
&\le \int_\Omega g(x, u_n(x))(u_n(x) - u(x))\,dx + \int_{\Gamma_3} \phi(x, u(x))\,d\Gamma - \int_{\Gamma_3} \phi(x, u_n(x))\,d\Gamma \\
&\quad - \int_\Omega f(x)(u(x) - u_n(x))\,dx + \int_{\Gamma_2} b(x)(u(x) - u_n(x))\,d\Gamma.
\end{aligned}$$

Passing to the upper limit as $n \to \infty$ to the inequality above and using the compactness of embeddings of $V$ to $L^\varsigma(\Omega)$ and of $V$ to $L^p(\Gamma_2)$, it gives

$$\begin{aligned}
&\limsup_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot\nabla(u_n-u)\,dx\\
&\quad\le \limsup_{n\to\infty}\int_\Omega g(x,u_n(x))(u(x)-u_n(x))\,dx\\
&\qquad+\int_{\Gamma_3}\phi(x,u(x))\,d\Gamma-\liminf_{n\to\infty}\int_{\Gamma_3}\phi(x,u_n(x))\,d\Gamma\\
&\qquad-\lim_{n\to\infty}\int_\Omega f(x)(u(x)-u_n(x))\,dx+\lim_{n\to\infty}\int_{\Gamma_2}b(x)(u(x)-u_n(x))\,d\Gamma\le 0,
\end{aligned}$$

where we have applied the weak lower semicontinuity of $\varphi$. The latter together with Lemma 6 implies that $u_n\to u$ in $V$ as $n\to\infty$.

Furthermore, we verify that $u$ is the unique solution of Problem 2. Let $w\in K$ be arbitrary. We insert $v=w$ into (3.8), use the fact $\theta_\varepsilon(w)=0$ and the monotonicity of $\theta_\varepsilon$ to confess that

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_n)\cdot\nabla(w-u_n)\,dx+\int_\Omega g(x,u_n(x))(w(x)-u_n(x))\,dx\\
&\qquad+\int_{\Gamma_3}\phi(x,w(x))\,d\Gamma-\int_{\Gamma_3}\phi(x,u_n(x))\,d\Gamma\\
&\qquad\ge\int_\Omega f(x)(w(x)-u_n(x))\,dx-\int_{\Gamma_2}b(x)(w(x)-u_n(x))\,d\Gamma.
\end{aligned}$$

Letting $\limsup_{n\to\infty}$ to the inequality above, it finds

$$\begin{aligned}
&\int_\Omega a(x,\nabla u)\cdot\nabla(w-u)\,dx+\int_\Omega g(x,u(x))(w(x)-u(x))\,dx\\
&\qquad+\int_{\Gamma_3}\phi(x,w(x))\,d\Gamma-\int_{\Gamma_3}\phi(x,u(x))\,d\Gamma\\
&\quad\ge\limsup_{n\to\infty}\left[\int_\Omega a(x,\nabla u_n)\cdot\nabla(w-u_n)\,dx+\int_\Omega g(x,u_n(x))(w(x)-u_n(x))\,dx\right.\\
&\qquad\left.+\int_{\Gamma_3}\phi(x,w(x))\,d\Gamma-\int_{\Gamma_3}\phi(x,u_n(x))\,d\Gamma\right]\\
&\quad\ge\limsup_{n\to\infty}\left[\int_\Omega f(x)(w(x)-u_n(x))\,dx-\int_{\Gamma_2}b(x)(w(x)-u_n(x))\,d\Gamma\right]\\
&\quad=\int_\Omega f(x)(w(x)-u(x))\,dx-\int_{\Gamma_2}b(x)(w(x)-u(x))\,d\Gamma.
\end{aligned}$$

Because $w\in K$ is arbitrary, so, we conclude that $u$ is the unique solution of Problem 2, namely, $u=u_0$.

Keeping in mind that $\{u_n\}$ is bounded and every convergent subsequence of $\{u_n\}$ converges to the same limit $u_0$. Therefore, we conclude that the whole sequence $\{u_n\}$ converges strongly to $u_0$ in $V$. □

Particularly, if $\Gamma_2=\Gamma_3=\emptyset$, i.e., $\Gamma_1=\Gamma$, then Problem 1 reduces the following nonlinear complementarity problem with Dirichlet boundary condition:

**Problem 11** *Find a pair functions $u\colon \Omega \to \mathbb{R}$ and $\mu\colon \Omega \to \mathbb{R}$ such that*

$$\begin{aligned}
&Z(x) = -\operatorname{div} a(x, \nabla u) + g(x, u) - f(x) + \mu(x) && \text{in } \Omega,\\
&\mu(x) \ge 0,\ u(x) - u^*(x) \le 0,\ \mu(x)\big(u(x) - u^*(x)\big) = 0 && \text{in } \Omega,\\
&Z(x) \ge 0,\ u_*(x) - u(x) \le 0,\ Z(x)\,(u_*(x) - u(x)) = 0 && \text{in } \Omega,\\
&u = 0 && \text{on } \Gamma.
\end{aligned}$$

Then, the corresponding weak variational formulation of Problem 11 is written by the following inequality:

**Problem 12** *Find function $u \in K$ such that*

$$\int_\Omega a(x, \nabla u) \cdot \nabla(v - u)\, dx + \int_\Omega g(x, u(x))(v(x) - u(x))\, dx \ge \int_\Omega f(x)(v(x) - u(x))\, dx$$

*for all $v \in K$.*

Likewise, we introduce the approximating problem of Problem 11 by using the power penalty method (see (3.6)) as follows:

**Problem 13** *Find function $u_\lambda \in V$ such that*

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_\lambda) \cdot \nabla(v - u_\lambda)\, dx + \int_\Omega (g(x, u_\lambda(x)) + \lambda\theta_\varepsilon(u_\lambda(x)))\,(v(x) - u_\lambda(x))\, dx\\
&\ge \int_\Omega f(x)(v(x) - u_\lambda(x))\, dx
\end{aligned}$$

*for all $v \in V$.*

So, we have the following corollary.

**Corollary 14** *Assume that $H(a)$ and $H(g)$ are satisfied. If, $f \in L^{p'}(\Omega)$, then, we have*

(i) *Problem 12 has a unique solution $u_0 \in K$.*
(ii) *For each $\lambda > 0$, Problem 13 has a unique solution $u_\lambda \in V$.*
(iii) *Let sequence $\{\lambda_n\} \subset (0, +\infty)$ be such that $\lambda_n \to \infty$ as $n \to \infty$. Then the unique solution of Problem 13 corresponding to $\lambda_n$ converges strongly to the unique solution $u_0$ of Problem 12 in $V$ as $n \to \infty$.*

***Remark 15*** Indeed, Problems 11, 12 and 13 have been studied by Duan-Wang-Zhou [8], recently, when function $a$ is assumed to be independent of $x$ and to satisfy the following stronger conditions:
$\underline{H(a')}$: $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is such that

(i) $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is continuous with $a(0) = 0$ and satisfies the inequality

$$(a(\xi) - a(\eta)) \cdot (\xi - \eta) \ge m_a|\xi - \eta|^p \tag{3.15}$$

for all $\xi, \eta \in \mathbb{R}^N$, where $m_a > 0$ and $2 \le N < p < +\infty$;
(ii) $|a(\xi)| \le \alpha_a|\xi|^{p-1}$ for all $\xi \in \mathbb{R}^N$ with some $\alpha_a > 0$.

It is obvious that our result, Corollary 14, extends Theorems 4.1 and 4.2 of Duan-Wang-Zhou [8].

## 4 Simultaneous distributed-boundary optimal control problems

The main contribution of this section is twofold. The first goal of this section is to investigate the existence of an optimal solution to a nonlinear simultaneous distributed-boundary optimal control problem, Problem 16, driven by the nonlinear complementarity system, Problem 1. More precisely, the control variable to optimal control problem is the vector $(f, b)$ in which $f$ is internal energy within $\Omega$ and $b$ is the boundary data on $\Gamma_2$. The second intention of this paper is to consider a perturbated optimal control problem governed by Problem 9, (see Problem 19) corresponding to Problem 16, and to explore the asymptotic behavior of optimal solutions (i.e., control-state pairs) and of minimal values for perturbated optimal control problem, Problem 19.

The simultaneous distributed and Neumann boundary optimal control problem is formulated by the following nonlinear optimization problem:

**Problem 16** *Find* $(f^*, b^*) \in \Lambda := L^{p'}(\Omega) \times L^{p'}(\Gamma_2)$ *such that*

$$L(S(f^*, b^*), f^*, b^*) = \inf_{(f,b)\in\Lambda} L(S(f, b), f, b), \tag{4.1}$$

*where* $L\colon V \times \Lambda \to \mathbb{R}$ *is a given cost functional and* $S\colon \Lambda \to K$ *is the solution mapping of Problem 2, i.e.,* $S(f, b) = u_{f,b}$ *here* $u_{f,b}$ *is the unique solution of Problem 2 corresponding to* $(f, b) \in \Lambda$.

In order to establish the existence of optimal solutions to optimal control problems, we suppose that the cost function $L\colon V \times \Lambda \to \mathbb{R}$ satisfies the following conditions:
$\underline{H(L)}$: $L\colon V \times \Lambda \to \mathbb{R}$ is bounded from below such that

(i) For every $(f, b) \in \Lambda$ the function $V \ni u \mapsto L(u, f, b) \in \mathbb{R}$ is continuous;
(ii) The inequality holds

$$\liminf_{n\to\infty} L(u_n, f_n, b_n) \geq L(u, f, b)$$

whether sequences $\{u_n\} \subset V$ and $\{(f_n, b_n)\} \subset \Lambda$ are such that $u_n \to u$ in $V$ and $(f_n, b_n) \xrightarrow{w} (f, b)$ in $\Lambda$ as $n \to \infty$ for some $u \in V$ and $(f, b) \in \Lambda$;
(iii) There exists a coercive function $\rho\colon \Lambda \to \mathbb{R}$, i.e., $\rho(f, b) \to +\infty$ as $\|f\|_{p',\Omega} + \|b\|_{p',\Gamma_2} \to +\infty$, satisfying

$$L(u, f, b) \geq \rho(f, b) \ \text{ for all } u \in V \text{ and } (f, b) \in \Lambda.$$

***Example 17*** The following function as a concrete example for cost functional satisfies hypotheses $H(L)$

$$L(u, f, b) = \frac{1}{p}\|\nabla u_n - z\|_{p,\Omega}^p + \frac{1}{p'}\|f\|_{p',\Omega}^{p'} + \frac{1}{p'}\|b\|_{p',\Gamma}^{p'} \ \text{ for all } u \in V \text{ and } (f, b) \in \Lambda,$$

where $z \in L^p(\Omega; \mathbb{R}^N)$ is the known observed or measured datum.

The existence theorem to Problem 16 is provided as follows.

**Theorem 18** *Suppose that* $H(a)$, $H(g)$, $H(\phi)$ *and* $H(L)$ *are fulfilled. Then, Problem 16 admits an optimal solution.*

***Proof*** It follows from hypotheses H($L$) that $L$ is bounded from below, i.e., there exists a constant $m_L \in \mathbb{R}$ such that

$$L(u, f, b) \geq m_L \ \text{ for all } u \in V \text{ and} (f, b) \in \Lambda.$$

Let sequence $\{(f_n, b_n)\} \subset \Lambda$ be a minimizing sequence of problem (4.1), namely,

$$\delta_0 := \inf_{(f,b)\in\Lambda} L(S(f,b), f, b) = \lim_{n\to\infty} L(u_n, f_n, b_n), \tag{4.2}$$

where $u_n := S(f_n, b_n)$, thus,

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n) \cdot \nabla(v - u_n)\,dx + \int_\Omega g(x, u_n(x))(v(x) - u_n(x))\,dx \\
&+ \int_{\Gamma_3} \phi(x, v(x))\,d\Gamma - \int_{\Gamma_3} \phi(x, u_n(x))\,d\Gamma \\
&\geq \int_\Omega f_n(x)(v(x) - u_n(x))\,dx - \int_{\Gamma_2} b_n(x)(v(x) - u_n(x))\,d\Gamma
\end{aligned} \tag{4.3}$$

for all $v \in K$.

By virtue of H($L$)(iii), we have

$$\delta_0 \geq \liminf_{n\to\infty} \rho(f_n, b_n).$$

This indicates that sequence $\{(f_n, b_n)\}$ is bounded in $\Lambda$. Without any loss of generality, we are able to find $(f^*, b^*) \in \Lambda$ such that

$$(f_n, b_n) \xrightarrow{w} (f^*, b^*) \text{ in } \Lambda \text{ as } n \to \infty. \tag{4.4}$$

Arguing as in the proof of Theorem 10(ii) (see (3.13)), we have

$$\begin{aligned}
&\|f_n\|_{p',\Omega}\|u_n\|_{p,\Omega} + \|b_n\|_{p',\Gamma_2}\|u_n\|_{p,\Gamma} + \int_{\Gamma_3} \phi(x, 0)\,d\Gamma + \beta_\varphi \\
&\geq \begin{cases} \frac{a_3}{p-1}\|\nabla u_n\|^p_{p,\Omega} + a_g\|u_n\|^p_{p,\Omega} - \|b_g\|_{1,\Omega} - \alpha_\varphi\|u_n\|_V & \text{if } \varsigma = p, \\ \frac{a_3}{p-1}\|\nabla u_n\|^p_{p,\Omega} + \|u_n\|^p_{p,\Omega} - \|b_g\|_{1,\Omega} - m_3 - \alpha_\varphi\|u_n\|_V & \text{if } \varsigma > p, \end{cases}
\end{aligned}$$

with some $m_3 > 0$. This, obviously, reveals that the sequence $\{u_n\}$ is bounded in $V$. Passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} \widetilde{u} \text{ in } V \text{ as } n \to \infty$$

for some $\widetilde{u} \in K$ (due to the closedness and convexity of $K$). Putting $v = \widetilde{u}$ into (4.3) and passing to the upper limit as $n \to \infty$, it gives

$$\limsup_{n\to\infty} \int_\Omega a(x, \nabla u_n) \cdot \nabla(u_n - \widetilde{u})\,dx \leq 0,$$

where we have used the compactness of the embeddings of $V$ to $L^\varsigma(\Omega)$ and of $V$ to $L^p(\Gamma_2)$. Using the inequality above and Lemma 6, we conclude that $u_n \to \widetilde{u}$ in $V$. Passing to the upper limit as $n \to \infty$ for (4.3), it deduces that $\widetilde{u} = S(f^*, b^*)$.

From (4.2), we use H($L$)(ii) to find

$$\begin{aligned}
\delta_0 = \inf_{(f,b)\in\Lambda} L(S(f,b), f, b) &= \lim_{n\to\infty} L(u_n, f_n, b_n) \\
&= \liminf_{n\to\infty} L(u_n, f_n, b_n) \\
&\geq L(\widetilde{u}, f^*, b^*) = L(S(f^*, b^*), f^*, b^*) \\
&\geq \inf_{(f,b)\in\Lambda} L(S(f,b), f, b).
\end{aligned}$$

This means that $(f^*, b^*) \in \Lambda$ is a solution of Problem 16. $\square$

For any $\lambda > 0$, let us move our attention to consider the following perturbated optimal control problem:

**Problem 19** *Find* $(f_\lambda^*, b_\lambda^*) \in \Lambda := L^{p'}(\Omega) \times L^{p'}(\Gamma_2)$ *such that*

$$L(S_\lambda(f_\lambda^*, b_\lambda^*), f_\lambda^*, b_\lambda^*) = \inf_{(f,b)\in\Lambda} L(S_\lambda(f,b), f, b), \tag{4.5}$$

*where* $S_\lambda \colon \Lambda \to K$ *is the solution mapping of Problem 9, i.e.,* $S_\lambda(f,b) = u_{f,b}$ *here* $u_{f,b}$ *is the unique solution of Problem 9 corresponding to* $(f,b) \in \Lambda$ *and* $\lambda > 0$.

The second contribution of this paper is to provide the following theorem which contains the existence of solutions to Problem 19, and the asymptotic behavior of optimal controls, system states as well as minimal values to Problem 19.

**Theorem 20** *Assume that* $H(a)$, $H(g)$, $H(\phi)$ *and* $H(L)$ *are satisfied. Then, we have*

(i) *For each* $\lambda > 0$, *Problem 19 has at least one solution.*
(ii) *Let sequence* $\{\lambda_n\}$ *be such that* $\lambda_n \to \infty$ *as* $n \to \infty$. *For any solution sequence of state-control* $\{(u_n, f_n, b_n)\}$ *of Problem 19 with* $u_n = S_{\lambda_n}(f_n, b_n)$, *there exists a subsequence of* $\{(u_n, f_n, b_n)\}$, *still denoted by the same way, such that*

$$\begin{cases} S_{\lambda_n}(f_n, b_n) = u_n \to u = S(f,b) \ \text{in}\ V, \\ (f_n, b_n) \xrightarrow{w} (f,b) \ \text{in}\ \Lambda & \text{as}\ n \to \infty, \\ L(u_n, f_n, b_n) \to L(u, f, b), \end{cases} \tag{4.6}$$

*where* $(f,b) \in \Lambda$ *is a solution of Problem 16.*

***Proof*** (i) The assertion is a consequence of Theorem 18.
(ii) Let sequence $\{\lambda_n\}$ be such that $\lambda_n \to \infty$ as $n \to \infty$, and let $\{(u_n, f_n, b_n)\}$ be a solution sequence of state-control of Problem 19 with $u_n = S_{\lambda_n}(f_n, b_n)$ and $\lambda = \lambda_n$. Then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} &\int_\Omega a(x, \nabla u_n) \cdot \nabla(v - u_n)\,dx + \int_\Omega (g(x, u_n(x)) + \lambda_n \theta_\varepsilon(u_n(x)))\,(v(x) - u_n(x))\,dx \\ &\quad + \int_{\Gamma_3} \phi(x, v(x))\,d\Gamma - \int_{\Gamma_3} \phi(x, u_n(x))\,d\Gamma \\ &\quad \ge \int_\Omega f_n(x)(v(x) - u_n(x))\,dx - \int_{\Gamma_2} b_n(x)(v(x) - u_n(x))\,d\Gamma \end{aligned} \tag{4.7}$$

for all $v \in V$. Let $(f^*, g^*) \in \Lambda$ be a solution of Problem 16. Hence, one has

$$L(S_{\lambda_n}(f_n, b_n), f_n, b_n) \le L(S_{\lambda_n}(f^*, b^*), f^*, b^*). \tag{4.8}$$

It follows from Theorem 10(ii) that $S_{\lambda_n}(f^*, b^*) \to S(f^*, b^*)$ in $V$ as $n \to \infty$. Employing the continuity of $u \mapsto L(u, f, b)$, we get

$$\lim_{n\to\infty} L(S_{\lambda_n}(f^*, b^*), f^*, b^*) = L(S(f^*, b^*), f^*, b^*). \tag{4.9}$$

Taking into account (4.8), (4.9) and hypothesis H($L$)(iii), it yields

$$\begin{aligned} \liminf_{n\to\infty} \rho(f_n, b_n) &\le \lim_{n\to\infty} L(S_{\lambda_n}(f_n, b_n), f_n, b_n) \\ &\le \lim_{n\to\infty} L(S_{\lambda_n}(f^*, b^*), f^*, b^*) \\ &= L(S(f^*, b^*), f^*, b^*). \end{aligned}$$

Therefore, we can observe that sequence $\{(f_n, b_n)\}$ is bounded in $\Lambda$. Without any loss of generality, we may assume that

$$(f_n, b_n) \xrightarrow{w} (\widetilde{f}, \widetilde{b}) \text{ in } \Lambda \text{ as} n \to \infty \tag{4.10}$$

for some $(\widetilde{f}, \widetilde{b}) \in \Lambda$.

Inserting $v = 0$ into the inequality (4.7) and using the fact that $\theta_\varepsilon(0) = 0$, so, a simple calculating gives

$$\begin{aligned}
&\|f_n\|_{p',\Omega}\|u_n\|_{p,\Omega} + \|b_n\|_{p',\Gamma_2}\|u_n\|_{p,\Gamma} + \int_{\Gamma_3} \phi(x, 0)\, d\Gamma + \beta_\varphi \\
&\geq \begin{cases} \frac{a_3}{p-1}\|\nabla u_n\|_{p,\Omega}^p + a_g\|u_n\|_{p,\Omega}^p - \|b_g\|_{1,\Omega} - \alpha_\varphi\|u_n\|_V & \text{if } \varsigma = p, \\ \frac{a_3}{p-1}\|\nabla u_n\|_{p,\Omega}^p + \|u_n\|_{p,\Omega}^p - \|b_g\|_{1,\Omega} - m_4 - \alpha_\varphi\|u_n\|_V & \text{if } \varsigma > p, \end{cases}
\end{aligned}$$

for some $m_4 > 0$. This turns out that $\{u_n\}$ is bounded in $V$. Passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} \widetilde{u} \text{ in } V \text{ as } n \to \infty$$

for some $\widetilde{u} \in V$. From (4.7), we have

$$\int_\Omega \theta_\varepsilon(\widetilde{u}(x))(v(x) - \widetilde{u}(x))\, dx = \lim_{n\to\infty} \int_\Omega \theta_\varepsilon(u_n(x))(v(x) - u_n(x))\, dx \leq \frac{1}{\lambda_n} m_5(v) = 0$$

for some $m_5(v) > 0$ which relies on $v$ but independent of $n$ (owing to the boundedness of $\{u_n\}$). So, it is true that $\theta_\varepsilon(\widetilde{u}) = 0$, i.e., $\widetilde{u} \in K$ (see Lemma 8).

Putting $v = \widetilde{u} \in K$ into (4.7) and passing to the upper limit as $n \to \infty$, we infer

$$\limsup_{n\to\infty} \int_\Omega a(x, \nabla u_n) \cdot \nabla(u_n - \widetilde{u})\, dx \leq 0.$$

Whereas, applying Lemma 6, it concludes that $u_n \to \widetilde{u}$ in $V$ as $n \to \infty$. For any $w \in K$, we insert $v = w$ into (4.7), use the fact $\theta_\varepsilon(w) = 0$, and pass to the upper limit as $n \to \infty$ for the resulting inequality that

$$\begin{aligned}
&\int_\Omega a(x, \nabla\widetilde{u}) \cdot \nabla(w - \widetilde{u})\, dx + \int_\Omega g(x, \widetilde{u}(x))(w(x) - \widetilde{u}(x))\, dx \\
&+ \int_{\Gamma_3} \phi(x, w(x))\, d\Gamma - \int_{\Gamma_3} \phi(x, \widetilde{u}(x))\, d\Gamma \\
\geq &\int_\Omega \widetilde{f}(x)(w(x) - \widetilde{u}(x))\, dx - \int_{\Gamma_2} \widetilde{b}(x)(w(x) - \widetilde{u}(x))\, d\Gamma
\end{aligned}$$

for all $w \in K$. This means that $\widetilde{u} = S(\widetilde{f}, \widetilde{b})$.

Passing to the lower limit as $n \to \infty$ for inequality (4.8) and using (4.9), we have

$$\begin{aligned}
L(S(f^*, b^*), f^*, b^*) &\geq \lim_{n\to\infty} L(S_{\lambda_n}(f^*, b^*), f^*, b^*) \\
&\geq \liminf_{n\to\infty} L(S_{\lambda_n}(f_n, b_n), f_n, b_n) \\
&\geq L(S(\widetilde{f}, \widetilde{b}), \widetilde{f}, \widetilde{b}) \\
&\geq \inf_{(f,b)\in\Lambda} L(S(f, b), f, b).
\end{aligned}$$

Recall that $(f^*, b^*) \in \Lambda$ is a solution of Problem 16, so, we conclude that $(\widetilde{f}, \widetilde{b}) \in \Lambda$ is also a solution of Problem 16. Consequently, we can observe that

$$\begin{cases} S_{\lambda_n}(f_n, b_n) = u_n \to u = S(\widetilde{f}, \widetilde{b}) \text{ in } V, \\ (f_n, b_n) \xrightarrow{w} (\widetilde{f}, \widetilde{b}) \text{ in } \Lambda \\ L(u_n, f_n, b_n) \to L(\widetilde{u}, \widetilde{f}, \widetilde{b}), \end{cases} \quad \text{as } n \to \infty.$$

This completes the proof of the theorem. □

**Acknowledgements** The authors wish to thank the two knowledgeable referees for their remarks in order to improve the paper. This project has received funding from the NNSF of China Grant Nos. 12001478, 12026255 and 12026256, and the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 823731 CONMECH, National Science Center of Poland under Preludium Project No. 2017/25/N/ST1 /00611, and the Startup Project of Doctor Scientific Research of Yulin Normal University No. G2020ZK07. It is also supported by Natural Science Foundation of Guangxi Grant Nos. 2021GXNSFFA196004 and 2020GXNSFBA297137, and the Ministry of Science and Higher Education of Republic of Poland under Grants Nos. 4004/GGPJII/H2020/2018/0 and 440328/PnH2/2019.

## References

1. Bai, Y.R., Migórski, S., Zeng, S.D.: Generalized vector complementarity problem in fuzzy environment. Fuzzy Sets Syst. **347**, 142–151 (2018)
2. Boukrouche, M., Tarzia, D.A.: Convergence of distributed optimal control problems governed by elliptic variational inequalities. Comput. Optim. Appl. **53**, 375–393 (2012)
3. Boukrouche, M., Tarzia, D.A.: Existence, uniqueness, and convergence of optimal control problems associated with parabolic variational inequalities of the second kind. Nonlinear Anal. **12**, 2211–2224 (2011)
4. Boukrouche, M., Tarzia, D.A.: Convergence of optimal control problems governed by second kind parabolic variational inequalities. J. Control Theory Appl. **11**, 422–427 (2013)
5. Brézis, H.: Functional Analysis. Sobolev Spaces and Partial Differential Equations, Springer, New York (2011)
6. Denkowski, Z., Migórski, S.: Control problems for parabolic and hyperbolic equations via the theory of $G$- and $\Gamma$-convergence. Ann. Mat. Pura Appl. **149**, 23–39 (1987)
7. Denkowski, Z., Migórski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Applications. Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York (2003)
8. Duan, Y., Wang, S., Zhou, Y.Y.: A power penalty approach to a mixed quasilinear elliptic complementarity problem. J. Global Optim. **81**, 901–918 (2021)
9. Feng, F., Han, W., Huang, J.G.: The virtual element method for an obstacle problem of a Kirchhoff-Love plate. Commun. Nonlinear Sci. Numer. Simulat. **103**, 106008 (2021)
10. Gasiński, L., Migórski, S., Ochal, A.: Existence results for evolutionary inclusions and variational-hemivariational inequalities. Appl. Anal. **94**, 1670–1694 (2015)
11. Huang, N.J., Fang, Y.P.: On vector variational inequalities in reflexive Banach spaces. J. Global Optim. **32**, 495–505 (2005)
12. Huang, N.J., Yang, X.Q., Chan, W.K.: Vector complementarity problems with a variable ordering relation. Eur. J. Oper. Res. **176**, 15–26 (2007)
13. Konnov, I.V.: An extension of the Jacobi algorithm for multi-valued mixed complementarity problems. Optimization **56**, 399–416 (2007)
14. Konnov, I.V., Dyabilkin, D.A.: Nonmonotone equilibrium problems: coercivity conditions and weak regularization. J. Global Optim. **49**, 575–587 (2011)
15. Li, X.W., Liu, Z.H.: Sensitivity analysis of optimal control problems described by differential hemivariational inequalities. SIAM J. Control. Optim. **56**, 3569–3597 (2018)
16. Liu, J.J., Yang, X.M., Zeng, S.D.: Optimal control and approximation for elliptic bilateral obstacle problems. Commun. Nonlinear Sci. Numer. Simulat. **102**, 105938 (2021)
17. Liu, Y.J., Migórski, S., Nguyen, V.T., Zeng, S.D.: Existence and convergence results for an elastic frictional contact problem with nonmonotone subdifferential boundary conditions. Acta Math. Sci. **41**, 1151–1168 (2021)
18. Liu, Z.H., Li, X.W., Motreanu, D.: Approximate controllability for nonlinear evolution hemivariational inequalities in Hilbert spaces. SIAM J. Control. Optim. **53**, 3228–3244 (2015)
19. Liu, Z.H., Migórski, S., Zeng, S.D.: Partial differential variational inequalities involving nonlocal boundary conditions in Banach spaces. J. Differ. Eq. **263**, 3989–4006 (2017)
20. Liu, Z.H., Motreanu, D., Zeng, S.D.: Positive solutions for nonlinear singular elliptic equations of $p$-Laplacian type with dependence on the gradient. Calc. Var. Partial Differ. Eq. **58**, 22 (2019)

21. Mermri, E.B., Han, W.: Numerical approximation of a unilateral obstacle problem. J. Optim. Theory Appl. **153**, 177–194 (2012)
22. Migórski, S.: Sensitivity analysis of distributed-parameter optimal control problems for nonlinear parabolic equations. J. Optim. Theory Appl. **87**, 595–613 (1995)
23. Migórski, S., Khan, A.A., Zeng, S.D.: Inverse problems for nonlinear quasi-variational inequalities with an application to implicit obstacle problems of $p$-Laplacian type. Inverse Probl. **35**, 035004 (2019)
24. Migórski, S., Khan, A.A., Zeng, S.D.: Inverse problems for nonlinear quasi-hemivariational inequalities with application to mixed boundary value problems. Inverse Probl. **36**, 024006 (2020)
25. Migórski, S., Ochal, A.: Optimal control of parabolic hemivariational inequalities. J. Global Optim. **17**, 285–300 (2000)
26. Migórski, S., Ochal, A., Sofonea, M.: Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics. Springer, New York (2013)
27. Papageorgiou, N.S.: On parametric evolution inclusions of the subdifferential type with applications to optimal control problems. T. Am. Math. Soc. **347**, 203–231 (1995)
28. Papageorgiou, N.S.: Properties of the relaxed trajectories of evolution equations and optimal control. SIAM J. Control. Optim. **27**, 267–288 (1989)
29. Papageorgiou, N.S.: Optimal programs and their price characterization in a multisector growth model with uncertainty. P. Am. Math. Soc. **122**, 227–240 (1994)
30. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Sensitivity analysis for optimal control problems governed by nonlinear evolution inclusions. Adv. Nonlinear Anal. **6**, 199–235 (2017)
31. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Relaxation methods for optimal control problems. B. Math. Sci. **10**, 2050004 (2020)
32. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Sensitivity analysis for optimal control problems governed by nonlinear evolution inclusions. Adv. Nonlienar Anal. **6**, 199–235 (2017)
33. Peng, Y., Liu, Z.H.: A derivative-free filter algorithm for nonlinear complementarity problem. Appl. Math. Comput. **182**, 846–853 (2006)
34. Peng, Z.J.: Optimal obstacle control problems involving nonsmooth cost functionals and quasilinear variational inequalities. SIAM J. Control. Optim. **58**, 2236–2255 (2020)
35. Ramadurai, G., Ukkusuri, S.V., Zhao, J., Pang, J.S.: Linear complementarity formulation for single bottleneck model with heterogeneous commuters. Transport. Res. Part B-Meth. **44**, 193–214 (2010)
36. Schaible, S., Yao, J.C.: On the equivalence of nonlinear complementarity problems and least-element problems. Math. Program. **70**, 191–200 (1995)
37. Tang, G.J., Huang, N.J.: Korpelevich's method for variational inequality problems on Hadamard manifolds. J. Global Optim. **54**, 493–509 (2012)
38. Wang, F., Han, W., Cheng, X.L.: Discontinuous Galerkin methods for solving elliptic variational inequalities. SIAM J. Numer. Anal. **48**, 708–733 (2010)
39. Wang, S., Zhang, K.: An interior penalty method for a finite-dimensional linear complementarity problem in financial engineering. Optim. Lett. **12**, 1161–1178 (2018)
40. Zeng, L.C., Yao, J.C.: Existence of solutions of generalized vector variational inequalities in reflexive Banach spaces. J. Global Optim. **36**, 483–497 (2006)
41. Zeng, S.D., Liu, Z.H., Migórski, S.: Positive solutions to nonlinear nonhomogeneous inclusion problems with dependence on the gradient. J. Math. Appl. Anal. **463**, 432–448 (2018)
42. Zeng, S.D., Bai, Y.R., Gasiński, L., Winkert, P.: Convergence analysis for double phase obstacle problems with multivalued convection term. Adv. Nonlinear Anal. **10**, 659–672 (2021)
43. Zeng, S.D., Bai, Y.R., Gasiński, L., Winkert, P.: Existence results for double phase implicit obstacle problems involving multivalued operators. Calc. Var. Partial Differ. Eq. **59**, 18 (2020)
44. Zeng, S.D., Migórski, S., Liu, Z.H.: Well-posedness, optimal control, and sensitivity analysis for a class of differential variational-hemivariational inequalities. SIAM J. Optim. **31**, 2829–2862 (2021)
45. Zeng, S.D., Migórski, S., Khan, A.A.: Nonlinear quasi-hemivariational inequalities: existence and optimal control. SIAM J. Control. Optim. **59**, 1246–1274 (2021)
46. Zeng, S.D., Rădulescu, V.D., Winkert, P.: Double phase implicit obstacle problems with convection and multivalued mixed boundary value conditions. SIAM J. Math. Anal. accepted (2021)
47. Zhou, Y.Y., Wang, S., Yang, X.Q.: A penalty approximation method for a semilinear parabolic double obstacle problem. J. Global Optim. **60**, 531–550 (2014)