Regrets of Proximal Method of Multipliers for Online Non-convex Optimization with Long Term Constraints

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Abstract

The online optimization problem with non-convex loss functions over a closed convex set, coupled with a set of inequality (possibly nonconvex) constraints is a challenging online learning problem. A proximal method of multipliers with quadratic approximations (named as OPMM) is presented to solve this online non-convex optimization with long term constraints. Regrets of the violation of Karush-Kuhn-Tucker conditions of OPMM for solving online non-convex optimization problems are analyzed. Under mild conditions, it is shown that this algorithm exhibits $\mathcal{O}(T^{-1/8})$ Lagrangian gradient violation regret, $\mathcal{O}(T^{-1/8})$ constraint violation regret and $\mathcal{O}(T^{-1/4})$ complementarity residual regret if parameters in the algorithm are properly chosen, where Tdenotes the number of time periods. For the case that the objective is a convex quadratic function, we demonstrate that the regret of the objective reduction can be established even the feasible set is non-convex. For the case when the constraint functions are convex, if the solution of the subproblem in OPMM is obtained by solving its dual, OPMM is proved to be an implementable projection method for solving the online non-convex optimization problem.

Keywords: Online Non-convex Optimization, Proximal Method of Multipliers with Quadratic Approximations, Lagrangian Gradient Violation Regret, Constraint Violation Regret, Complementarity Residual Regret

1 Introduction

In recent years, a number of efficient algorithms have been developed for online optimization. Convexity of the loss functions and the constraint sets has played a central role in the development of many of these algorithms. In this paper, we consider a more general setting, where the sequence of loss functions encountered by the learner could be non-convex and the constraint set is defined by a set of (possibly non-convex) inequalities. Such a setting has various applications in machine learning [1–3], especially in adversarial training [4] and training of Generative Adversarial Networks (GANs) [5].

Most of the existing works about online optimization have focused on convex loss functions. A number of computationally efficient approaches have been proposed for regret minimization in this setting. Among them the famous ones include Follow-the-leader [6], Follow-the-Regularized-Leader [7, 8], Exponentiated Online Gradient [9], Online Mirror Descent, Perceptron [10] and Winnow [11]. There are also a lot of publications concerning algorithms for online convex optimization, see [12, Chapter 7], [13, Chapter 21], and survey papers [14, 15] and references cited in these two papers.

However, when the loss functions are non-convex or the constraint sets are non-convex, minimizing the regret is computationally prohibitive. In the last years, there have been several papers about learning with non-convex losses over simple convex constraint sets. A few heuristic algorithms have been proposed in [16, 17] without establishing the regret bounds. In [18], the regret of online projection gradient method for a restricted class of loss functions is analyzed. The notion of local regret and the regret of online gradient method for a class of continuously differentiable non-convex loss functions are presented in [19]. DC (difference of convex functions) programming and DCA method for online learning problems with non-convex loss functions are investigated in [20]. In [21], a recursive exponential weighted algorithm that attains a regret of $\mathcal{O}(T^{-1/2})$ for non-convex Lipschitz continuous loss functions is proposed. It is shown in [22] that the classical Follow-the-Perturbed-Leader (FTPL) algorithm achieves $\mathcal{O}(T^{-1/3})$ regret for general non-convex losses which are Lipschitz continuous. Moreover, in [23], it is proved that FTPL achieves optimal regret rate $\mathcal{O}(T^{-1/2})$ for the problem of online learning with non-convex losses. An online cubic-regularized Newton method for non-convex online optimization is studied in [24].

In this paper, we consider a more complicated non-convex online optimization problem, which has a complex constraint set defined by

$$\Phi = \{ x \in \mathcal{C} : g_i(x) \le 0, \ i = 1, \dots, p \}.$$
(1)

Here, $\mathcal{C} \subset \mathbb{R}^n$ is a nonempty convex compact set with diameter $D_0 := \sup_{x,x' \in \mathcal{C}} ||x - x'||$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$ are continuous (possibly non-convex) functions.

In order to alleviate the computational challenge of the projection $\Pi_{\Phi}(\cdot)$ with Φ defined by (1), in [25] the authors considered to relax the constraints $g_i(x) \leq 0$ to be long term constraints. That is, the decision x^t is not required to satisfy $g_i(x_t) \leq 0$ at each round, but only require that $\sum_{t=1}^{T} g_i(x^t) \leq 0$. There are some recent works related to online convex optimization with long term constraints. In [25], a gradient based algorithm is designed to achieve $\mathcal{O}(T^{-1/2})$ regret bound and $\mathcal{O}(T^{-1/4})$ violation of constraints for an online optimization problem whose constraint set is defined by a set of inequalities of smooth convex functions. In [26, 27] new algorithms are developed to improve the performance in comparison with [25]. However, for non-convex online optimization problems with non-convex loss functions and constraint sets of the form (1), the research has been very limited until recently.

At round t, we consider the following non-convex optimization problem

$$\min_{\substack{x \in \mathcal{C} \\ \text{s.t.} }} f_t(x)$$
(2)

where $g(x) := (g_1(x), \ldots, g_p(x))^T$. Since Problem (2) is non-convex, it is unrealistic to analyze the regrets in both objective reduction and constraint violation. Just like the offline non-convex optimization, it is natural to consider the Karush-Kuhn-Tucker (KKT) conditions which are given by

$$0 \in \nabla_x L^t(x, \lambda) + N_{\mathcal{C}}(x),$$

$$0 \ge g(x) \perp \lambda \ge 0,$$
(3)

where $L^t(x,\lambda) = f_t(x) + \lambda^T g(x)$ and $N_{\mathcal{C}}(x)$ is the normal cone of \mathcal{C} at x. Conditions (3) are equivalent to the following equalities:

dist
$$\left(0, \nabla_x L^t(x, \lambda) + N_{\mathcal{C}}(x)\right) = 0,$$

 $\lambda - [\lambda + g(x)]_+ = 0,$
(4)

where $[\cdot]_+ := \max\{0, \cdot\}$. Therefore, it is reasonable to consider the regret of violation for the equalities in (4).

In this paper, we extend the proximal method of multipliers, a classical algorithm proposed in [28] to solve convex programming, for online nonconvex optimization problem, and analyze its regret bounds for KKT violation consisting of Lagrangian residual, constraint violation and complementarity residual. Let $q_i^t(x)$, $i = 0, 1, \ldots, p$ be the quadratic approximations of f_t and

 $g_i, i = 1, \ldots, p$ at x^t , respectively, defined by

$$\begin{aligned} q_0^t(x) &:= f_t(x^t) + \langle \nabla f_t(x^t), x - x^t \rangle + \frac{1}{2} \langle \Theta_0^t(x - x^t), x - x^t \rangle , \\ q_i^t(x) &:= g_i(x^t) + \langle \nabla g_i(x^t), x - x^t \rangle + \frac{1}{2} \langle \Theta_i^t(x - x^t), x - x^t \rangle , \ i = 1, \dots, p, \end{aligned}$$

where $\Theta_0^t \in \mathbb{S}^n$ and $\Theta_i^t \in \mathbb{S}^n$ are properly selected symmetric $n \times n$ matrices. The corresponding augmented Lagrangian function is defined by

$$\mathcal{L}_{\sigma}^{t}(x,\lambda) := q_{0}^{t}(x) + \frac{1}{2\sigma} \left[\sum_{i=1}^{p} [\lambda_{i} + \sigma q_{i}^{t}(x)]_{+}^{2} - \|\lambda\|^{2} \right]$$
(5)

for $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^p$ and $\sigma > 0$. At each round t, we let x^{t+1} be the optimal solution of the following problem

$$\min_{x \in \mathcal{C}} \mathcal{L}_{\sigma}^{t}(x, \lambda^{t}) + \frac{\alpha}{2} \|x - x^{t}\|^{2},$$

and update the multipliers by $\lambda_i^{t+1} = [\lambda_i^t + \sigma q_i^t(x^{t+1})]_+, i = 1, \ldots, p$, where $\alpha > 0$ is some parameter. Let $q^t(x) := (q_1^t(x), \ldots, q_p^t(x))^T$, then $\lambda^{t+1} = [\lambda^t + \sigma q^t(x^{t+1})]_+$. In detail, the online proximal method of multipliers (OPMM) with quadratic approximations for the non-convex online optimization problem with constraint set (1) can be described in Algorithm 1.

Algorithm 1: An online proximal method of multipliers (OPMM) with quadratic approximations.

Input: $\lambda^1 = 0, x^1 \in \mathcal{C}, \sigma > 0$ and $\alpha > 0$, receive a cost function $f_1(\cdot)$.

1 for $t \leftarrow 1$ to T do

2 Choose $\Theta_0^t \in \mathbb{S}^n$ and $\Theta_i^t \in \mathbb{S}^n$, i = 1, ..., p such that $q_0^t(\cdot)$ and $q_i^t(\cdot)$ are proper quadratic approximations of $f_t(\cdot)$ and $g_i(\cdot)$ at x^t , respectively.

3 Compute

$$x^{t+1} = \operatorname*{arg\,min}_{x \in \mathcal{C}} \left\{ \mathcal{L}^t_{\sigma}(x, \lambda^t) + \frac{\alpha}{2} \|x - x^t\|^2 \right\}.$$
(6)

4 Update

$$\lambda_i^{t+1} = [\lambda_i^t + \sigma q_i^t(x^{t+1})]_+, \ i = 1, \dots, p.$$

5 Receive a cost function $f_{t+1}(\cdot)$.

The main results of this paper can be summarized as follows.

• When we choose $\sigma = T^{-1/4}$ and $\alpha = T^{1/4}$, under mild assumptions, there exists $w^{t+1} \in N_{\mathcal{C}}(x^{t+1})$ for any $t = 1, \ldots, T$ such that the regret of Lagrangian residual is

$$\left\|\frac{1}{T}\sum_{t=1}^{T} \left[\nabla f_{t+1}(x^{t+1}) + \sum_{i=1}^{p} \lambda_i^{t+1} \nabla g_i(x^{t+1}) + w^{t+1}\right]\right\| \le \mathcal{O}(T^{-1/8}),$$

the regret of constraint violation is

$$\frac{1}{T}\sum_{t=1}^{T}g_i(x^t) \le \mathcal{O}(T^{-1/8}), \quad i = 1, \dots, p_i$$

and the regret of complementarity residual is

$$\frac{1}{T} \sum_{t=1}^{T} \|\lambda^{t+1} - [\lambda^{t+1} + \sigma g(x^{t+1})]_+\| \le \mathcal{O}(T^{-1/4}).$$

• For the case that the objective function f_t is convex quadratic, if $\sigma = T^{-1/2}$ and $\alpha = T^{1/2}$, the regret of objective reduction is

$$\frac{1}{T}\sum_{t=1}^{T} f_t(x^t) - \inf_{z \in \Phi} \frac{1}{T}\sum_{t=1}^{T} f_t(z) \le \mathcal{O}(T^{-1/2}).$$

• For the case that g_1, \ldots, g_p are convex functions, if the solution of the subproblem in OPMM is obtained by solving the dual of the subproblem, OPMM can be reformulated as an implementable projection method.

The remaining parts of this paper are organized as follows. In Section 2, we develop properties of OPMM, which play a key role in the regret analysis of OPMM. In Section 3, we establish regret bounds of Lagrangian residual, constraint violation and complementarity residual of OPMM for Problem (2). In Section 4, for the convex constraint set, OPMM is explained as an implementable projection method for solving the online optimization problem with long term constraints. We draw a conclusion in Section 5.

2 Auxiliary Properties of OPMM

In this section, we focus on establishing a variety of auxiliary properties of OPMM under some reasonable assumptions. We begin by introducing two classes of assumptions, in which the first class is about the structure of Problem (2) and the second class is to ensure that the quadratic approximations $q_i^t(x)$, $i = 0, 1, \ldots, p$ are well-defined.

Assumption A1 There exist constants $\kappa_f > 0$, $\kappa_g > 0$ and $\nu_g > 0$ such that for all $x, x' \in C$ and i = 1, ..., p,

 $|f_t(x) - f_t(x')| \le \kappa_f ||x - x'||, \ |g_i(x) - g_i(x')| \le \kappa_g ||x - x'||,$

and $||g(x)|| \leq \nu_g$.

Assumption A2 The functions f_t and g_i , i = 1, ..., p are continuously differentiable over C. There exist constants $L_f > 0$ and $L_g > 0$ such that for all $x, x' \in C$ and i = 1, ..., p,

 $\|\nabla f_t(x) - \nabla f_t(x')\| \le L_f \|x - x'\|, \ \|\nabla g_i(x) - \nabla g_i(x')\| \le L_g \|x - x'\|.$

Assumption A3 The Slater condition holds, that is, there exist a constant $\varepsilon_0 > 0$ and a vector $\hat{x} \in C$ such that

$$g_i(\widehat{x}) \leq -\varepsilon_0, \ i = 1, \dots, p_i$$

Note that the set \mathcal{C} is bounded, if Assumption A2 holds true, we have that Assumption A1 is satisfied. Indeed, if Assumption A2 holds, it follows that $g(x), \nabla f_t(x)$ and $\nabla g_i(x)$ are bounded over \mathcal{C} , and hence $f_t(x)$ and $g_i(x)$ are Lipschitz continuous. From Assumption A3 and $||g(x)|| \leq \nu_g$ in Assumption A1 it follows that $\nu_g \geq ||g(\hat{x})|| \geq \sqrt{p}\varepsilon_0$, which implicitly implies that $\nu_g \geq \varepsilon_0$.

Assumption B1 The matrix Θ_0^t is positively semidefinite.

Assumption B2 It holds that $q_i^t(x) \leq g_i(x), i = 1, ..., p$ for all $x \in C$.

Assumption B3 There exists a constant $\kappa_q > 0$ such that $\|\Theta_i^t\| \leq \kappa_q$ for $i = 0, 1, \ldots, p$.

Assumption B4 The augmented Lagrangian function $\mathcal{L}_{\sigma}^{t}(\cdot, \lambda^{t})$ is convex.

Roughly speaking, the role of Assumptions B1–B4 is to let the functions q_i^t be conservatively convex approximations to f_t and g_i , i = 1, ..., p, respectively, and let the subproblem (6) in OPMM be easily solvable. We remark that Assumption B4 is satisfied if all matrices Θ_i^t , i = 0, 1, ..., p are positively semidefinite.

Lemma 1 Let Assumptions A1, B3 be satisfied. Then, for i = 1, ..., p,

$$\sum_{t=1}^{T} g_i(x^t) \le \frac{1}{\sigma} \lambda_i^{T+1} + \gamma \kappa_g^2 T + \left[\frac{1}{4\gamma} + \frac{\kappa_q}{2}\right] \sum_{t=1}^{T} \|x^{t+1} - x^t\|^2,$$

where $\gamma > 0$ is an arbitrary scalar.

Proof From the relation $\lambda_i^{t+1} = [\lambda_i^t + \sigma q_i^t(x^{t+1})]_+$ and the fact that $[a]_+ \ge a$ for any scalar a, we have

$$\begin{split} \lambda_i^{t+1} &\geq \lambda_i^t + \sigma \left(g_i(x^t) + \langle \nabla g_i(x^t), x^{t+1} - x^t \rangle + \frac{1}{2} \left\langle \Theta_i^t(x^{t+1} - x^t), x^{t+1} - x^t \right\rangle \right) \\ &\geq \lambda_i^t + \sigma \left(g_i(x^t) - \| \nabla g_i(x^t) \| \| x^{t+1} - x^t \| - \frac{1}{2} \| \Theta_i^t \| \| x^{t+1} - x^t \|^2 \right), \end{split}$$

which, together with Assumptions A1, B3, implies for any $\gamma > 0$ that

$$\frac{1}{\sigma}(\lambda_i^{t+1} - \lambda_i^t) \ge g_i(x^t) - \gamma \kappa_g^2 - \left(\frac{1}{4\gamma} + \frac{\kappa_q}{2}\right) \|x^{t+1} - x^t\|^2.$$

Summing up the above inequality from t = 1 to T, rearranging terms and noticing that $\lambda^1 = 0$, we derive the claim.

In order to obtain a bound of $\sum_{t=1}^{T} g_i(x^t)$ in Lemma 1, we need to estimate an upper bound of $\sum_{t=1}^{T} ||x^{t+1} - x^t||^2$, which is given in the following lemma.

Lemma 2 Let Assumptions A1, B1, B2 be satisfied. Then, for any $\alpha > 0$,

$$\sum_{t=1}^{T} \|x^{t+1} - x^{t}\|^{2} \le \frac{4}{\alpha} \left[\frac{T}{\alpha} \kappa_{f}^{2} + \nu_{g} \sum_{t=1}^{T} \|\lambda^{t}\| + \frac{\sigma}{2} \nu_{g}^{2} T \right].$$

 $\begin{aligned} &Proof \text{ In view of (6), it follows from Assumption B2 that} \\ &\langle \nabla f_t(x^t), x^{t+1} - x^t \rangle + \frac{1}{2} \left\langle \Theta_0^t(x^{t+1} - x^t), x^{t+1} - x^t \right\rangle + \frac{1}{2\sigma} \|\lambda^{t+1}\|^2 + \frac{\alpha}{2} \|x^{t+1} - x^t\|^2 \\ &\leq \frac{1}{2\sigma} \sum_{i=1}^p [\lambda_i^t + \sigma q_i^t(x^t)]_+^2 \leq \frac{1}{2\sigma} \sum_{i=1}^p [\lambda_i^t + \sigma g_i(x^t)]_+^2 \leq \frac{1}{2\sigma} \|\lambda^t + \sigma g(x^t)\|^2, \\ \text{which, together with Assumption A1 and B1, implies that} \\ &\frac{\alpha}{4} \|x^{t+1} - x^t\|^2 \\ &\leq \langle \nabla f_t(x^t), x^t - x^{t+1} \rangle - \frac{\alpha}{4} \|x^{t+1} - x^t\|^2 - \frac{1}{2} \left\langle \Theta_0^t(x^{t+1} - x^t), x^{t+1} - x^t \right\rangle \\ &\quad + \frac{1}{2\sigma} [\|\lambda^t\|^2 - \|\lambda^{t+1}\|^2] + \langle \lambda^t, g(x^t) \rangle + \frac{\sigma}{2} \|g(x^t)\|^2 \\ &\leq \left(\langle \nabla f_t(x^t), x^t - x^{t+1} \rangle - \frac{\alpha}{4} \|x^{t+1} - x^t\|^2 \right) + \frac{1}{2\sigma} [\|\lambda^t\|^2 - \|\lambda^{t+1}\|^2] + \nu_g \|\lambda^t\| + \frac{\sigma}{2} \nu_g^2 \\ &\leq \frac{1}{\alpha} \kappa_f^2 + \frac{1}{2\sigma} [\|\lambda^t\|^2 - \|\lambda^{t+1}\|^2] + \nu_g \|\lambda^t\| + \frac{\sigma}{2} \nu_g^2. \end{aligned}$

The claim is obtained by summing up the above inequality from t = 1 to T, rearranging terms and noticing that $\lambda^1 = 0$.

Combining Lemma 1 and Lemma 2, we obtain the following result which plays an important role in estimating the constraint violation regret.

Proposition 3 Let Assumptions A1, B1, B2, B3 be satisfied. Then, for any scalar $\gamma > 0$, the following results hold:

$$\sum_{t=1}^{T} g_i(x^t) \leq \frac{1}{\sigma} \lambda_i^{T+1} + \gamma \kappa_g^2 T + \frac{1}{\alpha} \left[\frac{1}{\gamma} + 2\kappa_q \right] \left[\frac{\kappa_f^2}{\alpha} T + \nu_g \sum_{t=1}^{T} \|\lambda^t\| + \frac{\sigma}{2} \nu_g^2 T \right].$$

We next focus our attention on examining the bound of Lagrangian multiplier λ^t .

Lemma 4 Let Assumptions A1, B3 be satisfied. Then

$$\|\lambda^t\| - \sigma\beta_0 \le \|\lambda^{t+1}\| \le \|\lambda^t\| + \sigma\beta_0,$$

where

$$\beta_0 := \left[\nu_g + \sqrt{p} \left(\kappa_g D_0 + \frac{1}{2} \kappa_q D_0^2 \right) \right]. \tag{7}$$

Proof It follows from the nonexpansion property of $[\cdot]_+$, Assumption A1 and Assumption B3 that

$$\begin{aligned} \|\lambda^{t+1} - \lambda^{t}\| &= \|[\lambda^{t} + \sigma q^{t}(x^{t+1})]_{+} - [\lambda^{t}]_{+}\| \leq \sigma \|q^{t}(x^{t+1})\| \\ &\leq \sigma \|g(x^{t})\| + \sigma \|q^{t}(x^{t+1}) - g(x^{t})\| \\ &\leq \sigma \nu_{g} + \sigma \left(\sum_{i=1}^{p} \left(\|\nabla g_{i}(x^{t})\|\|x^{t+1} - x^{t}\| + \frac{1}{2}\|\Theta_{i}^{t}\|\|x^{t+1} - x^{t}\|^{2}\right)^{2}\right)^{1/2} \\ &\leq \sigma \nu_{g} + \sigma \left(\sum_{i=1}^{p} \left(\kappa_{g} D_{0} + \frac{1}{2}\kappa_{q} D_{0}^{2}\right)^{2}\right)^{1/2} \\ &\leq \sigma \left[\nu_{g} + \sqrt{p} \left(\kappa_{g} D_{0} + \frac{1}{2}\kappa_{q} D_{0}^{2}\right)\right], \end{aligned}$$
which completes the proof.

which completes the proof.

Lemma 5 Let Assumptions A1, A3, B1, B2, B3, B4 be satisfied. Let s > 0 be an arbitrary integer and

$$\vartheta(\sigma, \alpha, s) := \frac{\varepsilon_0 \sigma s}{2} + \beta_0 \sigma(s-1) + \frac{\alpha D_0^2}{\varepsilon_0 s} + \frac{\left(2\kappa_f D_0 + \kappa_q D_0^2\right)}{\varepsilon_0} + \frac{\sigma \nu_g^2}{\varepsilon_0}.$$
 (8)

Then, it follows that

$$|\|\lambda^{t+1}\| - \|\lambda^t\|| \le \sigma\beta_0,\tag{9}$$

where β_0 is defined by (7). Moreover, if $\|\lambda^t\| \ge \vartheta(\sigma, \alpha, s)$, it holds that

$$\|\lambda^{t+s}\| - \|\lambda^t\| \le -s\frac{\sigma\varepsilon_0}{2}.$$
(10)

Proof Inequality (9) follows directly from Lemma 4. It remains to prove

$$\|\lambda^{t+s}\| - \|\lambda^t\| \le -s\frac{\sigma\varepsilon_0}{2}$$

under the condition that $\|\lambda^t\| \ge \vartheta(\sigma, \alpha, s)$.

In the sequel, for given positive integer s, we suppose that $\|\lambda^t\| \ge \vartheta(\sigma, \alpha, s)$. For any $l \in \{t, t+1, \ldots, t+s-1\}$, under Assumption B4, it follows from (6), i.e.,

$$x^{l+1} = \underset{x \in \mathcal{C}}{\operatorname{arg\,min}} \left\{ \mathcal{L}_{\sigma}^{l}(x, \lambda^{l}) + \frac{\alpha}{2} \|x - x^{l}\|^{2} \right\},$$

and its optimality conditions that x^{l+1} is also a minimizer of $\mathcal{L}_{\sigma}^{l}(x,\lambda^{l}) + \frac{\alpha}{2}[||x - x^{l}||^{2} - ||x - x^{l+1}||^{2}]$ over \mathcal{C} . Therefore, $\langle \nabla f_{l}(x^{l}), x^{l+1} - x^{l} \rangle + \frac{1}{2} \left\langle \Theta_{0}^{l}(x^{l+1} - x^{l}), x^{l+1} - x^{l} \right\rangle + \frac{1}{2\sigma} ||\lambda^{l+1}||^{2} + \frac{\alpha}{2} ||x^{l+1} - x^{l}||^{2}$ $\leq \langle \nabla f_{l}(x^{l}), \hat{x} - x^{l} \rangle + \frac{1}{2} \left\langle \Theta_{0}^{l}(\hat{x} - x^{l}), \hat{x} - x^{l} \right\rangle + \frac{1}{2\sigma} ||\lambda^{l} + \sigma q^{l}(\hat{x})|_{+} ||^{2}$ $+ \frac{\alpha}{2} \left[||\hat{x} - x^{l}||^{2} - ||\hat{x} - x^{l+1}||^{2} \right]$ $\leq \langle \nabla f_{l}(x^{l}), \hat{x} - x^{l} \rangle + \frac{1}{2} \left\langle \Theta_{0}^{l}(\hat{x} - x^{l}), \hat{x} - x^{l} \right\rangle + \frac{1}{2\sigma} ||\lambda^{l}||^{2} + \sigma g(\hat{x})|_{+} ||^{2}$ $+ \frac{\alpha}{2} \left[||\hat{x} - x^{l}||^{2} - ||\hat{x} - x^{l+1}||^{2} \right]$ $\leq \langle \nabla f_{l}(x^{l}), \hat{x} - x^{l} \rangle + \frac{1}{2} \left\langle \Theta_{0}^{l}(\hat{x} - x^{l}), \hat{x} - x^{l} \right\rangle + \frac{1}{2\sigma} ||\lambda^{l}||^{2} + \langle \lambda^{l}, g(\hat{x}) \rangle + \frac{\sigma}{2} ||g(\hat{x})||^{2}$ $+ \frac{\alpha}{2} \left[||\hat{x} - x^{l}||^{2} - ||\hat{x} - x^{l+1}||^{2} \right],$

in which \hat{x} is given in Assumption A3 and the second inequality above is obtained from Assumption B2. Reorganizing terms and using Assumptions A1, B1, B3, we obtain

$$\frac{1}{2\sigma} \left[\|\lambda^{l+1}\|^2 - \|\lambda^l\|^2 \right] \\
\leq \langle \nabla f_l(x^l), \hat{x} - x^{l+1} \rangle + \frac{1}{2} \left\langle \Theta_0^l(\hat{x} - x^l), \hat{x} - x^l \right\rangle + \langle \lambda^l, g(\hat{x}) \rangle + \frac{\sigma}{2} \|g(\hat{x})\|^2 \\
+ \frac{\alpha}{2} \left[\|\hat{x} - x^l\|^2 - \|\hat{x} - x^{l+1}\|^2 \right] \\
\leq \kappa_f D_0 + \frac{1}{2} \kappa_q D_0^2 + \langle \lambda^l, g(\hat{x}) \rangle + \frac{\sigma}{2} \nu_g^2 + \frac{\alpha}{2} \left[\|\hat{x} - x^l\|^2 - \|\hat{x} - x^{l+1}\|^2 \right].$$
(11)

Noting that for $l \in \{t, t+1, \ldots, t+s-1\}$, one has from Assumption A3 that

$$\langle \lambda^l, g(\hat{x}) \rangle = \sum_{j=1}^p \lambda_j^l g_j(\hat{x}) \le -\varepsilon_0 \sum_{j=1}^p \lambda_j^l \le -\varepsilon_0 \|\lambda^l\|.$$
(12)

Thus, making a summation of (11) over $\{t, t+1, \ldots, t+s-1\}$, noticing (12) and the fact that $\|\lambda^{t+l}\| \ge \|\lambda^t\| - \sigma\beta_0 l$, we obtain

$$\begin{aligned} &\frac{1}{2\sigma} \left[\|\lambda^{t+s}\|^2 - \|\lambda^t\|^2 \right] \\ &\leq \left(\kappa_f D_0 + \frac{1}{2} \kappa_q D_0^2 \right) s + \frac{\sigma}{2} \nu_g^2 s + \sum_{l=t}^{t+s-1} \langle \lambda^l, g(\hat{x}) \rangle + \frac{\alpha}{2} \left[\left(\|\hat{x} - x^t\|^2 - \|\hat{x} - x^{t+s}\|^2 \right) \right] \\ &\leq \left(\kappa_f D_0 + \frac{1}{2} \kappa_q D_0^2 \right) s + \frac{\sigma}{2} \nu_g^2 s - \varepsilon_0 \sum_{l=0}^{s-1} \|\lambda^{t+l}\| + \frac{\alpha}{2} D_0^2 \\ &\leq \left(\kappa_f D_0 + \frac{1}{2} \kappa_q D_0^2 \right) s + \frac{\sigma}{2} \nu_g^2 s + \frac{\alpha}{2} D_0^2 - \varepsilon_0 \sum_{l=0}^{s-1} \left[\|\lambda^t\| - \sigma\beta_0 l \right] \\ &\leq \left(\kappa_f D_0 + \frac{1}{2} \kappa_q D_0^2 \right) s + \frac{\sigma}{2} \nu_g^2 s + \frac{\alpha}{2} D_0^2 + \varepsilon_0 \sigma\beta_0 \frac{s(s-1)}{2} - \varepsilon_0 s \|\lambda^t\|, \end{aligned}$$

which, together with $\|\lambda^t\| \ge \vartheta(\sigma, \alpha, s)$, further implies that $\|\lambda^{t+s}\|^2$

$$\leq \|\lambda^t\|^2 + 2\sigma \left(\kappa_f D_0 + \frac{1}{2}\kappa_q D_0^2\right) s + \sigma^2 \nu_g^2 s + \alpha \sigma D_0^2 + \varepsilon_0 \sigma^2 \beta_0 s(s-1) - 2\varepsilon_0 \sigma s \|\lambda^t\|$$

$$= \left(\|\lambda^t\| - \frac{\varepsilon_0 \sigma}{2}s\right)^2 - \frac{3\varepsilon_0^2 \sigma^2}{4}s^2 + \varepsilon_0 \sigma s \vartheta(\sigma, \alpha, s) - \varepsilon_0 \sigma s \|\lambda^t\|$$

$$\leq \left(\|\lambda^t\| - \frac{\varepsilon_0 \sigma}{2}s\right)^2.$$

Noticing that $\|\lambda^t\| \ge \vartheta(\sigma, \alpha, s) \ge \frac{\varepsilon_0 \sigma}{2} s$, we have $\|\lambda^{t+s}\| \le \|\lambda^t\| - \frac{\varepsilon_0 \sigma}{2} s$. The proof is completed. \Box

The following lemma is a simple variation of [29, Lemma 5], which shall be used to deal with KKT violation regret of OPMM. The proof is provided in Appendix A.

Lemma 6 Let $\{Z_t\}$ be a sequence with $Z_0 = 0$. Suppose there exist an integer $t_0 > 0$, real constants $\theta > 0$, $\delta_{\max} > 0$ and $0 < \zeta \leq \delta_{\max}$ such that $|Z_{t+1} - Z_t| \leq \delta_{\max}$ and

$$Z_{t+t_0} - Z_t \le -t_0 \zeta, \quad \text{if } Z_t \ge \theta \tag{13}$$

hold for all $t \in \{1, 2, \ldots\}$. Then,

$$Z_t \le \theta + t_0 \delta_{\max} + t_0 \frac{4\delta_{\max}^2}{\zeta} \log\left[\frac{8\delta_{\max}^2}{\zeta^2}\right], \forall t \in \{1, 2, \ldots\}.$$
 (14)

If we take $\theta = \vartheta(\sigma, \alpha, s)$, $\delta_{\max} = \sigma \beta_0$, $\zeta = \frac{\sigma}{2} \varepsilon_0$ and $t_0 = s$, we can observe from $\beta_0 \ge \nu_g \ge \varepsilon_0$ and Lemma 5 that the conditions in Lemma 6 are satisfied in terms of $\|\lambda^t\|$. For convenience, let us introduce

$$\psi(\sigma, \alpha, s) := \vartheta(\sigma, \alpha, s) + \left[\beta_0 + \frac{8\beta_0^2}{\varepsilon_0} \log \frac{32\beta_0^2}{\varepsilon_0^2}\right] \sigma s.$$

We can verify that the right-hand side of (14) equals exactly to $\psi(\sigma, \alpha, s)$, that is,

$$\theta + t_0 \delta_{\max} + t_0 \frac{4\delta_{\max}^2}{\zeta} \log\left[\frac{8\delta_{\max}^2}{\zeta^2}\right] = \psi(\sigma, \alpha, s)$$

Therefore, from Lemma 5 and Lemma 6 we directly derive the following useful result.

Proposition 7 Let Assumptions A1, A3, B1, B2, B3, B4 be satisfied. Then, for any arbitrary integer s > 0, the following inequality holds

$$\|\lambda^{\iota}\| \le \psi(\sigma, \alpha, s).$$

Finally, if we define

$$\kappa_0 = \frac{\left(2\kappa_f D_0 + \kappa_q D_0^2\right)}{\varepsilon_0}, \ \kappa_1 = \frac{D_0^2}{\varepsilon_0}, \ \kappa_2 = \frac{\nu_g^2}{\varepsilon_0} - \beta_0,$$

$$\kappa_3 = \left[2\beta_0 + \frac{\varepsilon_0}{2} + \frac{8\beta_0^2}{\varepsilon_0}\log\frac{32\beta_0^2}{\varepsilon_0^2}\right],$$
(15)

then $\psi(\sigma, \alpha, s)$ can be rewritten as

$$\psi(\sigma, \alpha, s) = \kappa_0 + \kappa_1 \frac{\alpha}{s} + \kappa_2 \sigma + \kappa_3 \sigma s.$$

3 Regret Analysis of OPMM

In this section, we establish the regret bounds of the proposed algorithm. In particular, we focus on estimating the following three regrets: regret of Lagrangian residual, regret of constraint violation and regret of complementarity residual. The following proposition establishes an upper bound of the so-called Lagrangian residual.

Proposition 8 Let Assumptions A1, A2, B3 be satisfied. Then, there exists a vector $w^{t+1} \in N_{\mathcal{C}}(x^{t+1})$ such that

$$\left\|\sum_{t=1}^{T} \mathcal{H}_{t}\right\| \leq 2\kappa_{f} + \frac{\kappa_{q}^{2}}{2\beta}T + \frac{(1+p)\beta}{2}\sum_{t=1}^{T} \|x^{t+1} - x^{t}\|^{2} + \frac{(L_{g} + \kappa_{q})^{2}}{2\beta}\sum_{t=1}^{T} \|\lambda^{t+1}\|^{2} + \alpha D_{0},$$
(16)

where $\beta > 0$ is an arbitrary scalar and

$$\mathcal{H}_t := \nabla f_{t+1}(x^{t+1}) + \sum_{i=1}^p \lambda_i^{t+1} \nabla g_i(x^{t+1}) + w^{t+1}.$$

Proof It follows from the optimality conditions of (6) that

$$0 \in \nabla q_0^t(x^{t+1}) + \mathcal{J}q^t(x^{t+1})^T \lambda^{t+1} + \alpha(x^{t+1} - x^t) + N_{\mathcal{C}}(x^{t+1}).$$

Equivalently, from the definitions of q_0^t and q^t , there exists $w^{t+1} \in N_{\mathcal{C}}(x^{t+1})$ such that

$$0 = \nabla f_t(x^t) + \Theta_0^t(x^{t+1} - x^t) + \sum_{i=1}^p \lambda_i^{t+1} [\nabla g_i(x^t) + \Theta_i^t(x^{t+1} - x^t)] + \alpha(x^{t+1} - x^t) + w^{t+1}.$$
(17)

In view of definitions of q_i^t , i = 0, 1, ..., p and \mathcal{H}_t , we may rewrite (17) as

$$0 = \mathcal{H}_t + [\nabla f_t(x^t) - \nabla f_{t+1}(x^{t+1})] + \sum_{i=1}^p \lambda_i^{t+1} (\nabla g_i(x^t) - \nabla g_i(x^{t+1})) \\ + \Theta_0^t(x^{t+1} - x^t) + \sum_{i=1}^p \lambda_i^{t+1} \Theta_i^t(x^{t+1} - x^t) + \alpha(x^{t+1} - x^t).$$

Making a summation, we obtain

m

$$-\sum_{t=1}^{T} \mathcal{H}_{t}$$

$$= [\nabla f_{1}(x^{1}) - \nabla f_{T+1}(x^{T+1})] + \sum_{t=1}^{T} \left[\sum_{i=1}^{p} \lambda_{i}^{t+1} (\nabla g_{i}(x^{t}) - \nabla g_{i}(x^{t+1})) \right]$$

$$+ \sum_{t=1}^{T} \Theta_{0}^{t}(x^{t+1} - x^{t}) + \sum_{t=1}^{T} \left[\sum_{i=1}^{p} \lambda_{i}^{t+1} \Theta_{i}^{t}(x^{t+1} - x^{t}) \right] + \alpha(x^{T+1} - x^{1}).$$

Therefore, it follows from Assumptions A1, A2, B3 that

$$\begin{split} \left\| \sum_{t=1}^{T} \mathcal{H}_{t} \right\| \\ &\leq [\|\nabla f_{1}(x^{1})\| + \|\nabla f_{T+1}(x^{T+1})\|] + \sum_{t=1}^{T} \left[\sum_{i=1}^{p} \lambda_{i}^{t+1} \|\nabla g_{i}(x^{t}) - \nabla g_{i}(x^{t+1})\| \right] \\ &+ \sum_{t=1}^{T} \|\Theta_{0}^{t}\| \|x^{t+1} - x^{t}\| + \sum_{t=1}^{T} \left[\sum_{i=1}^{p} \lambda_{i}^{t+1} \|\Theta_{i}^{t}\| \|x^{t+1} - x^{t}\| \right] + \alpha D_{0} \\ &\leq 2\kappa_{f} + \kappa_{q} \sum_{t=1}^{T} \|x^{t+1} - x^{t}\| + (L_{g} + \kappa_{q}) \sum_{t=1}^{T} \left[\sum_{i=1}^{p} \lambda_{i}^{t+1} \|x^{t+1} - x^{t}\| \right] + \alpha D_{0} \\ &\leq 2\kappa_{f} + \frac{\kappa_{q}^{2}}{2\beta}T + \frac{\beta}{2} \sum_{t=1}^{T} \|x^{t+1} - x^{t}\|^{2} + \frac{(L_{g} + \kappa_{q})^{2}}{2\beta} \sum_{t=1}^{T} \|\lambda^{t+1}\|^{2} \\ &+ \frac{p\beta}{2} \sum_{t=1}^{T} \|x^{t+1} - x^{t}\|^{2} + \alpha D_{0} \\ &= 2\kappa_{f} + \frac{\kappa_{q}^{2}}{2\beta}T + \frac{(1+p)\beta}{2} \sum_{t=1}^{T} \|x^{t+1} - x^{t}\|^{2} + \frac{(L_{g} + \kappa_{q})^{2}}{2\beta} \sum_{t=1}^{T} \|\lambda^{t+1}\|^{2} + \alpha D_{0}, \\ &\text{which proves (16). \\ \Box$$

Combining the results in Proposition 3, Proposition 7 and Proposition 8, we present the main theorem of this section.

Theorem 9 Let Assumptions A1, A2, A3, B1, B2, B3, B4 be satisfied. Then, for $\sigma = T^{-1/4}$ and $\alpha = T^{1/4}$, the following assertions hold.

(i) There exists a vector $w^{t+1} \in N_{\mathcal{C}}(x^{t+1})$ such that the regret of Lagrangian residual is bounded by

$$\left\|\frac{1}{T}\sum_{t=1}^{T} \left[\nabla f_{t+1}(x^{t+1}) + \sum_{i=1}^{p} \lambda_i^{t+1} \nabla g_i(x^{t+1}) + w^{t+1}\right]\right\| \le \varrho_0 T^{-1/8} + o(T^{-1/8}),$$

where

$$\varrho_0 = \frac{\kappa_q^2}{2} + 2(1+p)\nu_g(\kappa_0 + \kappa_1 + \kappa_3) + \frac{(L_g + \kappa_q)^2}{2}(\kappa_0 + \kappa_1 + \kappa_3)^2.$$

(ii) The regret of constraint violation is

$$\frac{1}{T}\sum_{t=1}^{T}g_i(x^t) \le (\nu_g(\kappa_0 + \kappa_1 + \kappa_3) + \kappa_g^2)T^{-1/8} + o(T^{-1/8}).$$

(iii) The regret of complementarity residual is

$$\frac{1}{T}\sum_{t=1}^{T} \|\lambda^{t+1} - [\lambda^{t+1} + \sigma g(x^{t+1})]_+\| \le \beta_0 T^{-1/4} + o(T^{-1/4})$$

Proof It follows from Proposition 7 that

$$\|\lambda^t\| \le \psi(\sigma, \alpha, s) = \kappa_0 + \kappa_1 \frac{\alpha}{s} + \kappa_2 \sigma + \kappa_3 \sigma s,$$

where $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ are defined by (15). For $\sigma = T^{-1/4}$ and $\alpha = T^{1/4}$, we take $s = T^{1/4}$ and hence

$$\|\lambda^t\| \le \kappa_0 + \kappa_1 + \kappa_3 + \kappa_2 T^{-1/4}.$$
(18)

Combining the results in Proposition 8 and Lemma 2, we have

$$\begin{aligned} \left\| \frac{1}{T} \sum_{t=1}^{T} \mathcal{H}_{t} \right\| &\leq \frac{2\kappa_{f}}{T} + \frac{\kappa_{q}^{2}}{2\beta} + \frac{2(1+p)\beta}{\alpha T} \left[\frac{T}{\alpha} \kappa_{f}^{2} + \nu_{g} \sum_{t=1}^{T} \|\lambda^{t}\| + \frac{\sigma}{2} \nu_{g}^{2} T \right] \\ &+ \frac{(L_{g} + \kappa_{q})^{2}}{2\beta T} \sum_{t=1}^{T} \|\lambda^{t+1}\|^{2} + \frac{\alpha D_{0}}{T}. \end{aligned}$$

Taking $\beta = T^{1/8}$ and using (18), we obtain $\left\| \frac{1}{T} \sum_{t=1}^{T} \mathcal{H}_t \right\|$

which yields item (i).

From Proposition 3, one has

$$\frac{1}{T}\sum_{t=1}^{T}g_i(x^t) \leq \frac{1}{\sigma T}\lambda_i^{T+1} + \gamma\kappa_g^2 + \frac{1}{\alpha T}\left[\frac{1}{\gamma} + 2\kappa_q\right]\left[\frac{\kappa_f^2}{\alpha}T + \nu_g\sum_{t=1}^{T}\|\lambda^t\| + \frac{\sigma}{2}\nu_g^2T\right].$$

Taking $\gamma = T^{-1/8}$ and using (18), we obtain

$$\frac{1}{T} \sum_{t=1}^{T} g_i(x^t) \\
\leq \left[T^{-1/8} + 2\kappa_q T^{-1/4} \right] \left[\kappa_f^2 T^{-1/4} + \nu_g(\kappa_0 + \kappa_1 + \kappa_3 + \kappa_2 T^{-1/4}) + \frac{\nu_g^2}{2} T^{-1/4} \right] \\
+ T^{-3/4}(\kappa_0 + \kappa_1 + \kappa_3 + \kappa_2 T^{-1/4}) + \kappa_g^2 T^{-1/8},$$

which proves item (ii).

Finally, we consider item (iii). First of all, we estimate $||g(x^{t+1}) - q^t(x^{t+1})||$. From Assumption A2 and Assumption B3, we have

$$\begin{aligned} \|g(x^{t+1}) - q^t(x^{t+1})\| &= \left(\sum_{i=1}^p (g_i(x^{t+1}) - q_i^t(x^{t+1}))^2\right)^{1/2} \\ &\leq \left(\sum_{i=1}^p \left(\frac{L_g + \kappa_g}{2} \|x^{t+1} - x^t\|^2\right)^2\right)^{1/2} \\ &= \frac{\sqrt{p}(L_g + \kappa_g)}{2} \|x^{t+1} - x^t\|^2. \end{aligned}$$

Then, from the definition of λ^{t+1} and Lemma 4 we obtain

$$\begin{aligned} \|\lambda^{t+1} - [\lambda^{t+1} + \sigma g(x^{t+1})]_+\| &= \|[\lambda^t + \sigma q^t(x^{t+1})]_+ - [\lambda^{t+1} + \sigma g(x^{t+1})]_+\| \\ &\leq \|\lambda^{t+1} - \lambda^t + \sigma [g(x^{t+1}) - q^t(x^{t+1})]\| \\ &\leq \beta_0 \sigma + \frac{\sqrt{p}(L_g + \kappa_q)\sigma}{2} \|x^{t+1} - x^t\|^2. \end{aligned}$$

Taking a summation and using Lemma 2, one has

$$\frac{1}{T} \sum_{t=1}^{T} \|\lambda^{t+1} - [\lambda^{t+1} + \sigma g(x^{t+1})]_{+}\|$$

$$\leq \beta_0 \sigma + \frac{2\sqrt{p}(L_g + \kappa_q)\sigma}{\alpha T} \left[\frac{T}{\alpha} \kappa_f^2 + \nu_g \sum_{t=1}^T \|\lambda^t\| + \frac{\sigma}{2} \nu_g^2 T \right]$$

Therefore, it follows from (18) that

$$\begin{split} &\frac{1}{T} \sum_{t=1}^{T} \|\lambda^{t+1} - [\lambda^{t+1} + \sigma g(x^{t+1})]_{+}\| \\ &\leq 2\sqrt{p} (L_g + \kappa_q) T^{-1/2} [\kappa_f^2 T^{-1/4} + \nu_g (\kappa_0 + \kappa_1 + \kappa_3 + \kappa_2 T^{-1/4}) + \frac{\nu_g^2}{2} T^{-1/4}] \\ &+ \beta_0 T^{-1/4}, \end{split}$$

which completes the proof of item (iii).

In the rest of this section, we analyze the objective reduction regret of the proposed algorithm under a setting where the objective function is a quadratic convex function, that is, $f_t(x) = q_0^t(x)$ and Assumption B1 holds true. We emphasize that although the objective function is assumed to be convex, the feasible set Φ may still be non-convex.

Proposition 10 Let Assumptions A1, B1, B2 be satisfied and $f_t(x) = q_0^t(x)$ for all $x \in C$. Let $\sigma = T^{-1/2}$ and $\alpha = T^{1/2}$. The following estimation holds:

$$\frac{1}{T}\sum_{t=1}^{T} f_t(x^t) - \inf_{z \in \Phi} \frac{1}{T}\sum_{t=1}^{T} f_t(z) \le \left(\kappa_f^2 + \frac{1}{2}\nu_g^2 + \frac{1}{2}\text{dist}^2(x^1, S^*)\right) T^{-1/2},$$

where S^* is the set of optimal solutions given by $S^* := \arg \min_{x \in \Phi} \sum_{t=1}^T f_t(x)$.

 $\begin{aligned} &Proof \text{ In view of (6), we have from Assumption B2 that, for all } z \in \mathcal{C}, \\ &q_0^t(x^{t+1}) + \frac{\alpha}{2} \|x^{t+1} - x^t\|^2 \\ &\leq q_0^t(z) + \frac{1}{2\sigma} \left[\|[\lambda^t + \sigma q^t(z)]_+\|^2 - \|[\lambda^t + \sigma q^t(x^{t+1})]_+\|^2 \right] \\ &\quad + \frac{\alpha}{2} \left[\|z - x^t\|^2 - \|z - x^{t+1}\|^2 \right] \\ &\leq q_0^t(z) + \frac{1}{2\sigma} \left[\|[\lambda^t + \sigma g(z)]_+\|^2 - \|\lambda^{t+1}\|^2 \right] + \frac{\alpha}{2} \left[\|z - x^t\|^2 - \|z - x^{t+1}\|^2 \right]. \end{aligned}$ Rearranging terms and noticing $q_0^t(z) = f_t(z)$ we obtain

$$f_{t}(x^{t}) + \frac{\alpha}{4} \|x^{t+1} - x^{t}\|^{2}$$

$$\leq f_{t}(z) + \left(f_{t}(x^{t}) - q_{0}^{t}(x^{t+1}) - \frac{\alpha}{4} \|x^{t+1} - x^{t}\|^{2}\right) + \frac{1}{2\sigma} (\|\lambda^{t}\|^{2} - \|\lambda^{t+1}\|^{2}) \quad (19)$$

$$+ \langle \lambda^{t}, g(z) \rangle + \frac{\sigma}{2} \|g(z)\|^{2} + \frac{\alpha}{2} \left[\|z - x^{t}\|^{2} - \|z - x^{t+1}\|^{2}\right].$$

From Assumption B1 and Assumption A1 one has

$$\begin{aligned} f_t(x^t) &- q_0^t(x^{t+1}) - \frac{\alpha}{4} \|x^{t+1} - x^t\|^2 \\ &= \langle -\nabla f_t(x^t), x^{t+1} - x^t \rangle - \frac{1}{2} \langle \Theta_0^t(x^{t+1} - x^t), x^{t+1} - x^t \rangle - \frac{\alpha}{4} \|x^{t+1} - x^t\|^2 \\ &\leq \frac{1}{\alpha} \|\nabla f_t(x^t)\|^2 \leq \frac{1}{\alpha} \kappa_f^2. \end{aligned}$$

Therefore, from (19) and the fact that $\langle \lambda^t, g(z) \rangle \leq 0$ for all $z \in \Phi$ we have $f_t(x^t) \le f_t(z) + \frac{1}{\alpha} \kappa_f^2 + \frac{1}{2\sigma} (\|\lambda^t\|^2 - \|\lambda^{t+1}\|^2) + \frac{\sigma}{2} \nu_g^2 + \frac{\alpha}{2} \left[\|z - x^t\|^2 - \|z - x^{t+1}\|^2 \right].$

Making a summation, we obtain

$$\frac{1}{T}\sum_{t=1}^{T}f_t(x^t) \le \frac{1}{T}\sum_{t=1}^{T}f_t(z) + \frac{\kappa_f^2}{\alpha} + \frac{\sigma\nu_g^2}{2} + \frac{\alpha}{2T}||z - x^1||^2.$$

Hence, the claim is derived by noting that $\sigma = T^{-1/2}$ and $\alpha = T^{1/2}$.

4 OPMM for Online Optimization with **Convex Constraints**

In this section, we consider the online optimization problem with convex functional constraints, namely, the case that g_1, \ldots, g_p are all convex functions. Moreover, we choose $\Theta_i^t = 0, i = 1, \dots, p$ in OPMM. In this case, Assumption B2 is naturally satisfied and Assumption B3 is reduced to the condition $\|\Theta_0^t\| \leq \kappa_q$. Further, under Assumption B1, the subproblem (6) is reduced to the following convex optimization problem

$$\min_{x \in \mathcal{C}} \left\{ q_0^t(x) + \frac{1}{2\sigma} \| [\lambda^t + \sigma g(x^t) + \sigma \mathcal{J}g(x^t)(x - x^t)]_+ \|^2 + \frac{\alpha}{2} \|x - x^t\|^2 \right\}.$$
(20)

For positively definite matrix $G \in \mathbb{S}^n$ and $x \in \mathbb{R}^n$, we use dist $\mathcal{C}^G(x)$ to denote the weighted distance of x from \mathcal{C} , which is defined by

$$\operatorname{dist}_{\mathcal{C}}^{G}(x) := \inf_{u \in \mathcal{C}} \|x - u\|_{G},$$

where $||x||_G = \sqrt{x^T G x}$ is the *G*-weighted norm of *x*. The *G*-weighted projection of x onto \mathcal{C} , denoted by $\Pi^G_{\mathcal{C}}(x)$, is defined by

$$\Pi^G_{\mathcal{C}}(x) := \operatorname*{arg\,min}_{u \in \mathcal{C}} \|x - u\|_G.$$

The following lemma is well-known, see, e.g., [30, Example 3.31].

Lemma 11 For a closed convex set $C \subset \mathbb{R}^n$ and a positively definite matrix $G \in \mathbb{S}^n$, let $\pi(x) := \frac{1}{2} \text{dist}_{\mathcal{C}}^G(x)^2$. Then, π is continuously differentiable and

$$\nabla \pi(x) = G(x - \Pi_{\mathcal{C}}^G(x)).$$

15

 \square

By introducing artificial vectors z and w, we can express Problem (20) as the following equivalent convex quadratic programming problem

$$\min_{\substack{x,z,w\\ x,z,w}} q_0^t(x) + \frac{\alpha}{2} \|x - x^t\|^2 + \frac{1}{2\sigma} \|z\|^2$$
s.t. $\lambda^t + \sigma g(x^t) + \sigma \mathcal{J}g(x^t)(x - x^t) - z + w = 0,$ (21)
 $x \in \mathcal{C}, \ z \ge 0, \ w \ge 0.$

The Lagrangian function of Problem (21) is given by

$$L^{t}(x, z, w, y) = q_{0}^{t}(x) + \frac{\alpha}{2} ||x - x^{t}||^{2} + \frac{1}{2\sigma} ||z||^{2} + \langle y, \lambda^{t} + \sigma g(x^{t}) + \sigma \mathcal{J}g(x^{t})(x - x^{t}) - z + w \rangle.$$

Then, the dual of Problem (21) is expressed as

$$\begin{split} \max_{y \in \mathbb{R}^{p}} \inf_{x \in \mathcal{C}} \inf_{z \ge 0} \inf_{w \ge 0} L^{t}(x, z, w, y) \\ &= \max_{y \in \mathbb{R}^{p}} \left\{ \langle y, \lambda^{t} + \sigma g(x^{t}) \rangle + \inf_{x \in \mathcal{C}} [q_{0}^{t}(x) + \frac{\alpha}{2} \| x - x^{t} \|^{2} + \langle y, \sigma \mathcal{J}g(x^{t})(x - x^{t}) \rangle] \right. \\ &+ \inf_{w \ge 0} y^{T} w + \inf_{z \ge 0} \left[\frac{1}{2\sigma} \| z \|^{2} - \langle y, z \rangle \right] \right\} \\ &= \max_{y \ge 0} \left\{ \langle y, \lambda^{t} + \sigma g(x^{t}) \rangle - \frac{\sigma}{2} \| y \|^{2} + f_{t}(x^{t}) - \frac{1}{2} \| \nabla f_{t}(x^{t}) + \sigma \mathcal{J}g(x^{t})^{T} y \|_{[H^{t}]^{-1}}^{2} \right. \\ &+ \inf_{x \in \mathcal{C}} \left[\frac{1}{2} \left\| x - \left(x^{t} - [H^{t}]^{-1} (\nabla f_{t}(x^{t}) + \sigma \mathcal{J}g(x^{t})^{T} y) \right) \right\|_{H^{t}}^{2} \right] \right\} \\ &= \max_{y \ge 0} \left\{ \langle y, \lambda^{t} + \sigma g(x^{t}) \rangle - \frac{\sigma}{2} \| y \|^{2} + f_{t}(x^{t}) - \frac{1}{2} \| \nabla f_{t}(x^{t}) + \sigma \mathcal{J}g(x^{t})^{T} y \|_{[H^{t}]^{-1}}^{2} \right. \\ &+ \left[\frac{1}{2} \mathrm{dist}_{\mathcal{C}}^{H^{t}} \left(x^{t} - [H^{t}]^{-1} (\nabla f_{t}(x^{t}) + \sigma \mathcal{J}g(x^{t})^{T} y) \right)^{2} \right] \right\}, \end{split}$$

where $H^t := \Theta_0^t + \alpha I$. Therefore, when we derive the optimal solution y^t by solving the dual problem, from the duality theory, the solution of the subproblem (6) is given by

$$x^{t+1} = \Pi_{\mathcal{C}}^{H^t} \left(x^t - [H^t]^{-1} (\nabla f_t(x^t) + \sigma \mathcal{J}g(x^t)^T y^t) \right).$$

Based on the above analysis, OPMM for the online problem with convex constraints can be rewritten as in Algorithm 2.

We now discuss the relationship between y^t and λ^{t+1} .

Proposition 12 Let $\omega_t(y)$ denote the objective function of Problem (22). Then,

$$\lambda^{t+1} = [\nabla \omega_t(y^t) + \sigma y^t]_+.$$
⁽²⁴⁾

Algorithm 2: Projection version of OPMM for online non-convex optimization with convex constraints.

Input: $\lambda^1 = 0, x^1 \in \mathcal{C}, \sigma > 0$ and $\alpha > 0$, receive a cost function $f_1(\cdot)$.

1 for $t \leftarrow 1$ to T do

2 Choose $\Theta_0^t \in \mathbb{S}^n$ and set $H^t := \Theta_0^t + \alpha I$. Solve the following convex optimization problem to obtain y^t :

$$\max_{y\geq 0} \left\{ -\frac{\sigma}{2} \|y\|^2 + \langle y, \lambda^t + \sigma g(x^t) \rangle - \frac{1}{2} \|\nabla f_t(x^t) + \sigma \mathcal{J}g(x^t)^T y\|_{[H^t]^{-1}}^2 + \left[\frac{1}{2} \operatorname{dist}_{\mathcal{C}}^{H^t} \left(x^t - [H^t]^{-1} (\nabla f_t(x^t) + \sigma \mathcal{J}g(x^t)^T y) \right)^2 \right] \right\}.$$
(22)

3 Compute

$$x^{t+1} = \Pi_{\mathcal{C}}^{H^t} \left(x^t - [H^t]^{-1} (\nabla f_t(x^t) + \sigma \mathcal{J}g(x^t)^T y^t) \right).$$
(23)

4 Update $\lambda_i^{t+1} = \left[\lambda_i^t + \sigma(g_i(x^t) + \langle \nabla g_i(x^t), x^{t+1} - x^t \rangle)\right]_+, \ i = 1, \dots, p.$ 5 Receive a cost function $f_{t+1}(\cdot)$.

 $Proof \ {\rm It}$ follows from Lemma 11 that

$$= -\sigma y + \lambda^{t} + \sigma g(x^{t}) - \sigma \mathcal{J}g(x^{t}) \left[[H^{t}]^{-1} (\nabla f_{t}(x^{t}) + \sigma \mathcal{J}g(x^{t})^{T}y) \right]$$
$$-\sigma \mathcal{J}g(x^{t})[H^{t}]^{-1}H^{t} \left[x^{t} - [H^{t}]^{-1} (\nabla f_{t}(x^{t}) + \sigma \mathcal{J}g(x^{t})^{T}y) -\Pi_{\mathcal{C}}^{H^{t}} \left(x^{t} - [H^{t}]^{-1} (\nabla f_{t}(x^{t}) + \sigma \mathcal{J}g(x^{t})^{T}y) \right) \right]$$
$$= -\sigma y + \lambda^{t} + \sigma g(x^{t}) - \sigma \mathcal{J}g(x^{t}) \left[x^{t} - \Pi_{\mathcal{C}}^{H^{t}} \left(x^{t} - [H^{t}]^{-1} (\nabla f_{t}(x^{t}) + \sigma \mathcal{J}g(x^{t})^{T}y) \right) \right].$$
Hence, from (22) we have

Hence, from (23) we have

$$\nabla \omega_t(y^t) = -\sigma y^t + \lambda^t + \sigma \Big[g(x^t) + \mathcal{J}g(x^t)(x^{t+1} - x^t) \Big],$$

which implies

 $\nabla \omega_t(y)$

$$\lambda^t + \sigma \Big[g(x^t) + \mathcal{J}g(x^t)(x^{t+1} - x^t) \Big] = \nabla \omega_t(y^t) + \sigma y^t.$$

Π

The claim is derived by noticing the definition of λ^{t+1} .

Furthermore, if we choose a scalar $\eta_t > 0$ such that $\Theta_0^t = \eta_t I$ satisfies the required assumptions, we obtain that $H^t = (\alpha + \eta_t)I$, $[H^t]^{-1} = (\alpha + \eta_t)^{-1}I$

and Problem (22) is equivalent to

$$\max_{y\geq 0} \left\{ -\frac{\sigma}{2} \|y\|^2 + \langle y, \lambda^t + \sigma g(x^t) \rangle - \frac{1}{2(\alpha+\eta_t)} \|\nabla f_t(x^t) + \sigma \mathcal{J}g(x^t)^T y\|^2 + \left[\frac{\alpha+\eta_t}{2} \operatorname{dist} \left(x^t - [\alpha+\eta_t]^{-1} (\nabla f_t(x^t) + \sigma \mathcal{J}g(x^t)^T y), \mathcal{C} \right)^2 \right] \right\}.$$
(25)

Hence, the formula (23) is reduced to

$$x^{t+1} = \Pi_{\mathcal{C}} \Big(x^t - [\alpha + \eta_t]^{-1} (\nabla f_t(x^t) + \sigma \mathcal{J}g(x^t)^T y^t) \Big),$$

where y^t is the solution to Problem (25). In this case, at each iteration of Algorithm 2 the main calculations are computing a projection and solving a relatively simple convex minimization problem which make the algorithm easy to be implemented.

5 Conclusion

In this paper, we present a proximal method of multipliers with quadratic approximations (OPMM) for solving an online non-convex optimization with (possibly non-convex) inequality constraints. We show that, this algorithm exhibits $\mathcal{O}(T^{-1/8})$ Lagrangian residual regret, $\mathcal{O}(T^{-1/8})$ regret of constraint violation and $\mathcal{O}(T^{-1/4})$ complementarity residual regret if parameters in the algorithm are properly chosen, where T denotes the number of iterations. We also show that, for the case when the constraint functions are all convex, the projection version of OPMM provides a practical way for finding a decision sequence $\{x^1, x^2, \ldots, x^T\}$. To the best of our knowledge, the regret analysis of numerical methods for online non-convex optimization with long term constraints has not been studied in the literature yet.

We note that, even for the simple bounded box set $\mathcal{C} = [a, b] \subset \mathbb{R}^n$, the analysis of regrets requires the exact solution to the subproblem (25). How to obtain the regret bounds when the subproblem is inexactly solved is an important future research topic worth considering.

Acknowledgments. The authors would like to thank the two reviewers for the valuable suggestions. The research is supported by the National Natural Science Foundation of China (No. 11731013, No. 11971089 and No. 11871135).

Appendix A Proof of Lemma 6

Proof Let

$$r = \frac{\zeta}{4t_0 \delta_{\max}^2}, \quad \rho = 1 - \frac{\zeta^2}{8\delta_{\max}^2},$$

then it yields that $\rho = 1 - \frac{rt_0}{2} \zeta$. Define $\eta(t) = Z_{t+t_0} - Z_t$, then from $|Z_{t+1} - Z_t| \leq \delta_{\max}$ we have $|\eta(t)| \leq t_0 \delta_{\max}$ and hence

$$|r\eta(t)| \le \frac{\zeta}{4t_0 \delta_{\max}^2} \cdot t_0 \delta_{\max} = \frac{\zeta}{4\delta_{\max}} \le 1.$$
(A1)

From (A1) and the following inequality

$$e^{\tau} \leq 1 + \tau + 2\tau^2$$
 when $|\tau| < 1$,

we obtain

$$e^{rZ_{t+t_0}} = e^{rZ_t} e^{r\eta(t)}$$

$$\leq e^{rZ_t} [1 + r\eta(t) + 2r^2 t_0^2 \delta_{\max}^2]$$

$$= e^{rZ_t} [1 + r\eta(t) + rt_0 \zeta/2].$$

Case 1: $Z_t \ge \theta$. In this case, one has from (13) that $\eta(t) \le -t_0 \zeta$ and hence

$$e^{rZ_{t+t_0}} \leq e^{rZ_t} [1 - rt_0\zeta + rt_0\zeta/2]$$

= $e^{rZ_t} [1 - rt_0\zeta/2]$ (A2)
= ρe^{rZ_t} .

Case 2: $Z_t < \theta$. In this case, one has $\eta(t) \leq t_0 \delta_{\max}$ and hence

$$e^{rZ_{t+t_0}} = e^{rZ_t} e^{r\eta(t)}$$

$$\leq e^{rZ_t} e^{rt_0\delta_{\max}}$$

$$\leq e^{r\theta} e^{rt_0\delta_{\max}}.$$
(A3)

Combining (A2) and (A3), we obtain

$$e^{rZ_{t+t_0}} \le \rho e^{rZ_t} + e^{r\theta} e^{rt_0 \delta_{\max}}.$$
 (A4)

We next prove the following inequality by induction,

$$e^{rZ_t} \le \frac{1}{1-\rho} e^{r\theta} e^{rt_0 \delta_{\max}}, \ t \in \{0, 1, \ldots\}.$$
 (A5)

We first consider the case $t \in \{0, 1, ..., t_0\}$. From $|Z_{t+1} - Z_t| \leq \delta_{\max}$ and $Z_0 = 0$ we have $Z_t \leq t \delta_{\max}$. This, together with the fact that $\frac{e^{r\theta}}{1-\rho} \geq 1$, implies

$$e^{rZ_t} \le e^{rt\delta_{\max}} \le e^{rt_0\delta_{\max}} \le e^{rt_0\delta_{\max}} \frac{e^{r\theta}}{1-\rho}$$

Hence, (A5) is satisfied for all $t \in \{0, 1, ..., t_0\}$. We now assume that (A5) holds true for all $t \in \{t_0 + 1, ..., \tau\}$ with arbitrary $\tau > t_0$. Consider $t = \tau + 1$. By (A4), we have

$$e^{rZ_{\tau+1}} \le \rho e^{rZ_{\tau+1-t_0}} + e^{r\theta} e^{rt_0\delta_{\max}}$$
$$\le \rho e^{rt_0\delta_{\max}} \frac{e^{r\theta}}{1-\rho} + e^{r\theta} e^{rt_0\delta_{\max}} = e^{rt_0\delta_{\max}} \frac{e^{r\theta}}{1-\rho}.$$

Therefore, the inequality (A5) holds for all $t \in \{0, 1, \ldots\}$. Taking logarithm on both sides of (A5) and dividing by r yields

$$Z_t \le \theta + t_0 \delta_{\max} + \frac{1}{r} \log\left(\frac{1}{1-\rho}\right) = \theta + t_0 \delta_{\max} + t_0 \frac{4\delta_{\max}^2}{\zeta} \log\left(\frac{8\delta_{\max}^2}{\zeta^2}\right).$$

The proof is completed.

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