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Computing the first eigenpair of the p-Laplacian via inverse iteration of sublinear supersolutions

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Abstract

We introduce an iterative method for computing the first eigenpair (λ_p, e_p) for the p-Laplacian operator with homogeneous Dirichlet data as the limit of (μ_q, u_q) as $q \to p^-$, where u_q is the positive solution of the sublinear Lane-Emden equation $-\Delta_p u_q = \mu_q u_q^{q-1}$ with the same boundary data. The method is shown to work for any smooth, bounded domain. Solutions to the Lane-Emden problem are obtained through inverse iteration of a supersolution which is derived from the solution to the torsional creep problem. Convergence of u_q to e_p is in the C^1-norm and the rate of convergence of μ_q to λ_p is at least $O(p-q)$. Numerical evidence is presented.

Keywords: p-Laplacian, first eigenvalue and eigenfunction, inverse iteration, Lane-Emden problem, torsional creep problem.

1 Introduction

In this paper we develop an iterative method to obtain the first eigenpair (λ_p, e_p) of the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $p > 1$, is the p-Laplacian operator and $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is any smooth, bounded domain. The p-Laplacian equation appears in several mathematical models in fluid dynamics, such as in the modelling of non-Newtonian fluids and glaciology [5, 16, 26, 41], turbulent flows [22], climatology [21] nonlinear diffusion (where it is called the N-diffusion equation; see [42] for the original article and [28] for some current developments), flow through porous media [45], power law materials [6] and in the study of torsional creep [34].

The first eigenvalue λ_p of (1) is variationally characterized by

$$\lambda_p = \min_{u \in W_0^{1,p}(\Omega)/\{0\}} R(u) > 0$$

where R is the Rayleigh quotient

$$R(u) = \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}.$$

The first eigenfunction e_p of (1) is characterized by the fact that the minimum of R is attained at e_p, so that

$$\lambda_p = \frac{\int_\Omega |\nabla e_p|^p \, dx}{\int_\Omega e_p^p dx}.$$

It is well-known that λ_p is isolated and simple, and that the corresponding eigenfunction $e_p \in C^{1,\alpha}\left(\overline{\Omega}\right)$ can be taken positive. Since R is homogeneous, we may assume $\|e_p\|_\infty = 1$, where $\|\cdot\|_\infty$ stands for the L^∞-norm.

In the one-dimensional case the first eigenpair (λ_p, e_p) is explicitly determined by solving the corresponding ODE boundary value problem. If $\Omega = (a, b)$, then $\lambda_p = (\pi_p/(b-a))^{p-1}$ and $e_p = (p-1)^{-1/p} \sin_p(\pi_p(x-a)/(b-a))$, where $\pi_p := 2(p-1)^{1/p}\int_0^1 (1-s^p)^{-1/p} ds$ and $\sin_p$ is a $2\pi_p$-periodic function that generalizes the classical sine function (see [11, 40]).

When $p = 2$, we have $\Delta_p = \Delta$, the Laplacian operator, whose first eigenpair (λ_p, e_p) is well-known for domains with simple geometry (that is, domains which admit some kind of symmetry); for more general domains it can be determined by several numerical methods (see [12, 20, 27, 30, 36]). However, if $p \neq 2$ and $N \geqslant 2$, the first eigenpair is not explicitly known even for simple symmetric domains such as a square or a ball, and there are few available numerical methods to deal directly with the eigenproblem (1) in these domains (see [10, 14, 15, 16, 38, 47]).

On the other hand, several numerical methods are available to solve homogeneous Dirichlet problems for the (Poisson) p-Laplacian equation in the form

$$-\Delta_p u = f(x)$$

when f depends only on $x \in \Omega$ (see [2, 8, 9, 23, 25, 46]). This fact motivated the development of an inverse iteration method by some of the authors for finding the first eigenpair in [10]. If Ω is a N-dimensional ball, the convergence of the method was established and numerical evidence for its applicability when Ω is a 2-dimensional square were also presented. In the special case of the Laplacian operator, the method was proved to work in general domains and can be also used to obtain other eigenpairs (see [12]). However, since the method was based on the iteration

of the nonlinear p-Laplacian equation in (1), the difficulties in dealing with the nonlinearity on the right-hand side of the equation prevented showing that the method works in any domain and any $p > 1$.

In this work we consider a different inverse iteration approach, also based on the solution of the Poisson p-Laplacian equation, but built around an eigenproblem which has a sublinear nonlinearity on its right-hand side. This type on nonlinearity is more manageable and we are able to prove that the iterative method works for any smooth, bounded domain. It is based on obtaining positive solutions $v_{\mu,q}$ for the Lane-Emden type problem

$$\begin{cases} -\Delta_p v = \mu \left|v\right|^{q-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

After rescaling, μ and $v_{\mu,q}$ produce a family of pairs $\{(\mu_q, u_q)\}_{1<q<p}$ converging to the first eigenpair (λ_p, e_p) when $q \to p^-$, the convergence $u_q \to e_p$ being in $C^1\left(\overline{\Omega}\right)$. We will now describe the method in more detail.

It is well known that for each fixed $\mu > 0$, problem (2) has a unique solution $v_{\mu,q}$, if $1 < q < p$ (see [32]). If $q = p$, we have the p-Laplacian eigenvalue problem. If $q > p$, positive solutions of (2) usually are not unique. A nonuniqueness result for ring-shaped domains is given in [7] when q is close to the Sobolev critical exponent p^* ($p^* = Np/\left(N-p\right)$, if $1 < p < N$, and $p^* = \infty$, if $p \geqslant N$). On the other hand, as proved in [1], positive solutions are unique when Ω is a ball, while for general bounded domains the uniqueness of positive solutions that reach the minimum energy (ground states) was established in [24] under the conditions $1 < p < N$ and $1 < q < p^*$.

Now, in order to construct the approximating sequence to the first eigenpair, first choose any $\mu > 0$ and a sequence (q_n), $1 < q_n < p$, such that $q_n \to p^-$. It is important to notice that μ need not to be taken close to λ_p. This point is crucial, since good *a priori* estimates for λ_p are hard to obtain. For each q_n we need to solve the Lane-Emden problem (2) in order to find v_{μ,q_n}, which is a degenerate nonlinear problem almost as hard to solve as the eigenvalue problem for the p-Laplacian (1) itself. In order to obtain the solutions v_{μ,q_n} we first solve the much easier *torsional creep problem*

$$\begin{cases} -\Delta_p \phi = 1 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Then compute $k_p = \|\phi\|_\infty^{1-p}$ and set

$$\phi_0 = \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q_n}} \frac{\phi}{\|\phi\|_\infty}. \tag{4}$$

ϕ_0 is a supersolution to (2). One immediately sees that the easiest choice is $\mu = k_p$, so that $\phi_0 = \phi/\|\phi\|_\infty$. Now apply an inverse iteration to ϕ_0, finding a sequence of iterates (ϕ_m) which satisfy

$$\begin{cases} -\Delta_p \phi_{m+1} = \mu \phi_m^{q_n - 1} & \text{in } \Omega, \\ \phi_{m+1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

This can be done by a number of numerical methods. Finite volume based methods are presented in [4, 25]; finite element based methods are also available (see [31] and the references therein).

After a pre-established tolerance limit has been reached at some ϕ_m, where m is a function of μ and q_n, set

$$v_{\mu,q_n} = \phi_m$$

and define u_{q_n} and μ_{q_n} as

$$\mu_{q_n} := \frac{\mu}{\|v_{\mu,q_n}\|_\infty^{p-q_n}} \quad \text{and} \quad u_{q_n} := \frac{v_{\mu,q_n}}{\|v_{\mu,q_n}\|_\infty}.$$

In Theorem 8 we show that $\mu_{q_n} \to \lambda_p$ and $u_{q_n} \to e_p$ in $C^1\left(\overline{\Omega}\right)$ when $q_n \to p^-$. Choosing a value for q close to p will give an approximation for the first eigenpair of the p-Laplacian. The procedure is summarized in Algorithm 1 below.

Algorithm 1 Inverse iteration for the first p-Laplacian eigenpair (λ_p, e_p)

set μ	(an arbitrary positive number)
set q	(q should be chosen close to p)
solve $-\Delta_p \phi = 1$ in Ω, $\phi = 0$ on $\partial\Omega$	(torsion function)
set $\phi_0 = (\mu/k_p)^{\frac{1}{p-q}} \phi / \|\phi\|_\infty$	(supersolution)
for $m = 0, 1, 2, \ldots$ **do**	
solve $-\Delta_p \phi_{m+1} = \mu \phi_m^{q-1}$ in Ω, $\phi_{m+1} = 0$ on $\partial\Omega$	(Inverse iteration sequence)
end for	
return $\mu / \|\phi_{m+1}\|_\infty^{p-q}$	(first eigenvalue λ_p)
return $\phi_{m+1} / \|\phi_{m+1}\|_\infty$	(first eigenfunction e_p)

However, we are able to produce a much more robust algorithm which is also easier to apply in practice. In Algorithm 2 below one does not need to use an arbitrary parameter μ nor to compute the value of the constant k_p. Normalization of the iterates at each step increases robustness and thus it should be the algorithm of choice.

Algorithm 2 Inverse iteration for the first p-Laplacian eigenpair (λ_p, e_p) with normalization

set q	(q should be chosen close to p)
solve $-\Delta_p \phi_0 = 1$ in Ω, $\phi_0 = 0$ on $\partial\Omega$	(torsion function)
for $m = 0, 1, 2, \ldots$ **do**	
solve $-\Delta_p \phi_{m+1} = (\phi_m / \|\phi_m\|_\infty)^{q-1}$ in Ω, $\phi_{m+1} = 0$ on $\partial\Omega$	(Inverse iteration sequence)
end for	
return $1 / \|\phi_{m+1}\|_\infty^{p-1}$	(first eigenvalue λ_p)
return $\phi_{m+1} / \|\phi_{m+1}\|_\infty$	(first eigenfunction e_p)

The outline of the paper is as follows. In Section 2 we present some preliminary results that will be used in the sequel. The sequences of approximates for both algorithms are built in Section 3 and the proof of their convergence to the first eigenpair is given in Section 4. In Section 5 we present some numerical results for the unit ball of dimensions $N = 2, 3, 4$ using the first algorithm, and for the two-dimensional square and the three-dimensional cube and torus using the second algorithm.

The main advantage of the method presented here, besides its applicability to general domains, is that approximations to both λ_p and e_p are obtained with the desired precision by an iterative process which is numerically simple and, in the case of a ball, also explicit.

2 Preliminary results

In this section we state simple versions of some results on the p-Laplacian. We begin with the following comparison principle (see [19] for a more general version).

Lemma 1 *For $i \in \{1,2\}$, let $h_i \in C\left(\overline{\Omega}\right)$ and $u_i \in W^{1,p}(\Omega)$ be such that $-\Delta_p u_i = h_i$ in Ω. If $h_1 \leq h_2$ in Ω and $u_1 \leqslant u_2$ on $\partial\Omega$, then $u_1 \leqslant u_2$ in Ω.*

The following result is a simple version of a general result proved in the classical paper [39] of Lieberman.

Theorem 2 *[39, Thm 1] Suppose that $u \in W^{1,p}(\Omega)$ is a weak solution of the Dirichlet problem*

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where f is a continuous function such that

$$|f(x,\xi)| \leqslant \Lambda \quad \text{for all } (x,\xi) \in \Omega \times [-M, M]$$

for positive constants Λ and M.

If $\|u\|_\infty \leqslant M$, then there exists $0 < \alpha < 1$, depending only on Λ, p and the dimension N, such that $u \in C^{1,\alpha}\left(\overline{\Omega}\right)$; moreover we have

$$\|u\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \leqslant C,$$

where C is a positive constant that depends only on Λ, p, N and M.

Thus, denoting by ϕ is the solution of the torsional creep problem (3) in the domain Ω, one can easily verify using (12) and the comparison principle in balls that $0 < \phi \leqslant M$ in Ω for some positive constant M. Hence, Theorem 2 implies that $\phi \in C^{1,\alpha}\left(\overline{\Omega}\right)$ for some $0 < \alpha < 1$.

For the next lemma set

$$k_p := \|\phi\|_\infty^{1-p} > 0. \tag{6}$$

Lemma 3 $k_p \leqslant \lambda_p$.

Proof. Let e_p be the first eigenfuncion associated with λ_p satisfying $\|e_p\|_\infty = 1$ in Ω. Since

$$\begin{cases} -\Delta_p e_p = \lambda_p e_p^{p-1} \leqslant \lambda_p = -\Delta_p \left(\lambda_p^{\frac{1}{p-1}} \phi\right) & \text{in } \Omega, \\ e_p = 0 = \lambda_p^{\frac{1}{p-1}} \phi & \text{on } \partial\Omega, \end{cases}$$

it follows from the comparison principle that

$$0 < e_p \leqslant \lambda_p^{\frac{1}{p-1}} \phi \quad \text{in } \Omega.$$

Hence,

$$1 = \|e_p\|_\infty \leqslant \lambda_p^{\frac{1}{p-1}} \|\phi\|_\infty ,$$

from what follows our claim. ■

Remark 4 *It follows from Picone's identity (see [3]) that, in fact, the inequality is strict, that is, $k_p < \lambda_p$ (for details, see [17, Lemma 8.1]).*

The following result is well-known and follows from Theorem 2.

Theorem 5 *Let $-\Delta_p^{-1} : C^1\left(\overline{\Omega}\right) \to W_0^{1,p}(\Omega)$ be the operator defined as follows: for each $v \in C^1\left(\overline{\Omega}\right)$ let $-\Delta_p^{-1} v := u \in W_0^{1,p}(\Omega)$ be the unique solution of the Dirichlet problem*

$$\begin{cases} -\Delta_p u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $-\Delta_p^{-1}$ is continuous and compact. Moreover, $-\Delta_p^{-1} v \in C^{1,\alpha}\left(\overline{\Omega}\right)$ for each $v \in C^1\left(\overline{\Omega}\right)$.

In the remainder of the paper (λ_p, e_p) denotes the first eigenpair of (1), ϕ denotes the torsion function of Ω and $k_p := \|\phi\|_\infty^{1-p}$.

3 Construction of the sequence of approximates

As mentioned before, if $1 < q < p$, then for each $\mu > 0$ the Lane-Emden problem

$$\begin{cases} -\Delta_p v = \mu |v|^{q-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

has a unique positive solution $v_{\mu,q}$, which can be obtained via standard variational, and therefore *non-constructive*, arguments. The existence and uniqueness of solutions of (7) in the case $1 < q < p$ implies that the map $\mu \mapsto v_{\mu,q}$ is well-defined and monotone, in the sense that $\mu_1 < \mu_2$ implies $v_{\mu_1,q} < v_{\mu_2,q}$ in Ω, since $v_{\mu_1,q} = (\mu_1/\mu_2)^{1/(p-q)} v_{\mu_2,q}$ for any μ_1, $\mu_2 > 0$.

The basis of our constructive method is given by

Theorem 6 *Suppose $1 < q < p$. For each $\mu > 0$ the unique positive solution $v_{\mu,q} \in C^{1,\alpha}\left(\overline{\Omega}\right) \cap W_0^{1,p}(\Omega)$ of (7) satisfies*

$$0 < \left(\frac{\mu}{\lambda_p}\right)^{\frac{1}{p-q}} e_p \leqslant v_{\mu,q} \leqslant \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q}} \frac{\phi}{\|\phi\|_\infty} \quad \text{in } \Omega. \tag{8}$$

Moreover, $v_{\mu,q}$ is the limit, in the $C^1\left(\overline{\Omega}\right)$ norm, of the sequence $\{v_n\} \subset C^{1,\alpha}\left(\overline{\Omega}\right) \cap W_0^{1,p}(\Omega)$ iteratively defined by

$$v_1 := \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q}} \frac{\phi}{\|\phi\|_\infty} \tag{9}$$

and, for $n \geqslant 1$,

$$\begin{cases} -\Delta_p v_{n+1} = \mu v_n^{q-1} & \text{in } \Omega, \\ v_{n+1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Proof. Define $\underline{v}_{\mu,q} := me_p$ and $\overline{v}_{\mu,q} := \dfrac{M\phi}{\|\phi\|_\infty}$ where

$$m := \left(\frac{\mu}{\lambda_p}\right)^{\frac{1}{p-q}} \quad \text{and} \quad M := \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q}}.$$

We have

$$-\Delta_p \underline{v}_{\mu,q} \leqslant \mu \underline{v}_{\mu,q}^{q-1} \quad \text{and} \quad -\Delta_p \overline{v}_{\mu,q} \geqslant \mu \overline{v}_{\mu,q}^{q-1} \quad \text{in } \Omega. \tag{11}$$

Indeed, in Ω in we have

$$-\Delta_p \underline{v}_{\mu,q} = \lambda_p \underline{v}_{\mu,q}^{p-1} = \lambda_p \underline{v}_{\mu,q}^{p-q} \underline{v}_{\mu,q}^{q-1} = \lambda_p (me_p)^{p-q} \underline{v}_{\mu,q}^{q-1} \leqslant \lambda_p m^{p-q} \underline{v}_{\mu,q}^{q-1} = \mu \underline{v}_{\mu,q}^{q-1}$$

and

$$-\Delta_p \overline{v}_{\mu,q} = k_p M^{p-1} = k_p M^{p-q} M^{q-1} \geqslant k_p M^{p-q} \left(\frac{M\phi}{\|\phi\|_\infty}\right)^{q-1} = \mu \overline{v}_{\mu,q}^{q-1}.$$

Since $\underline{v}_{\mu,q} = 0 = \overline{v}_{\mu,q}$ on $\partial\Omega$ the inequalities in (11) mean that $\underline{v}_{\mu,q}$ and $\overline{v}_{\mu,q}$ are, respectively, sub- and supersolutions for (7).

Moreover, $\underline{v}_{\mu,q}$ and $\overline{v}_{\mu,q}$ are ordered, that is $\underline{v}_{\mu,q} \leqslant \overline{v}_{\mu,q}$ in Ω. For, since $k_p \leqslant \lambda_p$, we have

$$\begin{aligned} \lambda_p m^{p-1} &= \lambda_p \left(\frac{\mu}{\lambda_p}\right)^{\frac{p-1}{p-q}} \\ &= \mu^{\frac{p-1}{p-q}} \left(\frac{1}{\lambda_p}\right)^{\frac{q-1}{p-q}} \leqslant \mu^{\frac{p-1}{p-q}} \left(\frac{1}{k_p}\right)^{\frac{q-1}{p-q}} = k_p \left(\frac{\mu}{k_p}\right)^{\frac{p-1}{p-q}} = k_p M^{p-1}, \end{aligned}$$

whence

$$-\Delta_p \underline{v}_{\mu,q} = \lambda_p \underline{v}_{\mu,q}^{p-1} \leqslant \lambda_p m^{p-1} \leqslant k_p M^{p-1} = -\Delta_p \overline{v}_{\mu,q}$$

in Ω. Thus, since $\underline{v}_{\mu,q} = \overline{v}_{\mu,q} = 0$ on $\partial\Omega$, we obtain $\underline{v}_{\mu,q} \leqslant \overline{v}_{\mu,q}$ in Ω by applying the comparison principle.

Since $u \mapsto \mu u^{q-1}$ is increasing and $\underline{v}_{\mu,q} \leqslant \overline{v}_{\mu,q}$ in Ω, the comparison principle also implies that the sequence $\{v_n\}$ defined by the iterative process (10) starting with the supersolution $\overline{v}_{\mu,q}$ satisfies

$$\underline{v}_{\mu,q} \leqslant v_{n+1} \leqslant v_n \leqslant \overline{v}_{\mu,q} \quad \text{in } \Omega.$$

Hence, v_n converges to a function $v_{\mu,q}$ a.e. in Ω. Since $\|v_n\|_\infty \leqslant \|\overline{v}_{\mu,q}\|_\infty = M$, it follows from Theorem 2 that $\{v_n\} \subset C^{1,\alpha}\left(\overline{\Omega}\right)$ for some $0 < \alpha < 1$ (which does not depend on n) and that

$$\|v_n\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \leqslant C$$

for some positive constant C which is independent of n.

Thus, from Arzela-Ascoli theorem we conclude that $v_n \to v$ in the C^1 norm.

Now, the continuity of the operator $-\Delta_p^{-1} : C^1\left(\overline{\Omega}\right) \to W_0^{1,p}(\Omega)$ permits passing to the limit in (10), which yields that $v_{\mu,q} \in C^1\left(\overline{\Omega}\right) \cap W_0^{1,p}(\Omega)$ is a solution of (7) satisfying

$$0 < \underline{v}_{\mu,q} \leqslant v_{\mu,q} \leqslant \overline{v}_{\mu,q} \text{ in } \Omega,$$

proving (8). The regularity $v_{\mu,q} \in C^{1,\alpha}\left(\overline{\Omega}\right)$ follows from Theorem 2. ■

This iterative process is also known as inverse iteration since $v_{n+1} = -\Delta_p^{-1}(\mu v_n^{q-1})$. It is essentially the sub- and supersolution method starting with the supersolution $\overline{v}_{\mu,q}$; the solution $v_{\mu,q}$ that it produces is characterized as the maximal solution between $\underline{v}_{\mu,q}$ and $\overline{v}_{\mu,q}$.

If one starts the iteration with the subsolution then one obtains an increasing sequence converging to the minimal solution between $\underline{v}_{\mu,q}$ and $\overline{v}_{\mu,q}$. Because of the uniqueness this minimal solution coincides with $v_{\mu,q}$. However, in order to compute the minimal solution from this iterative process, it is necessary to know *a priori* a subsolution, which is exactly one of the unknowns that we wish to find by applying the method.

On the other hand the supersolution $\overline{v}_{\mu,q}$ is easily obtainable since it involves the solution of the simpler problem (3).

For example, if $\Omega = B_R(x_0)$, the ball centered at $x_0 \in \mathbb{R}^N$ with radius $R > 0$, then it is easy to verify (see also below) that the *torsion function* ϕ is the radial function

$$\phi(r) = \frac{p-1}{pN^{\frac{1}{p-1}}}\left(R^{\frac{p}{p-1}} - |r|^{\frac{p}{p-1}}\right), \quad r = |x - x_0| \leqslant R. \tag{12}$$

We then obtain that

$$k_p = \|\phi\|_\infty^{1-p} = \frac{N}{R^p}\left(\frac{p}{p-1}\right)^{p-1} \tag{13}$$

and

$$\overline{v}_{\mu,q}(r) = \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q}}\left(1 - |r|^{\frac{p}{p-1}}\right) = \mu^{\frac{1}{p-q}}\left(\frac{p-1}{pN^{\frac{1}{p-1}}}\right)^{\frac{p-1}{p-q}}\left(1 - |r|^{\frac{p}{p-1}}\right)$$

where $r = |x - x_0|$.

In this case the sequence v_n converging to $v_{\mu,q}$ is given recursively by the formula

$$v_{n+1}(r) = \int_r^R \left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \mu v_n(s)^{q-1} ds\right)^{\frac{1}{p-1}} d\theta \tag{14}$$

where $v_0(r) = \overline{v}_{\mu,q}(r)$.

This integral formula follows from the more general fact: the Poisson problem

$$\begin{cases} -\Delta_p u = f(|x|) \ \text{ in } \ B_R(x_0) \\ u = 0 \ \text{ on } \ \partial B_R(x_0) \end{cases}$$

is equivalent to the ODE boundary value problem

$$\begin{cases} -\left(r^{N-1}\left|u'\right|^{p-2}u'\right)' = r^{N-1}f(r), \ 0 < r < R \\ u'(0) = 0 = u(R) \end{cases}$$

for radial solutions $u = u(r)$, $r = |x - x_0|$. Hence, after two integrations of the ODE taking into account the boundary conditions one obtains the following integral expression

$$u(r) = \int_r^R \left(\int_0^\theta (\frac{s}{\theta})^{N-1} f(s)ds \right)^{\frac{1}{p-1}} d\theta \tag{15}$$

for the solution $u = u(|x - x_0|)$ of the Poisson problem. In particular, when $f(r) \equiv 1$ this integral form might be simplified in order to find the expression (12) for the torsion function of B_R.

In our method, in order to compute the first eigenpair (λ_p, e_p), we fix a positive value $\mu > 0$ and choose q close to p^-. Then, we apply the inverse iteration of Theorem 6 starting with the supersolution

$$\overline{v}_{\mu,q} = \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q}} \frac{\phi}{\|\phi\|_\infty}$$

to obtain approximations for the function $v_{\mu,q}$. Hence,

$$\frac{\mu}{\|v_{\mu,q}\|_\infty^{p-q}} \to \lambda_p \quad \text{and} \quad \frac{v_{\mu,q}}{\|v_{\mu,q}\|_\infty} \to e_p \text{ (in the } C^1 \text{ norm)}$$

as $q \to p$, a result that we prove in the next section.

For the construction of the normalized sequence of Algorithm 2 one needs the following result:

Theorem 7 *Suppose $1 < q < p$. Then the normalized sequence $\{w_n / \|w_n\|_\infty\}$ where w_n is defined by*

$$w_0 := 1 \quad \textit{and} \quad \begin{cases} -\Delta_p w_{n+1} = \left(\dfrac{w_n}{\|w_n\|_\infty}\right)^{q-1} & \textit{in } \Omega \\ w_{n+1} = 0 & \textit{on } \partial\Omega \end{cases}$$

converges in the $C^1(\overline{\Omega})$ norm to $v_q / \|v_q\|_\infty$ where $v_q \in C^{1,\alpha}(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ is the solution of (7) with $\mu = k_p$.

Proof. Let $\{v_n\}$ be the sequence defined by

$$v_1 := \frac{\phi}{\|\phi\|_\infty} \quad \text{and} \quad \begin{cases} -\Delta_p v_{n+1} = k_p v_n^{q-1} & \text{in } \Omega \\ v_{n+1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

It follows from Theorem 6 that the sequence $\{v_n\}$ is decreasing and converges in the $C^1(\overline{\Omega})$ norm to the solution $v_q \in C^{1,\alpha}(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ of the Lane-Emden problem

$$\begin{cases} -\Delta_p v = k_p v^{q-1} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{17}$$

Since $w_1 = \phi$ we have that

$$w_2 = k_p^{-\frac{1}{p-1}} v_2$$

and, in particular $\dfrac{w_2}{\|w_2\|_\infty} = \dfrac{v_2}{\|v_2\|_\infty}$. In fact, this follows from the comparison principle:

$$\begin{aligned} -\Delta_p(k_p^{-\frac{1}{p-1}} v_2) &= k_p^{-1}\left(-\Delta_p v_2\right) \\ &= k_p^{-1} k_p v_1^{q-1} = \left(\frac{\phi}{\|\phi\|}\right)^{q-1} = \left(\frac{w_1}{\|w_1\|_\infty}\right)^{q-1} = -\Delta_p w_2. \end{aligned}$$

Repeating this procedure we obtain

$$\begin{aligned} -\Delta_p\left(k_p^{-\frac{1}{p-1}} \|v_2\|_\infty^{-\frac{q-1}{p-1}} v_3\right) &= k_p^{-1} \|v_2\|_\infty^{-(q-1)} \left(-\Delta_p v_3\right) \\ &= k_p^{-1} \|v_2\|_\infty^{-(q-1)} k_p v_2^{q-1} \\ &= \left(\frac{v_2}{\|v_2\|_\infty}\right)^{q-1} = \left(\frac{w_2}{\|w_2\|_\infty}\right)^{q-1} = -\Delta_p w_3, \end{aligned}$$

that is

$$w_3 = k_p^{-\frac{1}{p-1}} \|v_2\|_\infty^{-\frac{q-1}{p-1}} v_3$$

and

$$\frac{w_3}{\|w_3\|_\infty} = \frac{v_3}{\|v_3\|_\infty}.$$

Therefore, by an induction argument we conclude that

$$w_{n+1} = k_p^{-\frac{1}{p-1}} \|v_n\|_\infty^{-\frac{q-1}{p-1}} v_{n+1} \quad \text{and} \quad \frac{w_{n+1}}{\|w_{n+1}\|_\infty} = \frac{v_{n+1}}{\|v_{n+1}\|_\infty}$$

for all $n \geqslant 2$. Hence, it follows from Theorem 6 that

$$\frac{w_{n+1}}{\|w_{n+1}\|_\infty} = \frac{v_{n+1}}{\|v_{n+1}\|_\infty} \to \frac{v_q}{\|v_q\|_\infty}.$$

■

4 Convergence of the method

Theorem 8 *For $\mu > 0$ and for each $1 < q < p$ set*

$$u_q := \frac{v_{\mu,q}}{\|v_{\mu,q}\|_\infty}, \tag{18}$$

where $v_{\mu,q} \in C^{1,\alpha}\left(\overline{\Omega}\right)$ is the unique positive solution of (7), and

$$\mu_q := \frac{\mu}{\|v_{\mu,q}\|_\infty^{p-q}}. \tag{19}$$

Then $\mu_q \to \lambda_p$ and $u_q \to e_p$ in $C^1\left(\overline{\Omega}\right)$ as $q \to p^-$.

Proof. Since $\|u_q\|_\infty = 1$ and

$$-\Delta_p u_q = \frac{\mu}{\|v_{\mu,q}\|_\infty^{p-1}} v_{\mu,q}^{q-1} = \frac{\mu}{\|v_{\mu,q}\|_\infty^{p-q}} u_q^{q-1} = \mu_q u_q^{q-1},$$

we have that u_q is the unique solution of the problem

$$\begin{cases} -\Delta_p u_q = \mu_q u_q^{q-1} & \text{in } \Omega, \\ u_q = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

As a consequence of (8) we have

$$\frac{\mu}{\lambda_p} \leqslant \|v_{\mu,q}\|_\infty^{p-q} \leqslant \frac{\mu}{k_p},$$

$$0 < \left(\frac{k_p}{\lambda_p}\right)^{\frac{1}{p-q}} e_p \leqslant u_q \leqslant \left(\frac{\lambda_p}{k_p}\right)^{\frac{1}{p-q}} \frac{\phi}{\|\phi\|_\infty} \quad \text{in } \Omega \tag{21}$$

and

$$k_p \leqslant \mu_q \leqslant \lambda_p. \tag{22}$$

Since

$$0 \leqslant \mu_q u_q^{q-1} \leqslant \lambda_p,$$

it follows from Theorem 2 the existence of constants $0 < \alpha < 1$ and $C > 0$ independent of q such that $u_q \in C^{1,\alpha}\left(\overline{\Omega}\right)$ and

$$\|u_q\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \leqslant C \quad \text{for all } 1 < q < p.$$

Using the compactness of the immersion $C^{1,\alpha}\left(\overline{\Omega}\right) \hookrightarrow C^1\left(\overline{\Omega}\right)$, letting $q_n \to p$ we get, up to a subsequence, $\mu_{q_n} \to \lambda \in [k_p, \lambda_p]$ and $u_{q_n} \to u$ in $C^1\left(\overline{\Omega}\right)$. Taking the limit in (20), we conclude from Theorem 5 that u must satisfy

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\|u\|_\infty = 1$, whence $\lambda = \lambda_p$ and $u = e_p$ because λ is an eigenvalue and $u \neq 0$ is a corresponding eigenfuntion that does not change the signal in Ω (note from (21) that $u > 0$ in Ω). Since these limits are always the same, that is, do not depend on particular subsequences, this ends the proof. ■

Corollary 9 *Using the notation of Theorem 7, it follows that*

$$\frac{w_n}{\|w_n\|_\infty} \to e_p$$

in the $C^1(\overline{\Omega})$ *norm and*

$$\|w_n\|_\infty^{1-p} \to \lambda_p$$

as $n \to \infty$ *and* $q \to p^-$.

Proof. For each $1 < q < p$ let v_q denote the solution of the Lane-Emden problem (17). It follows from Theorem 8 that $v_q / \|v_q\|_\infty$ converges in the C^1-norm to the first eigenfunction e_p and that $k_p \|v_q\|_\infty^{q-p} \to \lambda_p$.

Using the notation of Theorem 7 we have

$$\lim_{q\to p^-} \lim_{n\to\infty} \frac{w_n}{\|w_n\|_\infty} = \lim_{q\to p^-} \lim_{n\to\infty} \frac{v_n}{\|v_n\|_\infty} = \lim_{q\to p^-} \frac{v_q}{\|v_q\|_\infty} = e_p.$$

Moreover,

$$\lim_{n\to\infty} \|w_{n+1}\|_\infty^{p-1} = \lim_{n\to\infty} k_p^{-1} \|v_{n+1}\|_\infty^{p-1} \|v_n\|^{1-q} = k_p^{-1} \|v_q\|_\infty^{p-q},$$

and hence

$$\lim_{q\to p^-} \lim_{n\to\infty} \|w_{n+1}\|_\infty^{1-p} = \lambda_p.$$

■

Next we prove an error estimate in the approximation of λ_p by μ_q or, alternatively, by the scaled quotient

$$\Lambda_q := \mu \frac{\|v_{\mu,q}\|_q^q}{\|v_{\mu,q}\|_p^p},$$

where $\|\cdot\|_r$ denotes the norm of the $L^r(\Omega)$, that is, $\|w\|_r = \left(\int_\Omega |w|^r \, dx\right)^{\frac{1}{r}}$.

The upper bound Λ_q together with the lower bound μ_q allows one to better control the accuracy of the approximation to λ_p.

Theorem 10 *There holds:*

(i) $\lambda_p \leqslant \Lambda_q$.

(ii) $\Lambda_q \to \lambda_p$ *as* $q \to p^-$.

(iii) There exists a positive constant K *which does not depend on* q *such that*

$$0 \leqslant \max\{(\lambda_p - \mu_q), (\Lambda_q - \lambda_p)\} \leqslant K(p-q) \tag{23}$$

for all q *sufficiently close to* p, $q < p$.

Proof. (i) follows directly from the variational characterization of λ_p and (2), since

$$\lambda_p \leqslant \frac{\|\nabla v_{\mu,q}\|_p^p}{\|v_{\mu,q}\|_p^p} = \frac{\mu \|v_{\mu,q}\|_q^q}{\|v_{\mu,q}\|_p^p} = \Lambda_q.$$

In order to prove (ii) we note from Theorem 8 that

$$\lim_{q\to p^-} \|u_q\|_q^q = \lim_{q\to p^-} \|u_q\|_p^p = \|e_p\|_p^p, \tag{24}$$

since u_q converges uniformly to e_p when $q \to p^-$. Thus, since

$$\Lambda_q = \mu \frac{\|v_{\mu,q}\|_q^q}{\|v_{\mu,q}\|_p^p} = \frac{\mu}{\|v_{\mu,q}\|_\infty^{p-q}} \frac{\|u_q\|_q^q}{\|u_q\|_p^p} = \mu_q \frac{\|u_q\|_q^q}{\|u_q\|_p^p}, \tag{25}$$

we obtain

$$\lim_{q\to p^-} \Lambda_q = \left(\lim_{q\to p^-} \mu_q\right)\left(\lim_{q\to p^-} \frac{\|u_q\|_q^q}{\|u_q\|_p^p}\right) = \lambda_p.$$

Now we prove error estimate (23). It follows from (i) and (22) that

$$\mu_q \leqslant \lambda_p \leqslant \Lambda_q.$$

Hence,

$$0 \leqslant \max\left\{(\lambda_p - \mu_q), (\Lambda_q - \lambda_p)\right\} \leqslant \Lambda_q - \mu_q.$$

Thus, in order to prove (iii) we need only to bound $\Lambda_q - \mu_q$. It follows from (25) that

$$\Lambda_q - \mu_q = \mu_q \left(\frac{\|u_q\|_q^q}{\|u_q\|_p^p} - 1\right) = \mu_q \frac{\int_\Omega \left(u_q^q - u_q^p\right) dx}{\int_\Omega u_q^p dx}.$$

Therefore,

$$\begin{aligned}
\Lambda_q - \mu_q &\leqslant \lambda_p \frac{\int_\Omega \left(u_q^q - u_q^p\right) dx}{\int_\Omega u_q^p dx} \\
&\leqslant \frac{\lambda_p}{\int_\Omega u_q^p dx} \int_\Omega \left[\max_{0\leqslant t\leqslant 1} \left(t^q - t^p\right)\right] dx \\
&= \frac{\lambda_p |\Omega|}{\int_\Omega u_q^p dx} \left(\frac{q}{p}\right)^{\frac{q}{p-q}} \frac{p-q}{p} \\
&\leqslant \frac{\lambda_p |\Omega|}{\int_\Omega u_q^p dx} (p-q).
\end{aligned}$$

Taking into account (24), there exists $R > 0$ such that $\int_\Omega u_q^p dx \geqslant R$ for all q near to p^-. Thus,

$$0 \leqslant \mu \frac{\|v_{\mu,q}\|_q^q}{\|v_{\mu,q}\|_p^p} - \mu_q \leqslant \frac{\lambda_p |\Omega|}{R} (p-q) = K(p-q).$$

■

5 Some numerical results

5.1 Unit Balls

In this section we present some numerical results in the unit ball of dimensions $N = 2, 3, 4$ applying Algorithm 1, since in this case k_p is explicitly known. Computations were performed on a Windows 7/ i5 - 4.0 GHz platform, using the GCC compiler. The numerical approximations for the first eigenpair were obtained choosing $\mu = k_p$ and taking $q = p - 0.01$. Thus, according to (13)

$$\mu = k_p = N\left(\frac{p}{p-1}\right)^{p-1}.$$

We recall from (15) that for the unit ball the functions in the sequence of iterates are radially ($r = |x|$) given by

$$v_{n+1}(r) = \int_r^1 \left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} k_p v_n(s)^{q-1} ds\right)^{\frac{1}{p-1}} d\theta, \quad \text{with} \quad v_0(r) \equiv \left(1 - |r|^{\frac{p}{p-1}}\right).$$

Thus, starting with the function

$$v_0(r) \equiv \left(1 - |r|^{\frac{p}{p-1}}\right)$$

we have implemented the sequence of iterates

$$v_{n+1}(r) = \frac{pN^{\frac{1}{p-1}}}{p-1} \int_r^1 \left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} v_n(s)^{(p-1.01)} ds\right)^{\frac{1}{p-1}} d\theta \tag{26}$$

which, after normalized by the sup norm, should be close to the normalized first eigenfunction e_p.

The first eigenvalue was approximated by the sequence

$$\frac{N}{\|v_n\|_\infty^{0.01}} \left(\frac{p}{p-1}\right)^{p-1}$$

given by (19).

In order to compute sequence (26) we mixed the composite Simpson and trapezoidal methods on a 101 points mesh for computation of the associated integrals. We adopted

$$\frac{\|v_{n+1} - v_n\|_\infty}{\|v_n\|_\infty} < 10^{-9} \tag{27}$$

as a stopping criterion.

At Table 1, the results for the first eigenvalue of the p-Laplacian for values of p ranging from 1.1 to 4.0 for the unit balls of dimensions $N = 2$, 3 and 4 are displayed and truncated at the fourth decimal place. The results compare very well with the ones presented in [10] up to the second decimal digit.

Table 1: First eigenvalues of the p-Laplacian on the unit ball.

p	$N=2$	$N=3$	$N=4$	p	$N=2$	$N=3$	$N=4$
1.1	2.5666	3.8665	5.17607	2.6	8.08856	14.9747	23.8345
1.2	2.9601	4.5026	6.0797	2.7	8.50354	15.9521	25.672
1.3	3.3182	5.1098	6.97306	2.8	8.92654	16.9646	27.6004
1.4	3.6637	5.71889	7.89478	2.9	9.35759	18.013	29.6225
1.5	4.0053	6.3419	8.86046	3.0	9.79673	19.0977	31.7409
1.6	4.3477	6.98495	9.87865	3.1	10.244	20.2194	33.9581
1.7	4.6932	7.65165	10.955	3.2	10.6994	21.3785	36.2769
1.8	5.0434	8.34438	12.094	3.3	11.163	22.5755	38.6999
1.9	5.3993	9.06487	13.2991	3.4	11.6347	23.8111	41.2298
2.0	5.7616	9.81443	14.5735	3.5	12.1146	25.0856	43.8694
2.1	6.1308	10.5942	15.9202	3.6	12.6027	26.3997	46.6213
2.2	6.5071	11.405	17.3421	3.7	13.099	27.7539	49.4884
2.3	6.8909	12.2478	18.8418	3.8	13.6034	29.1486	52.4734
2.4	7.2823	13.1232	20.422	3.9	14.1161	30.5844	55.5792
2.5	7.6815	14.0319	22.0855	4.0	14.6369	32.0618	58.8085

Graphs of some eigenfunctions generated by the inverse iteration of sublinear supersolutions are presented in Figures 1, 2 and 3 for $N = 2, 3$ and 4, respectively. In these graphs it is possible to observe the asymptotic behavior of the L^∞-normalized eigenfunctions e_p with respect to p for both cases: $p \to 1^-$ and $p \to \infty$. The eigenfunctions e_p converge to the characteristic function of the ball as $p \to 1^+$ (see [35]). On the other hand (see [33]), as $p \to \infty$ these functions converge to the distance function to the boundary, which in this case is $1 - |x|$.

Figure 4 illustrates the log concavity of the eigenfunctions e_p. Note from Figures 1, 2 and 3 that each eigenfunction e_p seems to be convex near the boundary ($r = 1$). However, $\log(e_p)$ is surely concave for convex domains, as proved in [43].

In Figure 5 we see that $\sqrt[p]{\lambda_p}$ approaches 1 as p increases, which is coherent with the following known asymptotic behavior (see [33]): $\lim_{p\to\infty} \sqrt[p]{\lambda_p} = 1/R$ where R is the inradius of the domain (that is, the radius of the largest ball that lies within the domain). Moreover, one observes from Table 1 that λ_p approaches the value N of the dimension as $p \to 1^-$. It is known (see [35]) that λ_p tends to N/R, if the domain is a ball of radius R.

Finally, for comparison we show in Figure 6 graphs of p versus λ_p obtained in two ways: one of them through the method proposed in [10] which is directly based on the inverse power method (IPM), while the other is the inverse iteration of sublinear supersolutions (IISS) as developed in the present work.

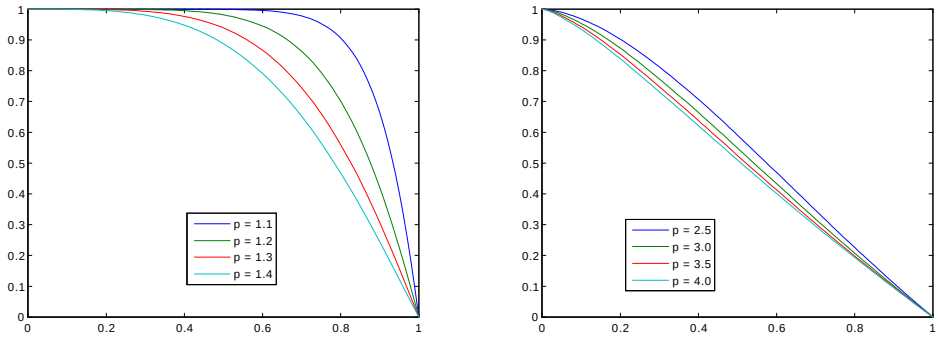

Figure 1: Radial profiles of the first eigenfunction for unit ball when $N = 2$, $p = 1.1, 1.2, 1.3, 1.4$ (left) and $p = 2.5, 3.0, 3.5, 4.0$ (right).

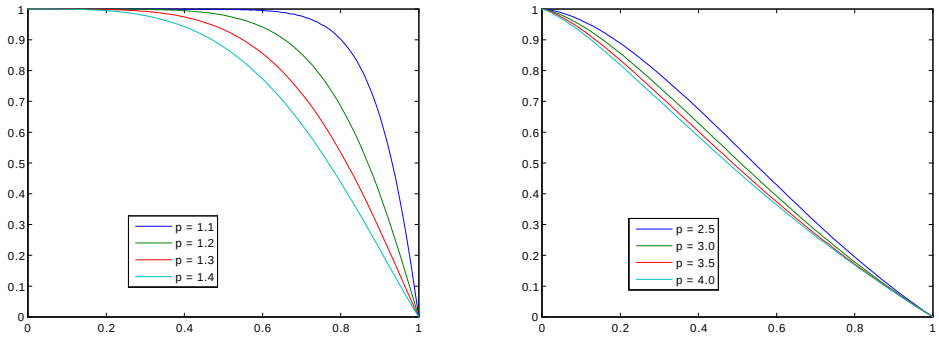

Figure 2: Radial profiles of the first eigenfunction for unit ball when $N = 3$, $p = 1.1, 1.2, 1.3, 1.4$ (left) and $p = 2.5, 3.0, 3.5, 4.0$ (right).

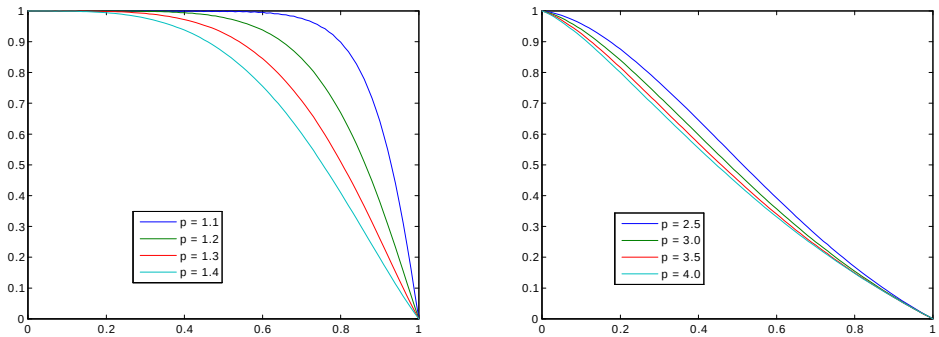

Figure 3: Radial profiles of the first eigenfunction for unit ball when $N = 4$, $p = 1.1, 1.2, 1.3, 1.4$ (left) and $p = 2.5, 3.0, 3.5, 4.0$ (right).

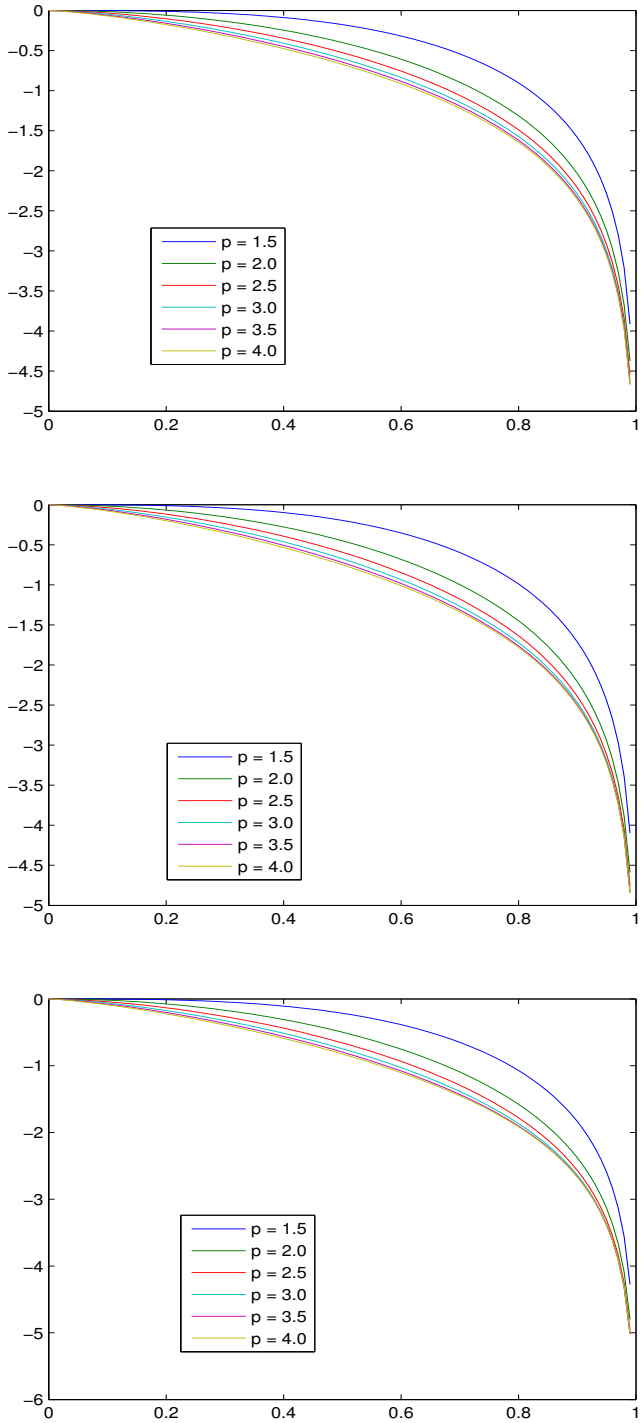

Figure 4: Graphs of $\log(e_p)$ versus p for the N-dimensional unit ball, and $N = 2$ (above), $N = 3$ (center), $N = 4$ (below), and $p = 1.5, 2.0, 2.5, 3.0, 3.5, 4.0$.

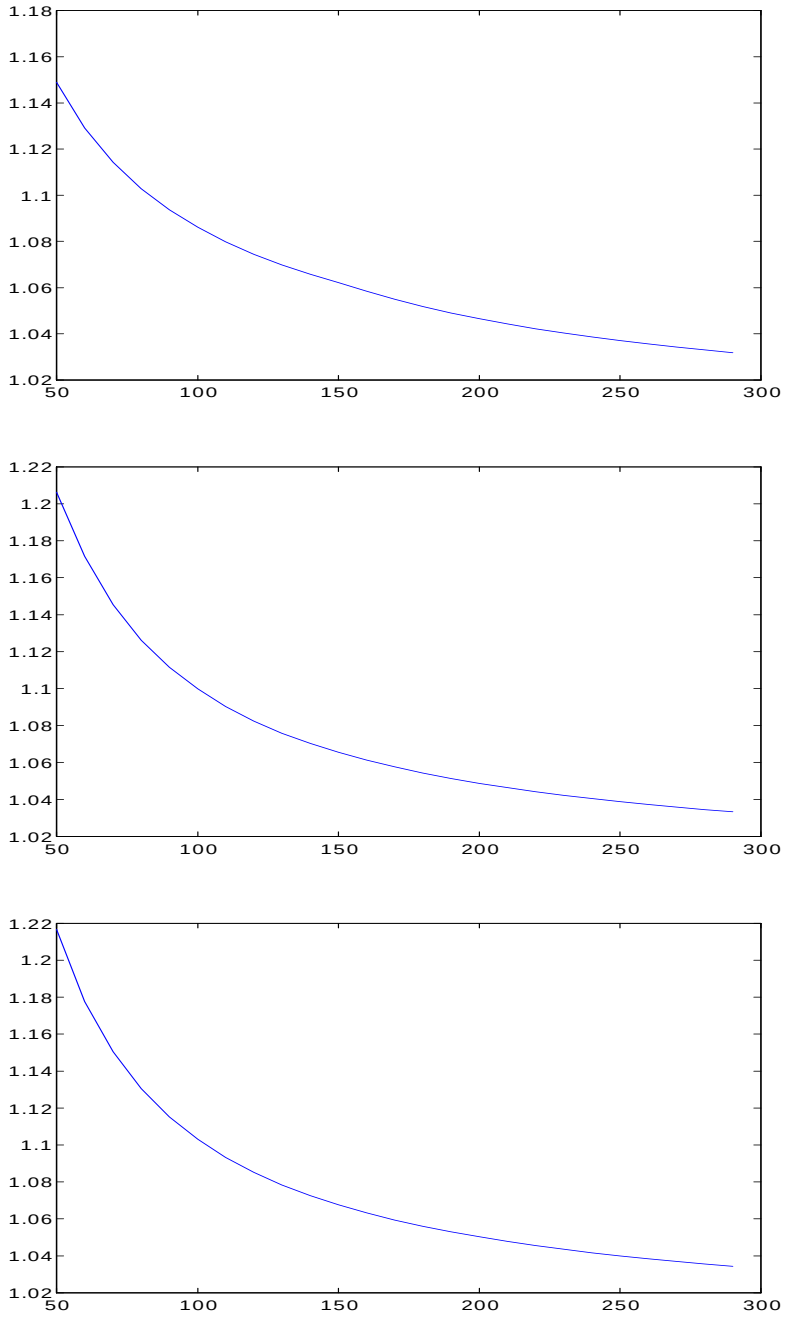

Figure 5: Graphs of $\sqrt[p]{\lambda_p}$ versus p for the N-dimensional unit ball, and $N = 2$ (above), $N = 3$ (center), $N = 4$ (below), from $p = 50$ to $p = 290$, step 10.

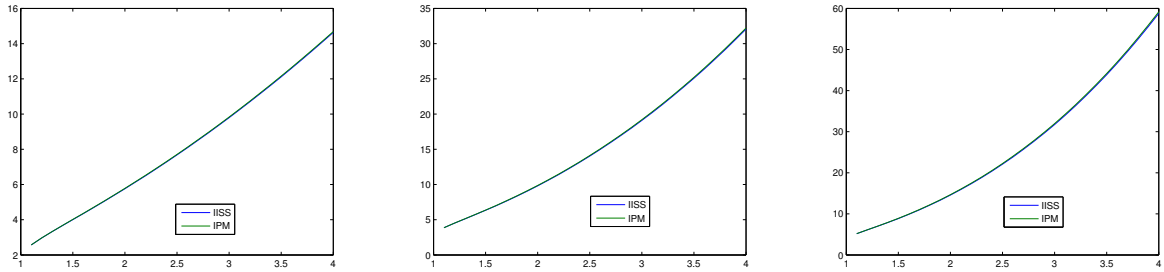

Figure 6: Graphs of λ_p versus p for the N-dimensional unit ball and $N = 2$ (left), $N = 3$ (center), $N = 4$ (right).

5.2 Square, Cube and Torus

To compute eigenvalues on more general domains, we use a p-version finite element discretization on unstructured hexahedral meshes. The discrete equations are solved with PETSc [44] using a Newton-Krylov method in which a matrix associated with a lowest-order discretization is assembled for preconditioning, while the high-order operator is applied in unassembled form (see [18] for details). For these more complicated domains we apply Algorithm 2. This produces the system

$$-\nabla \cdot \left(\left(\epsilon^2 + |\nabla \phi_{m+1}|^2 \right)^{\frac{p-2}{2}} \nabla \phi_{m+1} \right) = \left(\frac{\phi_m}{\|\phi_m\|_\infty} \right)^{q-1}$$

where $\epsilon = 10^{-5}$ is the regularization used to avoid the singularity or degeneracy at $\nabla \phi = 0$. The initial guess for the Newton iteration is taken to be ϕ_m which leads to very fast convergence in the terminal phase. To solve 2D problems with the 3D discretization, homogeneous Neumann boundary conditions are imposed on both faces in the z direction. The source code is publicly available from `https://github.com/jedbrown/dohp`.

Table 2 shows computed eigenvalues for the unit square and unit cube. These solutions were computed using Q_5 elements and are as accurate as double precision rounding error for the smooth solutions in the $p = 2$ case. The accuracy of the discretization for a given smooth solution has been verified to be essentially independent of p using the method of manufactured solutions. This indicates that the primary source of error in Table 2 is interpolation error, as usual for finite element methods.

Figure 7 shows computed eigenfunctions for $p = 1.2$ and $p = 5$ on a torus. The unstructured hexahedral mesh was created with CUBIT version 13.0 [13] using the commands

```
create torus major radius 1 minor radius 0.4
webcut volume all with plane xplane offset 0
mesh volume 1 2
```

and a Q_2 discretization was used. The computed eigenvalues are $\lambda_p = 7.800846$ for $p = 1.2$ and $\lambda_p = 2064.08$ for $p = 5$.

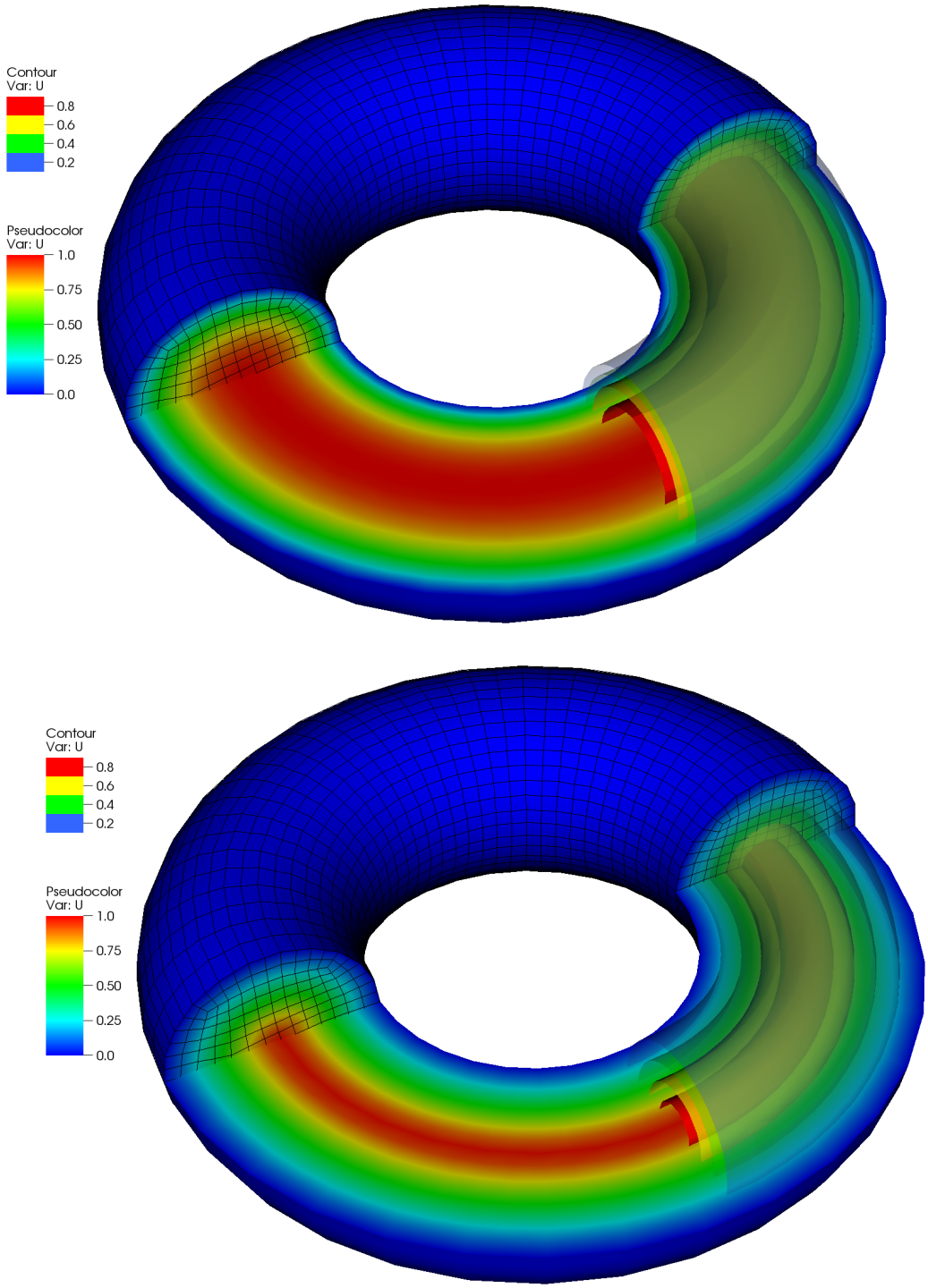

Figure 7: Eigenfunctions of the p-Laplacian for $p = 1.2$ (top) and $p = 5$ (bottom) computed on a torus with major radius 1 and minor radius 0.4.

Table 2: Eigenvalues of the p-Laplacian on the unit square and cube. The 2D results use a 10×10 mesh of Q_5 elements, the 3D results use a $6 \times 6 \times 6$ mesh of Q_5 elements. For $p = 2$, the exact solutions $2\pi^2$ and $3\pi^2$ are available so we show the error in the Reference column; these cases are denoted by [*].

	$2D$		$3D$	
p	Computed	Reference	Computed	Reference
1.2	6.195550328210643		8.642315135978254	
1.5	10.07201415299496	10.0722 [14]	14.47791516619582	
1.75	14.28146165697044	14.2815 [14]	20.96672431961172	
2	19.73920880217817	5.36×10^{-13}[*]	29.60881320326431	3.77×10^{-12}[*]
2.2	25.24862830212583	25.2412 [10]	38.51651963302274	
2.5	35.94868349730170	35.9493 [14]	56.19031685699854	
3	62.75762286200781	62:7633 [14]	101.8697977481977	
4	176.5980821441738	176.693 [14]	306.1647710559179	
5	463.8206306371868		849.9777670614186	

Table 3: Convergence of the computed eigenvalue $\lambda_{p,q} \to \lambda_p^-$ as $q \to p^-$ for the unit cube using a $4 \times 4 \times 4$ mesh with Q_5 discretization.

	$p = 1.5$		$p = 3$	
$p - q$	$\lambda_{p,q}$	$\lambda_p - \lambda_{p,q}$	$\lambda_{p,q}$	$\lambda_p - \lambda_{p,q}$
10^{-1}	13.797661713072	6.7943×10^{-1}	96.414190427672	5.4559
10^{-2}	14.405694866696	7.1393×10^{-2}	101.31034449471	5.5975×10^{-1}
10^{-3}	14.469912150762	7.1756×10^{-3}	101.81397339451	5.6119×10^{-2}
10^{-4}	14.476369813850	7.1792×10^{-4}	101.86447907670	5.6133×10^{-3}
10^{-5}	14.477015941408	7.1796×10^{-5}	101.86953107554	5.6135×10^{-4}
10^{-6}	14.477080557778	7.1796×10^{-6}	101.87003628973	5.6135×10^{-5}
0	14.477087737416	-	101.87009242480	-

Experimental evidence suggests that Algorithm 2 converges with $q = p$, but we have only been able to prove convergence for $q < p$. It is unknown whether the iteration will break down for some domain when $q = p$, but one can always compute with $q < p$ in which case Theorem 8 guarantees convergence with an error less than $K(p-q)$ for some positive constant K depending only on the domain and p. Table 3 shows numerical evidence of this result and quantifies K for the unit cube with $p = 1.5$ and $p = 3$.

In practice, the total computational cost to solve the eigenvalue problem is about twice that of only solving the torsion creep problem. Table 4 shows the convergence of inverse iteration when the Newton iteration at each step is started using the solution at the last iteration. The initial guess for the torsion creep problem is zero, which leads to a difficult nonlinear solve. The Newton iteration is guarded by a cubic backtracking line search which is sufficient in this case; a parameter continuation or grid sequencing is more robust. The torsion creep problem is significantly easier to solve for less extreme values of p or for larger values of the regularization ϵ.

Table 4: Convergence rate for inverse iteration applied to the torus with $p = 1.2$ and $p = 5$. Each nonlinear solve is converged to a relative tolerance of 10^{-8}.

$p = 1.2$		$p = 5$	
Newton its.	λ_p	Newton its.	λ_p
37	torsion	18	torsion
5	7.7670871	6	1628.81
4	7.7965212	4	1975.40
3	7.8003037	4	2043.11
3	7.8007802	3	2057.15
2	7.8008389	3	2061.17
2	7.8008456	3	2062.74
2	7.8008462	3	2063.35
		3	2063.38
		3	2063.98
		3	2064.08

After the torsion creep problem has been solved, a line search is no longer necessary and accurate estimates of the eigenvalue can be obtained in a few more Newton iterations.

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