LOWER BOUNDS FOR EIGENVALUES OF ELLIPTIC OPERATORS — BY NONCONFORMING FINITE ELEMENT METHODS

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ABSTRACT. The aim of the paper is to introduce a new systematic method that can produce lower bounds for eigenvalues. The main idea is to use nonconforming finite element methods. The general conclusion herein is that if local approximation properties of nonconforming finite element spaces V_h are better than global continuity properties of V_h, corresponding methods will produce lower bounds for eigenvalues. More precisely, under three conditions on continuity and approximation properties of nonconforming finite element spaces we first show abstract error estimates of approximate eigenvalues and eigenfunctions. Subsequently, we propose one more condition and prove that it is sufficient to guarantee nonconforming finite element methods to produce lower bounds for eigenvalues of symmetric elliptic operators. As one application, we show that this condition hold for most nonconforming elements in literature. As another important application, this condition provides a guidance to modify known nonconforming elements in literature and to propose new nonconforming elements. In fact, we enrich locally the Crouzeix-Raviart element such that the new element satisfies the condition; we propose a new nonconforming element for second order elliptic operators and prove that it will yield lower bounds for eigenvalues. Finally, we prove the saturation condition for most nonconforming elements.

1. INTRODUCTION

Finding eigenvalues of partial differential operators is important in the mathematical science. Since exact eigenvalues are almost impossible, many papers and books investigate their bounds from above and below. It is well known that the variational principle (including conforming finite element methods) provides upper bounds. But the problem of obtaining lower bounds is generally considerably more difficult. Moreover, a simple combination of lower and upper bounds will produce intervals to which exact eigenvalue belongs. This in turn gives reliable a posteriori error estimates of approximate eigenvalues, which is essential for the design of the coefficient of safety in practical engineering. Therefore, it is a fundamental problem to achieve lower bounds for eigenvalues of elliptic operators. In fact, the study of lower bounds for eigenvalues can date back to remarkable works of [17, 18] and [38, 39], where lower bounds of eigenvalues are derived by finite difference methods for second order elliptic eigenvalue problems. Since that finite difference methods in some sense coincide with standard linear finite element methods with mass lumping, one could expect that finite element methods with mass lumping give lower bounds for eigenvalues of operators, we refer interested readers to [1, 20] for this aspect.

Nonconforming finite element methods are alternative possible ways to produce lower bounds for eigenvalues of operators. In deed, the lower bound property of eigenvalues by nonconforming

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elements are observed in numerics, see, Zienkiewicz et al.[47], for the nonconforming Morley element, Rannacher [32], for the nonconforming Morley and Adini elements, Liu and Yan [30], for the nonconforming Wilson [35, 41], EQ_1^{rot} [28], and Q_1^{rot} [33] elements. See, Boffi [7], for further remarks on possible properties of discrete eigenvalues produced by nonconforming methods.

However, there are a few results to study the lower bound property of eigenvalues by nonconforming elements. The first result in this direction is analyzed in a remarkable paper by Armentano and Duran [2] for the Laplacian operator. The analysis is based on an identity for errors of eigenvalues. It is proved that the nonconforming linear element of [13] leads to lower bounds for eigenvalues provided that eigenfunctions $u \in H^{1+r}(\Omega) \cap H_0^1(\Omega)$ with $0 < r < 1$. The idea is generalized to the enriched nonconforming rotated Q_1 element of [28] in Li [24], and to the Wilson element in Zhang et al. [46]. See [44] for a survey of earlier works. The extension to the Morley element can be found in [45]. However, all of those papers are based on the saturation condition of approximations by piecewise polynomials for which a rigorous proof is missed in literature. We refer interested readers to [26, 27, 29, 42, 46, 44] for expansion methods based on superconvergence or extrapolation, which analyzes the lower bound property of eigenvalues by nonconforming elements on uniform rectangular meshes.

The aim of our paper is to introduce a new systematic method that can produce lower bounds for eigenvalues. The main idea is to use nonconforming finite element methods. However, some numerics from the literature demonstrate that some nonconforming elements produce upper bounds of eigenvalues though some other nonconforming elements yield lower bounds, see[30, 32]. We find that the general condition lies in that local approximation properties of nonconforming finite element spaces V_h should be better than global continuity properties of V_h. Then corresponding nonconforming methods will produce lower bounds for eigenvalues of elliptic operators. More precisely, first, we shall analyze errors of discrete eigenvalues and eigenfunctions. Second, we shall propose a condition on nonconforming element methods and then under the saturation condition prove that it is sufficient for lower bounds for eigenvalues. With this result, to obtain lower bound for eigenvalue is to design nonconforming element spaces with enough local degrees of freedom when compared to global continuity. This in fact results in a systematic method for the lower bounds of eigenvalues. As one application of our method, we check that this condition holds for most used nonconforming elements, e.g., the Wilson element [35, 41], the nonconforming linear element by Crouzeix and Raviart [13], the nonconforming rotated Q_1 element by Rannacher and Turek [33, 35], and the enriched nonconforming rotated Q_1 element by Lin, Tobiska and Zhou [28] for second order elliptic operators, the Morley element [31, 35] and the Adini element [25, 35] for fourth order elliptic operators, and the Morley-Wang-Xu element [37] for $2m$-th order elliptic operators. As another important application, we follow this guidance to enrich locally the Crouzeix-Raviart element such that the new element satisfies the sufficient condition and to propose a new nonconforming element method for second order elliptic operators and show that it actually produces lower bounds for eigenvalues. As an indispensable and important part of the paper, we prove the saturation condition for most of these nonconforming elements.

The paper is organized as follows. In the following section, we shall present symmetric elliptic eigenvalue problems and their nonconforming element methods in an abstract setting. In Section 3, based on three conditions on discrete spaces, we analyze error estimates for both discrete eigenvalues and eigenfunctions. In Section 4, under one more condition, we prove an abstract result that eigenvalues produced by nonconforming methods are smaller than exact ones. In Sections 5-6, we check these conditions for various nonconforming methods in literature and we also propose two new nonconforming methods that admit lower bounds for eigenvalues in Section 7. We end this paper by Section 8 where we give some comments, which is followed by appendixes where we analyze the saturation condition for piecewise polynomial approximations.

2. Eigenvalue problems and nonconforming finite element methods

Let $V \subset H^m(\Omega)$ denote some standard Sobolev space on some bounded Lipschitz domain Ω in $\mathbb{R}^n$ with a piecewise flat boundary $\partial\Omega$. $2m$-th order elliptic eigenvalue problems read: Find $(\lambda, u) \in \mathbb{R} \times V$ such that

$$a(u,v) = \lambda(\rho u, v)_{L^2(\Omega)} \text{ for any } v \in V \text{ and } \|\rho^{1/2}u\|_{L^2(\Omega)} = 1, \tag{2.1}$$

with some positive function $\rho \in L^\infty(\Omega)$. The bilinear form $a(u,v)$ is symmetric, bounded, and coercive in the following sense:

$$a(u,v) = a(v,u), |a(u,v)| \lesssim \|u\|_V\|v\|_V, \text{ and } \|v\|_V^2 \lesssim a(v,v) \text{ for any } u, v \in V, \tag{2.2}$$

with the norm $\|\cdot\|_V$ over the space V. Throughout the paper, an inequality $A \lesssim B$ replaces $A \le C\,B$ with some multiplicative mesh-size independent constant $C > 0$ that depends only on the domain Ω, the shape (e.g., through the aspect ratio) of elements, and possibly some norm of eigenfunctions u. Finally, $A \approx B$ abbreviates $A \lesssim B \lesssim A$.

Under the conditions (2.2), we have that the eigenvalue problem (2.1) has a sequence of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \nearrow +\infty,$$

and corresponding eigenfunctions

$$u_1, u_2, u_3, \cdots, \tag{2.3}$$

which can be chosen to satisfy

$$(\rho u_i, u_j)_{L^2(\Omega)} = \delta_{ij}, i, j = 1, 2, \cdots. \tag{2.4}$$

We define

$$E_\ell = \operatorname{span}\{u_1, u_2, \cdots, u_\ell\}. \tag{2.5}$$

Then, eigenvalues and eigenfunctions satisfy the following well-known minimum-maximum principle:

$$\lambda_k = \min_{\dim V_k = k, V_k \subset V} \max_{v \in V_k} \frac{a(v,v)}{(\rho v, v)_{L^2(\Omega)}} = \max_{u \in E_k} \frac{a(u,u)}{(\rho u, u)_{L^2(\Omega)}}. \tag{2.6}$$

For any eigenvalue λ of (2.1), we define

$$M(\lambda) := \{u : u \text{ is an eigenfunction of (2.1) to } \lambda\}. \tag{2.7}$$

We shall be interested in approximating the eigenvalue problem (2.1) by finite element methods. To this end, we suppose we are given a discrete space V_h defined over a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ into (closed) simplexes or n-rectangles [9].

We need the piecewise counterparts of differential operators with respect to $\mathcal{T}_h$. For any differential operator $\mathcal{L}$, we define its piecewise counterpart $\mathcal{L}_h$ in the following way: we suppose that v_K is defined over $K \in \mathcal{T}_h$ and that the differential action $\mathcal{L}v_K$ is well-defined on K which is denoted by $\mathcal{L}_K v_K$ for any $K \in \mathcal{T}_h$; then we define v_h by $v_h|_K = v_K$ where $v_h|_K$ denotes its restriction of v_h over K; finally we define $\mathcal{L}_h v_h$ by $(\mathcal{L}_h v_h)|_K = \mathcal{L}_K v_K$.

We consider the discrete eigenvalue problem: Find $(\lambda_h, u_h) \in \mathbb{R} \times V_h$ such that

$$a_h(u_h, v_h) = \lambda_h(\rho u_h, v_h)_{L^2(\Omega)} \text{ for any } v_h \in V_h \text{ and } \|\rho^{1/2}u_h\|_{L^2(\Omega)} = 1. \tag{2.8}$$

Here and throughout of this paper, $a_h(\cdot,\cdot)$ is the piecewise counterpart of the bilinear form $a(\cdot,\cdot)$ where differential operators are replaced by their discrete counterparts. Conditions on the approximation and continuity properties of discrete spaces V_h are assumed as follows, respectively.

(H1) $\|\cdot\|_h := a_h(\cdot,\cdot)^{1/2}$ is a norm over discrete spaces V_h.

(H2) Suppose $v \in V \cap H^{m+\mathcal{S}}(\Omega)$ with $0 < \mathcal{S} \leq 1$. Then,

$$\inf_{v_h \in V_h} \|v - v_h\|_h \lesssim h^{\mathcal{S}} |v|_{H^{m+\mathcal{S}}(\Omega)}.$$

(H3) Suppose $v \in V \cap H^{m+s}(\Omega)$ with $0 < s \leq \mathcal{S} \leq 1$. Then,

$$\sup_{0 \neq v_h \in V_h} \frac{a_h(v, v_h) - (Av, v_h)_{L^2(\Omega)}}{\|v_h\|_h} \lesssim h^s |v|_{H^{m+s}(\Omega)}\,.$$

(H4) Let u and u_h be eigenfunctions of problems (2.1) and (2.8), respectively. Assume that there exists an interpolation $\Pi_h u \in V_h$ with the following properties:

$$\begin{aligned} &a_h(u - \Pi_h u, u_h) = 0,\\ &\|\rho^{1/2}u\|^2_{L^2(\Omega)} - \|\rho^{1/2}\Pi_h u\|^2_{L^2(\Omega)} \lesssim h^{2s+\triangle s},\\ &\|\rho^{1/2}(\Pi_h u - u)\|_{L^2(\Omega)} \lesssim h^{\mathcal{S}+\triangle\mathcal{S}}\,, \end{aligned} \tag{2.9}$$

when the meshsize h is small enough and $u \in V \cap H^{m+\mathcal{S}}(\Omega)$ with two positive constants $\triangle s$ and $\triangle\mathcal{S}$.

Let $N = \dim V_h$. Under the condition (H1), the discrete problem (2.8) admits a sequence of discrete eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{N,h},$$

and corresponding eigenfunctions

$$u_{1,h}, u_{2,h}, \cdots, u_{N,h}\,.$$

In the case where V_h is a conforming approximation in the sense $V_h \subset V$, it immediately follows from the minimum-maximum principle (2.6) that

$$\lambda_k \leq \lambda_{k,h}, k = 1, 2, \cdots, N,$$

which indicates that $\lambda_{k,h}$ is an approximation above λ_k.

We define the discrete counterpart of E_ℓ by

$$E_{\ell,h} = \text{span}\{u_{1,h}, u_{2,h}, \cdots, u_{\ell,h}\}. \tag{2.10}$$

Then, we have the following discrete minimum-maximum principle:

$$\lambda_{k,h} = \min_{\dim V_{k,h}=k, V_{k,h}\subset V_h} \max_{v \in V_{k,h}} \frac{a_h(v,v)}{(\rho v, v)_{L^2(\Omega)}} = \max_{u \in E_{k,h}} \frac{a_h(u,u)}{(\rho u, u)_{L^2(\Omega)}}. \tag{2.11}$$

3. Error estimates of eigenvalues and eigenfunctions

In this section, we shall analyze errors of discrete eigenvalues and eigenfunctions by nonconforming methods. We refer to [5, 32] for some alternative analysis in the functional analysis setting. We would like to stress the analysis is a nontrivial extension to nonconforming methods of the analysis for conforming methods in [36]. For simplicity of presentation, we only consider the case where λ_ℓ is an eigenvalue of multiplicity 1 and also note that the extension to the multiplicity ≥ 2 case follows by using notations and concepts, for instance, from [10, Page 406].

Associated with the bilinear form $a(\cdot,\cdot)$, we define the operator A by

$$a(u, v) = (Au, v)_{L^2(\Omega)} \text{ for any } v \in V\,. \tag{3.1}$$

Given any $f \in L^2(\Omega)$, let u_f be the solution to the dual problem: Find $u_f \in V$ such that

$$a(u_f, v) = (\rho f, v)_{L^2(\Omega)} \text{ for any } v \in V\,. \tag{3.2}$$

Generally speaking, the regularity of u_f depends on, among others, regularities of f and ρ, elliptic operators under consideration, the shape of the domain Ω and the boundary condition imposed. To fix the main idea and therefore avoid too technical notation, throughout this paper, without loss of generality, assume that $u_f \in V \cap H^{m+s}(\Omega)$ with $0 < s \leq 1$ in the sense that

$$\|u_f\|_{H^{m+s}(\Omega)} \lesssim \|\rho^{1/2} f\|_{L^2(\Omega)}\,. \tag{3.3}$$

In order to analyze L^2 error estimates of eigenfunctions, define quasi-Ritz-projections $P_h' u_\ell \in V_h$ by

$$a_h(P_h' u_\ell, v_h) = \lambda_\ell(\rho u_\ell, v_h)_{L^2(\Omega)} \text{ for any } v_h \in V_h. \tag{3.4}$$

The analysis also needs Galerkin projection operators $P_h : V \to V_h$ by

$$a_h(P_h v, w_h) = a_h(v, w_h) \text{ for any } w_h \in V_h, v \in V. \tag{3.5}$$

Remark 3.1. *We note that P_h' is identical to P_h for conforming methods, which indicates the difference between conforming elements analyzed in* [36] *and nonconforming elements under consideration.*

Under the conditions (H1), (H2), and (H3), a standard argument for nonconforming finite element methods, see, for instance, [9], proves

$$\|\rho^{1/2}(v - P_h v)\|_{L^2(\Omega)} + h^s \|v - P_h v\|_h \lesssim h^{2s} |v|_{H^{m+s}(\Omega)}\,, \tag{3.6}$$

provided that $v \in V \cap H^{m+s}(\Omega)$ with $0 < s \leq 1$.

Throughout this paper, u_ℓ, u_j, and u_i are eigenfunctions of the problem (2.1), while $u_{\ell,h}$, $u_{j,h}$, and $u_{i,h}$ are discrete eigenfunctions of the discrete eigenvalue problem. Note that $P_h' u_\ell$ is the finite element approximation of u_ℓ. Under conditions (H1)-(H3), a standard argument for nonconforming finite element methods, see, for instance, [9], proves

Lemma 3.2. *Suppose that the conditions (H1)-(H3) hold. Then,*

$$\|\rho^{1/2}(u_\ell - P_h' u_\ell)\|_{L^2(\Omega)} + h^s \|u_\ell - P_h' u_\ell\|_h \lesssim h^{2s} |u_\ell|_{H^{m+s}(\Omega)}\,, \tag{3.7}$$

provided that $u_\ell \in V \cap H^{m+s}(\Omega)$ with $0 < s \leq 1$.

From $P_h' u_\ell \in V_h$ we have

$$P_h' u_\ell = \sum_{j=1}^{N} (\rho P_h' u_\ell, u_{j,h}) u_{j,h} \,. \tag{3.8}$$

For the projection operator P_h', we have the following important property

$$(\lambda_{j,h} - \lambda_\ell)(\rho P_h' u_\ell, u_{j,h})_{L^2(\Omega)} = \lambda_\ell(\rho(u_\ell - P_h' u_\ell), u_{j,h})_{L^2(\Omega)} \,. \tag{3.9}$$

In fact, we have

$$\lambda_{j,h}(\rho P_h' u_\ell, u_{j,h})_{L^2(\Omega)} = a_h(u_{j,h}, P_h' u_\ell) = \lambda_\ell(\rho u_\ell, u_{j,h})_{L^2(\Omega)} \,. \tag{3.10}$$

Suppose that $\lambda_\ell \neq \lambda_j$ if $\ell \neq j$. Then there exists a separation constant d_ℓ with

$$\frac{\lambda_\ell}{|\lambda_{j,h} - \lambda_\ell|} \leq d_\ell \text{ for any } j \neq \ell, \tag{3.11}$$

provided that the meshsize h is small enough.

Theorem 3.3. *Let u_ℓ and $u_{\ell,h}$ be eigenfunctions of* (2.1) *and* (2.8)*, respectively. Suppose that the conditions (H1)-(H3) hold. Then,*

$$\|\rho^{1/2}(u_\ell - u_{\ell,h})\|_{L^2(\Omega)} \lesssim h^{2s} |u_\ell|_{H^{m+s}(\Omega)} \,, \tag{3.12}$$

provided that $u_\ell \in V \cap H^{m+s}(\Omega)$ with $0 < s \leq 1$.

Proof. We denote the key coefficient $(\rho P_h' u_\ell, u_{\ell,h})_{L^2(\Omega)}$ by β_ℓ. The rest can be bounded as follows:

$$\begin{aligned} \|\rho^{1/2}(P_h' u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)}^2 &= \sum_{j\neq\ell} (\rho P_h' u_\ell, u_{j,h})_{L^2(\Omega)}^2 \leq d_\ell^2 \sum_{j\neq\ell} (\rho(u_\ell - P_h' u_\ell), u_{j,h})_{L^2(\Omega)}^2 \\ &\leq d_\ell^2 \|\rho^{1/2}(u_\ell - P_h' u_\ell)\|_{L^2(\Omega)}^2 \,. \end{aligned} \tag{3.13}$$

This leads to

$$\begin{aligned} &\|\rho^{1/2}(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)} \leq \|\rho^{1/2}(u_\ell - P_h' u_\ell)\|_{L^2(\Omega)} + \|\rho^{1/2}(P_h' u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)} \\ &\leq (1 + d_\ell)\|\rho^{1/2}(u_\ell - P_h' u_\ell)\|_{L^2(\Omega)} \lesssim h^{2s} \lambda_\ell^{(m+s)/2m} \,. \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\|\rho^{1/2} u_\ell\|_{L^2(\Omega)} - \|\rho^{1/2}(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)} \leq \|\rho^{1/2} \beta_\ell u_{\ell,h}\|_{L^2(\Omega)} \\ &\leq \|\rho^{1/2} u_\ell\|_{L^2(\Omega)} + \|\rho^{1/2}(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)}. \end{aligned} \tag{3.15}$$

Since both u_ℓ and $u_{\ell,h}$ are unit vectors, we can choose them such that $\beta_\ell \geq 0$. Hence we have $|\beta_\ell - 1| \leq \|\rho^{1/2}(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)}$. Thus, we obtain

$$\begin{aligned} \|\rho^{1/2}(u_\ell - u_{\ell,h})\|_{L^2(\Omega)} &\leq \|\rho^{1/2}(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)} + |\beta_\ell - 1| \|\rho^{1/2} u_{\ell,h}\|_{L^2(\Omega)} \\ &\leq 2\|\rho^{1/2}(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)} \lesssim h^{2s} |u_\ell|_{H^{m+s}(\Omega)} \,. \end{aligned} \tag{3.16}$$

This completes the proof. □

Next we analyze errors of eigenvalues. To this end, define $\tilde{u}_{\ell,h} \in V$ by

$$a(\tilde{u}_{\ell,h}, v) = \lambda_{\ell,h}(\rho u_{\ell,h}, v)_{L^2(\Omega)} \text{ for any } v \in V. \tag{3.17}$$

It follows from (2.1) and (2.8) that

$$(\rho(\tilde u_{\ell,h}-u_{\ell,h}),u_\ell)_{L^2(\Omega)} = \lambda_\ell^{-1}\lambda_{\ell,h}(\rho u_{\ell,h},u_\ell)_{L^2(\Omega)} - (\rho u_{\ell,h},u_\ell)_{L^2(\Omega)} \tag{3.18}$$
$$= \frac{(\lambda_{\ell,h}-\lambda_\ell)(\rho u_{\ell,h},u_\ell)_{L^2(\Omega)}}{\lambda_\ell}.$$

Thus we have

$$\lambda_{\ell,h}-\lambda_\ell = \frac{\lambda_\ell(\rho(\tilde u_{\ell,h}-u_{\ell,h}),u_\ell)_{L^2(\Omega)}}{(\rho u_{\ell,h},u_\ell)_{L^2(\Omega)}}. \tag{3.19}$$

Assume that $\tilde u_{\ell,h}\in V\cap H^{m+s}(\Omega)$ with $0<s\le 1$ in the sense that

$$\|\tilde u_{\ell,h}\|_{H^{m+s}(\Omega)} \lesssim \lambda_{\ell,h}\|\rho^{1/2}u_{\ell,h}\|_{L^2(\Omega)}. \tag{3.20}$$

Note that $u_{\ell,h}$ is the finite element approximation of $\tilde u_{\ell,h}$. A standard argument for nonconforming finite element methods, see, for instance, [9], proves

Lemma 3.4. *Suppose that the conditions (H1)-(H3) hold. Then,*

$$\|\rho^{1/2}(u_{\ell,h}-\tilde u_{\ell,h})\|_{L^2(\Omega)} + h^s\|u_{\ell,h}-\tilde u_{\ell,h}\|_h \lesssim \lambda_{\ell,h}h^{2s}\|\rho^{1/2}u_{\ell,h}\|_{L^2(\Omega)}\,. \tag{3.21}$$

Inserting the above estimate into (3.19) proves:

Theorem 3.5. *Let λ_ℓ and $\lambda_{\ell,h}$ be eigenvalues of* (2.1) *and* (2.8)*, respectively. Suppose that (H1)-(H3) hold. Then,*

$$|\lambda_{\ell,h}-\lambda_\ell| \lesssim h^{2s}|u_\ell|_{H^{m+s}(\Omega)}\,, \tag{3.22}$$

provided that h is small enough and that $u_\ell\in V\cap H^{m+s}(\Omega)$ with $0<s\le 1$..

Finally we can have error estimates in the energy norm of eigenfunctions.

Theorem 3.6. *Let u_ℓ and $u_{\ell,h}$ be eigenfunctions of* (2.1) *and* (2.8)*, respectively. Suppose that the conditions (H1)-(H3) hold. Then,*

$$\|u_\ell-u_{\ell,h}\|_h \lesssim h^s|u_\ell|_{H^{m+s}(\Omega)}\,, \tag{3.23}$$

provided that $u_\ell\in V\cap H^{m+s}(\Omega)$ with $0<s\le 1$.

Proof. In order to bound errors of eigenfunctions in the energy norm, we need the following decomposition:

$$a_h(u_\ell-u_{\ell,h},u_\ell-u_{\ell,h}) = a(u_\ell,u_\ell)+a_h(u_{\ell,h},u_{\ell,h})-2a_h(u_\ell,u_{\ell,h}) \tag{3.24}$$
$$= \lambda_\ell\|\rho^{1/2}(u_\ell-u_{\ell,h})\|^2_{L^2(\Omega)} + \lambda_{\ell,h}-\lambda_\ell + 2\lambda_\ell(\rho u_\ell,u_{\ell,h}-u_\ell) - 2a_h(u_\ell,u_{\ell,h}-u_\ell)\,.$$

Then, the desired result follows from Theorem 3.5, (3.16), and the condition (H3). □

4. Lower bounds for eigenvalues: an abstract theory

This section proves that the conditions (H1)-(H4) are sufficient conditions to guarantee nonconforming finite element methods to yield lower bounds for eigenvalues of elliptic operators.

Theorem 4.1. *Let (λ, u) and (λ_h, u_h) be solutions of problems (2.1) and (2.8), respectively. Assume that $u \in V \cap \in H^{m+\mathcal{S}}(\Omega)$ and that $h^{2s} \lesssim \|u-u_h\|_h^2$ with $0 < s \leq \mathcal{S} \leq 1$. If the conditions (H1)–(H4) hold, then*

$$\lambda_h \leq \lambda, \tag{4.1}$$

provided that h is small enough.

Proof. Let Π_h be the operator in the condition (H4). A similar argument of [2] proves

$$\begin{aligned} \lambda - \lambda_h &= \|u-u_h\|_h^2 - \lambda_h \|\rho^{1/2}(\Pi_h u - u_h)\|_{L^2(\Omega)}^2 \\ &\quad + \lambda_h(\|\rho^{1/2}\Pi_h u\|_{L^2(\Omega)}^2 - \|\rho^{1/2}u\|_{L^2(\Omega)}^2). \end{aligned} \tag{4.2}$$

(We refer interested readers to Zhang et al. [46] for an identity with full terms). From the abstract error estimate (3.12) it follows that

$$\|\rho^{1/2}(u-u_h)\|_{L^2(\Omega)} \lesssim h^{2s}\,. \tag{4.3}$$

Hence the triangle inequality and (H4) plus the saturation condition $h^{2s} \lesssim \|u-u_h\|_h^2$ show that the second third term on the right-hand side of (4.2) is of higher order than the first term. If $\|\rho^{1/2}\Pi_h u\|_{L^2(\Omega)}^2 \leq \|\rho^{1/2}u\|_{L^2(\Omega)}^2$, then the condition states that the third term is of higher order than the first term; otherwise, it will be positive. This completes the proof. □

The condition that $h^{2s} \lesssim \|u-u_h\|_h^2$ is usually referred to as the saturation condition in the literature. The condition is closely related to the inverse theorem in the context of the approximation theory by trigonometric polynomials or splines. For the approximation by conforming piecewise polynomials, the inverse theorem was analyzed in [3, 40]. For nonconforming finite element methods, the saturation condition was first analyzed in Shi [34] for the Wilson element by an example, which was developed by Chen and Li [12] by an expansion of the error. See [23] for lower bounds of discretization errors by conforming linear/bilinear finite elements. Babuska and Strouboulis [4] analyzed Lagrange finite element methods for elliptic problems in one dimension. In appendixes, we shall analyze the saturation condition for most of nonconforming finite element methods under consideration. To our knowledge, it is the first time to analyze systematically this condition for nonconforming methods.

Since Galerkin projection operators P_h from (3.5) or their high order perturbations of nonconforming spaces V_h are taken as interpolation operators Π_h, their error estimates are dependent on only local approximation properties but not global continuity properties of spaces V_h while errors $\|u-u_h\|_h$ generally depend on both properties (see Theorems 3.3 and 3.6). On the other hand, the term $\|\rho^{1/2}u\|_{L^2(\Omega)}^2 - \|\rho^{1/2}\Pi_h u\|_{L^2(\Omega)}^2$ will be either of high order or negative when we have enough many local degrees of freedom (compared to the continuity) and therefore consistency errors in $\|u-u_h\|_h^2$ will be dominant in the sense that

$$\|u-u_h\|_h^2 \geq \|\rho^{1/2}u\|_{L^2(\Omega)}^2 - \|\rho^{1/2}\Pi_h u\|_{L^2(\Omega)}^2.$$

If this happens we say local approximation properties of spaces V_h are better than global continuity properties of V_h. Hence, Theorem 4.1 states that corresponding methods of eigenvalue problems will produce lower bounds for eigenvalues for this situation. Thus, to get a lower bound for an eigenvalue is to design nonconforming finite element spaces with enough local degrees of freedom when compared to global continuity properties of V_h. This in fact provides a systematic tool for the construction of lower bounds for eigenvalues of operators in mathematical science.

5. Nonconforming elements of second order elliptic operators

This section presents some nonconforming schemes of second order elliptic eigenvalue problems that the conditions (H1)-(H4) proposed in Section 2 are satisfied. Let the boundary $\partial\Omega$ be divided into two parts: Γ_D and Γ_N with $|\Gamma_D|>0$, and $\Gamma_D\cup\Gamma_N=\partial\Omega$. For ease of presentation, assume that (2.1) is the Poisson eigenvalue problem imposed general boundary conditions.

Let $\mathcal{T}_h$ be regular n-rectangular triangulations of domains $\Omega\subset\mathbb{R}^n$ with $2\le n$ in the sense that $\bigcup_{K\in\mathcal{T}_h}K=\bar\Omega$, two distinct elements K and K' in $\mathcal{T}_h$ are either disjoint, or share an ℓ-dimensional hyper-plane, $\ell=0,\cdots,n-1$. Let $\mathcal{H}_h$ denote the set of all $n-1$ dimensional hyper-planes in $\mathcal{T}_h$ with the set of interior $n-1$ dimensional hyper-planes $\mathcal{H}_h(\Omega)$ and the set of boundary $n-1$ dimensional hyper-planes $\mathcal{H}_h(\partial\Omega)$. $\mathcal{N}_h$ is the set of nodes of $\mathcal{T}_h$ with the set of internal nodes $\mathcal{N}_h(\Omega)$ and the set of boundary nodes $\mathcal{N}_h(\partial\Omega)$.

For each $K\in\mathcal{T}_h$, introduce the following affine invertible transformation

$$F_K:\hat K\to K, x_i=h_{x_i,K}\xi_i+x_i^0$$

with the center $(x_1^0,x_2^0,\cdots,x_n^0)$ and the lengths $2h_{x_i,K}$ of K in the directions of the x_i-axis, and the reference element $\hat K=[-1,1]^n$. In addition, set $h=\max_{1\le i\le n}h_{x_i}$.

Over the above mesh $\mathcal{T}_h$, we shall consider two classes of nonconforming element methods for the eigenvalue problem (2.1), namely, the Wilson element in any dimension, the enriched nonconforming rotated Q_1 element in any dimension.

Let V_h be discrete spaces of aforementioned nonconforming element methods. The finite element approximation of Problem (2.1) is defined as in (2.8).

For all the elements, one can use continuity and boundary conditions for discrete spaces V_h given below to verify the conditions (H1)-(H3), see [28, 33, 35, 41] for further details. Let (λ,u) and (λ_h,u_h) be solutions to problems (2.1) and (2.8), by Theorems 3.3, 3.5, and 3.6 we have

$$|\lambda-\lambda_h|+h^{\mathfrak{s}}\|u-u_h\|_h+\|\rho^{1/2}(u-u_h)\|_{L^2(\Omega)}\lesssim h^{2\mathfrak{s}}\,, \tag{5.1}$$

provided that $u\in V\cap H^{1+\mathfrak{s}}(\Omega)$ with $0<\mathfrak{s}\le 1$.

We shall analyze the key condition (H4) for these elements in the subsequent subsections.

5.1. **The Wilson element in any dimension.**

Denote by $Q_{Wil}(\hat K)$ the nonconforming Wilson element space [35, 41] on the reference element defined by

$$Q_{Wil}(\hat K)=Q_1(\hat K)+\operatorname{span}\{\xi_1^2-1,\xi_2^2-1,\cdots,\xi_n^2-1\}, \tag{5.2}$$

where $Q_1(\hat K)$ is the space of polynomials of degree≤ 1 in each variable. The nonconforming Wilson element space V_h is then defined as

$$\begin{aligned}V_h:=\ &\big\{v\in L^2(\Omega):v|_K\circ F_K\in Q_{Wil}(\hat K)\text{ for each }K\in\mathcal{T}_h, v\text{ is continuous}\\ &\quad\text{at internal nodes, and vanishes at boundary nodes on }\Gamma_D\big\}.\end{aligned}$$

The degrees of freedom read

$$v(a_j), 1\le j\le 2^n\text{ and }\frac{1}{|K|}\int_K\frac{\partial^2 v}{\partial x_i^2}dx, 1\le i\le n,$$

where a_j denote vertexes of element K.

In order to show the condition (H4), let P_h be the Galerkin projection operator defined in (3.5). The approximation property of the operator P_h reads

$$h\|u-P_hu\|_h+\|u-P_hu\|_{L^2(\Omega)}\lesssim h^2|u|_{H^2(\Omega)}\,, \tag{5.3}$$

provided that $u \in V \cap H^2(\Omega)$. This plus (5.1) lead to

$$\lambda_h(\rho(P_h u - u), P_h u + u)_{L^2(\Omega)} = \lambda(\rho(P_h u - u), P_h u + u)_{L^2(\Omega)} + \mathcal{O}(h^4) \\ = 2\lambda(\rho(P_h u - u), u)_{L^2(\Omega)} + \mathcal{O}(h^4). \tag{5.4}$$

To analyze the term $\lambda(\rho(P_h u - u), u)_{L^2(\Omega)}$, let I_h be the canonical interpolation operator for the Wilson element, which admits the following error estimates:

$$h\|u - I_h u\|_h + \|u - I_h u\|_{L^2(\Omega)} \lesssim h^{2+\mathfrak{s}}|u|_{H^{2+\mathfrak{s}}(\Omega)}, \tag{5.5}$$

provided that $u \in V \cap H^{2+\mathfrak{s}}(\Omega)$ with $0 < \mathfrak{s} \le 1$. Since $\|u - P_h u\|_h \lesssim h^{1+\mathfrak{s}}$ provided that $u \in V \cap H^{2+\mathfrak{s}}(\Omega)$ with $0 < \mathfrak{s} \le 1$,

$$\lambda(\rho u, P_h u - I_h u)_{L^2(\Omega)} - a_h(u, P_h u - I_h u) \lesssim h|u|_{H^2(\Omega)}\|P_h u - I_h u\|_h \lesssim h^{2+\mathfrak{s}}\|u\|^2_{H^{2+\mathfrak{s}}(\Omega)}.$$

This and (5.5) state

$$\lambda(\rho(P_h u - u), u)_{L^2(\Omega)} = \lambda(\rho u, P_h u - u)_{L^2(\Omega)} - a_h(u, P_h u - u) + a_h(u - P_h u, P_h u - u) \\ = a_h(u - I_h u, u) + \mathcal{O}(h^{2+\mathfrak{s}}). \tag{5.6}$$

To analyze the term $a_h(u - I_h u, u)$, let I_K denote the restriction of I_h on element K. Then we have the following result.

Lemma 5.1. *For any $u \in P_3(K)$ and $v \in P_1(K)$, it holds that*

$$(\nabla(u - I_K u), \nabla v)_{L^2(K)} = -\sum_{i=1}^n \sum_{j\neq i} \frac{h^2_{x_i,K}}{3} \int_K \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial v}{\partial x_j} dx. \tag{5.7}$$

Proof. The definition of the interpolation operator I_K leads to

$$u - I_K u = \sum_{i=1}^n \frac{h^3_{x_i,K}}{6} \frac{\partial^3 u}{\partial x_i^3}(\xi_i^3 - \xi_i) + \sum_{i=1}^n \sum_{j\neq i} \frac{h^2_{x_i,K} h_{x_j,K}}{2} \frac{\partial^3 u}{\partial x_i^2 \partial x_j}(\xi_i^2 \xi_j - \xi_j).$$

A direct calculation proves

$$(\nabla(u - I_K u), \nabla v)_{L^2(K)} = -\sum_{i=1}^n \sum_{j\neq i} \frac{h^2_{x_i,K}}{3} \int_K \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial v}{\partial x_j} dx,$$

which completes the proof. □

Given any element K, define $J_{\ell,K} v \in P_\ell(K)$ by

$$\int_K \nabla^i J_{\ell,K} v dx = \int_K \nabla^i v dx, i = 0, \cdots, \ell, \tag{5.8}$$

for any $v \in H^\ell(K)$. Note that the operator $J_{\ell,K}$ is well-defined. Let Π_K^0 denote the constant projection operator over K, namely,

$$\Pi_K^0 v := \frac{1}{|K|} \int_K v dx \text{ for any } v \in L^2(K).$$

The property of operator $J_{\ell,K}$ reads

$$\|\nabla^i(v - J_{\ell,K} v)\|_{L^2(K)} \lesssim h_K^{\ell-i}\|\nabla^\ell(v - J_{\ell,K} v)\|_{L^2(K)} \text{ and } \nabla^\ell J_{\ell,K} = \Pi_K^0 \nabla^\ell v \text{ for any } v \in H^\ell(K). \tag{5.9}$$

Lemma 5.2. *For uniform meshes, it holds that*

$$(\nabla_h(u-I_hu),\nabla u)_{L^2(\Omega)}=\sum_{i=1}^n\sum_{j\neq i}\frac{h_{x_i,K}^2}{3}\int_K\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2dx+o(h^2), \tag{5.10}$$

provided that $u\in H^3(\Omega)$ and the meshsize is small enough.

Proof. A combination of (5.5) and (5.9) leads to

$$\begin{aligned}(\nabla_h(u-I_hu),\nabla u)_{L^2(\Omega)}&=\sum_{K\in\mathcal{T}_h}(\nabla(u-I_Ku),\nabla u)_{L^2(K)}\\ &=\sum_{K\in\mathcal{T}_h}(\nabla(u-I_Ku),\nabla J_{1,K}u)_{L^2(K)}+\mathcal{O}(h^3).\end{aligned}$$

The operator $J_{3,K}$ yields the following decomposition

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}(\nabla(u-I_Ku),\nabla J_{1,K}u)_{L^2(K)}=&\sum_{K\in\mathcal{T}_h}(\nabla(I-I_K)J_{3,K}u,\nabla J_{1,K}u)_{L^2(K)}\\ &+\sum_{K\in\mathcal{T}_h}(\nabla(I-I_K)(I-J_{3,K})u,\nabla J_{1,K}u)_{L^2(K)}.\end{aligned} \tag{5.11}$$

It follows from (5.5) and (5.9) that the second term on the right-hand side of the above equation can be estimated as

$$\sum_{K\in\mathcal{T}_h}(\nabla(I-I_K)(I-J_{3,K})u,\nabla J_{1,K}u)_{L^2(K)}\lesssim\sum_{K\in\mathcal{T}_h}h_K^2\|(I-\Pi_K^0)\nabla^3u\|_{L^2(K)}\|\nabla u\|_{L^2(K)}=o(h^2),$$

since piecewise constant functions are dense in the space $L^2(\Omega)$ when the meshsize is small enough. The first term on the right-hand side of (5.11) can be analyzed by (5.7), which reads

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}(\nabla(I-I_K)J_{3,K}u,\nabla J_{1,K}u)_{L^2(K)}&=-\sum_{K\in\mathcal{T}_h}\sum_{i=1}^n\sum_{j\neq i}\frac{h_{x_i,K}^2}{3}\int_K\frac{\partial^3J_{3,K}u}{\partial x_i^2\partial x_j}\frac{\partial J_{1,K}u}{\partial x_j}dx\\ &=-\sum_{K\in\mathcal{T}_h}\sum_{i=1}^n\sum_{j\neq i}\frac{h_{x_i,K}^2}{3}\int_K\frac{\partial^3u}{\partial x_i^2\partial x_j}\frac{\partial u}{\partial x_j}dx+o(h^2),\end{aligned}$$

when the meshsize is small enough. Since the mesh is uniform and $\frac{\partial^2u}{\partial x_i\partial x_j}\frac{\partial u}{\partial x_j}$ vanish on the boundary which is perpendicular to x_i axises, elementwise integrations by parts complete the proof. $\square$

A summary of (5.4), (5.6) and (5.10) proves that

$$\lambda_h(\rho(P_hu-u),P_hu+u)_{L^2(\Omega)}\geq 0 \tag{5.12}$$

when the meshsize is small enough and $u\in H^3(\Omega)$. In appendix A, we prove that $h\lesssim\|u-u_h\|_h$ when $u\in H^{2+\mathfrak{s}}(\Omega)$. Therefore, the condition (H4) holds for the Wilson element when $u\in H^3(\Omega)$ and the mesh is uniform.

5.2. **The enriched nonconforming rotated Q_1 element in any dimension.** Denote by $Q_{EQ}(K)$ the enriched nonconforming rotated Q_1 element space defined by [28]

$$Q_{EQ}(K) := P_1(K) + \operatorname{span}\{x_1^2, x_2^2, \cdots, x_n^2\}. \tag{5.13}$$

The enriched nonconforming rotated Q_1 element space V_h is then defined by

$$\begin{aligned} V_h := \ & \big\{v \in L^2(\Omega) : v|_K \in Q_{EQ}(K) \text{ for each } K \in \mathcal{T}_h, \textstyle\int_f [v] df = 0, \\ & \text{for all internal } n-1 \text{ dimensional hyper-planes } f\,, \text{ and } \textstyle\int_f v df = 0 \text{ for all } f \text{ on } \Gamma_D \big\}. \end{aligned}$$

Here and throughout this paper, $[v]$ denotes the jump of v across f. For the enriched nonconforming rotated Q_1 element, we define the interpolation operator $\Pi_h : H_D^1(\Omega) \to V_h$ by

$$\begin{aligned} & \int_f \Pi_h v df = \int_f v df \text{ for any } v \in H_D^1(\Omega),\ f \in \mathcal{H}_h\,, \\ & \int_K \Pi_h v dx = \int_K v dx \text{ for any } K \in \mathcal{T}_h. \end{aligned} \tag{5.14}$$

For this interpolation operator, we have

Lemma 5.3. *There holds that*

$$\|u - \Pi_h u\|_{L^2(K)} \lesssim h^2 |u|_{H^2(K)} \textit{for any } u \in H^2(K) \textit{ and } K \in \mathcal{T}_h, \tag{5.15}$$

$$\|u - \Pi_h u\|_{L^2(K)} \lesssim h^{1+\mathfrak{s}} |u|_{H^{1+\mathfrak{s}}(K)} \textit{for any } u \in H^{1+\mathfrak{s}}(K) \textit{ with } 0 < \mathfrak{s} < 1 \textit{ and } K \in \mathcal{T}_h. \tag{5.16}$$

Proof. Since $u - \Pi_h u$ has vanishing mean on $n-1$ dimensional hyper-plane of K, it follows from the Poincare inequality that

$$\|u - \Pi_h u\|_{L^2(K)} \lesssim h_K \|\nabla(u - \Pi_h u)\|_{L^2(K)}.$$

Then the desired result follows from the usual interpolation theory and the interpolation space theory for the singular case $u \in H^{1+\mathfrak{s}}(K)$. $\square$

Lemma 5.4. *For the enriched nonconforming rotated Q_1 element, it holds the condition (H4).*

Proof. We define the space $Q_K = \begin{pmatrix} a_{11} + a_{12}x_1 \\ a_{21} + a_{22}x_2 \\ \cdots \\ a_{n1} + a_{n2}x_n \end{pmatrix}$ with free parameters $a_{11},\ a_{12}, \cdots, a_{n1}, a_{n2}$. From the definition of the operator Π_h, we have

$$(\nabla(u - \Pi_h u), \boldsymbol{\psi})_{L^2(K)} = 0, \text{ for any } \boldsymbol{\psi} \in Q_K. \tag{5.17}$$

Let ∇_h be the piecewise gradient operator which is defined element by element. Since $\nabla_h \Pi_h u|_K \in Q_K$, this leads to

$$(\nabla_h \Pi_h u)|_K = P_K(\nabla u|_K), \tag{5.18}$$

with the L^2 projection operator P_K from $L^2(K)$ onto Q_K. This proves $a_h(u - \Pi_h u, u_h) = 0$. It remains to show estimates in (H4). To this end, let Π^0 be the piecewise constant projection operator (defined by $\Pi^0|_K = \Pi^0_K$ for element K). Without loss of generality, we assume that ρ is piecewise constant. It follows from the definition of the interpolation operator Π_h that

$$
\begin{aligned}
\|\rho^{1/2}\Pi_h u\|^2_{L^2(\Omega)} - \|\rho^{1/2}u\|^2_{L^2(\Omega)} &= (\rho(\Pi_h u - u), \Pi_h u + u)_{L^2(\Omega)} \\
&= (\rho(\Pi_h u - u), \Pi_h u + u - \Pi^0(\Pi_h u + u))_{L^2(\Omega)} \\
&\lesssim h\|\rho^{1/2}(\Pi_h u - u)\|_{L^2(\Omega)}\|\nabla_h(\Pi_h u + u)\|_{L^2(\Omega)},
\end{aligned}
\tag{5.19}
$$

which completes the proof of (H4) with $s = \triangle s = \mathcal{S} = \triangle\mathcal{S} = 1$ for the case $u \in H^1_D(\Omega) \cap H^2(\Omega)$; with $s = \mathcal{S} = \mathfrak{s}$, $\triangle s = 2 - \mathfrak{s}$, and $\triangle\mathcal{S} = 1$, for the case $u \in H^1_D(\Omega) \cap H^{1+\mathfrak{s}}(\Omega)$ with $0 < \mathfrak{s} < 1$. $\square$

In appendixes A and B, we show that $h \lesssim \|u - u_h\|_h$ when $u \in H^2(\Omega)$ and that there exist meshes such that $h^{\mathfrak{s}} \lesssim \|u - u_h\|_h$ holds when $u \in H^{1+\mathfrak{s}}(\Omega)$ with $0 < \mathfrak{s} < 1$. Therefore, we have that the result in Theorem 4.1 holds for this class of elements.

6. Morley-Wang-Xu elements for $2m$-th order operators

This section studies $2m$-th order elliptic eigenvalue problems defined over the bounded domain $\Omega \subset \mathbb{R}^n$ with $1 < n$ and $m \le n$. Let $\kappa = (\kappa_1, \cdots, \kappa_n)$ be the multi-index with $|\kappa| = \sum_{i=1}^{n} \kappa_i$, we define the space

$$
V := \{v \in L^2(\Omega), \frac{\partial^\kappa v}{\partial x^\kappa} \in L^2(\Omega), |\kappa| \le m, v|_{\partial\Omega} = \frac{\partial^\ell v}{\partial \nu^\ell}|_{\partial\Omega} = 0, \ell = 1, \cdots, m-1\},
\tag{6.1}
$$

with ν the unit normal vector to $\partial\Omega$. The partial derivatives $\frac{\partial^\kappa v}{\partial x^\kappa}$ are defined as

$$
\frac{\partial^\kappa v}{\partial x^\kappa} := \frac{\partial^{|\kappa|} v}{\partial x_1^{\kappa_1} \cdots \partial x_n^{\kappa_n}}.
\tag{6.2}
$$

Let $D^\ell v$ denote the m-th order tensor of all ℓ-th order derivatives of v, for instance, $\ell = 1$ the gradient, and $\ell = 2$ the Hessian matrix. Let $\mathcal{C}$ be a positive definite operator with the same symmetry as $D^m v$, the bilinear form $a(u, v)$ reads

$$
a(u, v) := (\sigma, D^m v)_{L^2(\Omega)} \text{ and } \sigma := \mathcal{C}D^m u,
\tag{6.3}
$$

which gives rise to the energy norm

$$
\|u\|^2_V := a(u, u) \text{ for any } u \in V,
\tag{6.4}
$$

which is equivalent to the usual $|u|_{H^m(\Omega)}$ norm for any $u \in V$.

$2m$-th order elliptic eigenvalue problems read: Find $(\lambda, u) \in \mathbb{R} \times V$ with

$$
a(u, v) = \lambda(\rho u, v)_{L^2(\Omega)} \text{ for any } v \in V \text{ and } \|\rho^{1/2}u\|_{L^2(\Omega)} = 1,
\tag{6.5}
$$

with some positive function $\rho \in L^\infty(\Omega)$.

Consider Morley-Wang-Xu elements in [37] and apply them to eigenvalue problems under consideration. Let $\mathcal{T}_h$ be some shape regular decomposition into n-simplex of the domain Ω. Denote by $\mathcal{H}_{n-i,h}$, $i = 1, \cdots, n$, all n-i dimensional subsimplexes of $\mathcal{T}_h$ with $\nu_{n-i,f}$ any one of unit normal vectors to $f \in \mathcal{H}_{n-i,h}$. Let $[\cdot]$ denote the jump of piecewise functions over f. For any n-i dimensional boundary sub-simplex f, the jump $[\cdot]$ denotes the trace restricted to f. As usual, h_K

is the diameter of $K \in \mathcal{T}_h$, and h_f the diameter of $f \in \mathcal{H}_{n-i,h}$. Given $K \in \mathcal{T}_h$, let ∂K denote the boundary of K. Morley-Wang-Xu element spaces are defined in [37], which read

$$V_h := \{v \in L^2(\Omega), v|_K \in P_m(K), \int_f [\frac{\partial^{m-i} v}{\partial \nu_{n-i,f}^{m-i}}] df = 0, \forall f \in \mathcal{H}_{n-i,h}, i = 1, \cdots, m\}. \tag{6.6}$$

Define the discrete stress $\sigma_h = \mathcal{C} D_h^m u_h$, the broken versions $a_h(\cdot,\cdot)$ and $\|\cdot\|_{\mathcal{C}_h}$ follow, respectively,

$$\begin{aligned} a_h(u_h, v_h) &:= (\sigma_h, D_h^m v_h)_{L^2(\Omega)}, \text{ for any } u_h\,, v_h \in V + V_h\,, \\ \|u_h\|_h^2 &:= a_h(u_h, u_h) \text{ for any } u_h \in V + V_h\,, \end{aligned}$$

where D_h^m is defined elementwise with respect to the partition $\mathcal{T}_h$.

The discrete eigenvalue problem reads: Find $(\lambda_h, u_h) \in \mathbb{R} \times V_h$, such that

$$a_h(u_h, v_h) = \lambda_h (\rho u_h, v_h)_{L^2(\Omega)} \text{ for any } v_h \in V_h \text{ and } \|\rho^{1/2} u_h\|_{L^2(\Omega)} = 1. \tag{6.7}$$

The canonical interpolation operator for the spaces V_h is defined by: Given any $v \in V$, the interpolation $\Pi_h v \in V_h$ is defined as

$$\int_f \frac{\partial^{m-i} \Pi_h v}{\partial \nu_{n-i,f}^{m-i}} df = \int_f \frac{\partial^{m-i} v}{\partial \nu_{n-i,f}^{m-i}} df, \text{ for any } f \in \mathcal{H}_{n-i,h}\,, i = 1\,, \cdots, m\,. \tag{6.8}$$

For this interpolation, we have the following approximation

$$\|\rho^{1/2}(u - \Pi_h u)\|_{L^2(\Omega)} \lesssim h^{m+\mathfrak{s}} |u|_{H^{m+\mathfrak{s}}(\Omega)} \text{ for any } u \in V \cap H^{m+\mathfrak{s}}(\Omega) \text{ with } 0 < \mathfrak{s} \leq 1\,. \tag{6.9}$$

It is straightforward to see that conditions (H1)-(H3) hold for this class of elements, see [35, 37]. Then, it follows from Theorems 3.3, 3.5, and 3.6 that

$$\|u - u_h\|_h \lesssim h^{\mathfrak{s}} \text{ and } \|u - u_h\|_{L^2(\Omega)} \lesssim h^{2\mathfrak{s}}\,, \tag{6.10}$$

provided that eigenfunctions $u \in V \cap H^{m+\mathfrak{s}}(\Omega)$ with $0 < \mathfrak{s} \leq 1$.

Theorem 6.1. *Let (λ, u) and (λ_h, u_h) be solutions of problems* (6.5) *and* (6.7)*, respectively. Then,*

$$\lambda_h \leq \lambda, \tag{6.11}$$

provided that h is small enough.

Proof. The definition of Π_h in (6.8) yields $a_h(u - \Pi_h u, v_h) = 0$ for any $v_h \in V_h$. The condition (H4) follows immediately from (6.9). In addition, in appendixes A and B, we show that $h \lesssim |u - u_h|_h$ when $u \in V \cap H^{m+1}(\Omega)$ and that there exist meshes such that $h^{\mathfrak{s}} \lesssim \|u - u_h\|_h$ holds when $u \in H^{m+\mathfrak{s}}(\Omega)$ with $0 < \mathfrak{s} < 1$. Then, the desired result follows from Theorem 4.1 for $m \geq 2$. □

7. New nonconforming elements

In this section, we shall follow the condition (H4) and the saturation condition in Theorem 4.1 to propose two new nonconforming finite elements for second order operators. This is of two fold, one is to modify a nonconforming element in literature such that the modified one will meet the condition (H4), the other is to construct a new nonconforming element.

7.1. **The enriched Crouzeix-Raviart element.** To fix the idea, we only consider the case where $n = 2$ and note that the results can be generalized to any dimension. Let $\mathcal{T}_h$ be some shape regular decomposition into triangles of the polygonal domain $\Omega \subset \mathbb{R}^2$. Here we restrict ourselves to the case where the bilinear form $a(u, v) = (\nabla u, \nabla v)_{L^2(\Omega)}$ with the mixed boundary condition $|\Gamma_N| \neq 0$.

Note that the original Crouzeix-Raviart element can only guarantee theoretically lower bounds of eigenvalues for the singular case in the sense that $u \in H^{1+\mathfrak{s}}(\Omega)$ with $0 < \mathfrak{s} < 1$. To produce lower bounds of eigenvalues for both the singular case $u \in H^{1+\mathfrak{s}}(\Omega)$ and the smooth case $u \in H^2(\Omega)$, we propose to enrich the shape function space by $x_1^2 + x_2^2$ on each element. This leads to the following shape function space

$$Q_{ECR}(K) = P_1(K) + \mathrm{span}\{x_1^2 + x_2^2\} \quad \text{for any } K \in \mathcal{T}_h. \tag{7.1}$$

The enriched Crouzeix-Raviart element space V_h is then defined by

$$\begin{aligned} V_h := \ &\big\{ v \in L^2(\Omega) : v|_K \in Q_{ECR}(K) \text{ for each } K \in \mathcal{T}_h, \textstyle\int_f [v] df = 0, \\ &\text{for all internal edges } f, \text{ and } \textstyle\int_f v df = 0 \text{ for all edges } f \text{ on } \Gamma_D \big\}. \end{aligned}$$

For the enriched Crouzeix-Raviart element, we define the interpolation operator $\Pi_h : H^1_D(\Omega) \to V_h$ by

$$\begin{aligned} &\int_f \Pi_h v df = \int_f v df \text{ for any } v \in H^1_D(\Omega) \text{ for any edge } f, \\ &\int_K \Pi_h v dx = \int_K v dx \text{ for any } K \in \mathcal{T}_h. \end{aligned} \tag{7.2}$$

For this interpolation operator, a similar argument of Lemma 5.3 leads to:

Lemma 7.1. *There holds that*

$$\|u - \Pi_h u\|_{L^2(K)} \lesssim h^2 |u|_{H^2(K)} \text{for any } u \in H^2(K) \text{ and } K \in \mathcal{T}_h, \tag{7.3}$$

$$\|u - \Pi_h u\|_{L^2(K)} \lesssim h^{1+s} |u|_{H^{1+\mathfrak{s}}(K)} \text{for any } u \in H^{1+\mathfrak{s}}(K) \text{ with } 0 < \mathfrak{s} < 1 \text{ and } K \in \mathcal{T}_h. \tag{7.4}$$

Lemma 7.2. *For the enriched Crouzeix-Raviart element, it holds the condition (H4).*

Proof. We follow the idea in Lemma 5.4 to define the space $Q_K = \begin{pmatrix} a_{11} + a_{12}x_1 \\ a_{21} + a_{12}x_2 \end{pmatrix}$ with free parameters a_{11}, a_{21}, a_{12}. From the definition of the operator Π_h, we have

$$(\nabla(u - \Pi_h u), \boldsymbol{\psi})_{L^2(K)} = 0, \text{ for any } \boldsymbol{\psi} \in Q_K. \tag{7.5}$$

Indeed, we integrate by parts to get

$$(\nabla(u - \Pi_h u), \boldsymbol{\psi})_{L^2(K)} = -(u - \Pi_h u, \operatorname{div} \boldsymbol{\psi})_{L^2(K)} + \sum_{f \subset \partial K} \int_f (u - \Pi_h u) \boldsymbol{\psi} \cdot \nu_f ds.$$

Since $\operatorname{div} \boldsymbol{\psi}$ and $\boldsymbol{\psi} \cdot \nu_f$ (on each edge f) are constant, then (7.5) follows from (7.2). Since $\nabla_h \Pi_h u|_K \in Q_K$, the identity (7.5) leads to

$$(\nabla_h \Pi_h u)|_K = P_K(\nabla u|_K), \tag{7.6}$$

with the L^2 projection operator P_K from $L^2(K)$ onto Q_K. Then a similar argument of Lemma 5.4 completes the proof. □

In the appendixes A and C, we have proven that $h \lesssim \|u - u_h\|_h$ when $u \in H^2(\Omega)$ and that there exist meshes such that $h^{\mathfrak{s}} \lesssim \|u - u_h\|_h$ holds when $u \in H^{1+\mathfrak{s}}(\Omega)$ with $0 < \mathfrak{s} < 1$. Hence, the result in Theorem 4.1 holds for this class of elements.

7.2. **A new first order nonconforming element.** With the condition from Theorem 4.1, a systematic method obtaining the lower bounds for eigenvalues is to design nonconforming finite element spaces with good local approximation property but not so good global continuity property. To make the idea clearer, we propose a new nonconforming element that admits lower bounds for eigenvalues. Let $\mathcal{T}_h$ be some shape regular decomposition into triangles of the polygonal domain $\Omega \subset \mathbb{R}^2$. We define

$$\begin{aligned} V_h := \; & \big\{ v \in L^2(\Omega) : v|_K \in P_2(K) \text{ for each } K \in \mathcal{T}_h, \textstyle\int_f [v] df = 0, \\ & \text{for all internal edges } f \text{ , and } \textstyle\int_f v df = 0 \text{ for all edges } f \text{ on } \Gamma_D \big\} . \end{aligned}$$

Since the conforming quadratic element space on the triangle mesh is a subspace of V_h, the usual dual argument proves

$$\|u - P_h u\|_{L^2(\Omega)} \lesssim h^{2+\mathfrak{s}} |u|_{H^{2+\mathfrak{s}}(\Omega)},$$

provided that $u \in V \cap H^{2+\mathfrak{s}}(\Omega)$ with $0 < \mathfrak{s} \leq 1$. In the appendix A, it is shown that $h \lesssim \|\nabla_h(u - u_h)\|_{L^2(\Omega)}$, which in fact implies the condition (H4) for this case. For the singular case $u \in V \cap H^{1+\mathfrak{s}}(\Omega)$, a similar argument of the enriched Crouzeix-Raviart element is able to show the condition (H4).

8. Conclusion and comments

In this paper, we propose a systematic method that can produce lower bounds for eigenvalues of elliptic operators. With this method, to obtain lower bounds is to design nonconforming finite element spaces with enough local degrees of freedom when compared to the global continuity. We check that several nonconforming methods in literature possess this promising property. We also propose some new nonconforming methods with this feature. In addition, we study systematically the saturation condition for both conforming and nonconforming finite element methods.

Certainly, there are many other nonconforming finite elements which are not analyzed herein. Let mention several more elements and give some short comments on applications of the theory herein to them. The first one is the nonconforming rotated Q_1 element from [33]. For this element, discrete eigenvalues are smaller than exact ones when eigenfunctions are singular, see more details from [44]. The same comments applies for the Crouzeix–Raviart element of [13], see more details from [2, 43]. The last one is the Adini element [25, 35] for fourth order problems. For this element, by an expansion result of [22, Lemma] and a similar identity like that of Lemma 4.1 therein, a similar argument for the Wilson element is able to show that discrete eigenvalues are smaller than exact ones provided that eigenfunctions $u \in H^4(\Omega)$.

Appendix A. The saturation condition

In the following two sections, we shall prove, for some cases, the saturation condition which is used in Theorem 4.1. The error basically consists of two parts: approximation errors and the consistency errors. In this section, we analyze the case where approximation errors are dominant and the case where consistency errors are dominant; in the appendix B, we give some comments for the case where eigenfunctions are singular.

A.1. **The saturation condition where approximation error are dominant.** Let $u \in V \cap H^m(\Omega)$ be eigenfunctions of some $2m$-th order elliptic operator. Let V_h be some k-th order conforming or nonconforming approximation spaces to $H^m(\Omega)$ over the mesh $\mathcal{T}_h$ in the following sense:

$$\frac{\sup\limits_{0\neq v\in H^{m+k}(\Omega)\cap V} \inf\limits_{v_h\in V_h} \|D_h^m(v-v_h)\|_{L^2(\Omega)}}{|v|_{H^{m+k}}} \lesssim h^k \text{ for some positive integer } k. \tag{A.1}$$

Then the following condition is sufficient for the saturation condition:

H5 At least one fixed component of $D_h^{m+k}v_h$ vanishes for all $v_h \in V_h$ while the L^2 norm of the same component of $D^{m+k}u$ is nonzero.

Recall that $D^\ell v$ denote the ℓ-th order tensor of all ℓ-th order derivatives of v, for instance, $\ell = 1$ the gradient, and $\ell = 2$ the Hessian matrix, and that D_h^ℓ are the piecewise counterparts of D^ℓ defined element by element.

In order to achieve the desired result, we shall use the operator defined in (5.8). For readers' convenience, we recall its definition. Given any element K, define $J_{m+k,K}v \in P_{m+k}(K)$ by

$$\int_K D^\ell J_{m+k,K}v dxdy = \int_K D^\ell v dxdy, \ell = 0, 1, \cdots, m+k, \tag{A.2}$$

for any $v \in H^{m+k}(K)$. Note that the operator $J_{m+k,K}$ is well-defined. Since $\int_K D^\ell(v - J_{m+k,K}v)dxdy = 0$, $\ell = 0, \cdots, m+k$,

$$\|D^{\ell_1}(v - J_{m+k,K}v)\|_{L^2(K)} \leq Ch_K^{\ell_2-\ell_1}\|D^{\ell_2}(v - J_{m+k,K}v)\|_{L^2(K)} \text{ for any } 0 \leq \ell_1 \leq \ell_2 \leq m+k. \tag{A.3}$$

Finally, define the global operator J_{m+k} by

$$J_{m+k}|_K = J_{m+k,K} \text{ for any } K \in \mathcal{T}_h. \tag{A.4}$$

It follows from the very definition of $J_{m+k,K}$ in (A.2) that

$$D_h^{m+k} J_{m+k}v = \Pi^0 D^{m+k}v, \tag{A.5}$$

where Π^0 is the L^2 piecewise constant projection operator with respect to $\mathcal{T}_h$, which is defined in subsection 5.1. Since piecewise constant functions are dense in the space $L^2(\Omega)$,

$$\|D_h^{m+k}(v - J_{m+k}v)\|_{L^2(\Omega)} \to 0 \text{ when } h \to 0. \tag{A.6}$$

Theorem A.1. *Under the condition H5, there holds the following saturation condition:*

$$h^k \lesssim \|D_h^m(u - u_h)\|_{L^2(\Omega)}. \tag{A.7}$$

Proof. By the condition H5, we let $\mathfrak{N}$ denote the multi-index set such that $|\kappa| = m+k$ for any $\kappa \in \mathfrak{N}$ and that

$$\frac{\partial^\kappa v_h|_K}{\partial x^\kappa} \equiv 0 \text{ for any } K \in \mathcal{T}_h \text{ and } v_h \in V_h \text{ while } \|\frac{\partial^\kappa u}{\partial x^\kappa}\|_{L^2(\Omega)} \neq 0. \tag{A.8}$$

Let J_{m+k} be defined as in (A.2) and (A.4). It follows from the triangle inequality and the piecewise inverse estimate that

$$\begin{aligned}\sum_{\kappa\in\mathfrak{N}}\|\frac{\partial^\kappa u}{\partial x^\kappa}\|^2_{L^2(\Omega)} &= \sum_{\kappa\in\mathfrak{N}}\sum_{K\in\mathcal{T}_h}\|\frac{\partial^\kappa (u-u_h)}{\partial x^\kappa}\|^2_{L^2(K)}\\ &\leq 2\sum_{\kappa\in\mathfrak{N}}\sum_{K\in\mathcal{T}_h}\big(\|\frac{\partial^\kappa (u-J_{m+k}u)}{\partial x^\kappa}\|^2_{L^2(K)}+\|\frac{\partial^\kappa (J_{m+k}u-u_h)}{\partial x^\kappa}\|^2_{L^2(K)}\big)\\ &\lesssim \|D_h^{m+k}(u-J_{m+k}u)\|^2_{L^2(\Omega)}+h^{-2k}\|D_h^m(J_{m+k}u-u_h)\|^2_{L^2(\Omega)}.\end{aligned}\tag{A.9}$$

The estimate of (A.3) and the triangle inequality lead to

$$\sum_{\kappa\in\mathfrak{N}}\|\frac{\partial^\kappa u}{\partial x^\kappa}\|^2_{L^2(\Omega)}\lesssim \|D_h^{m+k}(u-J_{m+k}u)\|^2_{L^2(\Omega)}+h^{-2k}\|D_h^m(u-u_h)\|^2_{L^2(\Omega)}.\tag{A.10}$$

Finally it follows from (A.6) that

$$h^{2k}\sum_{\kappa\in\mathfrak{N}}\|\frac{\partial^\kappa u}{\partial x^\kappa}\|^2_{L^2(\Omega)}\lesssim \|D_h^m(u-u_h)\|^2_{L^2(\Omega)}\tag{A.11}$$

when the meshsize is small enough, which completes the proof. $\square$

Remark A.2. *Under the condition H5, a similar argument can prove the following general saturation conditions:*

$$h^{k+m-\ell}\lesssim \|D_h^\ell(u-u_h)\|_{L^2(\Omega)}, \ell=0,1,\cdots,m.\tag{A.12}$$

Next, we prove the condition H5 for various element in literature.

(1) The Morley-Wang-Xu element. Since $D_h^{m+1}v_h\equiv 0$ for all $v_h\in V_h$ for this family of elements and $v\equiv 0$ if $D^{m+1}v\equiv 0$ for any $v\in V\cap H^{m+1}(\Omega)$, the condition H5 holds.

(2) The enriched Crouzeix-Raviart element. Let $\partial_{12,h}$ denote the piecewise counterpart of the differential operator $\frac{\partial^2}{\partial x\partial y}$. We have $\partial_{12,h}v_h\equiv 0$ for any $v_h\in V_h$. We only consider the case where $\Omega=[0,1]^2$ and $u\in H_0^1(\Omega)$. If $\|\frac{\partial^2 v}{\partial x\partial y}\|_{L^2(\Omega)}$ vanishes for $v\in V\cap H_0^2(\Omega)$. Then, v should be of the form $v(x,y)=f(x)+g(y)$, where $f(x)$ is some function of the variable x and $g(y)$ is some function of the variable y. Now the homogenous boundary condition indicates that $f(x)\equiv C_1$ and $g(y)\equiv C_2$ for some constants C_1 and C_2, which in turn concludes that $v\equiv 0$. This proves the condition $H5$.

(3) The same argument applies to the nonconforming Q_1 element, the enriched nonconforming rotated Q_1 element, and the conforming Q_1 element in any dimension.

A.2. **The saturation condition where consistency errors are dominant.** In this subsection, we prove the saturation condition for the case where consistency errors are dominant. As usual it is very complicated to give an abstract estimate for consistency errors in a unifying way. Therefore, for ease of presentation, we shall only consider the new first order nonconforming element proposed in this paper. However, the idea can be extended to other nonconforming finite element methods.

In order to give lower bounds of consistency errors, given any edge (boundary and interior) e, we construct functions $v_e\in V_h$ such that:

(1) v_e vanishes on $\Omega\backslash\omega_e$;

(2) v_e vanishes on two Gauss-Legendre points of the other four edges than e of ω_e;

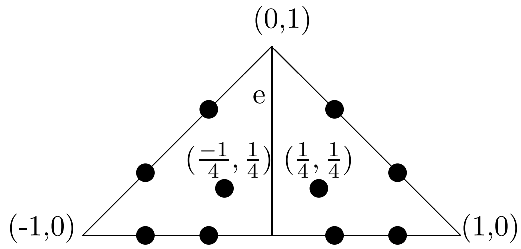

FIGURE 1. Reference Edge patch and degrees of freedom for v_e

(3) v_e vanishes at two interior points of ω_e, see points $(\frac{1}{4},\frac{1}{4})$ and $(\frac{-1}{4},\frac{1}{4})$ in Figure 1 for examples of the reference edge patch;
(4) $\int_e [v_e] s ds = \mathcal{O}(h^2) \neq 0$.

See Figure 1 for the reference edge patch and degrees of freedom for v_e. Note that such a function can be found. In fact, for the reference edge patch in Figure 1, a direct calculation shows that there exists a function $v_e \in V_h$ such that $\int_e [v_e] s ds = 0.1715 \neq 0$.

Let Π_e^1 be the L^2 projection from $L^2(e)$ to $P_1(e)$. Since $\int_e [v_h] ds = 0$ for any edge e of $\mathcal{T}_h$ and $v_h \in V_h$, it follows that

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial u}{\partial \nu} v_h ds = \sum_e \int_e \frac{\partial u}{\partial \nu}[v_h] ds = \sum_e \int_e \frac{\partial}{\partial \tau}(\Pi_e^1 \frac{\partial u}{\partial \nu})[v_h] s ds + \sum_e \int_e (I-\Pi_e^1)\frac{\partial u}{\partial \nu}[v_h] ds. \tag{A.13}$$

Define

$$v_h = \sum_e v_e \frac{\partial}{\partial \tau}(\Pi_e^1 \frac{\partial u}{\partial \nu}). \tag{A.14}$$

Since $\frac{\partial}{\partial \tau}(\Pi_e^1 \frac{\partial u}{\partial \nu})$ are constants, definitions of v_e yield

$$\sum_e \int_e \frac{\partial}{\partial \tau}(\Pi_e^1 \frac{\partial u}{\partial \nu})[v_h] s ds \geq Ch \sum_e \|\frac{\partial}{\partial \tau}(\Pi_e^1 \frac{\partial u}{\partial \nu})\|_{L^2(e)}^2,$$

and

$$\|\nabla_h v_h\|_{L^2(\Omega)} \leq Ch^{-1/2}\bigg(\sum_e \|\frac{\partial}{\partial \tau}(\Pi_e^1 \frac{\partial u}{\partial \nu})\|_{L^2(e)}^2\bigg)^{1/2}.$$

A substitution of these two inequalities into (A.13) leads to

$$\sup_{0\neq v_h\in V_h} \frac{\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial u}{\partial \nu} v_h ds}{\|\nabla_h v_h\|_{L^2(\Omega)}} \geq C_1 h \|\nabla^2 u\|_{L^2(\Omega)} - C_2 h^{1+s}|u|_{H^{2+s}(\Omega)}, \tag{A.15}$$

provided that $u \in H^{2+s}(\Omega)$ with $0 < s \leq 1$ for some positive constants C_1 and C_2. Since $\|\nabla^2 u\|_{L^2(\Omega)}$ can not vanish, this proves the saturation condition.

Remark A.3. *Thanks to two nonconforming bubble functions in each element, a similar argument is able to show a corresponding result for the Wilson element* [35, 41].

Appendix B. The comment for the saturation condition of the singular case

We need the concept of the interpolation space. Let X, Y be a pair of normed linear spaces. We shall assume that Y is continuously embedded in X with $Y \subset X$ and $\|\cdot\|_X \lesssim \|\cdot\|_Y$. For any $t \geq 0$, we define the $K-$functional

$$K(f,t) = K(f,t,X,Y) = \inf_{g \in Y} \|f-g\|_X + t|g|_Y, \tag{B.1}$$

where $\|\cdot\|_X$ is the norm on X and $|\cdot|_Y$ is a semi-norm on Y. The function $K(f,.)$ is defined on $\mathbb{R}_+$ and is monotone and concave (being the pointwise infimum of linear functions). If $0 < \theta < 1$ and $1 < q \leq \infty$, the interpolation space $(X,Y)_{\theta,q}$ is defined as the set of all functions $f \in X$ such that [6, 15, 16]

$$|f|_{(X,Y)_{\theta,q}} = \begin{cases} (\sum\limits_{k=0}^{\infty} [2^{(s+\epsilon)k\theta} K(f, 2^{-k(s+\epsilon)})]^q)^{1/q}, 0 < q < \infty, \\ \sup\limits_{k \geq 0} 2^{k(s+\epsilon)\theta} K(f, 2^{-k(s+\epsilon)}), q = \infty, \end{cases} \tag{B.2}$$

is finite for some $0 < s + \epsilon \leq 1$.

B.1. **An abstract theory.** We assume that $u \in H^{m+s}(\Omega)$ with $0 < s < 1$ and V_h is some nonconforming or conforming approximation space to the space $H^m(\Omega)$.

Then the following conditions imply in some sense the saturation condition for the singular case:

H6. There exists a piecewise polynomial space $V^c_{m+1,h} \subset H^{m+1}(\Omega)$ such that $V^c_{m+1,h} \subset V^c_{m+1,h/2}$ when $\mathcal{T}_{h/2}$ is some nested conforming refinement of $\mathcal{T}_h$;

H7. There holds the following Berstein inequality

$$|v|_{H^{m+s+\epsilon}(\Omega)} \lesssim h^{-(s+\epsilon)} |v|_{H^m(\Omega)} \text{ for any } v \in V^c_{m+1,h}; \tag{B.3}$$

H8. There exists some quasi-interpolation operator $\Pi^c : V_h \to V^c_{m+1,h}$ such that

$$\|D^m(u - \Pi^c u_h)\|_{L^2(\Omega)} \lesssim h^{s+\epsilon} \tag{B.4}$$

provided that $\|D^m_h(u-u_h)\|_{L^2(\Omega)} \lesssim h^{s+\epsilon}$ with $\epsilon > 0$ and $s + \epsilon \leq 1$.

Theorem B.1. *Suppose the eigenfunction $u \in H^{m+s}(\Omega)$ with $0 < s < 1$. Under conditions H6–H8, there exist meshes such that the following saturation condition holds*

$$h^s \lesssim \|D^m_h(u-u_h)\|_{L^2(\Omega)}. \tag{B.5}$$

Proof. We assume that the saturation condition $h^s \lesssim \|D^m_h(u-u_h)\|_{L^2(\Omega)}$ does not hold for any mesh $\mathcal{T}_h$ with the meshsize h. In other word, we have

$$\|D^m_h(u-u_h)\|_{L^2(\Omega)} \lesssim h^{s+\epsilon}, \tag{B.6}$$

for some $\epsilon > 0$. In the following, we assume that $s + \epsilon \leq 1$. By the condition H8, we have

$$\inf_{v \in V^c_{m+1,h}} \|D^m(u-v)\|_{L^2(\Omega)} \lesssim \|D^m_h(u - \Pi^c u_h)\|_{L^2(\Omega)} \lesssim h^{s+\epsilon}. \tag{B.7}$$

Take $X = H^m(\Omega)$ and $Y = H^{m+s+\epsilon}(\Omega)$. The inequality (B.7) is the usual Jackson inequality and the inequality (B.3) is the Berstein inequality in the context of the approximation theory [16, 15]. We can follow the idea of [16, Theorem 5.1 , Chapter 7] to estimate terms like $K(u, 2^{-\ell(s+\epsilon)})$ for any positive integer ℓ. In fact, let $\varphi_k \in V^c_{m+1,2^{-k(s+\epsilon)}}$ be the best approximation to u in the sense that $\|D^m(u-\varphi_k)\|_{L^2(\Omega)} = \inf_{v\in V^c_{m+1,2^{-k(s+\epsilon)}}} \|D^m(u-v)\|_{L^2(\Omega)}$, $k\geq 1$. Let $\psi_k = \varphi_k - \varphi_{k-1}$, $k=1,2,\cdots$, where $\psi_0=\varphi_0=0$. Then we have

$$\|D^m\psi_k\|_{L^2(\Omega)} \leq \|D^m(u-\varphi_k)\|_{L^2(\Omega)} + \|D^m(u-\varphi_{k-1})\|_{L^2(\Omega)} \lesssim 2^{-k(s+\epsilon)}. \tag{B.8}$$

Since $\varphi_\ell = \sum_{k=0}^{\ell}\psi_k$ and $|\psi_0|_{H^{m+s+\epsilon}(\Omega)}=0$, it follows from (B.7), (B.3) and (B.8) that

$$\begin{aligned} K(u,2^{-(s+\epsilon)\ell}) &\leq \|u-\varphi_\ell\|_{H^m(\Omega)} + 2^{-(s+\epsilon)\ell}|\varphi_\ell|_{H^{m+s+\epsilon}} \\ &\lesssim 2^{-(s+\epsilon)\ell} + 2^{-(s+\epsilon)\ell}\sum_{k=1}^{\ell} 2^{k(s+\epsilon)^2}\|\psi_k\|_{H^m(\Omega)} \\ &\lesssim \ell 2^{-(s+\epsilon)\ell}. \end{aligned} \tag{B.9}$$

$$|u|_{(H^m(\Omega),H^{m+s+\epsilon}(\Omega))_{\theta,2}} = \left(\sum_{k=0}^{\infty}\left[2^{k(s+\epsilon)\theta}K(u,2^{-k(s+\epsilon)})\right]^2\right)^{1/2} \lesssim \left(\sum_{k=0}^{\infty}\left[k2^{k(s+\epsilon)(\theta-1)}\right]^2\right)^{1/2}. \tag{B.10}$$

Let $\theta = 1-\epsilon_0$ with $\epsilon_0>0$ such that $\epsilon-(s+\epsilon)\epsilon_0>0$. This leads to

$$|u|_{(H^m(\Omega),H^{m+s+\epsilon}(\Omega))_{\theta,2}} \lesssim \left(\sum_{k=0}^{\infty}\left[k2^{-k(s+\epsilon)\epsilon_0}\right]^2\right)^{1/2} < \infty. \tag{B.11}$$

This proves that $u\in H^{m+(1-\epsilon_0)(s+\epsilon)}(\Omega)$ which is a proper subspace of $H^{m+s}(\Omega)$ since $\epsilon-(s+\epsilon)\epsilon_0>0$, which contradicts with the fact that we only have the regularity $u\in H^{m+s}(\Omega)$. $\square$

B.2. **Proofs for H6–H8.** It follows from [14] that there exist piecewise polynomial spaces $V^c_{m+1,h}$ with nodal basis over $\mathcal{T}_h$ such that $V^c_{m+1,h}$ are nested and conforming in the sense that $V^c_{m+1,h}\subset V^c_{m+1,h/2}\subset H^m(\Omega)$ for any $1\leq n$ and $m\leq n$.

This result actually proves the conditions H6 and H7. The proof of H8 needs the interpolation of V_h into the conforming finite element space. To this end, we introduce the projection average interpolation operator of [8, 35].

Let $V^c_{m+1,h}$ be a conforming finite element space defined by (K,P^c_K,D^c_K), where D^c_T is the vector functional and the components of D^c_K are defined as follows: for any $v\in C^\kappa(K)$

$$d_{i,K}(v) := \begin{cases} D_{i,K}v(a_{i,K}) & 1\leq i\leq k_1, \\ \dfrac{1}{|F_{i,K}|}\displaystyle\int_{F_{i,K}} D_{i,K}v\,\mathrm{d}\,s & k_1<i\leq k_2, \\ \dfrac{1}{|K|}\displaystyle\int_K D_{i,K}v\,\mathrm{d}\,x & k_2<i\leq L, \end{cases} \tag{$*$}$$

where $a_{i,K}$ are points in K, $F_{i,K}$ are non zero-dimensional faces of K. $\kappa := \max_{1\leq i\leq L} k(i)$ where $k(i)$ orders of derivatives used in degrees of freedom $D_{i,K} := \sum_{|\alpha|=k(i)}\eta_{i,\alpha,K}\partial^\alpha, 1\leq i\leq L$, $\eta_{i,\alpha,K}$ are constants which depend on $i,\alpha,$ and K.

Let $\omega(a)$ denote the union of elements that share point a and $\omega(F)$ the union of elements having in common the face F. Let $N(a)$ and $N(F)$ denote the number of elements in $\omega(a)$ and $\omega(F)$, respectively. For any $v\in V_h$, define the projection average interpolation operator $\Pi^c: V_h\to V^c_{m+1,h}$ by

(1) for $1 \le i \le k_1$, if $a_{i,K} \in \partial\Omega$ and $d_{i,K}(\phi) = 0$ for any $\phi \in C^\kappa(\overline{\Omega}) \cap V$, then $d_{i,K}(\Pi^c v|_K) := 0$; otherwise
$$d_{i,K}(\Pi^c v|_K) := \frac{1}{N(a_{i,K})} \sum_{K' \in \omega(a_{i,K})} D_{i,K}(v|_{K'})(a_{i,K});$$

(2) for $k_1 < i \le k_2$, if $F_{i,K} \subset \partial\Omega$ and $d_{i,K}(\phi) = 0$ for any $\phi \in C^\kappa(\overline{\Omega}) \cap V$, then $d_{i,K}(\Pi^c v|_K) := 0$; otherwise
$$d_{i,K}(\Pi^c v|_K) := \frac{1}{N(F_{i,K})} \sum_{K' \in \omega(F_{i,K})} \frac{1}{|F_{i,K}|} \int_{F_{i,K}} D_{i,K}(v|_{K'})(a_{i,K}) \, \mathrm{d}\, s;$$

(3) for $k_2 < i \le L$
$$d_{i,K}(\Pi^c v|_K) := \frac{1}{|K|} \int_K D_{i,K}(v|_K) \, \mathrm{d}\, x.$$

Lemma B.2. *For all nonconforming element spaces under consideration, there exists $r \in \mathbb{N}, r \geqslant m$ such that $V_h|_K \subset P_r(K) \subset P_K^c$. Then, for $m < k \leqslant min\{r+1, 2m\}$, $0 \leqslant l \leqslant m$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, it holds that*
$$\|D_h^m(v_h - \Pi^c v_h)\|_{L^2(\Omega)}^2 \lesssim \sum_{K \in \mathcal{T}_h} \left(\sum_{j=m}^{k-1} h_K^{2(j-m)+1} \sum_{F \subset \partial K / \partial\Omega} \sum_{|\alpha|=j} \|[\partial^\alpha v_h]\|_{0,F}^2 \right.$$
$$\left. + h_K \sum_{F \subset \partial K \cap \partial\Omega} \sum_{|\alpha|=m, \alpha_1 < m} \| \frac{\partial^{|\alpha|} v_h}{\partial \nu_F^{\alpha_1} \partial \tau_{F,2}^{\alpha_2} \cdots \partial \tau_{F,n}^{\alpha_n}} \|_{0,F}^2 \right),$$
where $\tau_{F,2}, \cdots, \tau_{F,n}$ are $n-1$ orthonormal tangent vectors of F.

Proof. Since $V_h|_K \subset P_r(K) \subset P_K^c$, a slight modification of the argument in [35, Lemma 5.6.4] can prove the desired result; see also [8] for the proof of the nonconforming linear element with $m = 1$. □

The remaining proof is based on bubble function techniques, see [11] for a posteriori error analysis of second order problems, see [19, 21] for a posteriori error analysis of fourth order problems. Let $v_h = u_h$ in the above lemma. Such an analysis leads to

$$\|D_h^m(\Pi u_h - u_h)\|_{L^2(\Omega)} \lesssim \|D_h^m(u - u_h)\|_{L^2(\Omega)} \lesssim h^{s+\epsilon}. \tag{B.12}$$

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