REMARKS ON THE CONSISTENCY OF UPWIND SOURCE AT INTERFACE SCHEMES ON NONUNIFORM GRIDS

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preliminary study of the *supra-convergence* of finite volume schemes for conservation laws with geometrical source terms

– *upwind interfacial discretizations* for nonuniform time/space grids
– inconsistent characteristics of the (local) truncation error
– notion of (global) consistency, related to the *well-balance property*
– convergence theory at optimal rates according to Wendroff &White
– computational performance as *adaptive meshing* yields an extra stabilization against the nonlinear response over shock regions

formal analysis for scalar (linear) advection/transport balance equations

$$\partial_t u + \partial_x A(u) + b(u)\, z'(x) = 0\,, \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}$$
$$u(0,x) = u_0(x) \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ \ 1 \leq p < +\infty$$

where $a(u) = A'(u) = a > 0$, $z' \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $b \in C^1(\mathbb{R})$

and the stationary solutions are described by

$$D(u(x)) + z(x) = C^{st}, \quad D'(u) = \frac{a(u)}{b(u)} \in L^\infty(\mathbb{R})$$

(with D strictly monotonic for the existence of a unique Lipschitz continuous steady state)

difficulties and limits : restriction to *geometrical source terms* for the extended notation as (non conservative) fluxes; physical applications *with negligible fluxes* (groundwater models, nonlinear age-dependent population dynamics, stochastic processes with multiplicative noise); improvements by mesh adaptivity may reduce because *modified schemes loose critical features* (conservation form, well-balancing, ...)

numerical issues : accurate computation of non-constant steady states, occurring for the balance between source term and internal forces; unstructured grids required for multi-dimensional problems incorporating composite physical geometries

- B.D. Rogers, A.G.L. Borthwick, P.H. Taylor, Mathematical balancing of flux gradient and source terms prior to using Roe's approximate Riemann solver, *J. Comput. Phys.* (2003)
- M.J. Castro, P. Garcia-Navarro, The application of a conservative grid adaptation technique to 1D shallow water equations, *Math. Comput. Modelling* (2001)
- S. Karni, A. Kurganov, G. Petrova, A smoothness indicator for adaptive algorithms for hyperbolic systems, *J. Comput. Phys.* (2002)

analytical studies : recent theoretical advances on adaptive techniques for mesh refinement; consistency properties of finite volume schemes setting on nonuniform grids with respect to (strong) convergence

- G. Puppo, M. Semplice, Numerical entropy and adaptivity for finite volume schemes, *Commun. Comput. Phys.* (2011)

- C. Arvanitis, A.I. Delis, Behavior of finite volume schemes for hyperbolic conservation laws on adaptive redistributed spatial grids, *J. Sci. Comput.* (2006)
- C. Arvanitis, Ch. Makridakis, N.I. Sfakianakis, Entropy conservative schemes and adaptive mesh selection for hyperbolic conservation laws, *J. Hyperbolic Differ. Equ.* (2010)

numerical simulations for smooth data with periodic boundary conditions

experimental errors at time $T=1.5$, with $a=0.5$, $b=1.0$ and $CFL=0.9$

the nonuniform meshes for the computation are *arbitrarily generated* (not compatible with some L^p-type regularity condition)

cells	$\|\|e(t)\|\|_{L^1}$		$\|\|e(t)\|\|_{L^2}$		$\|\|e(t)\|\|_{L^\infty}$	
		rates		rates		rates
30	0.313172E-01		0.365947E-01		0.596413E-01	
60	0.147558E-01	1.105	0.174315E-01	1.109	0.251566E-01	1.173
120	0.724135E-02	1.071	0.777632E-02	1.073	0.138610E-01	1.132
240	0.350149E-02	1.055	0.374801E-02	1.056	0.556219E-02	1.111
480	0.181983E-02	1.043	0.198378E-02	1.042	0.264091E-02	1.108
960	0.732448E-03	1.023	0.781343E-03	1.029	0.824566E-03	1.089

Table: standard (first order) scheme on highly nonuniform grids

– nonuniform mesh with strong inhomogeneity of the cells' size :

$$h>0\,,\ \gamma\gg 1\,,\quad \Delta x_i = \begin{cases} h & i=2\,k \\ \gamma\,h & i=2\,k+1 \end{cases}\,,\ k\in\mathbb{Z}$$

accuracy test for the Saint-Venant equations of shallow waters through the *kinetic scheme with reflections*

- B. Perthame, C. Simeoni, A kinetic scheme for the Saint-Venant system with a source term, *Calcolo* (2001)

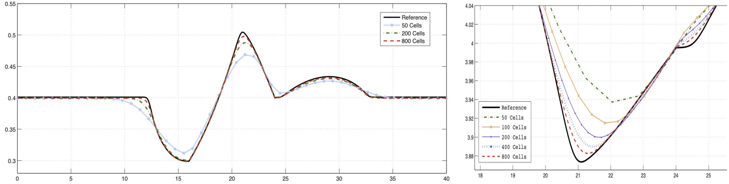

Figure: continuous solutions with discontinuity in the derivatives

cells	$\|\|e(t)\|\|_{L^1}$		$\|\|e(t)\|\|_{L^2}$		$\|\|e(t)\|\|_{L^\infty}$	
		rates		rates		rates
50	0.2251		0.0768		0.0460	
100	0.1870	0.2673	0.0637	0.2682	0.0466	-0.0198
200	0.0927	1.0126	0.0309	1.0456	0.0224	1.0590
400	0.0492	0.9134	0.0163	0.9218	0.0113	0.9849
800	0.0259	0.9254	0.00874	0.8976	0.00570	0.9863
1600	0.0157	0.7197	0.00627	0.4802	0.00388	0.558

Table: experimental errors at time $T = 1.0$ for parabolic source term finite volume schemes because of the *conservation property*, possible discrete versions of the *entropy inequalities*, and implementation with low regularity external fields (*integral formulation*)

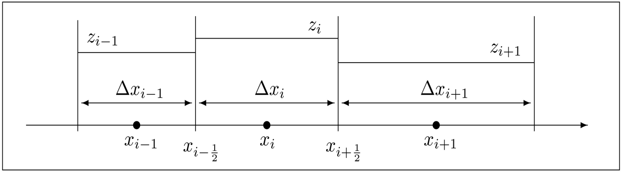

FIGURE 1. piecewise constant reconstruction on the spatial mesh

non-degeneracy constraint for nonuniform grids :

$$\exists\, \alpha\,,\beta > 0 \;/\; \alpha\,\Delta x_{i+1} \leq \Delta x_i \leq \beta\,\Delta x_{i+1}\,,\ \forall i \in \mathbb{Z}$$

variable time-step $\Delta t_n = t_{n+1} - t_n\,,\, n \in \mathbb{N}$

cell-centered discrete unknowns

$$v_i^n \approx \frac{1}{\Delta x_i} \int_{C_i} u(t_n, x)dx = u(t_n, x_i) + \mathcal{O}(h^2)$$

characteristic parameters $h = \max_{i\in\mathbb{Z}} \Delta x_i$ and $\Delta t = \sup_{n\in\mathbb{N}} \Delta t_n$

the simplest explicit (three points) interfacial upwind (and eventually well-balanced) scheme reads

$$\frac{v_i^{n+1} - v_i^n}{\Delta t_n} + a\,\frac{v_i^n - v_{i-1}^n}{\Delta x_i} + b(v_i^n)\,\frac{z_i - z_{i-1}}{\Delta x_i} = 0$$

together with initial data and boundary conditions

– the standard CFL-condition on the (local) ratio $\Delta t_n/\Delta x_i$ guarantees the numerical stability (adjusted to include the numerical source term)

• S. Osher, R. Sanders, Numerical approximations to nonlinear conservation laws with locally varying time and space grids, *Math. Comp.* (1983)

– extension to multi-dimensional finite volumes through the splitting of one-dimensional schemes (with application to linear systems)

• M. Ben-Artzi, J. Falcovitz, An upwind second-order scheme for compressible duct flows, *SIAM J. Sci. Statist. Comput.* (1986)
• P.L. Roe, Upwind differencing schemes for hyperbolic conservation laws with source terms, *Lecture Notes in Math.* (1987)
• A. Bermudez, M.E. Vazquez, Upwind methods for hyperbolic conservation laws with source terms, Computers & Fluids (1994)

for the cell-averages $g_i = \frac{1}{\Delta x_i}\int_{C_i} g(x)dx$ of any function $g \in W^{2,p}$, $1 \le p < +\infty$, thanks to the symmetry of cell-centered integrals

$$g_i = g(x_i) + \frac{1}{\Delta x_i}\int_{C_i} g''(\xi(x))\frac{(x-x_i)^2}{2}\,dx$$
$$= g(x_{i-\frac{1}{2}}) + g'(x_{i-\frac{1}{2}})\frac{\Delta x_i}{2} + \mathcal{O}(h, \|g''\|_{L^p})$$

$$g_i - g_{i-1} = g'(x_i)\left(\frac{\Delta x_{i-1}}{2} + \frac{\Delta x_i}{2}\right) + \frac{1}{\Delta x_i}\int_{C_i} g''(\eta(x))\,\Theta(x)\,dx$$

with $\Theta(x) = (\xi(x) - x_i)\big((x - x_{i-\frac{3}{2}}) - \frac{\Delta x_{i-1}}{\Delta x_i}(x - x_{i-\frac{1}{2}})\big)$
for some $\xi(x), \eta(x) \in C_i \cup C_{i-1}$

providing the correct approximation on nonuniform meshes

– formal estimations about the *consistency error*
– additional condition $b(u) > 0$ for the *stationary equations*
– *smooth solutions* for which expansions can be performed

the (local) truncation error is evaluated by returning the analytical solution into the discrete formulation, with $u_i^n = u(t_n, x_i)$

$$T_i^n = \partial_t u_i^n + \frac{\Delta x_{i-1} + \Delta x_i}{2\,\Delta x_i} \Big[\, a\,\partial_x u_i^n + b(u_i^n) z'(x_i) \,\Big] + \mathcal{O}(\Delta t, h)$$

revealing a lack of consistency with the underlying balance equation

the space-step Δx_i could be very different from the length of an interfacial interval $|x_i - x_{i-1}| = \frac{\Delta x_{i-1}}{2} + \frac{\Delta x_i}{2}$, and the pointwise consistency error does not vanish : unless the spatial mesh is quasi-uniform, namely $\Delta x_{i-1} = \Delta x_i + \mathcal{O}(h^2)$, it seems that convergence by the *Lax theorem* cannot be expected, or at least a significant reduction in the rates occurs...

- J.D. Hoffman, Relationship between the truncation errors of centered finite-difference approximations on uniform and nonuniform meshes, *J. Comput. Phys.* (1982)
- E. Turkel, Accuracy of schemes with nonuniform meshes for compressible fluid flows, *Appl. Numer. Math.* (1986)
- J. Pike, Grid adaptive algorithms for the solution of the Euler equations on irregular grids, *J. Comput. Phys.* (1987)

formal accuracy of finite volume schemes is actually maintained on nonuniform meshes, because of the supra-convergence phenomenon

– investigated first for homogeneous hyperbolic conservation laws

- A.N. Tikhonov, A.A. Samarsky, On the theory of homogeneous difference schemes, *Outlines Joint Sympos. Partial Differential Equations* (Novosibirsk, 1963)
- H.-O. Kreiss, T.A. Manteuffel, B. Swartz, B. Wendroff, A.B. White Jr., Supra-convergent schemes on irregular grids, *Math. Comp.* (1986)
- B. Wendroff, A.B. White Jr., A supraconvergent scheme for nonlinear hyperbolic systems, *Comput. Math. Appl.* (1989)

??? (fully discrete) convergence at optimal rates for smooth solutions of upwind schemes for linear equations on two-dimensional triangulations

- B. Després, Lax theorem and finite volume schemes, *Math. Comp.* (2004)
- B. Després, An explicit a priori estimate for a finite volume approximation of linear advection on non-Cartesian grids, *SIAM J. Numer. Anal.* (2004)

??? comprehensive interpretation of *a priori error estimates*, based on the *Kuznetsov's theory*, for (scalar) nonlinear problems with low regularity

- B. Cockburn, P.-A. Gremaud, A priori error estimates for numerical methods for scalar conservation laws. II. Flux-splitting monotone schemes on irregular Cartesian grids, *Math. Comp.* (1997)

the <u>discrete unknowns</u> (and external fields) are replaced by

$$v_i^n \approx \frac{1}{\Delta x_i} \int_{C_i} \Big[u(t_n, x) + \frac{\Delta x_i}{2} \partial_x u(t_n, x) \Big] dx \,, \quad n \in \mathbb{N}\,,\; i \in \mathbb{Z}$$

with a first order correction to compensate the truncation error, since it represents precisely the <u>discrepancy between cell-centered and interfacial averages</u>, which do not coincide for nonuniform grids

• D. Bouche, J.-M. Ghidaglia, F. Pascal, Error estimate and the geometric corrector for the upwind finite volume method applied to the linear advection equation, *SIAM J. Numer. Anal.* (2005)

the modified equation of the scheme for the new reconstruction reads

$$\begin{aligned} R_i^n = {} & \partial_t u_i^n + a\,\partial_x u_i^n + b(u_i^n) z'(x_i) + \\ & + \frac{\Delta x_{i-1}}{2} \Big[a\,\partial_{xx} u_i^n + b(u_i^n) z''(x_i) \Big] \frac{\Delta x_{i-1} + \Delta x_i}{2\,\Delta x_i} + \\ & + \frac{\Delta x_i}{2} \Big[\partial_{tx} u_i^n + b'(u_i^n) z'(x_i) \partial_x u_i^n \Big] + \mathcal{O}(\Delta t, h) \end{aligned}$$

for $u_i^n = u(t_n, x_i)$, with extra terms involving (admissible) higher regularity of the solution, and bounded through the *mesh's condition* first order approximation justifies to use

$$E_i^n = v_i^n - u(t_n, x_i) - \frac{\Delta x_i}{2} \partial_x u(t_n, x_i)\,, \quad S_i^n = z_i - z(x_i) - \frac{\Delta x_i}{2} z'(x_i)$$

as reference quantities for error analysis : fixing $b(u) = b > 0$ to avoid inessential technicality, the typical stability equation of the scheme

$$E_i^{n+1} = E_i^n - a \frac{\Delta t_n}{\Delta x_i}\left(E_i^n - E_{i-1}^n\right) - b \frac{\Delta t_n}{\Delta x_i}\left(S_i^n - S_{i-1}^n\right) - \Delta t_n R_i^n$$

provides convergence with optimal rates for (regular) nonuniform grids

– diffusive and dispersive characteristics of the numerical method

– the truncation error always vanishes for the simulation of steady states, for the well-balance property (besides an overall stability)

– *conservative schemes* perform substantially better on unstructured meshes in comparison to those not preserving some special structures even for uniform meshes

• O.V. Vasilyev, High order finite difference schemes on non-uniform meshes with good conservation properties, *J. Comput. Phys.* (2000)

for many applications to real systems, the advection may become negligible, so that the external fields dominate over the fluxes...

cells	$\|\|e(t)\|\|_{L^1}$		$\|\|e(t)\|\|_{L^2}$		$\|\|e(t)\|\|_{L^\infty}$	
		rates		rates		rates
30	0.208333E+01		0.143466E+01		0.151412E+01	
60	0.229167E+01	-0.138	0.152438E+01	-0.088	0.275635E+01	-0.860
120	0.260417E+01	-0.161	0.151382E+01	-0.039	0.253439E+01	-0.370
240	0.255208E+01	-0.098	0.155620E+01	-0.039	0.144264E+01	0.023
480	0.252604E+01	-0.069	0.157700E+01	-0.034	0.202358E+01	-0.105
960	0.248698E+01	-0.051	0.159362E+01	-0.030	0.176882E+01	-0.045

Table: standard (first order) scheme on highly nonuniform grid for $a = 0.005$

cells	$\|\|e(t)\|\|_{L^1}$		$\|\|e(t)\|\|_{L^2}$		$\|\|e(t)\|\|_{L^\infty}$	
		rates		rates		rates
30	0.308642E-01		0.156844E+00		0.254948E-01	
60	0.169753E-01	0.862	0.887122E-01	0.818	0.164315E-01	1.108
120	0.925926E-02	0.868	0.424803E-01	0.939	0.787522E-02	1.093
240	0.462963E-02	0.912	0.216954E-01	0.953	0.384303E-02	1.075
480	0.231481E-02	0.934	0.127825E-01	0.905	0.189368E-02	1.062
960	0.115138E-02	0.949	0.739794E-02	0.882	0.698813E-02	1.055

Table: modified (first order) scheme on highly nonuniform grid for $a = 0.005$

- Th. Katsaounis, C. Simeoni, Three-points interfacial quadrature for geometrical source terms on nonuniform grids, *Calcolo* (2011) steady states for sinusoidal source term : *adaptive techniques* customarily generate quasi-uniform grids, and the *well-balance property* significantly improve the numerical accuracy

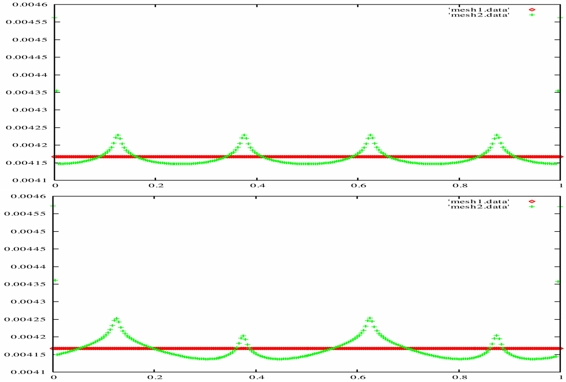

Figure: cells sizes after mesh refinement (green) over uniform mesh

• C. Arvanitis, Mesh redistribution strategies and finite element schemes for hyperbolic conservation laws, *J. Sci. Comput.* (2008)

cells	$\|\|e(t)\|\|_{L^1}$		$\|\|e(t)\|\|_{L^2}$		$\|\|e(t)\|\|_{L^\infty}$	
		rates		rates		rates
30	0.195000E-02		0.232276E-02		0.419508E-02	
60	0.569572E-01	1.820	0.679157E-01	1.819	0.123060E-02	1.814
120	0.218606E-01	1.619	0.260709E-01	1.618	0.470566E-01	1.619
240	0.960606E-00	1.484	0.114540E-01	1.484	0.206341E-01	1.485
480	0.450752E-00	1.391	0.537440E-00	1.391	0.967720E-00	1.392
960	0.221537E-00	1.234	0.294508E-00	1.233	0.461213E-00	1.234

Table: experimental well-balance error $= 0.224508E-02$

– the <u>well-balance error is predominant for finer grids</u>, and higher rates are observed at longer times for the <u>smoothing effects of mesh refinement</u>

– difficulties are undervalued in the simulation of stationary solutions, so that testing the convergence rates may not be really effective...

– <u>counter-examples</u> for well-balanced schemes <u>with centered fluxes</u> on strongly or adaptive nonuniform meshes (uniform block-structured grids)

– (theoretical) convergence does not mean accurate pointwise simulation the general (finite volume) Upwind Source at Interface scheme reads

$$\frac{\Delta x_i}{\Delta t_n}\left(v_i^{n+1}-v_i^n\right)+\left(A^n_{i+\frac{1}{2}}-A^n_{i-\frac{1}{2}}\right)+B^{n,+}_{i-\frac{1}{2}}+B^{n,-}_{i+\frac{1}{2}}=0$$

and the numerical source term does not take a conservative form,

$$B^{n,\pm}_{i+\frac{1}{2}}=B^{\pm}\left(\Delta x_i,\Delta x_{i+1},v_i^n,v_{i+1}^n,z_{i+1}-z_i\right)$$

the minimal (global) consistency requirement, for $K_B>0$ constant,

$$\left|\frac{B^-(h,k,u,u,\lambda)+B^+(h,k,u,u,\lambda)}{\lambda}-b(u)\right|\leq K_B\,\lambda$$

is actually a *structural property* (not derived from the modified equation) and thus it could fail the truncation error to vanish for nonuniform grids

• B. Perthame, C. Simeoni, Convergence of the upwind interface source method for hyperbolic conservation laws, Hyperbolic problems: theory, numerics, applications, 61-78, Springer, Berlin, 2003 (invited paper)

well-balanced schemes are consistent (Godunov solvers, VFRoe schemes, relaxation and central methods, kinetic schemes) for the supra-convergence

• F. Bouchut, Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources, Frontiers in Mathematics, Birkhäuser (2004)

??? ??? ???

influence of the non-uniformity of grids on the convergence's rates =
effect of grid irregularity on the accuracy of finite volume algorithms

– alternative technique for rigorous proofs, because *entropy methods* following the *Kružkov's theory* do not provide the rates of convergence (and BV-bounds for strong convergence hold uniquely for uniform grids)

- M.G. Crandall, A. Majda, Monotone difference approximations for scalar conservation laws, *Math. Comp.* (1980)
- R. Sanders, On convergence of monotone finite difference schemes with variable spatial differencing, *Math. Comp.* (1983)

– thanks to an explicit formulation in terms of numerical derivatives, mesh-dependent extensions of the *Lax-Wendroff method* (on staggered grids) retrieve conservation form and standard (local) consistency

- M. Dumbser, A. Hidalgo, M. Castro, C. Parés, E.F. Toro, FORCE schemes on unstructured meshes II : Non-conservative hyperbolic systems, *Comput. Methods Appl. Mech. Engrg.* (2010)