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Abstract

A pressure-stabilized characteristics finite element scheme for the Oseen equations is presented. Stability and convergence results with the optimal error estimates for the velocity and the pressure are proved. The scheme can deal with convection-dominated problems and leads to a symmetric coefficient matrix of the system of linear equations. A cheap P1/P1 finite element is employed and the degrees of freedom are smaller than that of other typical elements for the equations, e.g., P2/P1. Therefore, the scheme is efficient especially for three dimensional problems. Two and three dimensional numerical results are shown to recognize the theoretical convergence orders and applicability to the linear stability analysis of stationary flows for the Navier-Stokes equations.

Keywords Error estimates · The finite element method · The method of characteristics · Pressure-stabilization · The Oseen equations

Mathematics Subject Classification (2000) 65M12 · 65M60 · 65M25 · 76D07

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1 Introduction

In this paper we present a combined finite element scheme with a pressure-stabilization and a characteristics method for the Oseen equations, and prove its stability and error estimates. A corresponding scheme to the Navier-Stokes equations has been proposed by us [21, 23] and the theoretical analysis will be given in a forthcoming paper [25].

The system of the Oseen equations is well known as a linearized model of the Navier-Stokes equations and has been used to understand the incompressible fluid dynamics, e.g., the flows past a cylinder and a sphere. Although the system is linear, the numerical treatment is not so easy especially in the convection-dominant case. We focus on the following two issues in order to devise efficient schemes.

The one is what pair of finite elements should be employed for the velocity and the pressure. The well-known inf-sup condition [15] requires a restriction on the choice of elements. A typical pair of elements satisfying the inf-sup condition is the so-called P2/P1 (Hood-Taylor) finite element [15], i.e., continuous piecewise quadratic polynomial approximation to the velocity and continuous piecewise linear approximation to the pressure. The P2/P1 finite element is still expensive especially for three dimensional computations, while it has a second order approximation property. On the other hand, the P1/P1 finite element, i.e., continuous piecewise linear approximation to both the velocity and the pressure, is cheap and useful for three dimensional computations, while it does not satisfy the inf-sup condition and has a first order interpolation property. In [6] a pressure-stabilization technique has been originally proposed for the Stokes equations and a first order convergence has been proved.

The other is how to discretize the convection term especially for convection-dominated (small viscosity, high Reynolds number) problems. In order to deal with such problems a lot of ideas have been proposed, e.g., upwind methods [1, 4, 7, 13, 17, 18, 32, 34], characteristics(-based) methods [3, 9, 11, 12, 21–24, 26–29] and so on. We note that the approximation based on the method of characteristics is one of the least diffusive methods among them and has such a common advantage that the resulting matrix of the system of linear equations is symmetric, which enables us to use efficient linear iterative solvers for symmetric matrices, i.e., MINRES, CR and so on [2, 30].

In this paper we propose a combined finite element scheme with a pressure-stabilization and a characteristics method for the Oseen equations, and prove its stability and error estimates. The characteristics method works for convection-dominated problems, and the pressure-stabilization is employed for the use of the cheap P1/P1 finite element. Since the scheme is symmetric by virtue of the characteristics method, we can use efficient linear iterative solvers for symmetric matrices. The resulting matrix is identical with respect to the time step and it is enough to make the matrix only once at the first time step. The scheme is essentially unconditionally stable and has a first order convergence property both in time and space.

Let m be a non-negative integer and Ω be a bounded domain in $\mathbb{R}^d (d = 2,3)$. We use the Sobolev spaces $W^{m,\infty}(\Omega)$ and $H^m(\Omega)$ as well as $C^m(\bar{\Omega})$. For any normed space X with norm $\|\cdot\|_X$, we define function spaces $C^m([0,T];X)$ and $H^m(0,T;X)$ consisting of X-valued functions in $C^m([0,T])$ and $H^m(0,T)$, respectively. We use the same notation $(\cdot,\cdot)$ to represent the $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions. $L^2_0(\Omega)$ is a subspace of $L^2(\Omega)$ defined by

$$L^2_0(\Omega) \equiv \left\{q \in L^2(\Omega);\ (q,1) = 0\right\}.$$

We often omit $[0,T]$, Ω and/or d if there is no confusion, e.g., $C^0(H^1)$ in place of $C^0([0,T];H^1(\Omega)^d)$. For t_0 and $t_1 \in \mathbb{R}$ we introduce function spaces

$$Z^m(t_0,t_1) \equiv \{v \in H^j(t_0,t_1;H^{m-j}(\Omega)^d);\ j = 0,\cdots,m,\ \|v\|_{Z^m(t_0,t_1)} < \infty\},$$

and $Z^m \equiv Z^m(0,T)$, where the norm $\|v\|_{Z^m(t_0,t_1)}$ is defined by

$$\|v\|_{Z^m(t_0,t_1)} \equiv \max_{j=0,\cdots,m} \|v\|_{H^j(t_0,t_1;H^{m-j}(\Omega)^d)}.$$

The abbreviations LHS and RHS mean left- and right-hand sides, respectively.

2 A pressure-stabilized characteristics finite element scheme

In this section we present our pressure-stabilized characteristics finite element scheme for the Oseen equations.

Let Ω be a bounded domain in $\mathbb{R}^d (d=2,3)$, $\Gamma \equiv \partial\Omega$ be the boundary of Ω and T be a positive constant. We consider an initial boundary value problem; find $(u,p):\ \Omega\times(0,T)\to\mathbb{R}^d\times\mathbb{R}$ such that

$$\frac{Du}{Dt} - \nabla\big(2\nu D(u)\big) + \nabla p + \lambda u = f \qquad \text{in } \Omega\times(0,T), \tag{1a}$$

$$\nabla\cdot u = 0 \qquad \text{in } \Omega\times(0,T), \tag{1b}$$

$$u = 0 \qquad \text{on } \Gamma\times(0,T), \tag{1c}$$

$$u = u^0 \qquad \text{in } \Omega,\ \text{at } t=0, \tag{1d}$$

where u is the velocity, p is the pressure, $f:\ \Omega\times(0,T)\to\mathbb{R}^d$ is a given external force, $u^0:\ \Omega\to\mathbb{R}^d$ is a given initial velocity, $\lambda:\ \Omega\times(0,T)\to\mathbb{R}^{d\times d}$ is a given reaction rate, $\nu\in(0,\nu_0]$ is a viscosity for a fixed $\nu_0>0$, $D(u)$ is a strain-rate tensor defined by

$$D_{ij}(u) \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (i,j=1,\cdots,d),$$

D/Dt is a material derivation defined by

$$\frac{D}{Dt} \equiv \frac{D}{Dt_w} \equiv \frac{\partial}{\partial t} + w\cdot\nabla,$$

where $w:\ \Omega\times(0,T)\to\mathbb{R}^d$ is a given velocity.

Remark 1. *If w is replaced by u and $\lambda=0$, (1) becomes the Navier-Stokes problem. Here, we focus on the Oseen problem (1). The discussion of the Navier-Stokes problem will be presented in the forthcoming paper [25].*

We impose assumptions for the given velocity w and reaction rate λ.

Hypothesis 1. *The function w satisfies*

$$\begin{cases} w\in C^0([0,T];W^{1,\infty}(\Omega)^d), \\ w=0 \ \ on \ \ \Gamma\times[0,T]. \end{cases}$$

Hypothesis 2. *The function λ satisfies*

$$\lambda\in C^0([0,T];L^\infty(\Omega)^{d\times d}).$$

Let $V \equiv H_0^1(\Omega)^d$ and $Q \equiv L_0^2(\Omega)$ be function spaces. We define bilinear forms a on $V \times V$, b on $V \times Q$ and $\mathscr{A}$ on $(V \times Q) \times (V \times Q)$ by

$$a(u,v) \equiv 2\big(D(u), D(v)\big), \quad b(v,q) \equiv -(\nabla \cdot v, q),$$
$$\mathscr{A}((u,p),(v,q)) \equiv \nu a(u,v) + b(v,p) + b(u,q),$$

respectively. Then, we can write the weak formulation of (1); find $(u,p): \ (0,T) \to V \times Q$ such that, for $t \in (0,T)$,

$$\Big(\frac{Du}{Dt}(t), v\Big) + \mathscr{A}((u,p)(t),(v,q)) + (\lambda(t)u(t),v) = (f(t),v), \quad \forall (v,q) \in V \times Q, \tag{2}$$

with $u(0) = u^0$.

We introduce a basic idea of the method of characteristics. Let $X : (0,T) \to \mathbb{R}^d$ be a solution of the ordinary differential equation,

$$\frac{dX}{dt} = w(X,t). \tag{3}$$

Then, for a smooth function $u : \Omega \times (0,T) \to \mathbb{R}^d$ it holds that

$$\frac{Du}{Dt}(X(t),t) = \frac{d}{dt} u\big(X(t),t\big).$$

Let Δt be a time increment, $t^n \equiv n\Delta t$ for $n \in \mathbb{N} \cup \{0\}$ and $X(\cdot; x, t^n)$ be the solution of (3) satisfying an initial condition $X(t^n) = x$. Then, we can consider a first order approximation of the material derivative at $t = t^n (n \geq 1)$ as follows.

$$\begin{aligned}
\frac{Du}{Dt}(x,t^n) &= \frac{d}{dt} u\big(X(t;x,t^n),t\big)\Big|_{t=t^n} \\
&= \frac{u(X(t^n;x,t^n),t^n) - u(X(t^{n-1};x,t^n),t^{n-1})}{\Delta t} + O(\Delta t) \\
&= \frac{u^n - u^{n-1} \circ X_1^n}{\Delta t}(x) + O(\Delta t),
\end{aligned} \tag{4}$$

where $X_1^n(x)$ is a function defined by

$$X_1^n(x) \equiv x - w^n(x)\Delta t,$$

and we have used notations, $u^n \equiv u(\cdot, t^n)$ and

$$v \circ X_1^n(x) \equiv v(X_1^n(x)).$$

The point $X_1^n(x)$ is called an upwind point of x. The approximation (4) of Du/Dt is a basic idea to devise numerical schemes based on the method of characteristics. The idea has been combined with finite element and difference methods, cf. [9, 22, 24, 27, 29].

The next proposition proved in [29] gives a sufficient condition to guarantee all upwind points are in Ω.

Proposition 1 ([29], Proposition 1)**.** *Under Hypothesis 1 and the inequality*

$$\Delta t < \frac{1}{\|w\|_{C^0(W^{1,\infty}(\Omega))}}, \tag{5}$$

it holds that, for any $n = 0, \cdots, N_T$,

$$X_1^n(\Omega) = \Omega.$$

For the sake of simplicity we assume Ω is a polygonal ($d=2$) or polyhedral ($d=3$) domain. Let $\mathscr{T}_h=\{K\}$ be a triangulation of $\bar{\Omega}$ $(=\bigcup_{K\in\mathscr{T}_h}K)$, h_K be a diameter of $K\in\mathscr{T}_h$, and $h\equiv\max_{K\in\mathscr{T}_h}h_K$ be the maximum element size. We consider a regular family of subdivisions $\{\mathscr{T}_h\}_{h\downarrow 0}$ satisfying the inverse assumption [8], i.e., there exists a positive constant α_0, independent of h, such that

$$\frac{h}{h_K}\leq\alpha_0,\quad\forall K\in\bigcup_h\mathscr{T}_h. \tag{6}$$

We define function spaces X_h, M_h, V_h and Q_h by

$$\begin{aligned}X_h&\equiv\{v_h\in C^0(\bar{\Omega}_h)^d;\ v_h|_K\in P_1(K)^d,\ \forall K\in\mathscr{T}_h\},\\ M_h&\equiv\{q_h\in C^0(\bar{\Omega}_h);\ q_h|_K\in P_1(K),\ \forall K\in\mathscr{T}_h\},\end{aligned}$$

$V_h\equiv X_h\cap V$ and $Q_h\equiv M_h\cap Q$, respectively, where $P_1(K)$ is a polynomial space of linear functions on $K\in\mathscr{T}_h$. Let $N_T\equiv[T/\Delta t]$ be a total number of time steps, δ_0 be a positive constant and $(\cdot,\cdot)_K$ be the $L^2(K)^d$ inner product. We define bilinear forms $\mathscr{C}_h$ on $H^1(\Omega)\times H^1(\Omega)$ and $\mathscr{A}_h$ on $(V\times H^1(\Omega))\times(V\times H^1(\Omega))$ by

$$\begin{aligned}\mathscr{C}_h(p,q)&\equiv-\delta_0\sum_{K\in\mathscr{T}_h}h_K^2(\nabla p,\nabla q)_K,\\ \mathscr{A}_h((u,p),(v,q))&\equiv\nu a(u,v)+b(v,p)+b(u,q)+\frac{1}{\nu}\mathscr{C}_h(p,q),\end{aligned}$$

respectively. Suppose that $f\in C^0([0,T];L^2(\Omega)^d)$ and $u^0\in V$. Let an approximate function $u_h^0\in V_h$ of u^0 be given. Our pressure-stabilized characteristics finite element scheme for (1) is to find $\{(u_h^n,\,p_h^n)\}_{n=1}^{N_T}\subset V_h\times Q_h$ such that, for $n=1,\cdots,N_T$,

$$\begin{aligned}\Big(\frac{u_h^n-u_h^{n-1}\circ X_1^n}{\Delta t},v_h\Big)+\mathscr{A}_h((u_h^n,p_h^n),(v_h,q_h))+(\lambda^n u_h^{n-1},v_h)&\\ =(f^n,v_h),\quad\forall(v_h,q_h)\in V_h\times Q_h.&\end{aligned} \tag{7}$$

Remark 2. *(i) We can replace the third term by $(\lambda^n u_h^n,v_h)$ and prove the stability and convergence of the scheme. The scheme, however, loses such an advantage of the Galerkin characteristics method that the resulting matrix is symmetric unless λ is symmetric. That is the reason why we consider scheme* (7). *(ii) The choice of the coefficient $1/\nu$ before $\mathscr{C}_h(p,q)$ is natural from the theoretical point of view as shown in Lemma 5 below. (iii) Scheme* (7) *leads to the symmetric matrix of the form*

$$\begin{pmatrix}A & B^T\\ B & C\end{pmatrix},$$

where A, B and C are corresponding to $\frac{1}{\Delta t}(u_h^n,v_h)+\nu a(u_h^n,v_h)$, $b(u_h^n,q_h)$ and $\frac{1}{\nu}\mathscr{C}_h(p_h^n,q_h)$, respectively, and the matrix is independent of the given velocity w and time step n.

We also prepare the following problem which generalizes (7). Let $f_h=\{f_h^n\}_{n=1}^{N_T}\subset L^2(\Omega)^d$, $g_h=\{g_h^n\}_{n=1}^{N_T}\subset H^1(\Omega)$ and $u_h^0\in V_h$ be given. The problem is to find $\{(u_h^n,p_h^n)\}_{n=1}^{N_T}\subset V_h\times Q_h$ such that, for $n=1,\cdots,N_T$,

$$\begin{aligned}\Big(\frac{u_h^n-u_h^{n-1}\circ X_1^n}{\Delta t},v_h\Big)+\mathscr{A}_h((u_h^n,p_h^n),(v_h,q_h))+(\lambda^n u_h^{n-1},v_h)&\\ =(f_h^n,v_h)+\frac{1}{\nu}\mathscr{C}_h(g_h^n,q_h),\quad\forall(v_h,q_h)\in V_h\times Q_h.&\end{aligned} \tag{8}$$

Remark 3. *When $f_h=f$ and $g_h=0$, problem* (8) *reduces to* (7).

3 Main results

In this section we present the main results of stability and error estimates, which are proved in section 4.

We use c to represent the generic positive constant independent of the discretization parameters h and Δt, and it can take different values at different places. $c(A)$ means a positive constant, which monotonically increases as A increases. For $i = 0, 1, 2$ and 3 constants c_i have particular meanings in this paper,

$$c_0 = c(\|w\|_{C^0(L^\infty)}), \quad c_1 = c(\|w\|_{C^0(W^{1,\infty})}), \quad c_2 = c(\|\lambda\|_{C^0(L^\infty)}), \quad c_3 = \max\{c_1, c_2\},$$

respectively. We use norms and seminorms, $\|\cdot\|_k \equiv \|\cdot\|_{H^k(\Omega)}$ $(k = 0,1,2)$, $\|\cdot\|_{V_h} \equiv \|\cdot\|_V \equiv \|\cdot\|_1$, $\|\cdot\|_{Q_h} \equiv \|\cdot\|_Q \equiv \|\cdot\|_0$, $\|(v,q)\|_{V\times Q,\nu} \equiv \{\nu\|v\|_V^2 + (1/\nu)\|q\|_Q^2\}^{1/2}$, $\|(v,q)\|_{V\times Q} \equiv \|(v,q)\|_{V\times Q,1}$, $\|(v,q)\|_{H^2\times H^1,\nu} \equiv \{\nu\|v\|_2^2 + (1/\nu)\|q\|_1^2\}^{1/2}$,

$$\|u\|_{l^\infty(X)} \equiv \max_{n=0,\cdots,N_T} \|u^n\|_X, \quad \|u\|_{l^2(X)} \equiv \Big\{\Delta t \sum_{n=1}^{N_T} \|u^n\|_X^2\Big\}^{1/2},$$

$$|q|_h \equiv \Big\{\sum_{K\in\mathscr{T}_h} h_K^2 (\nabla q, \nabla q)_K\Big\}^{1/2}, \quad |p|_{l^2(|\cdot|_h)} \equiv \Big\{\Delta t \sum_{n=1}^{N_T} |p^n|_h^2\Big\}^{1/2},$$

$$\|(u,p)\|_{H^1(t_0,t_1;H^2\times H^1,\nu)} \equiv \Big\{\sum_{k=0,1} \Big\|\frac{\partial^k}{\partial t^k}(u,p)(t)\Big\|^2_{L^2(t_0,t_1;H^2\times H^1,\nu)}\Big\}^{1/2},$$

for $X = L^2(\Omega)$ and $H^1(\Omega)$. $\overline{D}_{\Delta t}$ is the backward difference operator defined by

$$\overline{D}_{\Delta t} a^n \equiv \frac{a^n - a^{n-1}}{\Delta t}.$$

First we show the stability result.

Theorem 1 (stability). *(i) Suppose that Hypotheses 1 and 2 hold. Let Δt_0 be any fixed positive number satisfying* (5)*. For any $\Delta t \in (0, \Delta t_0]$, $f_h = \{f_h^n\}_{n=1}^{N_T} \subset L^2(\Omega)^d$, $g_h = \{g_h^n\}_{n=1}^{N_T} \subset H^1(\Omega)$ and $u_h^0 \in V_h$, there exists a unique solution (u_h, p_h) of scheme* (8)*, and it holds that*

$$\|u_h\|_{l^\infty(L^2)}, \ \sqrt{\nu}\|u_h\|_{l^2(H^1)}, \ \frac{1}{\sqrt{\nu}}|p_h|_{l^2(|\cdot|_h)} \le c_3(\|u_h^0\|_0 + \|f_h\|_{l^2(L^2)} + \frac{1}{\sqrt{\nu}}|g_h|_{l^2(|\cdot|_h)}). \tag{9}$$

(ii) Moreover, suppose that there exist $p_h^0 \in Q_h$ and $g_h^0 \in H^1(\Omega)$ such that

$$b(u_h^0, q_h) + \frac{1}{\nu}\mathscr{C}_h(p_h^0, q_h) = \frac{1}{\nu}\mathscr{C}_h(g_h^0, q_h), \quad \forall q_h \in Q_h. \tag{10}$$

Then, it holds that

$$\begin{aligned}&\sqrt{\nu}\|u_h\|_{l^\infty(H^1)}, \ \|\overline{D}_{\Delta t} u_h\|_{l^2(L^2)}, \ \|p_h\|_{l^2(L^2)} \\ &\quad \le c_3(1/\nu)\Big(\|u_h^0\|_1 + |p_h^0|_h + \|f_h\|_{l^2(L^2)} + \frac{1}{\sqrt{\nu}}|g_h|_{l^2(|\cdot|_h)} + \frac{1}{\sqrt{\nu}}|\overline{D}_{\Delta t} g_h|_{l^2(|\cdot|_h)}\Big).\end{aligned} \tag{11}$$

Corollary 1. *(i) Suppose that Hypotheses 1 and 2 hold. Let $f \in C^0([0,T]; L^2(\Omega)^d)$ be given. Let Δt_0 be any fixed positive number satisfying* (5)*. For any $\Delta t \in (0, \Delta t_0]$ and $u_h^0 \in V_h$ there exists a unique solution (u_h, p_h) of scheme* (7)*, and it holds that*

$$\|u_h\|_{l^\infty(L^2)}, \ \sqrt{\nu}\|u_h\|_{l^2(H^1)}, \ \frac{1}{\sqrt{\nu}}|p_h|_{l^2(|\cdot|_h)} \le c_3(\|u_h^0\|_0 + \|f\|_{l^2(L^2)}). \tag{12}$$

(ii) Moreover, suppose there exists $p_h^0 \in Q_h$ such that

$$b(u_h^0, q_h) + \frac{1}{\nu}\mathscr{C}_h(p_h^0, q_h) = 0, \quad \forall q_h \in Q_h. \tag{13}$$

Then, it holds that

$$\sqrt{\nu}\|u_h\|_{l^\infty(H^1)},\ \|\overline{D}_{\Delta t} u_h\|_{l^2(L^2)},\ \|p_h\|_{l^2(L^2)} \le c_3(1/\nu)(\|u_h^0\|_1 + |p_h^0|_h + \|f\|_{l^2(L^2)}).$$

Remark 4. *(i) Since the constant c_3 in* (12) *is independent of ν, scheme* (7) *is stable even when ν tends to* 0. *(ii) Corollary 1 is obtained by setting $f_h = f$ and $g_h = 0$ in Theorem 1. (iii) The relation* (13) *is satisfied if $(u_h^0, p_h^0) \in V_h \times Q_h$ is chosen as Stokes projection of $(u^0, 0)$ (cf. Definition 1 below).*

We give error estimates after preparing a (pressure-stabilized) Stokes projection using P1/P1-element and two hypotheses.

Definition 1 (Stokes projection). *For $(u, p) \in V \times (Q \cap H^1(\Omega))$ we define the Stokes projection $(\hat{u}_h, \hat{p}_h) \in V_h \times Q_h$ of (u, p) by*

$$\mathscr{A}_h((\hat{u}_h, \hat{p}_h), (v_h, q_h)) = \mathscr{A}_h((u, p), (v_h, q_h)), \quad \forall (v_h, q_h) \in V_h \times Q_h. \tag{14}$$

Hypothesis 3. *The function u^0 satisfies compatibility conditions $\nabla \cdot u^0 = 0$ and $u^0 \in V$.*

Hypothesis 4. *The solution (u, p) of* (2) *satisfies $u \in C^0([0,T]; V \cap H^2(\Omega)^d) \cap Z^2 \cap H^1(0,T; H^2(\Omega)^d)$ and $p \in C^0([0,T]; Q \cap H^1(\Omega)) \cap H^1(0,T; H^1(\Omega))$.*

Theorem 2 (error estimate). *(i) Let (u, p) be the solution of* (2)*. Suppose Hypotheses 1–4 hold. Let Δt_0 be any fixed positive number satisfying* (5)*. Then, for any $\Delta t \in (0, \Delta t_0]$ the solution (u_h, p_h) of scheme* (7) *satisfies*

$$\begin{aligned} &\|u_h - u\|_{l^\infty(L^2)},\ \sqrt{\nu}\|u_h - u\|_{l^2(H^1)},\ \frac{1}{\sqrt{\nu}}|p_h - p|_{l^2(|\cdot|_h)} \\ &\le c_3\Big(\|u_h^0 - u^0\|_0 + \Delta t\|u\|_{Z^2} + \frac{h}{\sqrt{\nu}}\|(u,p)\|_{C^0([0,T]; H^2\times H^1, \nu)\cap H^1(0,T; H^2\times H^1,\nu)}\Big). \end{aligned} \tag{15}$$

(ii) Moreover, suppose u_h^0 is the first component of the Stokes projection of $(u^0, 0)$ by (14)*. Then, it holds that*

$$\begin{aligned} &\sqrt{\nu}\|u_h - u\|_{l^\infty(H^1)},\ \Big\|\overline{D}_{\Delta t} u_h - \frac{\partial u}{\partial t}\Big\|_{l^2(L^2)},\ \|p_h - p\|_{l^2(L^2)} \\ &\le c_3(1/\nu)\Big(\Delta t\|u\|_{Z^2} + \frac{h}{\sqrt{\nu}}\|(u,p)\|_{C^0([0,T]; H^2\times H^1, \nu)\cap H^1(0,T; H^2\times H^1,\nu)}\Big). \end{aligned} \tag{16}$$

Remark 5. *(i) RHS of* (15) *is of $O(\Delta t + h)$ if $u_h^0 \in V_h$ satisfies $\|u_h^0 - u^0\|_0 \le ch$, e.g., we can take $u_h^0 = \Pi_h u^0$ for the linear interpolation operator $\Pi_h : C^0(\bar{\Omega}) \to V_h$, or the first component of the Stokes projection of $(u^0, 0)$ by* (14)*. (ii) In* (15) *the constant c_3 is independent of ν.*

4 Proofs of Theorems 1 and 2

This section is devoted to the proofs of Theorems 1 and 2.

4.1 Preliminaries

First we prepare six lemmas and a proposition to be used in the proofs. We omit the proofs of the first three lemmas only by referring to the related bibliography.

Lemma 1 (discrete Gronwall inequality, [16, 33])**.** *Let a_0 and a_1 be non-negative numbers, $\Delta t \in (0, 1/(2a_0)]$ be a number, and $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be non-negative sequences. Suppose*

$$\overline{D}_{\Delta t} x_n + y_n \leq a_0 x_n + a_1 x_{n-1} + b_n, \quad \forall n \geq 1.$$

Then, it holds that

$$x_n + \Delta t \sum_{i=1}^{n} y_i \leq \exp\big\{3(a_0 + a_1) n\Delta t\big\} \Big(x_0 + \Delta t \sum_{i=1}^{n} b_i\Big), \quad \forall n \geq 1.$$

Lemma 2 (Korn inequality, [10])**.** *Let Ω be a bounded domain with a Lipschitz-continuous boundary. Then, we have the followings.*
(i) There exists a positive constant α_1 such that

$$(\|D(v)\|_0^2 + \|v\|_0^2)^{1/2} \geq \alpha_1 \|v\|_1, \quad \forall v \in H^1(\Omega)^d.$$

(ii) The norms $\|D(\cdot)\|_0$ and $\|\cdot\|_1$ are equivalent in $H_0^1(\Omega)^d$.

Lemma 3 ([29])**.** *Assume Hypothesis 1 and* (5)*. Then, it holds that, for any $n \in \{0, \cdots, N_T\}$,*

$$\|v \circ X_1^n\|_0 \leq (1 + c_1 \Delta t)\|v\|_0, \quad \forall v \in L^2(\Omega)^d. \tag{17}$$

The next lemma is proved easily by a scaling argument.

Lemma 4. *Let $\{\mathscr{T}_h\}_{h\downarrow 0}$ be a regular family of triangulations of $\bar{\Omega}$. Then, there exists a positive constant α_2 such that*

$$|q_h|_h \leq \alpha_2 \|q_h\|_0, \quad \forall q_h \in Q_h. \tag{18}$$

The next lemma shows a modified version of the stability inequality in [5, 14], and the lemma easily yields the following Proposition 2.

Lemma 5. *There exists a positive constant γ_0, independent of h and ν, such that*

$$\inf_{(u_h,p_h)\in V_h\times Q_h} \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{\mathscr{A}_h((u_h,p_h),(v_h,q_h))}{\|(u_h,p_h)\|_{V\times Q,\nu} \|(v_h,q_h)\|_{V\times Q,\nu}} \geq \gamma_0. \tag{19}$$

Proof. Introducing $(\tilde{u}_h, \tilde{p}_h) \equiv (\sqrt{\nu} u_h, (1/\sqrt{\nu}) p_h)$ and $(\tilde{v}_h, \tilde{q}_h) \equiv (\sqrt{\nu} v_h, (1/\sqrt{\nu}) q_h)$, we have

$$\begin{aligned}
&\text{LHS of (19)} \\
&= \inf_{(u_h,p_h)\in V_h\times Q_h} \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{\nu a(u_h,v_h) + b(v_h,p_h) + b(u_h,q_h) + \frac{1}{\nu}\mathscr{C}_h(p_h,q_h)}{\|(u_h,p_h)\|_{V\times Q,\nu} \|(v_h,q_h)\|_{V\times Q,\nu}} \\
&= \inf_{(\tilde{u}_h,\tilde{p}_h)\in V_h\times Q_h} \sup_{(\tilde{v}_h,\tilde{q}_h)\in V_h\times Q_h} \frac{a(\tilde{u}_h,\tilde{v}_h) + b(\tilde{v}_h,\tilde{p}_h) + b(\tilde{u}_h,\tilde{q}_h) + \mathscr{C}_h(\tilde{p}_h,\tilde{q}_h)}{\|(\tilde{u}_h,\tilde{p}_h)\|_{V\times Q} \|(\tilde{v}_h,\tilde{q}_h)\|_{V\times Q}} \\
&\geq \gamma_0,
\end{aligned}$$

where the last inequality has been proved in [5, 14]. □

Remark 6. *Although the conventional inf-sup condition [15],*

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_1 \|q_h\|_0} \geq \beta^* > 0,$$

does not hold for the pair V_h and Q_h of the P1/P1 finite element spaces, $\mathscr{A}_h$ satisfies the stability inequality (19) *for this pair.*

Proposition 2. *Suppose $(u, p) \in (V \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$, and u satisfies $\nabla \cdot u = 0$. Let $(\hat{u}_h, \hat{p}_h)$ be the Stokes projection of (u, p) by* (14). *Then, it holds that*

$$\sqrt{\nu}\|u - \hat{u}_h\|_1, \ \frac{1}{\sqrt{\nu}}\|p - \hat{p}_h\|_0, \ \frac{1}{\sqrt{\nu}}|p - \hat{p}_h|_h \leq ch\|(u, p)\|_{H^2 \times H^1, \nu}. \tag{20}$$

Lemma 6. *Assume Hypothesis 1. Let Δt_0 be any fixed positive number satisfying* (5). *Then, for any $\Delta t \in (0, \Delta t_0]$ and $n \in \{0, \cdots, N_T\}$ it holds that*

$$\|u - u \circ X_1^n\|_0 \leq c_1 \Delta t \|u\|_1, \qquad \forall u \in V, \tag{21a}$$

$$(u - u \circ X_1^n, v) \leq c_1' \Delta t \|u\|_0 \|v\|_1, \quad \forall u, v \in V. \tag{21b}$$

Proof. Let any $n \in \{0, \cdots, N_T\}$ be fixed. Let $y(x) \equiv X_1^n(x)$, and $J(x) \equiv \det(\partial y / \partial x) > 0$ be the Jacobian. Using $J(x)^{-1} \leq 1 + c_1 \Delta t$,

$$u(x) - u(y) = [u(x + s(y - x))]_{s=1}^0 = \Delta t \int_0^1 [\{w^n(x) \cdot \nabla\} u](x + s(y - x))\, ds,$$

and the Schwarz inequality, we obtain (21a).

(21b) is proved as follows,

$$\begin{aligned}
\text{LHS of (21b)} &= (u, v) - (u \circ X_1^n, v) = (u, v) - \int_\Omega u(y) v((X_1^n)^{-1}(y)) J(x)^{-1}\, dy \\
&\leq \|u\|_0 \Big\{ \int_\Omega \{v(y) - v((X_1^n)^{-1}(y)) J(x)^{-1}\}^2\, dy \Big\}^{1/2} \\
&= \|u\|_0 \Big\{ \int_\Omega \{v \circ X_1^n(x) - v(x) J(x)^{-1}\}^2 J(x)\, dx \Big\}^{1/2} \\
&\leq c_1 \|u\|_0 \{\|v \circ X_1^n - v\|_0 + \|v - vJ^{-1}\|_0\} \leq c_1 \Delta t \|u\|_0 \{\|v\|_1 + \|v\|_0\} \\
&\leq c_1 \Delta t \|u\|_0 \|v\|_1.
\end{aligned}$$

□

4.2 Proof of Theorem 1-(i)

Let $(u_h, p_h) = \{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ be the solution of scheme (8). Substituting $(u_h^n, -p_h^n) \in V_h \times Q_h$ into (v_h, q_h) in (8), we have

$$\begin{aligned}
\Big(\frac{u_h^n - u_h^{n-1} \circ X_1^n}{\Delta t}, u_h^n\Big) + \nu a(u_h^n, u_h^n) - \frac{1}{\nu}\mathscr{C}_h(p_h^n, p_h^n) + (\lambda^n u_h^{n-1}, u_h^n) \\
= (f_h^n, u_h^n) - \frac{1}{\nu}\mathscr{C}_h(g_h^n, p_h^n).
\end{aligned} \tag{22}$$

Let ε_i $(i = 1, 2)$ be any positive numbers. We have, by (21a),

$$\Big(\frac{u_h^n - u_h^{n-1} \circ X_1^n}{\Delta t}, u_h^n\Big) = \frac{1}{\Delta t}\Big\{\frac{1}{2}(\|u_h^n\|_0^2 - \|u_h^{n-1} \circ X_1^n\|_0^2) + \frac{1}{2}\|u_h^n - u_h^{n-1} \circ X_1^n\|_0^2\Big\}$$

$$\geq \overline{D}_{\Delta t}\big(\frac{1}{2}\|u_h^n\|_0^2\big) - c_1\|u_h^{n-1}\|_0^2, \tag{23a}$$

$$\nu a(u_h^n, u_h^n) = 2\nu\|D(u_h^n)\|_0^2, \tag{23b}$$

$$-\frac{1}{\nu}\mathscr{C}_h(p_h^n, p_h^n) = \frac{\delta_0}{\nu}|p_h^n|_h^2, \tag{23c}$$

$$(\lambda^n u_h^{n-1}, u_h^n) \geq -c_2\big(\varepsilon_1\|u_h^n\|_0^2 + \frac{1}{4\varepsilon_1}\|u_h^{n-1}\|_0^2\big), \tag{23d}$$

$$(f_h^n, u_h^n) \leq \varepsilon_2\|u_h^n\|_0^2 + \frac{1}{4\varepsilon_2}\|f_h^n\|_0^2, \tag{23e}$$

$$-\frac{1}{\nu}\mathscr{C}_h(g_h^n, p_h^n) \leq \frac{\delta_0}{\nu}|g_h^n|_h|p_h^n|_h \leq \frac{\delta_0}{2\nu}(|g_h^n|_h^2 + |p_h^n|_h^2). \tag{23f}$$

From (22) and (23) it holds that, for $n = 1, \cdots, N_T$,

$$\begin{aligned}
&\overline{D}_{\Delta t}\big(\frac{1}{2}\|u_h^n\|_0^2\big) + 2\nu\|D(u_h^n)\|_0^2 + \frac{\delta_0}{2\nu}|p_h^n|_h^2 \\
&\leq (c_2\varepsilon_1 + \varepsilon_2)\|u_h^n\|_0^2 + \big(c_1 + \frac{c_2}{4\varepsilon_1}\big)\|u_h^{n-1}\|_0^2 + \frac{1}{4\varepsilon_2}\|f_h^n\|_0^2 + \frac{\delta_0}{2\nu}|g_h^n|_h^2.
\end{aligned} \tag{24}$$

Applying Lemma 1 to (24) with proper ε_i $(i = 1, 2)$ satisfying $\Delta t_0 \leq 1/\{4(c_2\varepsilon_1 + \varepsilon_2)\}$, we obtain (9). □

4.3 Proof of Theorem 1-(ii)

From (8) with $v_h = 0 \in V_h$ and (10), it holds that, for $n = 0, \cdots, N_T$,

$$b(u_h^n, q_h) + \frac{1}{\nu}\mathscr{C}_h(p_h^n, q_h) = \frac{1}{\nu}\mathscr{C}_h(g_h^n, q_h), \quad \forall q_h \in Q_h,$$

which gives, for $n = 1, \cdots, N_T$,

$$b(\overline{D}_{\Delta t}u_h^n, q_h) + \frac{1}{\nu}\mathscr{C}_h(\overline{D}_{\Delta t}p_h^n, q_h) = \frac{1}{\nu}\mathscr{C}_h(\overline{D}_{\Delta t}g_h^n, q_h), \quad \forall q_h \in Q_h. \tag{25}$$

Substituting $(\overline{D}_{\Delta t}u_h^n, 0) \in V_h \times Q_h$ into (v_h, q_h) in (8) and using (25) with $q_h = -p_h^n$, we have, for $n = 1, \cdots, N_T$,

$$\begin{aligned}
\Big(\frac{u_h^n - u_h^{n-1}\circ X_1^n}{\Delta t}, \overline{D}_{\Delta t}u_h^n\Big) + \nu a(u_h^n, \overline{D}_{\Delta t}u_h^n) - \frac{1}{\nu}\mathscr{C}_h(\overline{D}_{\Delta t}p_h^n, p_h^n) + (\lambda^n u_h^{n-1}, \overline{D}_{\Delta t}u_h^n)& \\
= (f_h^n, \overline{D}_{\Delta t}u_h^n) - \frac{1}{\nu}\mathscr{C}_h(\overline{D}_{\Delta t}g_h^n, p_h^n).&
\end{aligned} \tag{26}$$

We evaluate each term in (26) as follows.

$$\begin{aligned}
\Big(\frac{u_h^n - u_h^{n-1}\circ X_1^n}{\Delta t}, \overline{D}_{\Delta t}u_h^n\Big) &= \Big(\overline{D}_{\Delta t}u_h^n + \frac{1}{\Delta t}(u_h^{n-1} - u_h^{n-1}\circ X_1^n), \overline{D}_{\Delta t}u_h^n\Big) \\
&= \|\overline{D}_{\Delta t}u_h^n\|_0^2 + \frac{1}{\Delta t}\big(u_h^{n-1} - u_h^{n-1}\circ X_1^n, \overline{D}_{\Delta t}u_h^n\big) \\
&\geq \|\overline{D}_{\Delta t}u_h^n\|_0^2 - \Big(c_1\|u_h^{n-1}\|_1^2 + \frac{1}{4}\|\overline{D}_{\Delta t}u_h^n\|_0^2\Big) \quad \text{(by (21a))} \\
&\geq \frac{3}{4}\|\overline{D}_{\Delta t}u_h^n\|_0^2 - c_1\|D\big(u_h^{n-1}\big)\|_0^2,
\end{aligned} \tag{27a}$$

$$
\begin{aligned}
\nu a(u_h^n, \overline{D}_{\Delta t} u_h^n) &= \overline{D}_{\Delta t}\big(\frac{\nu}{2} a(u_h^n, u_h^n)\big) + \frac{\nu \Delta t}{2} a(\overline{D}_{\Delta t} u_h^n, \overline{D}_{\Delta t} u_h^n) \\
&= \overline{D}_{\Delta t}\big(\nu \|D(u_h^n)\|_0^2\big) + \nu \Delta t \|D\big(\overline{D}_{\Delta t} u_h^n\big)\|_0^2, && (27b) \\
-\frac{1}{\nu}\mathscr{C}_h(\overline{D}_{\Delta t} p_h^n, p_h^n) &= \overline{D}_{\Delta t}\big(-\frac{1}{2\nu}\mathscr{C}_h(p_h^n, p_h^n)\big) - \frac{\Delta t}{2\nu}\mathscr{C}_h(\overline{D}_{\Delta t} p_h^n, \overline{D}_{\Delta t} p_h^n) \\
&= \overline{D}_{\Delta t}\big(\frac{\delta_0}{2\nu}|p_h^n|_h^2\big) + \frac{\delta_0 \Delta t}{2\nu}|\overline{D}_{\Delta t} p_h^n|_h^2, && (27c) \\
-(\lambda^n u_h^{n-1}, \overline{D}_{\Delta t} u_h^n) &\le c_2 \|u_h^{n-1}\|_0^2 + \frac{1}{8}\|\overline{D}_{\Delta t} u_h^n\|_0^2, && (27d) \\
(f_h^n, \overline{D}_{\Delta t} u_h^n) &\le 2\|f_h^n\|_0^2 + \frac{1}{8}\|\overline{D}_{\Delta t} u_h^n\|_0^2, && (27e) \\
-\frac{1}{\nu}\mathscr{C}_h(\overline{D}_{\Delta t} g_h^n, p_h^n) &\le \frac{\delta_0}{\nu}\Big(\varepsilon_3 |p_h^n|_h^2 + \frac{1}{4\varepsilon_3}|\overline{D}_{\Delta t} g_h^n|_h^2\Big), && (27f)
\end{aligned}
$$

for any positive number ε_3, where Lemma 2 has been used for (27a). Combining (27) with (26), we have, for $n = 1, \cdots, N_T$,

$$
\begin{aligned}
&\overline{D}_{\Delta t}\Big(\nu \|D(u_h^n)\|_0^2 + \frac{\delta_0}{2\nu}|p_h^n|_h^2\Big) + \frac{1}{2}\|\overline{D}_{\Delta t} u_h^n\|_0^2 \\
&\quad \le \varepsilon_3 \frac{\delta_0}{\nu}|p_h^n|_h^2 + \frac{c_1}{\nu}\nu \|D(u_h^{n-1})\|_0^2 + c_2 \|u_h^{n-1}\|_0^2 + 2\|f_h^n\|_0^2 + \frac{\delta_0}{4\varepsilon_3 \nu}|\overline{D}_{\Delta t} g_h^n|_h^2. \qquad (28)
\end{aligned}
$$

Hence, the first and second inequalities of (11) are obtained by applying Lemma 1 to (28) with a proper ε_3 satisfying $\Delta t_0 \le 1/(4\varepsilon_3)$ and estimating $\|u_h^{n-1}\|_0$ by (9).

Next we prove the third inequality of (11). From Lemmas 4, 5 and 6 it holds that

$$
\begin{aligned}
\|p_h^n\|_0 &\le \sqrt{\nu}\|(u_h^n, p_h^n)\|_{V\times Q,\nu} \le \frac{\sqrt{\nu}}{\gamma_0} \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{\mathscr{A}_h((u_h^n, p_h^n),(v_h,q_h))}{\|(v_h,q_h)\|_{V\times Q,\nu}} \\
&= \frac{\sqrt{\nu}}{\gamma_0} \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{(f_h^n, v_h) + \frac{1}{\nu}\mathscr{C}_h(g_h^n, q_h) - \frac{1}{\Delta t}(u_h^n - u_h^{n-1}\circ X_1^n, v_h) - (\lambda^n u_h^{n-1}, v_h)}{\|(v_h,q_h)\|_{V\times Q,\nu}} \\
&\le \frac{c}{\gamma_0}\Big\{\|f_h^n\|_0 + \delta_0 |g_h^n|_h + \|\overline{D}_{\Delta t} u_h^n\|_0 + \frac{1}{\Delta t}\sup_{v_h\in V_h}\frac{\big(u_h^{n-1} - u_h^{n-1}\circ X_1^n, v_h\big)}{\|v_h\|_1} + \|\lambda^n u_h^{n-1}\|_0\Big\} \\
&\le \frac{c_3}{\gamma_0}(\|f_h^n\|_0 + \delta_0 |g_h^n|_h + \|\overline{D}_{\Delta t} u_h^n\|_0 + \|u_h^{n-1}\|_0),
\end{aligned}
$$

which yields the third inequality of (11) by the first inequality of (9) and the second inequality of (11). □

4.4 Proof of Theorem 2

Let $\{(u,p)(t);\ t \in (0,T)\} \subset V \times Q$ and $(u_h, p_h) = \{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ be the solutions of (2) and scheme (7). Let $(\hat{u}_h, \hat{p}_h)(t) \in V_h \times Q_h$ be the Stokes projection of $(u,p)(t) \in H^2(\Omega)^d \times H^1(\Omega)$ by (14) and set

$$
e_h^n \equiv u_h^n - \hat{u}_h^n, \quad \varepsilon_h^n \equiv p_h^n - \hat{p}_h^n, \quad \eta_h(t) \equiv (u - \hat{u}_h)(t).
$$

For any $(v_h, q_h) \in V_h \times Q_h$, it holds that, from (2), (7), (14) and an identity,

$$
e_h^n = \eta_h^n - u^n + u_h^n,
$$

$$
\begin{aligned}
&\Big(\frac{e_h^n - e_h^{n-1}\circ X_1^n}{\Delta t}, v_h\Big) + \nu a(e_h^n, v_h) + b(v_h, \varepsilon_h^n) + b(e_h^n, q_h) + \frac{1}{\nu}\mathscr{C}_h(\varepsilon_h^n, q_h) + (\lambda^n e_h^{n-1}, v_h) \\
&= \Big(\frac{\eta_h^n - \eta_h^{n-1}\circ X_1^n}{\Delta t} - \frac{u^n - u^{n-1}\circ X_1^n}{\Delta t}, v_h\Big) + \big(\lambda^n(\eta_h^{n-1} - u^{n-1}), v_h\big) \\
&\quad + \Big(\frac{Du^n}{Dt} + \lambda^n u^n, v_h\Big) - \frac{1}{\nu}\mathscr{C}_h(p^n, q_h) \\
&= \Big(\frac{\eta_h^n - \eta_h^{n-1}\circ X_1^n}{\Delta t} + \Big(\frac{Du^n}{Dt} - \frac{u^n - u^{n-1}\circ X_1^n}{\Delta t}\Big) + \lambda^n(\eta_h^{n-1} + u^n - u^{n-1}), v_h\Big) \\
&\quad - \frac{1}{\nu}\mathscr{C}_h(p^n, q_h) \\
&= (\tilde{f}_h^n, v_h) + \frac{1}{\nu}\mathscr{C}_h(\tilde{g}_h^n, q_h),
\end{aligned} \tag{29}
$$

where

$$
\begin{aligned}
\tilde{f}_h^n &\equiv \frac{\eta_h^n - \eta_h^{n-1}\circ X_1^n}{\Delta t} + \Big(\frac{Du^n}{Dt} - \frac{u^n - u^{n-1}\circ X_1^n}{\Delta t}\Big) + \lambda^n(\eta_h^{n-1} + u^n - u^{n-1}), \\
\tilde{g}_h^n &\equiv -p^n.
\end{aligned}
$$

Applying Theorem 1-(i) to (29), we obtain

$$
\|e_h\|_{l^\infty(L^2)},\ \sqrt{\nu}\|e_h\|_{l^2(H^1)},\ \frac{1}{\sqrt{\nu}}|\varepsilon_h|_{l^2(|\cdot|_h)} \le c_3\Big(\|e_h^0\|_0 + \|\tilde{f}_h\|_{l^2(L^2)} + \frac{1}{\sqrt{\nu}}|\tilde{g}_h|_{l^2(|\cdot|_h)}\Big). \tag{30}
$$

We evaluate $\|\tilde{f}_h\|_{l^2(L^2)}$ and $|\tilde{g}_h|_{l^2(|\cdot|_h)}$. It holds that

$$
\begin{aligned}
\|\tilde{f}_h^n\|_0 &\le \Big\|\frac{\eta_h^n - \eta_h^{n-1}\circ X_1^n}{\Delta t}\Big\|_0 + \Big\|\frac{Du^n}{Dt} - \frac{u^n - u^{n-1}\circ X_1^n}{\Delta t}\Big\|_0 + \|\lambda^n(\eta_h^{n-1} + u^n - u^{n-1})\|_0 \\
&\equiv I_1^n + I_2^n + I_3^n.
\end{aligned}
$$

I_i^n $(i = 1,2,3)$ and $I_4^n \equiv |\tilde{g}_h^n|_h$ are evaluated as

$$
\begin{aligned}
I_1^n &= \|\frac{\eta_h^n - \eta_h^{n-1}\circ X_1^n}{\Delta t}\|_0 = \Big\|\overline{D}_{\Delta t}\eta_h^n + \frac{\eta_h^{n-1} - \eta_h^{n-1}\circ X_1^n}{\Delta t}\Big\|_0 \\
&\le \|\overline{D}_{\Delta t}\eta_h^n\|_0 + c_1\|\eta_h^{n-1}\|_1 \quad \text{(by (21a))} \\
&\le \frac{1}{\sqrt{\Delta t}}\|\eta_h\|_{H^1(t^{n-1},t^n;L^2)} + c_1\|\eta_h^{n-1}\|_1 \\
&\le \frac{c_1 h}{\sqrt{\nu}}\Big(\frac{1}{\sqrt{\Delta t}}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1,\nu)} + \|(u^{n-1},p^{n-1})\|_{H^2\times H^1,\nu}\Big),
\end{aligned} \tag{31a}
$$

$$
I_2^n = \Big\|\frac{Du^n}{Dt} - \frac{u^n - u^{n-1}\circ X_1^n}{\Delta t}\Big\|_0 \le c_1\sqrt{\Delta t}\|u\|_{Z^2(t^{n-1},t^n)}, \tag{31b}
$$

$$
\begin{aligned}
I_3^n &= \|\lambda^n(\eta_h^{n-1} - u^{n-1} + u^n)\|_0 \le c_2(\|\eta_h^{n-1}\|_0 + \|u^n - u^{n-1}\|_0) \\
&\le c_2\Big(\frac{h}{\sqrt{\nu}}\|(u^{n-1},p^{n-1})\|_{H^2\times H^1,\nu} + \sqrt{\Delta t}\|u\|_{H^1(t^{n-1},t^n;L^2)}\Big),
\end{aligned} \tag{31c}
$$

$$
I_4^n = |\tilde{g}_h^n|_h = |p^n|_h \le h\|p^n\|_1, \tag{31d}
$$

which imply

$$
\|\tilde{f}_h\|_{l^2(L^2)} \le c_3\Big(\Delta t\|u\|_{Z^2} + \frac{h}{\sqrt{\nu}}\|(u,p)\|_{H^1(0,T;H^2\times H^1,\nu)}\Big), \tag{32a}
$$

$$|\tilde{g}_h|_{l^2(|\cdot|_h)} \le h\|p\|_{l^2(H^1)}. \tag{32b}$$

Combining (32) with (30), we obtain (15).

Next we prove Theorem 2-(ii). Applying Theorem 1-(ii) to (29), we obtain

$$\begin{aligned} &\sqrt{\nu}\|e_h\|_{l^\infty(H^1)},\ \ \|\overline{D}_{\Delta t} e_h\|_{l^2(L^2)},\ \ \|\varepsilon_h\|_{l^2(L^2)} \\ &\quad \le c_3(1/\nu)\Big(\|e_h^0\|_1 + |\varepsilon_h^0|_h + \Delta t\|u\|_{Z^2} + \frac{h}{\sqrt{\nu}}\|(u,p)\|_{H^1(0,T;H^2\times H^1,\nu)}\Big), \end{aligned} \tag{33}$$

from (32) and the estimate

$$|\overline{D}_{\Delta t}\tilde{g}_h|_{l^2(|\cdot|_h)} = |\overline{D}_{\Delta t} p|_{l^2(|\cdot|_h)} \le h|\overline{D}_{\Delta t} p|_{l^2(H^1)} \le h\Big\|\frac{\partial p}{\partial t}\Big\|_{L^2(H^1)}.$$

Since (u_h^0, p_h^0) and $(\hat{u}_h^0, \hat{p}_h^0)$ are the Stokes projections of $(u^0, 0)$ and (u^0, p^0) by (14), respectively, it holds that

$$\|e_h^0\|_1 = \|u_h^0 - \hat{u}_h^0\|_1 \le \|u_h^0 - u^0\|_1 + \|u^0 - \hat{u}_h^0\|_1 \le \frac{ch}{\sqrt{\nu}}\|(u^0, p^0)\|_{H^2\times H^1,\nu}, \tag{34a}$$

$$|\varepsilon_h^0|_h = |p_h^0 - \hat{p}_h^0|_h \le |p_h^0 - 0|_h + |\hat{p}_h^0 - p^0|_h + |p^0|_h \le ch\sqrt{\nu}\|(u^0, p^0)\|_{H^2\times H^1,\nu}. \tag{34b}$$

Combining (34) with (33), we obtain (16). □

5 Numerical results

In this section two and three dimensional problems are computed by scheme (7).

A quadrature formulae of degree five (2D: seven points, 3D: fifteen points) [31] is employed for computation of the integral

$$\int_K u_h^{n-1} \circ X_1^n(x) v_h(x)\, dx$$

appearing in scheme (7). Let $Re \equiv 1/\nu$ be the Reynolds number. $\delta_0 = 0.05$ is chosen by some numerical experience. The system of linear equations is solved by MINRES.

5.1 Numerical convergence order

In order to observe the convergence order we prepare analytic solutions.

Example 1. *In problem* (1) *we set $\Omega = (0,\pi)^d$, $T = \pi$ and two values of ν,*

$$\nu = 1,\ 10^{-1}.$$

(i) In the case of $d = 2$ we set

$$\begin{aligned} w(x,t) &= (1+\sin t)\big(\sin^2 x_1 \sin(2x_2),\ -\sin^2 x_2 \sin(2x_1)\big)^T, \\ \lambda(x,t) &= (1+\sin t)\begin{pmatrix} \sin x_1 & \cos x_1 \\ \sin x_1 \sin x_2 & \sin x_1 \cos x_2 \end{pmatrix}. \end{aligned}$$

The functions f and u^0 are given so that the exact solution is

$$(u,p)(x,t) = \big(w(x,t), (1+\sin t)(\sin x_1 - \sin x_2)\big).$$

(ii) In the case of $d = 3$ we set

$$w(x,t) = (1+\sin t)\begin{pmatrix} -\sin^2 x_1 \sin(2x_2)\sin(2x_3) \\ 2\sin(2x_1)\sin^2 x_2 \sin(2x_3) \\ -\sin(2x_1)\sin(2x_2)\sin^2 x_3 \end{pmatrix},$$

$$\lambda(x,t) = (1+\sin t)\begin{pmatrix} \sin x_1 & \sin x_2 & \sin x_3 \\ \cos x_1 & \cos x_2 & \cos x_3 \\ \sin x_1 & \sin x_2 & \sin x_3 \end{pmatrix}.$$

The functions f and u^0 are given so that the exact solution is

$$(u,p)(x,t) = \big(w(x,t), (1+\sin t)(-\sin x_1 + 2\sin x_2 - \sin x_3)\big).$$

We solve Example 1 to recognize the theoretical convergence order. Let N be the division number of each side of the domain. We set $N = 16, 32, 64, 128$ and 256 for $d = 2$ and $N = 8, 16, 32$ and 64 for $d = 3$, and (re)define $h \equiv \pi/N$. Sample meshes are shown in Fig. 1 for $d = 2$ (left, $N = 16$) and 3 (right, $N = 8$). The time increment Δt is set to be $\Delta t = 1/N = h/\pi$. Let (u_h, p_h) be the solution of scheme (7). The initial function u_h^0 in scheme (7) is chosen as the first component of the Stokes projection of $(u^0, 0)$ by (14). We define Err by

$$Err \equiv \frac{\|u_h - \Pi_h u\|_{l^2(H^1)} + \|p_h - \Pi_h p\|_{l^2(L^2)}}{\|\Pi_h u\|_{l^2(H^1)} + \|\Pi_h p\|_{l^2(L^2)}}$$

as the relative error between (u,p) and (u_h,p_h). Fig. 2 shows graphs of Err versus h in logarithmic scale. We can see that Err is almost of first order in h for both $d = 2$ and 3, and the results are consistent with Theorem 2.

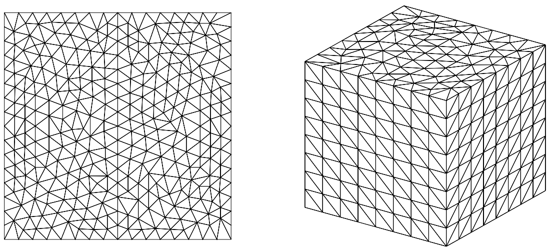

Figure 1: Sample meshes used for Example 1 in 2D (left, $N = 16$) and 3D (right, $N = 8$).

5.2 Application to the linear stability analysis for flows past a circular cylinder

Applying scheme (7) to linearized Navier-Stokes equations at stationary symmetric solutions past a circular cylinder, we reconfirm the fact that the onset of flow instability is around $Re = 50$. The critical Re is delicately dependent on the size of Ω and the outflow boundary conditions [19,20]. Let

$$\Omega \equiv \{x \in \mathbb{R}^2;\ -7.5 < x_1 < 22.5,\ -7.5 < x_2 < 7.5,\ |x| > 0.5\} \tag{35}$$

be the domain and $\mathscr{T}_h$ be the triangulation of $\bar{\Omega}$. Fig. 3 shows Ω (left) and $\mathscr{T}_h$ around the cylinder. The boundary conditions for the stationary flows are also put in the left figure, where $\tau \equiv (-pI +$

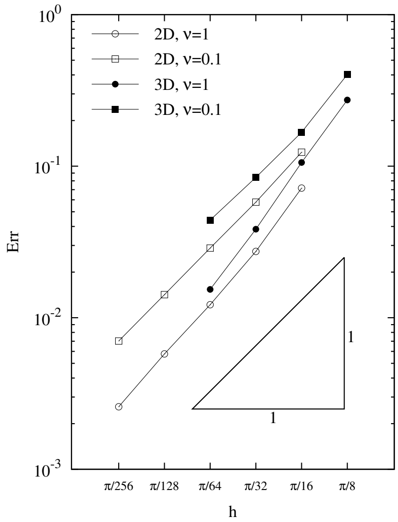

Figure 2: Err vs. h in two and three dimensional test problems.

$2\nu D(u))n$ for the identity matrix I and the outward unit normal vector n. The number of elements is $52,416$, the number of nodes is $26,608$ ($h_{\min} = 1.16 \times 10^{-2}$, $h = h_{\max} = 2.50 \times 10^{-1}$) and the number of degrees of freedom is $78,924$. The triangulation $\mathscr{T}_h$ is symmetric with respect to the x_1-axis, cf. Fig. 3 (right). Let $\Omega_+ \equiv \{x \in \Omega;\ x_2 > 0\}$ be the upper half domain. We impose the boundary conditions $\tau_1 = 0$ and $u_2 = 0$ on the x_1-axis and the conditions of Fig. 3 on the other boundaries. For each Re subject to the initial value $u = 0$ we solve the non-stationary Navier-Stokes equations in Ω_+ by a pressure-stabilized characteristics finite element scheme [21, 23] and obtain a numerically stationary solution. Extending the solution symmetrically to the domain $x_2 < 0$, we get a symmetric stationary solution $(u_h^{(\mathrm{NS})}, p_h^{(\mathrm{NS})}) \in X_h \times M_h$. Fig. 4 exhibits streamlines (left) and pressure contours (right) of the stationary solution $(u_h^{(\mathrm{NS})}, p_h^{(\mathrm{NS})})$ for $Re = 10$ (top) and 100 (bottom). Considering the perturbation of the velocity and the pressure for the Navier-Stokes equations, we set the following problem.

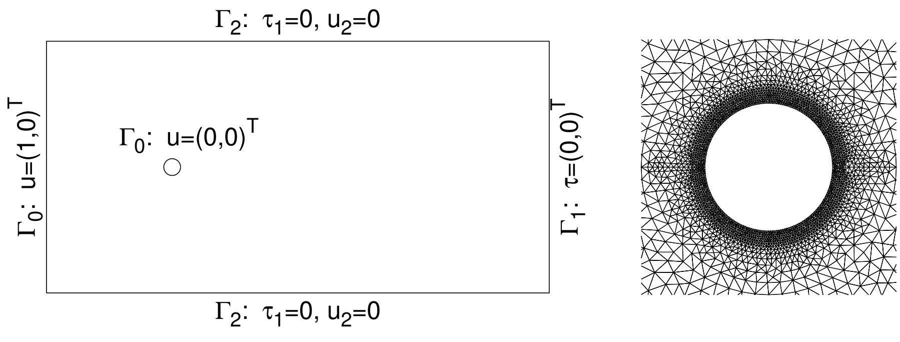

Figure 3: The domain Ω with the boundary conditions for the Navier-Stokes equations (left) and the used triangular mesh around the cylinder (right).

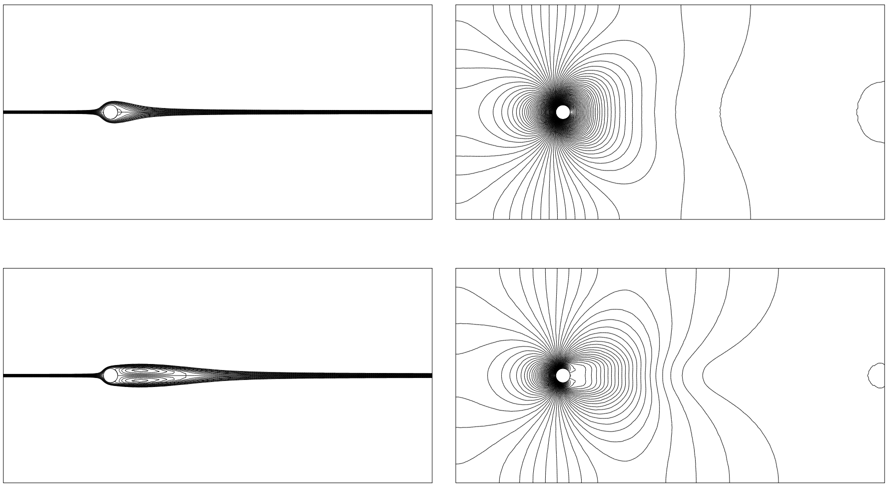

Figure 4: Streamlines (left, [-0.1, 0.1; 0.01]) and pressure contours (right, [-0.9, 0.9; 0.01]) of the stationary solution $(u_h^{(\mathrm{NS})}, p_h^{(\mathrm{NS})})$ for $Re = 10$ (top) and 100 (bottom).

Example 2. *In* (1) *we set Ω by* (35)*, $T = 100$, five values of ν,*

$$\nu = \frac{1}{Re}, \quad Re = 10, 40, 50, 60, 100,$$

$w = u_h^{(\mathrm{NS})}$*,* $\lambda_{ij} = \partial u_{hi}^{(\mathrm{NS})}/\partial x_j$ $(i, j = 1, 2)$*,* $f = 0$ *and* $u^0 \approx 0$*. The homogeneous boundary condition* (1c) *is replaced with the boundary conditions,* $u = 0$ *on* Γ_0*,* $\tau = 0$ *on* Γ_1 *and* $\tau_1 = 0$ *and* $u_2 = 0$ *on* Γ_2*, cf. Fig. 3 (left) for the definitions of* Γ_i $(i = 0, 1, 2)$*.*

We solve Example 2 by a slightly modified scheme of (7) with $\Delta t = 1/50$; find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset \tilde{V}_h \times \tilde{Q}_h$ such that, for $n = 1, \cdots, N_T$,

$$\begin{aligned} \Big(\frac{u_h^n - u_h^{n-1} \circ X_1^n}{\Delta t}, v_h\Big) + \mathscr{A}_h((u_h^n, p_h^n), (v_h, q_h)) + (\lambda^n u_h^{n-1}, v_h) \\ = (f^n, v_h), \quad \forall (v_h, q_h) \in \tilde{V}_h \times \tilde{Q}_h, \end{aligned} \tag{36}$$

where $\tilde{V}_h \equiv X_h \cap \{v \in H^1(\Omega)^d;\ v = 0 \text{ on } \Gamma_0,\ v_2 = 0 \text{ on } \Gamma_2\}$ and $\tilde{Q}_h \equiv M_h$. The small perturbation, $u_h^0(-1.36, 0) = (0.01, 0)^T$, is given while $u_h^0(P) = 0$ at the other nodes P.

We compute $\|(u_h^n, p_h^n)\|_{V \times Q}$ for $n = 1, \cdots, N_T$ and observe the behavior of the solutions. The graphs of $\|(u_h^n, p_h^n)\|_{V \times Q}$ versus t are shown in Fig. 5. For $Re = 10, 40$ and 50 the values of $\|(u_h^n, p_h^n)\|_{V \times Q}$ finally decrease and for $Re = 60$ and 100 the values of $\|(u_h^n, p_h^n)\|_{V \times Q}$ monotonically increase after $t = 3$. For $Re = 50$ we have additionally performed a computation on a fine mesh (♯elements: 195,200, ♯nodes: 98,400, $h_{\min} = 5.67 \times 10^{-3}$, $h = h_{\max} = 1.66 \times 10^{-1}$) and with a small $\Delta t (= 1/100)$. The result is shown in the dashed line in Fig. 5. The graph of $\|(u_h^n, p_h^n)\|_{V \times Q}$ is almost flat after $t = 50$, which implies the onset of the flow instability is around this Reynolds number. In order to obtain the critical Reynolds number a finer mesh and a smaller Δt should be employed. Continuing the computation for $Re = 100$ until $t = 140$, we obtain $(u_h, p_h)(t = 140)$. Fig. 6 shows streamlines (left) and pressure contours (right) of $(u_h^{(\mathrm{NS})}, p_h^{(\mathrm{NS})}) + (u_h, p_h)(t = 140)$ for $Re = 100$, where $\max_{x \in \bar{\Omega}} |u_h(x)(t = 140)| \approx 0.126$ and $\max_{x \in \bar{\Omega}} |u_h^{(\mathrm{NS})}(x)| \approx 1.230$. A non-symmetricity with respect to the x_1-axis caused by (u_h, p_h) is observed.

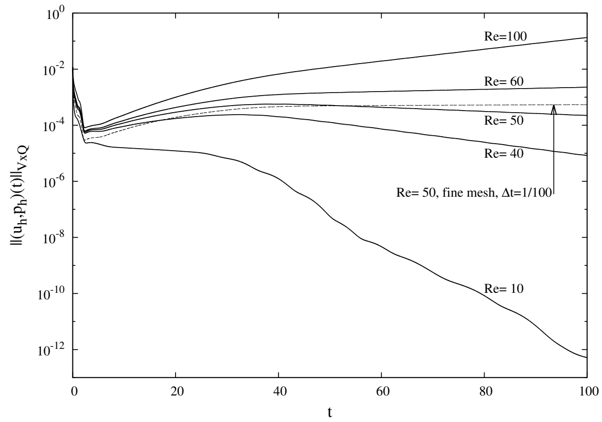

Figure 5: Graphs of $\|(u_h, p_h)(t)\|_{V\times Q}$ vs. t for $Re = 10, 40, 50, 60$ and 100.

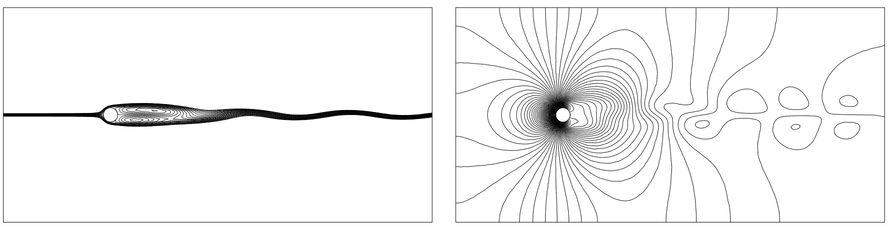

Figure 6: Streamlines (left, $[-0.1, 0.1; 0.01]$) and pressure contours (right, $[-0.6, 0.6; 0.01]$) of $(u_h^{(\mathrm{NS})}, p_h^{(\mathrm{NS})}) + (u_h, p_h)(t = 140)$ for $Re = 100$.

6 Conclusions

We have presented a pressure-stabilized characteristics finite element scheme for the Oseen equations. Stability (Theorem 1) and convergence (Theorem 2) results with the optimal error estimates for the velocity and the pressure have been proved. The scheme is based on the method of characteristics, which works well for convection-dominated problems and leads to a symmetric coefficient matrix of the system of linear equations. The system can be solved by efficient linear solvers for symmetric matrices, e.g., MINRES, CR and so on. Since a cheap P1/P1 finite element is employed, the degrees of freedom are smaller than that of other typical elements for the equations, e.g., P2/P1. These advantages, i.e., symmetry of the coefficient matrix and small degrees of freedom, reduces computation cost (time and memory). Two and three dimensional numerical results have been shown. The numerical convergence orders in Example 1 are consistent with the theoretical results. In Example 2 scheme (7) is applied to the linear stability analysis of stationary flows past a circular cylinder governed by the Navier-Stokes equations. The obtained results imply that the scheme is applicable to such linear stability analysis and reconfirmed that the critical *Re* is around 50. Theoretical analysis of stability and error estimates for a corresponding scheme to the Navier-Stokes equations will be shown in a forthcoming paper [25].

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References

[1] Baba, K., Tabata, M.: On a conservative upwind finite element scheme for convective diffusion equations. R.A.I.R.O. Analyse nuérique/Numerical Analysis. **15**, 3–25 (1981)

[2] Barrett, R., Berry, M., Chan, T.F., Demmel, J., Donato, J., Dongarra, J., Eijkhout, V., Pozo, R., Romine, C., van der Vorst, H.: Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods. SIAM, Philadelphia (1994)

[3] Boukir, K., Maday, Y., Métivet, B., Razafindrakoto, E.: A high-order characteristics/finite element method for the incompressible Navier-Stokes equations. International Journal for Numerical Methods in Fluids. **25**, 1421–1454 (1997)

[4] Braack, M., Burman, E., John, V., Lube, G.: Stabilized finite element methods for the generalized Oseen problem. Computer Methods in Applied Mechanics and Engineering. **196**, 853–866 (2007)

[5] Brezzi, F., Douglas Jr., J.: Stabilized mixed methods for the Stokes problem. Numerische Mathematik. **53**, 225–235 (1988)

[6] Brezzi, F., Pitkäranta, J.: On the stabilization of finite element approximations of the Stokes equations. In W. Hackbusch editor, Efficient solutions of Elliptic Systems. Vieweg, Wiesbaden, 11–19 (1984)

[7] Brooks, A.N., Hughes, T.J.R.: Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. Computer Methods in Applied Mechanics and Engineering. **32**, 199–259 (1982)

[8] Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)

[9] Douglas Jr., J., Russell, T.F.: Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures. SIAM Journal on Numerical Analysis. **19**, 871–885 (1982)

[10] Duvaut, G., Lions, J.L.: Inequalities in mechanics and physics. Springer, Berlin (1976)

[11] Ewing, R.E., Russell, T.F.: Multistep Galerkin methods along characteristics for convection-diffusion problems. In Vichnevetsky, R. and Stepleman, R.S. editors, Advances in Computer Methods for Partial Differential Equations. **4**, 28–36, IMACS (1981)

[12] Ewing, R.E., Russell, T.F., Wheeler, M.F.: Simulation of miscible displacement using mixed method and modified method of characteristics. Proceedings of SPE Reservoir Simulation Symposium (15-18 November, 1983, San Francisco), Society of Petroleum Engineers. 71–81 (1983)

[13] Franca, L.P., Frey, S.L.: Stabilized finite element methods: II. The incompressible Navier-Stokes equations. Computer Methods in Applied Mechanics and Engineering. **99**, 209–233 (1992)

[14] Franca, L.P., Stenberg, R.: Error analysis of some Galerkin least squares methods for the elasticity equations. SIAM Journal on Numerical Analysis. **28**, 1680–1697 (1991)

[15] Girault, V., Raviart, P.-A.: Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms. Springer, Berlin (1986)

[16] Heywood, J.G., Rannacher, R.: Finite-element approximation of the nonstationary Navier-Stokes problem. Part IV: Error analysis for second-order time discretization. SIAM Journal on Numerical Analysis. **27**, 353–384 (1990)

[17] Hughes, T.J.R., Franca, L.P., Mallet, M.: A new finite element formulation for computational fluid dynamics: VI. Convergence analysis of the generalized SUPG formulation for linear time-dependent multidimensional advective-diffusive systems. Computer Methods in Applied Mechanics and Engineering. **63**, 97–112 (1987)

[18] Johnson, C.: Numerical Solution of Partial Differential Equations by the Finite Element Method. Cambridge Univ. Press. Cambridge (1987)

[19] Kumar, B., Mittal, S.: Effect of blockage on critical parameters for flow past a circular cylinder. International Journal for Numerical Methods in Fluids. **50**, 987–1001 (2006)

[20] Norberg, C.: An experimental investigation of the flow around a circular cylinder: influence of aspect ratio. Journal of Fluid Mechanics. **258**, 287–316 (1994)

[21] Notsu, H.: Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme. Transactions of Japan Society for Computational Engineering and Science. ONLINE ISSN:1347-8826 (2008)

[22] Notsu, H., Rui, H., Tabata, M.: Characteristics finite difference schemes of second order in time for convection diffusion problems. to appear in Journal of Algorithms & Computational Technology.

[23] Notsu, H., Tabata, M.: A combined finite element scheme with a pressure stabilization and a characteristic-curve method for the Navier-Stokes equations. Transactions of the Japan Society for Industrial and Applied Mathematics. **18**, 427–445 (2008) (in Japanese)

[24] Notsu, H., Tabata, M.: A single-step characteristic-curve finite element scheme of second order in time for the incompressible Navier-Stokes equations, Journal of Scientific Computing. **38**, 1–14 (2009)

[25] Notsu, H., Tabata, M.: Error estimates of a pressure-stabilized characteristics finite element scheme for the Navier-Stokes equations (in preparation)

[26] Pironneau, O.: On the transport-diffusion algorithm and its applications to the Navier-Stokes equations. Numerische Mathematik. **38**, 309–332 (1982)

[27] Pironneau, O.: Finite Element Methods for Fluids. John Wiley & Sons, Chichester (1989)

[28] Pironneau, O., Tabata, M.: Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type. International Journal for Numerical Methods in Fluids. **64**, 1240–1253 (2010)

[29] Rui, H., Tabata, M.: A second order characteristic finite element scheme for convection-diffusion problems. Numerische Mathematik. **92**, 161–177 (2002)

[30] Saad, Y.: Iterative Methods for Sparse Linear Systems. SIAM, Philadelphia (2003)

[31] Stroud, A.H.: Approximate calculation of multiple integrals. Prentice-Hall, Englewood Cliffs, New Jersey (1971).

[32] Tabata, M.: A finite element approximation corresponding to the upwind finite differencing. Memoirs of Numerical Mathematics. **4**, 47–63 (1977)

[33] Tabata, M., Tagami, D.: Error estimates for finite element approximations of drag and lift in nonstationary Navier-Stokes flows. Japan Journal of Industrial and Applied Mathematics. **17**, 371–389 (2000)

[34] Tezduyar, T.E., Mittal, S., Ray, S.E., Shih, R.: Incompressible flow computations with stabilized bilinear and linear equal-order-interpolation velocity-pressure elements. Computer Methods in Applied Mechanics and Engineering. **95**, 221–242 (1992)