# Strong convergence analysis of the stochastic exponential Rosenbrock scheme for the finite element discretization of semilinear SPDEs driven by multiplicative and additive noise 

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#### Abstract

In this paper, we consider the numerical approximation of a general second order semilinear stochastic partial differential equation (SPDE) driven by multiplicative and additive noise. Our main interest is on such SPDEs where the nonlinear part is stronger than the linear part also called stochastic reactive dominated transport equations. Most numerical techniques, including current stochastic exponential integrators lose their good stability properties on such equations. Using finite element for space discretization, we propose a new scheme appropriated on such equations, called stochastic exponential Rosenbrock scheme (SERS) based on local linearization at every time step of the semi-discrete equation obtained after space discretization. We consider noise with finite trace and give a strong convergence proof of the new scheme toward the exact solution in the root-mean-square $L^{2}$ norm. Numerical experiments to sustain theoretical results are provided.


Keywords Exponential Rosenbrock-Euler method • Stochastic partial differential equations • Multiplicative \& additive noise • Strong convergence • Finite element method • Errors estimate • Stochastic reactive dominated transport equations.

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## 1 Introduction

The strong numerical approximation of an Itô stochastic partial differential equation defined in the bounded domain $\Lambda \subset \mathbb{R}^{d}(d=1,2,3)$ is analyzed. The domain $\Lambda$ is assumed to be a convex polygon, or has smooth boundary. Boundary conditions on the domain $\Lambda$ are typically Neumann, Dirichlet or Robin conditions. More precisely, we consider in the abstract setting the following stochastic partial differential equation

$$
d X(t)=[A X(t)+F(X(t))] d t+B(X(t)) d W(t), \quad X(0)=X_{0}, \quad t \in[0, T],(1)
$$

on $H=L^{2}(\Lambda), T>0$ is a final time, $F$ and $B$ are nonlinear functions, $X_{0}$ is the initial data which may be random, $A$ is a linear operator, unbounded, not necessarily self adjoint, and the generator of an analytic semigroup $S(t):=$ $e^{t A}, t \geq 0$. The noise $W(t)=W(x, t)$ is a $Q$-Wiener process defined in a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P},\left\{\mathcal{F}_{t}\right\}_{t \geq 0}\right)$. The filtration is assumed to fulfill the usual conditions (see [28, Definition 2.1.11]). We assume that the noise can be represented as

$$
\begin{equation*}
W(x, t)=\sum_{i \in \mathbb{N}^{d}} \sqrt{q_{i}} e_{i}(x) \beta_{i}(t), \quad t \in[0, T], \tag{2}
\end{equation*}
$$

where $q_{i}, e_{i}, i \in \mathbb{N}^{d}$ are respectively the eigenvalues and the eigenfunctions of the covariance operator $Q$, and $\beta_{i}$ are independent and identically distributed standard Brownian Motions. Precise assumptions on $F, B, X_{0}$ and $A$ will be given in the next section to ensure the existence of the unique mild solution $X$ of (1) which has the following representation (see 26,28$]$ ) for $t \in(0, T]$

$$
\begin{equation*}
X(t)=S(t) X_{0}+\int_{0}^{t} S(t-s) F(X(s)) d s+\int_{0}^{t} S(t-s) B(X(s)) d W(s) \tag{3}
\end{equation*}
$$

In few cases, exact solutions are explicitly available, so numerical techniques are the only tools to provide good approximations in more general cases (see for examples $[13,19,20,23,27,40,42]$ ). Approximations are done at two levels, spatial approximation and temporal approximation. For the spatial approximation, the finite difference, the finite element method and the Galerkin spectral method are usually used $13,20,23,32,41,42$. As for PDEs, standard explicit time stepping methods for SPDEs are usually unstable for stiff problems and therefore severe time step constraint is needed. To overcome that drawback, standard implicit Euler methods are usually used [20, 27, 39]. Although standard implicit Euler methods ${ }^{1}$ are stable, their implementation requires significantly more computational effort, specially full implicit methods, as Newton method is usually used to solve nonlinear algebraic equations

[^1]at each time step. Recently, stochastic exponential integrators [13, 23, 40] appeared as non standard explicit methods efficient for SPDE (1). All stochastic exponential integrators analyzed in the literature for SPDEs $13,23,40$ are bounded on the nonlinear problem as in (1) where the linear part $A$ and the nonlinear function $F$ are explicitly known a priori. Such approach is justified in situations where the nonlinear function $F$ is small. Indeed when $F$ is small the linear operator $A$ drives the SPDE (1) and the good stability of the exponential integrators and semi-implicit method are ensured. In fact, in more realistic applications the nonlinear function $F$ can be stronger. Typical examples are semilinear advection diffusion reaction equations with stiff reaction term. In such cases, the SPDE (1) is driven by the nonlinear operator $F$ and both exponential integrators 13, 23, 40 and semi-implicit Euler 27, 39 will behave as explicit Euler-Maruyama scheme (see Section 2.3), therefore their good stability properties are lost. To overcome this issue we propose in this work a new scheme called Stochastic Exponential Rosenbrock Scheme (SERS). Coupled with finite element for space discretization, the new scheme is based on a local linearization of the drift term at each time step in the corresponding semi-discrete problem of (1). The local linearization therefore weakens the nonlinear part of the drift such that the linearized semi-discrete problem is driven by its linear part, which change at each time step. The standard stochastic exponential scheme 23 is applied at the end to that linearized semi-discrete problem and the corresponding scheme is our new scheme. The challenge here is to deal with the new discrete semigroup which indeed is a semigroup process, called stochastic perturbed semigroup. The key idea comes from the deterministic exponential Rosenbrock method $10-12,24,29$. Note that similar schemes for stochastic differential equations in finite dimensions have been proposed in $[2,3]$. Using some deterministic tools from [24], we propose a strong convergence proof of the new schemes where the linear operator $A$ is not necessarily self adjoint. Note that the orders of convergence are the same with stochastic exponential schemes proposed in 23. The deterministic part of this scheme is of order 2 in time and has been proven to be efficient and robust in comparison to standard schemes in many applications 7,35 where the perturbed semigroup and related matrix functions have been computed using the Krylov subspace technique [9] and fast Leja points technique [1, 35]. For our new stochastic scheme, numerical simulations show its good stability behavior compared with a stochastic exponential scheme proposed in 23], where the stochastic perturbed semigroup and related matrix functions are computed using Krylov subspace technique.
The rest of this paper is organized as follows. Section 2 is devoted to the mathematical setting, the numerical method and the main result. In Section 3 some preparatory results and the proof of the main result are provided. In Section 4 we provide some numerical experiments to sustain our theoretical results. We end the paper in Section 5 by providing a concluding remark.

## 2 Mathematical setting and main results

2.1 Main assumptions and well posedness

Before we state the well posedness result, let us define key functional spaces, norms and notations that will be used in the rest of the paper. Let $\left(H,\langle., .\rangle_{H},\|\cdot\|\right)$ be a separable Hilbert space. For all $p \geq 2$ and for a Banach space $U$, we denote by $L^{p}(\Omega, U)$ the Banach space of all equivalence classes of $p$ integrable $U$-valued random variables. We denote by $L(U, H)$ the space of bounded linear mappings from $U$ to $H$ endowed with the usual operator norm $\|\cdot\|_{L(U, H)}$. By $\mathcal{L}_{2}(U, H):=H S(U, H)$, we denote the space of Hilbert-Schmidt operators from $U$ to $H$. We equip $\mathcal{L}_{2}(U, H)$ with the norm

$$
\begin{equation*}
\|l\|_{\mathcal{L}_{2}(U, H)}^{2}:=\sum_{i=1}^{\infty}\left\|l \psi_{i}\right\|^{2}, \quad l \in \mathcal{L}_{2}(U, H) \tag{4}
\end{equation*}
$$

where $\left(\psi_{i}\right)_{i=1}^{\infty}$ is an orthonormal basis of $U$. Note that this definition is independent of the orthonormal basis of $U$. For simplicity, we use the notations $L(U, U)=: L(U)$ and $\mathcal{L}_{2}(U, U)=: \mathcal{L}_{2}(U)$. It is well known that for all $l \in L(U, H)$ and $l_{1} \in \mathcal{L}_{2}(U), l l_{1} \in \mathcal{L}_{2}(U, H)$ and

$$
\begin{equation*}
\left\|l l_{1}\right\|_{\mathcal{L}_{2}(U, H)} \leq\|l\|_{L(U, H)}\left\|l_{1}\right\|_{\mathcal{L}_{2}(U)} . \tag{5}
\end{equation*}
$$

Throughout this paper $W(t)$ is a $Q$-wiener process. We assume that the covariance operator $Q: H \longrightarrow H$ is positive and self-adjoint. The space of HilbertSchmidt operators from $Q^{1 / 2}(H)$ to $H$ is denoted by $L_{2}^{0}:=\mathcal{L}_{2}\left(Q^{1 / 2}(H), H\right)=$ $H S\left(Q^{1 / 2}(H), H\right)$ with the corresponding norm $\|\cdot\|_{L_{2}^{0}}$ defined by

$$
\|l\|_{L_{2}^{0}}:=\left\|l Q^{1 / 2}\right\|_{H S}=\left(\sum_{i=1}^{\infty}\left\|l Q^{1 / 2} e_{i}\right\|^{2}\right)^{1 / 2}, \quad l \in L_{2}^{0}
$$

where $\left(e_{i}\right)_{i=1}^{\infty}$ is an orthonormal basis of $H$. This definition is independent of the orthonormal basis of $H$. In the rest of the paper, we take $H=L^{2}(\Lambda)$.
In order to ensure the existence and the uniqueness of solution of (1) and for the purpose of the convergence analysis, we make the following assumptions.
Assumption 1 [Linear operator $A] A: \mathcal{D}(A) \subset H \longrightarrow H$ is a negative generator of an analytic semigroup $S(t):=e^{A t}$.
Assumption 2 [Initial value $X_{0}$ ] We assume that $X_{0} \in L^{p}\left(\Omega, \mathcal{D}\left((-A)^{\beta / 2}\right)\right)$, $0 \leq \beta \leq 2, p \geq 2$.
As in the current literature on deterministic exponential Rosenbrock-Type methods $10,11,24,30,31$, we make the following assumption on the nonlinear term.
Assumption 3 [Nonlinear term $F]$ We assume that the nonlinear mapping $F: H \longrightarrow H$ is Fréchet differentiable with bounded derivative, i.e. there exists a constant $C>0$ such that

$$
\left\|F^{\prime}(v)\right\|_{L(H)} \leq C, \quad \forall v \in H
$$

Assumption 3 together with the mean value theorem show that there exists a constant $L>0$ such that

$$
\begin{equation*}
\|F(Y)-F(Z)\| \leq L\|Y-Z\|, \quad Y, Z \in H \tag{6}
\end{equation*}
$$

As a consequence of (6), there exists a positive constant $C$ such that

$$
\begin{aligned}
\|F(Z)\| & \leq\|F(0)\|+\|F(Z)-F(0)\| \\
& \leq\|F(0)\|+L\|Z\| \leq C(1+\|Z\|), \quad Z \in H
\end{aligned}
$$

Following [26, Chapter 7] or [15,20, 23, 41] we make the following assumption on the diffusion term.

Assumption 4 [Diffusion term ] We assume that the operator $B: H \longrightarrow$ $L_{2}^{0}$ satisfies the global Lipschitz condition, i.e. there exists a positive constant $C$ such that

$$
\|B(Y)-B(Z)\|_{L_{2}^{0}} \leq C\|Y-Z\|, \quad \forall \quad Y, Z \in H
$$

As a consequence, there exists a positive constant $L>0$ such that

$$
\begin{aligned}
\|B(Z)\|_{L_{2}^{0}} & \leq\|B(0)\|_{L_{2}^{0}}+\|B(Z)-B(0)\|_{L_{2}^{0}} \\
& \leq\|B(0)\|_{L_{2}^{0}}+C\|Z\| \leq L(1+\|Z\|), \quad \forall Z \in H .
\end{aligned}
$$

To establish our $L^{2}$ strong convergence result when dealing with multiplicative noise, we will also need the following further assumption on the diffusion term when $\beta \in[1,2)$, which was also used in $15,20,21,23$.

Assumption 5 We assume that there exist two positive constants $c>0$, and $\gamma \in\left(0, \frac{\beta}{10}\right]$ small enough such that $B\left(\mathcal{D}(-A)^{\gamma / 2}\right) \subset H S\left(Q^{1 / 2}(H), \mathcal{D}(-A)^{\gamma / 2}\right)$ and $\left\|(-A)^{\gamma / 2} B(v)\right\|_{L_{2}^{0}} \leq c\left(1+\left\|(-A)^{\gamma / 2} v\right\|\right)$ for all $v \in \mathcal{D}\left((-A)^{\gamma / 2}\right)$, where $\beta$ is the parameter defined in Assumption 2.
Typical examples satisfying Assumption 5 are stochastic reaction diffusion equations (see [15, Section 4]).

When dealing with additive noise, the strong convergence proof will make use of the following assumption on the noise.

Assumption 6 We assume that the covariance operator $Q: H \longrightarrow H$ satisfies the following estimate

$$
\begin{equation*}
\left\|(-A)^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}<\infty \tag{7}
\end{equation*}
$$

where $\beta$ is defined in Assumption 2.
We equip $V_{\alpha}:=\mathcal{D}\left((-A)^{\alpha / 2}\right), \alpha \in \mathbb{R}$ with the norm $\|v\|_{\alpha}:=\left\|(-A)^{\alpha / 2} v\right\|$, for all $v \in H$. It is well known that $\left(V_{\alpha},\|\cdot\|_{\alpha}\right)$ is a Banach space [8].

To achieve optimal order when dealing with additive noise, we require the nonlinear function $F$ to satisfy the following further assumption, also used in 36,3840 .

Assumption 7 We assume that the deterministic mapping $F: H \longrightarrow H$ is twice differentiable and there exists a positive constant $C$ such that

$$
\left\|F^{\prime \prime}(u)\left(v_{1}, v_{2}\right)\right\|_{-\eta} \leq C\left\|v_{1}\right\| \cdot\left\|v_{2}\right\|, \quad u, v_{1}, v_{2} \in H, \quad \text { for some } \eta \in[1,2)
$$

Let us recall in the following proposition some semigroup properties of the operator $S(t)$ generated by $A^{2}$ that will be useful in the rest of the paper.

Proposition 1 [Smoothing properties of the semigroup] [8] Let $\alpha>0$, $\delta \geq 0$ and $0 \leq \gamma \leq 1$, then there exists a constant $C>0$ such that

$$
\begin{aligned}
\left\|(-A)^{\delta} S(t)\right\|_{L(H)} & \leq C t^{-\delta}, \quad t>0 \\
\left\|(-A)^{-\gamma}(\mathbf{I}-S(t))\right\|_{L(H)} & \leq C t^{\gamma}, \quad t \geq 0 \\
(-A)^{\delta} S(t) & =S(t)(-A)^{\delta} \quad \text { on } \quad \mathcal{D}\left((-A)^{\delta}\right), \\
\left\|D_{t}^{l} S(t) v\right\|_{\delta} & \leq C t^{-l-(\delta-\alpha) / 2}\|v\|_{\alpha}, \quad t>0, \quad v \in \mathcal{D}\left((-A)^{\alpha / 2}\right),
\end{aligned}
$$

where $l=0,1$, and $D_{t}^{l}=\frac{d^{l}}{d t^{l}}$.
If $\delta \geq \gamma$ then $\mathcal{D}\left((-A)^{\delta}\right) \subset \mathcal{D}\left((-A)^{\gamma}\right)$.
Theorem 8 [Well posedness result] [26, Theorem 7.4]
Let Assumption 1, Assumption 3 and Assumption 4 be satisfied. If $X_{0}$ is a $\mathcal{F}_{0}-$ measurable $H$ valued random variable, then there exists a unique mild solution $X$ of problem (1) of the form (3) and satisfying the following

$$
\mathbb{P}\left[\int_{0}^{T}\|X(s)\|^{2} d s<\infty\right]=1
$$

and for any $p \geq 2$ there exists a constant $C=C(p, T)>0$ such that

$$
\sup _{t \in[0, T]} \mathbb{E}\|X(t)\|^{p} \leq C\left(1+\mathbb{E}\left\|X_{0}\right\|^{p}\right)
$$

Furthermore from [15, Theorem 1] or [23, Theorem 2.6] it holds that for all $\gamma \in[0,1)$, for all $p \geq 2$ there exists a positive constant $C$ such that

$$
\begin{equation*}
\left(\mathbb{E}\|X(t)\|_{\gamma}^{p}\right)^{1 / p} \leq C\left(1+\left(\mathbb{E}\left\|X_{0}\right\|_{\gamma}^{p}\right)^{1 / p}\right), \quad t \in[0, T] . \tag{9}
\end{equation*}
$$

### 2.2 Finite element discretization

In the rest of this paper, to simplify the presentation, we assume that the linear operator $A$ is of second order. More precisely, we assume that our SPDE (1) is a second order semilinear parabolic and takes the form

$$
\begin{align*}
d X(t, x) & =[\nabla \cdot(\mathbf{D} \nabla X(t, x))-\mathbf{q} \cdot \nabla X(t, x)+f(x, X(t, x))] d t \\
& +b(x, X(t, x)) d W(t, x), \quad x \in \Lambda, \quad t \in[0, T], \tag{10}
\end{align*}
$$

[^2]where the functions $f: \Lambda \times \mathbb{R} \longrightarrow \mathbb{R}$ and $b: \Lambda \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuously differentiable with globally bounded derivatives. In the abstract framework (1), the linear operator $A$ takes the form
\[

$$
\begin{align*}
A u & =\sum_{i, j=1}^{d} \frac{\partial}{\partial x_{i}}\left(D_{i j}(x) \frac{\partial u}{\partial x_{j}}\right)-\sum_{i=1}^{d} q_{i}(x) \frac{\partial u}{\partial x_{i}}  \tag{11}\\
\mathbf{D} & =\left(D_{i, j}\right)_{1 \leq i, j \leq d} \quad \mathbf{q}=\left(q_{i}\right)_{1 \leq i \leq d} \tag{12}
\end{align*}
$$
\]

where $D_{i j} \in L^{\infty}(\Lambda), q_{i} \in L^{\infty}(\Lambda)$. We assume that there is a positive constant $c_{1}>0$ such that

$$
\sum_{i, j=1}^{d} D_{i j}(x) \xi_{i} \xi_{j} \geq c_{1}|\xi|^{2}, \quad \forall \xi \in \mathbb{R}^{d}, \quad x \in \bar{\Omega}
$$

The functions $F: H \longrightarrow H$ and $B: H \longrightarrow H S\left(Q^{1 / 2}(H), H\right)$ are defined by

$$
\begin{equation*}
(F(v))(x)=f(x, v(x)) \quad \text { and } \quad(B(v) u)(x)=b(x, v(x)) \cdot u(x), \tag{13}
\end{equation*}
$$

for all $x \in \Lambda, v \in H, u \in Q^{1 / 2}(H)$, with $H=L^{2}(\Lambda)$. For an appropriate family of eigenfunctions $\left(e_{i}\right)$ such that $\sup _{i \in \mathbb{N}^{d}}\left[\sup _{x \in \Lambda}\left\|e_{i}(x)\right\|\right]<\infty$, it is well known 15 Section 4] that the Nemystskii operator $F$ related to $f$ and the multiplication operator $B$ associated to $b$ defined in (13) satisfy Assumption 3. Assumption 4 and Assumption 5. As in [6, 23], we introduce two spaces $\mathbb{H}$ and $V$, such that $\mathbb{H} \subset V$; the two spaces depend on the boundary conditions of $\Lambda$ and the domain of the operator $A$. For Dirichlet (or first-type) boundary conditions we take

$$
V=\mathbb{H}=H_{0}^{1}(\Lambda)=\left\{v \in H^{1}(\Lambda): v=0 \quad \text { on } \quad \partial \Lambda\right\} .
$$

For Robin (third-type) boundary conditions and Neumann (second-type) boundary condition, which is a special case of Robin boundary conditions, we take $V=H^{1}(\Lambda)$ and

$$
\mathbb{H}=\left\{v \in H^{2}(\Lambda): \partial v / \partial \mathrm{v}_{A}+\alpha_{0} v=0, \quad \text { on } \quad \partial \Lambda\right\}, \quad \alpha_{0} \in \mathbb{R},
$$

where $\partial v / \partial \mathrm{v}_{A}$ is the normal derivative of $v$ and $\mathrm{v}_{A}$ is the exterior pointing normal $n=\left(n_{i}\right)$ to the boundary of $\Lambda$ given by

$$
\partial v / \partial \mathrm{v}_{A}=\sum_{i, j=1}^{d} n_{i}(x) D_{i j}(x) \frac{\partial v}{\partial x_{j}}, \quad x \in \partial \Lambda
$$

Using the Green's formula and the boundary conditions, the corresponding bilinear form associated to $-A$ is given by

$$
a(u, v)=\int_{\Lambda}\left(\sum_{i, j=1}^{d} D_{i j} \frac{\partial u}{\partial x_{i}} \frac{\partial v}{\partial x_{j}}+\sum_{i=1}^{d} q_{i} \frac{\partial u}{\partial x_{i}} v\right) d x, \quad u, v \in V
$$

for Dirichlet and Neumann boundary conditions, and
$a(u, v)=\int_{\Lambda}\left(\sum_{i, j=1}^{d} D_{i j} \frac{\partial u}{\partial x_{i}} \frac{\partial v}{\partial x_{j}}+\sum_{i=1}^{d} q_{i} \frac{\partial u}{\partial x_{i}} v\right) d x+\int_{\partial \Lambda} \alpha_{0} u v d x, \quad u, v \in V$.
for Robin boundary conditions. Using the Gårding's inequality ( $\sqrt[33]{ }$ ), it holds that there exist two positive constants $c_{0}$ and $\lambda_{0}$ such that

$$
\begin{equation*}
a(v, v) \geq \lambda_{0}\|v\|_{H^{1}(\Lambda)}^{2}-c_{0}\|v\|^{2}, \quad \forall v \in V \tag{14}
\end{equation*}
$$

By adding and substracting $c_{0} X d t$ on the right hand side of (1), we have a new linear operator that we still call $A$ corresponding to the new bilinear form that we still call $a$ such that the following coercivity property holds

$$
\begin{equation*}
a(v, v) \geq \lambda_{0}\|v\|_{1}^{2}, \quad v \in V \tag{15}
\end{equation*}
$$

Note that the expression of the nonlinear term $F$ has changed as we included the term $-c_{0} X$ in the new nonlinear term that we still denote by $F$. The coercivity property implies that $A$ is sectorial on $L^{2}(\Lambda)$, i.e. there exist $C_{1}, \theta \in\left(\frac{1}{2} \pi, \pi\right)$ such that

$$
\begin{equation*}
\left\|(\lambda I-A)^{-1}\right\|_{L\left(L^{2}(\Lambda)\right)} \leq \frac{C_{1}}{|\lambda|}, \quad \lambda \in S_{\theta} \tag{16}
\end{equation*}
$$

where $S_{\theta}:=\left\{\lambda \in \mathbb{C}: \lambda=\rho e^{i \phi}, \rho>0,0 \leq|\phi| \leq \theta\right\}$ (see 8|). Then $A$ is the infinitesimal generator of a bounded analytic semigroup $S \overline{(t)}:=e^{t A}$ on $L^{2}(\Lambda)$ such that

$$
\begin{equation*}
S(t):=e^{t A}=\frac{1}{2 \pi i} \int_{\mathcal{C}} e^{t \lambda}(\lambda I-A)^{-1} d \lambda, \quad t>0 \tag{17}
\end{equation*}
$$

where $\mathcal{C}$ denotes a path that surrounds the spectrum of $A$. The coercivity property 15 also implies that $-A$ is a positive operator and its fractional powers are well defined for any $\alpha>0$, by

$$
\left\{\begin{align*}
(-A)^{-\alpha} & =\frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} \mathrm{e}^{t A} d t  \tag{18}\\
(-A)^{\alpha} & =\left((-A)^{-\alpha}\right)^{-1}
\end{align*}\right.
$$

where $\Gamma(\alpha)$ is the Gamma function (see [8]). Let's now turn to the discretization of our problem (1). We start by splitting the domain $\Lambda$ in finite triangles. Let $\mathcal{T}_{h}$ be the triangulation with maximal length $h$ satisfying the usual regularity assumptions, and $V_{h} \subset V$ the space of continuous functions that are piecewise linear over the triangulation $\mathcal{T}_{h}$. We consider the projection $P_{h}$ from $H=L^{2}(\Lambda)$ to $V_{h}$ defined for every $u \in H$ by

$$
\begin{equation*}
\left\langle P_{h} u, \chi\right\rangle_{H}=\langle u, \chi\rangle_{H}, \quad \forall \chi \in V_{h} . \tag{19}
\end{equation*}
$$

The discrete operator $A_{h}: V_{h} \longrightarrow V_{h}$ is defined by

$$
\begin{equation*}
\left\langle A_{h} \phi, \chi\right\rangle_{H}=\langle A \phi, \chi\rangle_{H}=-a(\phi, \chi), \quad \forall \phi, \chi \in V_{h}, \tag{20}
\end{equation*}
$$

Like $A, A_{h}$ is also a generator of a semigroup $S_{h}(t):=e^{t A_{h}}$. As any semigroup and its generator, $A_{h}$ and $S_{h}(t)$ satisfy the smoothing properties of Proposition 1 with a uniform constant $C$, independent of $h$. Following [4, 6, 22, 36, we characterize the domain of the operator $(-A)^{k / 2}, 1 \leq k \leq 2$ as follow

$$
\mathcal{D}\left((-A)^{k / 2}\right)=\mathbb{H} \cap H^{k}(\Lambda), \quad \text { (for Dirichlet boundary conditions), }
$$ $\mathcal{D}(-A)=\mathbb{H}, \quad \mathcal{D}\left((-A)^{1 / 2}\right)=H^{1}(\Lambda), \quad$ (for Robin boundary conditions).

The semi-discrete version of problem (1) consists to find $X^{h}(t) \in V_{h}, t \in(0, T]$ such that $X^{h}(0)=P_{h} X_{0}$ and

$$
\begin{equation*}
d X^{h}(t)=\left[A_{h} X^{h}(t)+P_{h} F\left(X^{h}(t)\right)\right] d t+P_{h} B\left(X^{h}(t)\right) d W(t), \quad t \in(0, T] .(21 \tag{21}
\end{equation*}
$$

We note that $A_{h}$ and $P_{h} F$ satisfy the same assumptions as $A$ and $F$ respectively. We also note that $P_{h} B$ satisfies Assumption 4. Therefore, Theorem 8 ensures the existence of the unique mild solution $X^{h}(t)$ of (21) such that

$$
\begin{equation*}
\left\|X^{h}(t)\right\| \leq C\left(1+\left\|P_{h} X_{0}\right\|\right) \leq C\left(1+\left\|X_{0}\right\|\right), \quad \forall t \in[0, T] \tag{22}
\end{equation*}
$$

The mild solution of (21) can be represented as follows

$$
\begin{align*}
X^{h}(t) & =S_{h}(t) X^{h}(0)+\int_{0}^{t} S_{h}(t-s) P_{h} F\left(X^{h}(s)\right) d s \\
& +\int_{0}^{t} S_{h}(t-s) P_{h} B\left(X^{h}(s)\right) d W(s) . \tag{23}
\end{align*}
$$

The following lemma will be useful in our convergence analysis.
Lemma 1 The following estimate holds

$$
\left\|\left(-A_{h}\right)^{\alpha} P_{h} v\right\| \leq C\left\|(-A)^{\alpha} v\right\|, \quad \forall 0 \leq \alpha \leq \frac{1}{2}, \quad \forall v \in \mathcal{D}\left((-A)^{\alpha}\right)
$$

Proof From the equivalence of norms (see [22, (3.12)]) we have

$$
\begin{equation*}
\left\|\left(-A_{h}\right)^{1 / 2} P_{h} v\right\| \leq C\left\|P_{h} v\right\|_{H^{1}(\Lambda)}, \quad v \in H^{1}(\Lambda) \tag{24}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\left\|P_{h} v\right\|_{H^{1}(\Lambda)}^{2}=\left\|P_{h} v\right\|_{L^{2}(\Lambda)}^{2}+\sum_{i=1}^{d}\left\|\frac{\partial\left(P_{h} v\right)}{\partial x_{i}}\right\|_{L^{2}(\Lambda)}^{2} \tag{25}
\end{equation*}
$$

where $\frac{\partial}{\partial x_{i}}$ stands for the weak derivative. Let $\mathcal{D}(\Lambda)$ be the set of functions $\varphi \in C^{\infty}(\Lambda)$ with compact support in $\Lambda$. Let $v \in L^{2}(\Lambda)$, for all $\varphi \in \mathcal{D}(\Lambda)$, we have

$$
\begin{equation*}
\left\langle\frac{\partial\left(P_{h} v\right)}{\partial x_{i}}, \varphi\right\rangle=-\left\langle P_{h} v, \frac{\partial \varphi}{\partial x_{i}}\right\rangle=-\left\langle v, P_{h}^{*} \frac{\partial \varphi}{\partial x_{i}}\right\rangle, \tag{26}
\end{equation*}
$$

where $\langle.,$.$\rangle is a duality pairing between \mathcal{D}^{\prime}(\Lambda)$ and $\mathcal{D}(\Lambda)$, and $\frac{\partial \varphi}{\partial x_{i}}$ is the derivative of $\varphi$ in the classical sense. From 24, Remark 2.1] we have

$$
P_{h}^{*} \frac{\partial \varphi}{\partial x_{i}}=\frac{\partial\left(P_{h}^{*} \varphi\right)}{\partial x_{i}}
$$

since $P_{h}^{*}$ is a linear operator. So from the equality 26 it holds that

$$
\begin{equation*}
\left\langle\frac{\partial\left(P_{h} v\right)}{\partial x_{i}}, \varphi\right\rangle=-\left\langle v, \frac{\partial\left(P_{h}^{*} \varphi\right)}{\partial x_{i}}\right\rangle=\left\langle\frac{\partial v}{\partial x_{i}}, P_{h}^{*} \varphi\right\rangle=\left\langle P_{h} \frac{\partial v}{\partial x_{i}}, \varphi\right\rangle \tag{27}
\end{equation*}
$$

Since 27 holds for all $\varphi \in \mathcal{D}(\Lambda)$, it follows that

$$
\begin{equation*}
\frac{\partial\left(P_{h} v\right)}{\partial x_{i}}=P_{h} \frac{\partial v}{\partial x_{i}} \quad \text { in the weak sense. } \tag{28}
\end{equation*}
$$

Inserting this latter relation in 25, using the fact that the projection $P_{h}$ is bounded with respect to the norm $\|\cdot\|_{L^{2}(\Lambda)}$ and again the equivalence of norm [22, (3.12)] yields

$$
\begin{align*}
\left\|P_{h} v\right\|_{H^{1}(\Lambda)}^{2} & =\left\|P_{h} v\right\|_{L^{2}(\Lambda)}^{2}+\sum_{i=1}^{d}\left\|P_{h} \frac{\partial v}{\partial x_{i}}\right\|_{L^{2}(\Lambda)}^{2} \\
& =\|v\|_{L^{2}(\Lambda)}^{2}+\sum_{i=1}^{d}\left\|\frac{\partial v}{\partial x_{i}}\right\|_{L^{2}(\Lambda)}^{2} \\
& =\|v\|_{H^{1}(\Lambda)}^{2} \leq C\left\|(-A)^{1 / 2} v\right\| . \tag{29}
\end{align*}
$$

We therefore have

$$
\begin{equation*}
\left\|\left(-A_{h}\right)^{1 / 2} P_{h} v\right\| \leq C\left\|(-A)^{1 / 2} v\right\| . \tag{30}
\end{equation*}
$$

Note that 30 remains true if we replace $\frac{1}{2}$ by 0 . By interpolation theory we have

$$
\begin{equation*}
\left\|\left(-A_{h}\right)^{\alpha} P_{h} v\right\| \leq C\left\|(-A)^{\alpha} v\right\|, \quad \forall 0 \leq \alpha \leq \frac{1}{2}, \quad \forall v \in \mathcal{D}\left((-A)^{\alpha}\right) \tag{31}
\end{equation*}
$$

Let us recall the following well known lemma.
Lemma 2 [Itô isometry] [28, Proposition 2.3.5]
For any $t \in[0, T]$ and for any $L_{2}^{0}$-valued predictable process $\phi(s), s \in[0, t]$ the following equality holds

$$
\mathbb{E}\left[\left\|\int_{0}^{t} \phi(s) d W(s)\right\|^{2}\right]=\mathbb{E}\left[\int_{0}^{t}\|\phi(s)\|_{L_{2}^{o}}^{2} d s\right] .
$$

The following two lemmas provide space and time regularity results of the mild solution of the semi-discrete problem (21). These lemmas play an important role in our convergence analysis. More results on the regularity of the mild solution of problem (1) can be found in 15, 21,26.

Lemma 3 [Space regularity of the mild solution $X^{h}(t)$ ]
Let Assumption 1, Assumption 2, Assumption 3 and Assumption 4 be fulfilled with $\beta \in[0,1)$, and $p \geq 2$. Then for all $t \in[0, T], X^{h}(t) \in L^{p}\left(\Omega, \mathcal{D}\left((-A)^{\beta / 2}\right)\right)$. Moreover, there exists a positive constant $C$ independent of $h$ such that

$$
\left\|\left(-A_{h}\right)^{\beta / 2} X^{h}(t)\right\|_{L^{p}(\Omega, H)} \leq C\left(1+\left\|(-A)^{\beta / 2} X_{0}\right\|_{L^{p}(\Omega, H)}\right), \quad t \in[0, T] .(32)
$$

Further, when dealing with additive noise $(B=\mathbf{I})$, if Assumption 6 is fulfilled with $\beta \in[0,2)$, then for all $t \in[0, T] X^{h}(t) \in L^{p}\left(\Omega, \mathcal{D}\left((-A)^{\beta / 2}\right)\right)$ and 32) holds for $\beta \in[0,2)$.

Proof The proof follows the sames lines as that of [23, Lemma 2.6] or [15 Theorem 1] or 21, Theorem 3.1] by making use of Lemma 1.

Lemma 4 [Time regularity of the mild solution $X^{h}(t)$ ] Let $X^{h}$ be the mild solution of (21). If Assumption 1, Assumption 2, Assumption 3 and Assumption 4 are fulfilled with the corresponding $0<\beta \leq 2$. For $0<\beta<1$, there exists a positive constant $C$ independent of $h$ such that for $t_{1}, t_{2} \in[0, T]$, $t_{1}<t_{2}$, we have

$$
\begin{equation*}
\left\|X^{h}\left(t_{2}\right)-X^{h}\left(t_{1}\right)\right\|_{L^{p}(\Omega, H)} \leq C\left(t_{2}-t_{1}\right)^{\beta / 2}\left(1+\left\|(-A)^{\beta / 2} X_{0}\right\|_{L^{p}(\Omega, H)}\right) . \tag{33}
\end{equation*}
$$

Moreover, if Assumption 5 is fulfilled with $1 \leq \beta \leq 2$, then there exists a positive constant $C$ such that

$$
\begin{equation*}
\left\|X^{h}\left(t_{2}\right)-X^{h}\left(t_{1}\right)\right\|_{L^{p}(\Omega, H)} \leq C\left(t_{2}-t_{1}\right)^{1 / 2}\left(1+\left\|(-A)^{\beta / 2} X_{0}\right\|_{L^{p}(\Omega, H)}\right) \tag{34}
\end{equation*}
$$

For additive noise $(B=\mathbf{I})$, if Assumption 1, Assumption 2, Assumption 3 and Assumption 6 are fulfilled, then the following time regularity holds
$\left\|X^{h}\left(t_{2}\right)-X^{h}\left(t_{1}\right)\right\|_{L^{p}(\Omega, H)} \leq C\left(t_{2}-t_{1}\right)^{\min (\beta, 1) / 2}\left(1+\left\|(-A)^{\beta / 2} X_{0}\right\|_{L^{p}(\Omega, H)}\right)$.
Proof The proof follows the same lines as that of [23, Lemma 2.7] or 15 Theorem 1] or 21, Theorem 4.1] by making use of Lemma 1.
2.3 Current stable and efficient schemes for semilinear SPDEs

Recall that the simple efficient standard semi-implicit Euler-Maruyama scheme for (11) is given by (see e.g. [27])

$$
\begin{align*}
Z_{m+1}^{h} & =S_{h, \Delta t}\left[\left(\mathbf{I}+\Delta t(1-\theta) A_{h}\right) Z_{m}^{h}+\Delta t P_{h} F\left(Z_{m}^{h}\right)+P_{h} B\left(Z_{m}^{h}\right) \Delta W_{m}\right]  \tag{35}\\
S_{h, \Delta t} & =\left(\mathbf{I}-\theta \Delta t A_{h}\right)^{-1}, \quad \theta \in[0,1] \tag{36}
\end{align*}
$$

The exponential integrators schemes developed in 23,33 are given by

$$
\begin{equation*}
K_{m+1}^{h}=S_{h}(\Delta t)\left(K_{m}^{h}+\Delta t P_{h} F\left(K_{m}^{h}\right)+P_{h} B\left(K_{m}^{h}\right) \Delta W_{m}\right), \tag{37}
\end{equation*}
$$

and

$$
\begin{align*}
& L_{m+1}^{h}=S_{h}(\Delta t)\left(L_{m}^{h}+P_{h} B\left(L_{m}^{h}\right) \Delta W_{m}\right)+\Delta t \varphi_{1}\left(\Delta t A_{h}\right) P_{h} F\left(L_{m}^{h}\right)  \tag{38}\\
& \Delta W_{m}:=W_{t_{m+1}}-W_{t_{m}}=\sqrt{\Delta t} \sum_{i \in \mathbb{N}} \sqrt{q_{i}} R_{i, m} e_{i}
\end{align*}
$$

where $R_{i, m}$ are independent, standard normally distributed random variables with mean 0 and variance 1 , and the function $\varphi_{1}$ is defined in 50 . Note that all the initial values in all the schemes are taken to be $P_{h} X_{0}$ and scheme (35) is more stable for $\theta \geq 1 / 2$. If the linear operator $A$ tends to the operator null ${ }^{3}$ the corresponding discrete operator $A_{h}$ tends also to null, $S_{h}(\Delta t), S_{h, \Delta t}$ and $\varphi_{1}\left(\Delta t A_{h}\right)$ tend to the identical operator $\mathbf{I}$. Therefore the numerical schemes (35), (37) and (38) become the unstable explicit Euler-Maruyama scheme.

### 2.4 Novel fully discrete scheme

Let us build a more stable scheme, robust when the operator $A$ tends to null. For the time discretization, we consider the one-step method which provides the numerical approximated solution $X_{m}^{h}$ of $X^{h}\left(t_{m}\right)$ at discrete time $t_{m}=$ $m \Delta t, m=0, \cdots, M$. The method is based on the continuous linearization of (21). More precisely we linearize (21) at each time step as

$$
d X^{h}(t)=\left[A_{h} X^{h}(t)+J_{m}^{h} X^{h}(t)+G_{m}^{h}\left(X^{h}(t)\right)\right] d t+P_{h} B\left(X^{h}(t)\right) d W(t),(39)
$$

for all $t_{m} \leq t \leq t_{m+1}$, where $J_{m}^{h}$ is the Fréchet derivative of $P_{h} F$ at $X_{m}^{h}$ and $G_{m}^{h}$ is the remainder at $X_{m}^{h}$. Both $J_{m}^{h}$ and $G_{m}^{h}$ are random functions and are defined for all $\omega \in \Omega$ by

$$
\begin{align*}
J_{m}^{h}(\omega) & :=\left(P_{h} F\right)^{\prime}\left(X_{m}^{h}(\omega)\right)=P_{h} F^{\prime}\left(X_{m}^{h}(\omega)\right),  \tag{40}\\
G_{m}^{h}(\omega)\left(X^{h}(t)\right) & :=P_{h} F\left(X^{h}(t)\right)-J_{m}^{h}(\omega) X^{h}(t) . \tag{41}
\end{align*}
$$

Before building the new numerical scheme, let us recall the following important lemma.

Lemma 5 For all $m \in \mathbb{N}$ and all $\omega \in \Omega$, the random linear operator $A_{h}+$ $J_{m}^{h}(\omega)$ is the generator of a strongly continuous semigroup $S_{m}^{h}(\omega)(t):=e^{\left(A_{h}+J_{m}^{h}(\omega)\right) t}$ called random (or stochastic) perturbed semigroup and uniformly bounded on $[0, T]$, i.e. there exists a positive constant $C_{1}$ independent of $h, m, \Delta t$ and the sample $\omega$ such that

$$
\left\|e^{\left(A_{h}+J_{m}^{h}(\omega)\right) t}\right\|_{L(H)} \leq C_{1}, \quad 0 \leq t \leq T
$$

Proof Using the boundedness of $P_{h}$ and Assumption 3, it holds that

$$
\begin{equation*}
\left\|J_{m}^{h}(\omega)\right\|_{L(H)} \leq\left\|F^{\prime}\left(X_{m}^{h}(\omega)\right)\right\|_{L(H)}<C, \quad m \in \mathbb{N}, \quad \omega \in \Omega \tag{42}
\end{equation*}
$$

[^3]Therefore $J_{m}^{h}(\omega)$ is a bounded linear operator. It follows then from 25, Theorem 1.1, Chapter 3, Page 76] that $A_{h}+J_{m}^{h}(\omega)$ is a generator of a strongly continuous semigroup denoted by $S_{m}^{h}(\omega)(t)=e^{\left(A_{h}+J_{m}^{h}(\omega)\right) t}$. Since $A_{h}$ is a generator of an analytic semigroup $S_{h}(t)=e^{A_{h} t}$, there exist two constants $K \geq 0$ and $C_{0} \in \mathbb{R}$ such that

$$
\begin{equation*}
\left\|e^{A_{h} t}\right\|_{L(H)} \leq K e^{C_{0} t}, \quad t \geq 0 \tag{43}
\end{equation*}
$$

Finally using (42) and (43) it holds by applying again 25, Theorem 1.1, Chapter 3, Page 76]) that

$$
\begin{align*}
\left\|e^{\left(A_{h}+J_{m}^{h}(\omega)\right) t}\right\|_{L(H)} & \leq K e^{\left(C_{0}+\left\|J_{m}^{h}(\omega)\right\|_{L(H)}\right) t} \\
& \leq K e^{\left(C_{0}+C\right) t} \leq C_{1}, \quad t \in[0, T], \tag{44}
\end{align*}
$$

where $C_{1}$ is a positive constant, independent of $h, m, \omega$ and $\Delta t$. This complete the proof of Lemma 5 .

Given the solution $X^{h}\left(t_{m}\right)$ and the numerical solution $X_{m}^{h}$ at $t_{m}$, we obtain from (39) the following mild representation form of $X^{h}\left(t_{m+1}\right)$

$$
\begin{align*}
X^{h}\left(t_{m+1}\right) & =e^{\left(A_{h}+J_{m}^{h}\right) \Delta t} X^{h}\left(t_{m}\right)+\int_{t_{m}}^{t_{m+1}} e^{\left(A_{h}+J_{m}^{h}\right)\left(t_{m+1}-s\right)} G_{m}^{h}\left(X^{h}(s)\right) d s \\
& +\int_{t_{m}}^{t_{m+1}} e^{\left(A_{h}+J_{m}^{h}\right)\left(t_{m+1}-s\right)} P_{h} B\left(X^{h}(s)\right) d W(s) \tag{45}
\end{align*}
$$

We note that 45 is the exact solution of (21) at $t_{m+1}$. To establish our numerical method we use the following approximations

$$
\begin{array}{r}
G_{m}^{h}\left(X^{h}\left(t_{m}+s\right)\right) \approx G_{m}^{h}\left(X_{m}^{h}\right) \\
e^{\left(A_{h}+J_{m}^{h}\right)\left(t_{m+1}-s\right)} P_{h} B(X(s)) \approx e^{\left(A_{h}+J_{m}^{h}\right) \Delta t} P_{h} B\left(X_{m}^{h}\right) \tag{47}
\end{array}
$$

Therefore the deterministic integral part of (45) can be approximated as follows

$$
\begin{align*}
& \int_{t_{m}}^{t_{m+1}} e^{\left(A_{h}+J_{m}^{h}\right)\left(t_{m+1}-s\right)} G_{m}^{h}\left(X^{h}(s)\right) d s \\
= & \int_{0}^{\Delta t} e^{\left(A_{h}+J_{m}^{h}\right)(\Delta t-s)} G_{m}^{h}\left(X^{h}\left(t_{m}+s\right)\right) d s \\
\approx & G_{m}^{h}\left(X_{m}^{h}\right)\left(A_{h}+J_{m}^{h}\right)^{-1}\left(e^{\left(A_{h}+J_{m}^{h}\right) \Delta t}-\mathbf{I}\right) . \tag{48}
\end{align*}
$$

Inserting (48) and (47) in 45) and using the approximation $X^{h}\left(t_{m}\right) \approx X_{m}^{h}$ give the following approximation $X_{m+1}^{h}$ of $X^{h}\left(t_{m+1}\right)$, called Stochastic Exponential Rosenbrock Scheme (SERS)

$$
\begin{align*}
X_{m+1}^{h} & =e^{\left(A_{h}+J_{m}^{h}\right) \Delta t} X_{m}^{h}+\left(A_{h}+J_{m}^{h}\right)^{-1}\left(e^{\left(A_{h}+J_{m}^{h}\right) \Delta t}-\mathbf{I}\right) G_{m}^{h}\left(X_{m}^{h}\right) \\
& +e^{\left(A_{h}+J_{m}^{h}\right) \Delta t} P_{h} B\left(X_{m}^{h}\right)\left(W_{t_{m+1}}-W_{t_{m}}\right) \tag{49}
\end{align*}
$$

with $X_{0}^{h}:=X^{h}(0)=P_{h} X_{0}$. The numerical scheme 49 can be rewritten in the following equivalent form, which is efficient for implementation

$$
\begin{aligned}
X_{m+1}^{h} & =X_{m}^{h}+P_{h} B\left(X_{m}^{h}\right) \Delta W_{m} \\
& +\varphi_{1}\left(\Delta t\left(A_{h}+J_{m}^{h}\right)\right)\left[\left(A_{h}+J_{m}^{h}\right)\left(X_{m}^{h}+P_{h} B\left(X_{m}^{h}\right) \Delta W_{m}\right)+G_{m}^{h}\left(X_{m}^{h}\right)\right]
\end{aligned}
$$

where

$$
\begin{align*}
\varphi_{1}\left(\Delta t\left(A_{h}+J_{m}^{h}\right)\right) & :=\left(A_{h}+J_{m}^{h}\right)^{-1}\left(e^{\Delta t\left(A_{h}+J_{m}^{h}\right)}-\mathbf{I}\right) \\
& =\int_{0}^{\Delta t} e^{(\Delta t-s)\left(A_{h}+J_{m}^{h}\right)} d s \tag{50}
\end{align*}
$$

Note that the operator $\varphi_{1}\left(\Delta t\left(A_{h}+J_{m}^{h}(\omega)\right)\right)$ is uniformly bounded (independently of $h, m$ and $\omega$ ), see e.g. 10, Lemma 2.4].

Remark 1 Note that the corresponding standard stochastic exponential scheme (38) presented in 23 can be written as

$$
\begin{align*}
L_{m+1}^{h} & =L_{m}^{h}+P_{h} B\left(L_{m}^{h}\right) \Delta W_{m} \\
& +\varphi_{1}\left(\Delta t A_{h}\right)\left[A_{h}\left(L_{m}^{h}+P_{h} B\left(L_{m}^{h}\right) \Delta W_{m}\right)+P_{h} F\left(L_{m}^{h}\right)\right] \tag{51}
\end{align*}
$$

This scheme will be called SETD1 and will be used in our numerical simulations for comparison with SERS scheme.

Remark 2 If the deterministic part is also approximated as the diffusion part 47), we will obtain the following new scheme

$$
\begin{equation*}
\left.U_{m+1}^{h}=e^{\left(A_{h}+J_{m}^{h}\right) \Delta t}\left[U_{m}^{h}+P_{h} B\left(U_{m}^{h}\right) \Delta W_{m}+G_{m}^{h}\left(U_{m}^{h}\right)\right)\right] . \tag{52}
\end{equation*}
$$

Our main result is also valid for scheme (52) and the extension of our proof to that scheme is done as in [23] without any issue.

Having the numerical method (49) in hand, our goal is to analyze its strong convergence toward the exact solution in the root-mean-square $L^{2}$ sense. In the following subsection we state our strong convergence results, which are in fact our main results.

### 2.5 Main results

Throughout this paper we take $t_{m}=m \Delta t \in[0, T]$, where $T=M \Delta t$ for $m, M \in \mathbb{N}, m \leq M, T$ is fixed, $C$ is a generic constant that may change from one place to another and $\epsilon>0$ is a positive constant small enough. The main results of this paper are formulated in the following theorems. For multiplicative noise we have the following result.

Theorem 9 Let $X\left(t_{m}\right)$ and $X_{m}^{h}$ be respectively the mild solution (3) and the numerical approximation given by 49) at $t_{m}=m \Delta t$. Let Assumption 1, Assumption 2 (with $0<\beta<1$ and $p=2$ ), Assumption 3 and Assumption 4 be fulfilled. Then the following error estimate holds

$$
\left(\mathbb{E}\left\|X\left(t_{m}\right)-X_{m}^{h}\right\|^{2}\right)^{1 / 2} \leq C\left(h^{\beta}+\Delta t^{\beta / 2}\right) .
$$

Moreover, under a strong regularity of the initial data, that is Assumption 2 (with $p=2$ ) and Assumption 5 are fulfilled with $\beta \in[1,2)$ and $\gamma=\beta-1$, the following error estimate holds

$$
\left(\mathbb{E}\left\|X\left(t_{m}\right)-X_{m}^{h}\right\|^{2}\right)^{1 / 2} \leq C\left(h^{\beta}+\Delta t^{1 / 2}\right) .
$$

If $\beta=2$ and Assumption 2 (with $p=2$ ) and Assumption 5 are fulfilled with $\gamma=1$, then the following error estimate holds

$$
\left(\mathbb{E}\left\|X\left(t_{m}\right)-X_{m}^{h}\right\|^{2}\right)^{1 / 2} \leq C\left(h^{2}\left(1+\max \left(0, \ln \left(t_{m} / h^{2}\right)\right)\right)+\Delta t^{1 / 2}\right) .
$$

As in 23, Remark 2.9], strong assumptions on the nonlinear function $F$ can allow to achieve a spatial error of order $\mathcal{O}\left(h^{2}\right)$. Note that Assumption 5 in Theorem 9 is key to obtain optimal order of convergence. The following remark provides the error estimate without Assumption 5.

Remark 3 If Assumption 1. Assumption 2 (with $p=2$ ), Assumption 3 and Assumption 4 are fulfilled with $\beta \in[1,2]$, then the following error estimate holds

$$
\left(\mathbb{E}\left\|X\left(t_{m}\right)-X_{m}^{h}\right\|^{2}\right)^{1 / 2} \leq C\left(h^{1-\epsilon}+\Delta t^{1 / 2-\epsilon}\right),
$$

where $\epsilon$ is a positive constant small enough.
For additive noise (that is $B=\mathbf{I}$ ), we have the following result.
Theorem 10 Let $X\left(t_{m}\right)$ and $X_{m}^{h}$ be respectively the mild solution (3) and the numerical approximation given by (49) at $t_{m}=m \Delta t$. For additive noise, if Assumption 1, Assumption 2 (with $0<\beta<2$ and $p=4$ ), Assumption 3. Assumption $\sqrt{6}$ and Assumption 7 are fulfilled, then the following error estimate holds

$$
\begin{equation*}
\left(\mathbb{E}\left\|X\left(t_{m}\right)-X_{m}^{h}\right\|^{2}\right)^{1 / 2} \leq C\left(h^{\beta}+\Delta t^{\beta / 2-\epsilon}\right) \tag{53}
\end{equation*}
$$

Moreover if $\beta=2$, the following error estimate holds

$$
\begin{equation*}
\left(\mathbb{E}\left\|X\left(t_{m}\right)-X_{m}^{h}\right\|^{2}\right)^{1 / 2} \leq C\left(h^{2}\left(1+\max \left(0, \ln \left(t_{m} / h^{2}\right)\right)\right)+\Delta t^{1-\epsilon}\right) . \tag{54}
\end{equation*}
$$

Remark 4 For additive noise, we achieved suboptimal order $\mathcal{O}\left(h^{2-\epsilon}+\Delta t^{1-\epsilon}\right)$ for $\beta=2$, where $\epsilon$ is a positive number, small enough. The suboptimal order $1-\epsilon$ in time was achieved in [14], where authors imposed a strong regularity on the drift function (namely 14 , Assumption 2]), but with less regular noise. The recent works in $[39,40]$ achieved optimal order 1 in time with less restrictive assumptions than [14, Assumption 2]. Here we have achieved suboptimal order $1-\epsilon$ in time with similar assumptions as in 39, 40. Note that the current work and the work in [39] use standard Brownian increments, while the works in 14,40 use the linear functionals of the noise to achieve optimal order with less regular noise.

Remark 5 Note that the semi-discrete problem (21) can be replaced by the following semi-discrete problem where the noise is truncated

$$
\begin{equation*}
d X^{h}(t)=\left[A_{h} X^{h}(t)+P_{h} F\left(X^{h}(t)\right)\right] d t+P_{h} B\left(X^{h}(t)\right) P_{h} d W(t), t \in[0, T] . \text {.55 } \tag{55}
\end{equation*}
$$

It was shown in 19 that in the case of additive noise with smooth covariance operator kernel, this truncation can be done severely without loosing the spatial accuracy of the finite element method. Applying our stochastic exponential Rosenbrock scheme to (55) yields

$$
\begin{align*}
Y_{m+1}^{h} & =e^{\left(A_{h}+J_{m}^{h}\right) \Delta t} Y_{m}^{h}+\left(A_{h}+J_{m}^{h}\right)^{-1}\left(e^{\left(A_{h}+J_{m}^{h}\right) \Delta t}-\mathbf{I}\right) G_{m}^{h}\left(Y_{m}^{h}\right) \\
& +e^{\left(A_{h}+J_{m}^{h}\right) \Delta t} P_{h} B\left(Y_{m}^{h}\right) P_{h}\left(W_{t_{m+1}}-W_{t_{m}}\right) \tag{56}
\end{align*}
$$

We note that Theorem 9 and Theorem 10 also hold for the numerical scheme (56). Parts of [36] can be used in the proof.

## 3 Proof of the main results

Before prove our main results, some preparatory results are needed.

### 3.1 Preparatory results

Lemma 6 The function $G_{m}^{h}(\omega)$ defined by (41) satisfies the global Lipschitz condition with a uniform constant, i.e. there exists a positive constant $C>0$, independent of $h, m$ and $\omega$ such that

$$
\left\|G_{m}^{h}(\omega)\left(u^{h}\right)-G_{m}^{h}(\omega)\left(v^{h}\right)\right\| \leq C\left\|u^{h}-v^{h}\right\|, \quad \forall m \in \mathbb{N}, \quad \forall u^{h}, v^{h} \in V_{h}
$$

Proof Using Assumption 3 and relations 40-41, the proof is straightforward.

We introduce the Riesz representation operator $R_{h}: V \longrightarrow V_{h}$ defined by

$$
\begin{equation*}
\left\langle-A R_{h} v, \chi\right\rangle_{H}=\langle-A v, \chi\rangle_{H}=a(v, \chi), \quad \forall v \in V, \quad \forall \chi \in V_{h} . \tag{57}
\end{equation*}
$$

It is well known (see 22,23 ) that $A$ and $A_{h}$ are related by $A_{h} R_{h}=P_{h} A$. Under the regularity assumptions on the triangulation and in view of the $V$ -
 errors estimates hold

$$
\begin{equation*}
\left\|R_{h} v-v\right\|+h\left\|R_{h} v-v\right\|_{H^{1}(\Omega)} \leq C h^{r}\|v\|_{H^{r}(\Omega)}, \quad v \in V \cap H^{r}(\Omega) \tag{58}
\end{equation*}
$$

Let us consider the following deterministic linear problem : find $u \in V$ such that

$$
\begin{equation*}
\frac{d u}{d t}=A u, \quad u(0)=v, \quad t \in(0, T] \tag{59}
\end{equation*}
$$

The corresponding semi-discrete problem in space consists to find $u_{h} \in V_{h}$ such that

$$
\begin{equation*}
\frac{d u_{h}}{d t}=A_{h} u_{h}, \quad u_{h}(0)=P_{h} v, \quad t \in(0, T] . \tag{60}
\end{equation*}
$$

Let us define the following operator

$$
\begin{equation*}
T_{h}(t):=S(t)-S_{h}(t) P_{h}=e^{A t}-e^{A_{h} t} P_{h} \tag{61}
\end{equation*}
$$

so that $u(t)-u_{h}(t)=T_{h}(t) v$. The estimate was used in 23 to prove the key part of the following lemma.

## Lemma 7 The following estimate holds

$$
\begin{equation*}
\left\|T_{h}(t) v\right\| \leq C h^{r} t^{-(r-\alpha) / 2}\|v\|_{\alpha}, \quad r \in[0,2], \quad \alpha \leq r, t \in(0, T] . \tag{62}
\end{equation*}
$$

Proof The proof of Lemma 7 for $r \in[1,2]$ can be found in 23, Lemma 3.1]. Using the stability property of $S(t)$ and $S_{h}(t)$, and the fact that the projection $P_{h}$ is bounded, it follows that

$$
\begin{equation*}
\left\|S(t) v-S_{h}(t) P_{h} v\right\| \leq C\|v\| \tag{63}
\end{equation*}
$$

Inequality (63) shows that 62) holds for $r=0$. Interpolating between $r=0$ and $r=2$ completes the proof of Lemma 7 .

Lemma 8 Let $X(t)$ and $X^{h}(t)$ be the mild solutions given respectively by (3) and (23).
(1) For multiplicative noise, assume that Assumption 1, Assumption 2, Assumption 3 and Assumption 4 are fulfilled. Then the following error estimate holds:
(i) For $0 \leq \beta<1$

$$
\left\|X(t)-X^{h}(t)\right\|_{L^{2}(\Omega, H)} \leq C h^{\beta}, \quad t \in(0, T] .
$$

(ii) For $1 \leq \beta<2$

$$
\left\|X(t)-X^{h}(t)\right\|_{L^{2}(\Omega, H)} \leq C h^{1-\epsilon}, \quad t \in(0, T]
$$

where $\epsilon$ is a positive constant small enough.
(iii) For $1 \leq \beta<2$, if moreover Assumption 5 is fulfilled with $\gamma=\beta-1$, we have

$$
\left\|X(t)-X^{h}(t)\right\|_{L^{2}(\Omega, H)} \leq C h^{\beta}, \quad t \in(0, T] .
$$

(iv) For $\beta=2$ and if Assumption 5 is fulfilled with $\gamma=1$, we have

$$
\left\|X(t)-X^{h}(t)\right\|_{L^{2}(\Omega, H)} \leq C h^{2}\left(1+\max \left(0, \ln \left(t / h^{2}\right)\right)\right), \quad t \in(0, T] .
$$

(2) For additive noise $(B=\mathbf{I})$, if Assumption 1, Assumption 2, Assumption 3 and Assumption 6 are fulfilled, then the following error estimate holds:
(i) For $0 \leq \beta<2$

$$
\begin{equation*}
\left\|X(t)-X^{h}(t)\right\|_{L^{2}(\Omega, H)} \leq C h^{\beta} . \tag{64}
\end{equation*}
$$

(ii) For $\beta=2$

$$
\left\|X(t)-X^{h}(t)\right\|_{L^{2}(\Omega, H)} \leq C h^{2}\left(1+\max \left(0, \ln \left(t / h^{2}\right)\right)\right), \quad t \in(0, T]
$$

Proof The proof of (1) (i) and (iii) can be found in [36, Theorem 6.1]. The proof of (1) (ii) is similar to that of 36 , Theorem 6.1] using Lemma 7 The proof of (2) (i) and (ii) can be found in [19, Proposition 3.3].

Lemma 9 Under Assumption 1, for all $\omega \in \Omega$, the stochastic perturbed semigroup $S_{m}^{h}(\omega)(t)$ satisfies the following stability properties
(i) For $\gamma_{1}, \gamma_{2} \leq 1$, such that $0 \leq \gamma_{1}+\gamma_{2} \leq 1$, we have

$$
\left\|\left(-A_{h}\right)^{-\gamma_{1}}\left(S_{m}^{h}(\omega)(t)-\mathbf{I}\right)\left(-A_{h}\right)^{-\gamma_{2}}\right\|_{L(H)} \leq C t^{\gamma_{1}+\gamma_{2}}, t \in(0, T]
$$

(ii) For $\gamma_{1} \geq 0$, we have

$$
\left\|S_{m}^{h}(\omega)(t)\left(-A_{h}\right)^{\gamma_{1}}\right\|_{L(H)} \leq C t^{-\gamma_{1}}, t \in(0, T], \gamma_{1} \geq 0
$$

(iii) For $\gamma_{1} \geq 0$ and $0 \leq \gamma_{2}<1$ such that $\gamma_{2}-\gamma_{1} \geq 0$, we have

$$
\left\|\left(-A_{h}\right)^{-\gamma_{1}} S_{m}^{h}(\omega)(t)\left(-A_{h}\right)^{\gamma_{2}}\right\|_{L(H)} \leq C t^{\gamma_{1}-\gamma_{2}}, t \in(0, T] .
$$

(iv) For $\gamma_{1}, \gamma_{2}>0$ such that $0 \leq \gamma_{2}-\gamma_{1} \leq 1$, then the following estimate holds

$$
\left\|\left(-A_{h}\right)^{-\gamma_{1}}\left(S_{m}^{h}(\omega)(t)-\mathbf{I}\right)\left(-A_{h}\right)^{\gamma_{2}}\right\|_{L(H)} \leq C t^{\gamma_{1}-\gamma_{2}}, t \in(0, T]
$$

where $C$ is a positive constant independent of $h, m, \Delta t$ and the sample $\omega$.
Proof We recall that the perturbed semigroup satisfies the following variation of parameters formula (see [5, Chapter 3, Corollary 1.7] or [25, Section 3.1, Page 77])

$$
\begin{equation*}
S_{m}^{h}(\omega)(t) v=S_{h}(t) v+\int_{0}^{t} S_{h}(t-s) J_{m}^{h}(\omega) S_{m}^{h}(\omega)(s) v d s \tag{65}
\end{equation*}
$$

for all $v \in H$ and all $t \geq 0$. Then it follows from that

$$
\begin{equation*}
\left(S_{m}^{h}(\omega)(t)-\mathbf{I}\right) v=\left(S_{h}(t)-\mathbf{I}\right) v+\int_{0}^{t} S_{h}(t-s) J_{m}^{h}(\omega) S_{m}^{h}(\omega)(s) v d s \tag{66}
\end{equation*}
$$

It is obvious that $\left(-A_{h}\right)^{-\gamma_{2}} v \in H$ for all $v \in H$. Then, replacing $v$ in (66) by $\left(-A_{h}\right)^{-\gamma_{2}} v$ and pre-multiplying both right-hand sides of $\sqrt[66]{ }$ by $\left(-A_{h}\right)^{-\gamma_{1}}$ yields

$$
\begin{align*}
& \left(-A_{h}\right)^{-\gamma_{1}}\left(S_{m}^{h}(\omega)(t)-\mathbf{I}\right)\left(-A_{h}\right)^{-\gamma_{2}} v \\
= & \left(S_{h}(t)-\mathbf{I}\right)\left(-A_{h}\right)^{-\gamma_{2}-\gamma_{1}} v  \tag{67}\\
+ & \int_{0}^{t}\left(-A_{h}\right)^{-\gamma_{1}} S_{h}(t-s) J_{m}^{h}(\omega) S_{m}^{h}(\omega)(s)\left(-A_{h}\right)^{-\gamma_{2}} v d s .
\end{align*}
$$

Taking the norm in both sides of (67) and using Proposition 1, the fact that $\left(-A_{h}\right)^{-\gamma_{2}}$ and $J_{m}^{h}(\omega)$ are uniformly bounded, it follows that

$$
\begin{aligned}
\left\|\left(-A_{h}\right)^{-\gamma_{1}}\left(S_{m}^{h}(\omega)(t)-\mathbf{I}\right)\left(-A_{h}\right)^{-\gamma_{2}} v\right\| & \leq C t^{\gamma_{2}+\gamma_{1}}\|v\|+C \int_{0}^{t}\|v\| d s \\
& \leq C t^{\gamma_{2}+\gamma_{1}}\|v\|
\end{aligned}
$$

Using the definition of the norm $\|\cdot\|_{L(H)}$ gives the desired result for (i). To prove (ii), we replace $v$ by $(-A)^{\gamma_{1}} v$ in (65) and obtain

$$
\begin{align*}
S_{m}^{h}(\omega)(t)\left(-A_{h}\right)^{\gamma_{1}} v & =S_{h}(t)\left(-A_{h}\right)^{\gamma_{1}} v \\
& +\int_{0}^{t} S_{h}(t-s) J_{m}^{h}(\omega) S_{m}^{h}(\omega)(s)\left(-A_{h}\right)^{\gamma_{1}} v d s \tag{68}
\end{align*}
$$

for all $v \in H$ and all $t \geq 0$. Taking the norm in both sides of 68) and using the stability property of $S_{h}(t), S_{m}^{h}(\omega)(t)$ with the uniformly boundedness of $J_{m}^{h}(\omega)$ gives

$$
\begin{align*}
\left\|S_{m}^{h}(\omega)(t)\left(-A_{h}\right)^{\gamma_{1}} v\right\| & \leq C t^{-\gamma_{1}}\|v\| \\
& +C \int_{0}^{t}\left\|S_{m}^{h}(\omega)(s)\left(-A_{h}\right)^{\gamma_{1}}\right\|_{L(H)}\|v\| d s \tag{69}
\end{align*}
$$

From (69) it holds that

$$
\begin{equation*}
\left\|S_{m}^{h}(\omega)(t)\left(-A_{h}\right)^{\gamma_{1}}\right\|_{L(H)} \leq C t^{-\gamma_{1}}+C \int_{0}^{t}\left\|S_{m}^{h}(\omega)(s)\left(-A_{h}\right)^{\gamma_{1}}\right\|_{L(H)} d s \tag{70}
\end{equation*}
$$

Applying the continuous Gronwall's lemma to (70) completes the proof of (ii). To prove (iii), we replace $v$ in (65) by $\left(-A_{h}\right)^{\gamma_{2}} v$ and pre-multiply both sides by $\left(-A_{h}\right)^{-\gamma_{1}}$. This yields

$$
\begin{align*}
\left(-A_{h}\right)^{-\gamma_{1}} S_{m}^{h}(\omega)(t)\left(-A_{h}\right)^{\gamma_{2}} v & =\left(-A_{h}\right)^{-\gamma_{1}} S_{h}(t)\left(-A_{h}\right)^{\gamma_{2}} v  \tag{71}\\
& +\int_{0}^{t}\left(-A_{h}\right)^{-\gamma_{1}} S_{h}(t-s) J_{m}^{h}(\omega) S_{m}^{h}(\omega)(s)\left(-A_{h}\right)^{\gamma_{2}} v d s .
\end{align*}
$$

Taking the norm in both sides of (71), using the stability properties of Proposition 1, the boundedness of $\left(-A_{h}\right)^{-\gamma_{1}}, J_{m}^{h}$ and applying Lemma 9 (ii), it holds that

$$
\begin{align*}
\left\|\left(-A_{h}\right)^{-\gamma_{1}} S_{m}^{h}(\omega)(t)\left(-A_{h}\right)^{\gamma_{2}} v\right\| & \leq\left\|\left(-A_{h}\right)^{-\gamma_{1}} S_{h}(t)\left(-A_{h}\right)^{\gamma_{2}} v\right\| \\
& +C \int_{0}^{t}\left\|S_{m}^{h}(\omega)(s)\left(-A_{h}\right)^{\gamma_{2}} v\right\| d s \\
& \leq C t^{\gamma_{1}-\gamma_{2}}\|v\|+C \int_{0}^{t} s^{-\gamma_{2}} d s\|v\| \\
& \leq C\left(t^{\gamma_{1}-\gamma_{2}}+t^{1-\gamma_{2}}\right)\|v\|, \tag{72}
\end{align*}
$$

and for $t \leq T$, since $\gamma_{1} \leq \gamma_{2} \leq 1, t^{1-\gamma_{2}} \leq C(T) t^{\gamma_{1}-\gamma_{2}}$. This ends the proof of (iii). The proof of (iv) is similar to that of (i).

The following lemma is similar to [30, Lemma 4], but its proof is easier than that of [30, Lemma 4] since we do not use any further lemmas in its proof.

Lemma 10 Under Assumption 1 and Assumption 3, the perturbed semigroup $S_{m}^{h}(\omega)$ satisfies the following stability property

$$
\left\|e^{\left(A_{h}+J_{m}^{h}(\omega)\right) \Delta t} \cdots e^{\left(A_{h}+J_{k}^{h}(\omega)\right) \Delta t}\left(-A_{h}\right)^{\nu}\right\|_{L(H)} \leq C t_{m+1-k}^{-\nu}, \quad 0 \leq \nu<1
$$

where $C$ is a positive constant independent of $m, k, h, \Delta t$ and the sample $\omega$.
Proof As in 24 we set

$$
\left\{\begin{array}{lr}
S_{m, k}^{h}(\omega):=e^{\left(A_{h}+J_{m}^{h}(\omega)\right) \Delta t} \cdots e^{\left(A_{h}+J_{k}^{h}(\omega)\right) \Delta t}, & \text { if } m \geq k \\
S_{m, k}^{h}(\omega):=\mathbf{I}, & \text { if } m<k
\end{array}\right.
$$

Using the telescoping sum, we can rewrite the composition of the perturbed semigroup $S_{m, k}^{h}(\omega)$ as follow

$$
\begin{align*}
S_{m, k}^{h}(\omega) & =e^{A_{h}\left(t_{m+1-k}\right)}+e^{A_{h}\left(t_{m+1}-t_{k+1}\right)}\left(e^{\left(A_{h}+J_{k}^{h}(\omega)\right) \Delta t}-e^{A_{h} \Delta t}\right) \\
& +\sum_{j=k+1}^{m} e^{A_{h}\left(t_{m+1}-t_{j+1}\right)}\left(e^{\left(A_{h}+J_{j}^{h}(\omega)\right) \Delta t}-e^{A_{h} \Delta t}\right) S_{j-1, k}^{h}(\omega) . \tag{73}
\end{align*}
$$

Multiplying both sides of 73 by $\left(-A_{h}\right)^{\nu}$ yields

$$
\begin{align*}
& S_{m, k}^{h}(\omega)\left(-A_{h}\right)^{\nu} \\
= & e^{A_{h} t_{m+1-k}}\left(-A_{h}\right)^{\nu}+e^{A_{h}\left(t_{m+1}-t_{k+1}\right)}\left(e^{\left(A_{h}+J_{k}^{h}(\omega)\right) \Delta t}-e^{A_{h} \Delta t}\right)\left(-A_{h}\right)^{\nu} \\
+ & \sum_{j=k+1}^{m} e^{A_{h}\left(t_{m+1}-t_{j+1}\right)}\left(e^{\left(A_{h}+J_{j}^{h}(\omega)\right) \Delta t}-e^{A_{h} \Delta t}\right) S_{j-1, k}^{h}(\omega)\left(-A_{h}\right)^{\nu} . \tag{74}
\end{align*}
$$

Using the variation of parameter formula 65), the fact that the Jacobian, the semigroup $S_{m}^{h}(\omega)(t)$ and $S_{h}(t)$ are uniformly bounded, we obtain

$$
\begin{equation*}
\left\|e^{\left(A_{h}+J_{m}^{h}(\omega)\right) \Delta t}-e^{A_{h} \Delta t}\right\|_{L(H)} \leq C \Delta t \tag{75}
\end{equation*}
$$

Taking the norm in both sides of (74) and using the stability property of $S_{h}(t)$ together with 75 gives

$$
\begin{align*}
& \left\|S_{m, k}^{h}(\omega)\left(-A_{h}\right)^{\nu}\right\|_{L(H)} \\
\leq & C t_{m+1-k}^{-\nu}+\left\|e^{A_{h}\left(t_{m+1}-t_{k+1}\right)}\right\|_{L(H)}\left\|\left(e^{\left(A_{h}+J_{k}^{h}(\omega)\right) \Delta t}-e^{A_{h} \Delta t}\right)\left(-A_{h}\right)^{\nu}\right\|_{L(H)} \\
+ & \sum_{j=k+1}^{m}\left\|e^{A_{h}\left(t_{m+1}-t_{j+1}\right)}\right\|_{L(H)}\left\|e^{\left(A_{h}+J_{m}^{h}(\omega)\right) \Delta t}-e^{A_{h} \Delta t}\right\|_{L(H)}\left\|S_{j-1, k}^{h}(\omega)\left(-A_{h}\right)^{\nu}\right\|_{L(H)} \\
\leq & C t_{m+1-k}^{-\nu}+C\left\|\left(e^{\left(A_{h}+J_{k}^{h}(\omega)\right) \Delta t}-e^{A_{h} \Delta t}\right)\left(-A_{h}\right)^{\nu}\right\|_{L(H)} \\
+ & C \Delta t \sum_{j=k+1}^{m}\left\|S_{j-1, k}^{h}(\omega)\left(-A_{h}\right)^{\nu}\right\|_{L(H)} \tag{76}
\end{align*}
$$

Rewriting (65) with $t=\Delta t$ yields

$$
\begin{equation*}
e^{\left(A_{h}+J_{m}^{h}(\omega)\right) \Delta t}-e^{A_{h} \Delta t}=\int_{0}^{\Delta t} e^{A_{h}(\Delta t-s)} J_{m}^{h}(\omega) e^{\left(A_{h}+J_{m}^{h}(\omega)\right) s} d s \tag{77}
\end{equation*}
$$

Multiplying both sides of 77 by $\left(-A_{h}\right)^{\nu}$ gives

$$
\begin{align*}
& \left(e^{\left(A_{h}+J_{m}^{h}(\omega)\right) \Delta t}-e^{A_{h} \Delta t}\right)\left(-A_{h}\right)^{\nu} \\
= & \int_{0}^{\Delta t} e^{A_{h}(\Delta t-s)} J_{m}^{h}(\omega) e^{\left(A_{h}+J_{m}^{h}(\omega)\right) s}\left(-A_{h}\right)^{\nu} d s . \tag{78}
\end{align*}
$$

Taking the norm in both sides of 78), using the stability property of $e^{A_{h} t}$, the uniform boundedness of $J_{m}^{h}$ and Lemma 9 (ii) with $\gamma_{1}=\nu$ gives

$$
\begin{align*}
& \left\|\left(e^{\left(A_{h}+J_{m}^{h}(\omega)\right) \Delta t}-e^{A_{h} \Delta t}\right)\left(-A_{h}\right)^{\nu}\right\|_{L(H)} \\
\leq & \int_{0}^{\Delta t}\left\|e^{A_{h}(\Delta t-s)}\right\|_{L(H)}\left\|J_{m}^{h}(\omega)\right\|_{L(H)}\left\|e^{\left(A_{h}+J_{m}^{h}(\omega)\right) s}\left(-A_{h}\right)^{\nu}\right\|_{L(H)} d s \\
\leq & C \int_{0}^{\Delta t} s^{-\nu} d s \leq C \Delta t^{1-\nu}=C t_{1}^{-\nu} \Delta t \tag{79}
\end{align*}
$$

Substituting (79) in (76) yields

$$
\begin{align*}
\left\|S_{m, k}^{h}(\omega)\left(-A_{h}\right)^{\nu}\right\|_{L(H)} & \leq C t_{m+1-k}^{-\nu}+C t_{1}^{-\nu} \Delta t\|\mathbf{I}\|_{L(H)} \\
& +C \Delta t \sum_{j=k+1}^{m}\left\|S_{j-1, k}^{h}(\omega)\left(-A_{h}\right)^{\nu}\right\|_{L(H)} \tag{80}
\end{align*}
$$

Applying the discrete Gronwall's inequality to 80 completes the proof of Lemma 10

Lemma 11 If Assumption 6 is fulfilled, then the following estimate holds

$$
\begin{equation*}
\left\|\left(-A_{h}\right)^{\frac{\beta-1}{2}} P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2}<C \tag{81}
\end{equation*}
$$

where $\beta$ is defined in Assumption 2 .
Proof The proof when $0 \leq \beta \leq 1$ can be found in [36, Proposition 4.1]. To prove (81) when $1<\beta \leq 2$, we use the definition of $\|\cdot\|_{\mathcal{L}_{2}(H)}$, apply Lemma 1 with $\alpha=\frac{\beta-1}{2}$ and Assumption 6 to get

$$
\begin{align*}
\left\|\left(-A_{h}\right)^{\frac{\beta-1}{2}} P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2} & =\sum_{i=1}^{\infty}\left\|\left(-A_{h}\right)^{\frac{\beta-1}{2}} P_{h} Q^{\frac{1}{2}} e_{i}\right\|^{2} \\
& \leq C \sum_{i=1}^{\infty}\left\|(-A)^{\frac{\beta-1}{2}} Q^{\frac{1}{2}} e_{i}\right\|^{2} \\
& =C\left\|(-A)^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2} \leq C . \tag{82}
\end{align*}
$$

Lemma 12 Under Assumption 3 and Assumption 7, for all $\omega \in \Omega$, the following estimates hold

$$
\begin{array}{r}
\left\|\left(G_{k}^{h}(\omega)\right)^{\prime}(u) v\right\| \leq C\|v\|, \quad u, v \in H, \quad k \in \mathbb{N}, \\
\left\|\left(-A_{h}\right)^{\frac{-\eta}{2}}\left(G_{k}^{h}(\omega)\right)^{\prime \prime}(u)\left(v_{1}, v_{2}\right)\right\| \leq C\left\|v_{1}\right\| \cdot\left\|v_{2}\right\|, \quad v_{1}, v_{2} \in H, \quad k \in \mathbb{N},
\end{array}
$$

for some $\eta \in[1,2)$.
Proof The proof follows the same lines as 36, Proposition 4.1]. Indeed since the Jacobian $J_{k}^{h}(\omega)$ is a linear operator, taking the differential in both sides of (41) yields

$$
\begin{equation*}
\left(G_{k}^{h}(\omega)\right)^{\prime}(u)=P_{h} F^{\prime}(u)-J_{k}^{h}(\omega)=P_{h} F^{\prime}(u)-P_{h} F^{\prime}\left(X_{k}^{h}(\omega)\right), \tag{85}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\left(G_{k}^{h}(\omega)\right)^{\prime}(u) v=P_{h} F^{\prime}(u) v-P_{h} F^{\prime}\left(X_{k}^{h}(\omega)\right) v, \quad v \in H \tag{86}
\end{equation*}
$$

Taking the norm in both sides of (86) and using Assumption 3 yields

$$
\begin{equation*}
\left\|\left(G_{k}^{h}(\omega)\right)^{\prime}(u) v\right\| \leq\left\|P_{h} F^{\prime}(u) v\right\|+\left\|P_{h} F^{\prime}\left(X_{k}^{h}(\omega)\right) v\right\| \leq C\|v\| \tag{87}
\end{equation*}
$$

which proves 83$)$. Taking the differential at the point $u \in H$ in both sides of (85) yields

$$
\begin{equation*}
\left(G_{k}^{h}(\omega)\right)^{\prime \prime}\left(v_{1}, v_{2}\right)=P_{h} F^{\prime \prime}(u)\left(v_{1}, v_{2}\right), \quad v_{1}, v_{2} \in H \tag{88}
\end{equation*}
$$

Taking the norm in both sides of (88), using 36, (70)] and Assumption 7 yields

$$
\begin{align*}
\left\|\left(-A_{h}\right)^{\frac{-\eta}{2}}\left(G_{k}^{h}(\omega)\right)^{\prime \prime}(u)\left(v_{1}, v_{2}\right)\right\| & =\left\|\left(-A_{h}\right)^{\frac{-\eta}{2}} P_{h} F^{\prime \prime}(u)\left(v_{1}, v_{2}\right)\right\| \\
& \leq\left\|(-A)^{\frac{-\eta}{2}} F^{\prime \prime}(u)\left(v_{1}, v_{2}\right)\right\| \\
& \leq C\left\|v_{1}\right\| \cdot\left\|v_{2}\right\| . \tag{89}
\end{align*}
$$

Gathering our preparatory results, we are now ready to prove our main result in Theorem 9 .

### 3.2 Proof of Theorem 9

Using the standard technique in the error analysis, we split the fully discrete error in two terms

$$
\begin{aligned}
\left\|X\left(t_{m}\right)-X_{m}^{h}\right\|_{L^{2}(\Omega, H)} & \leq\left\|X\left(t_{m}\right)-X^{h}\left(t_{m}\right)\right\|_{L^{2}(\Omega, H)}+\left\|X^{h}\left(t_{m}\right)-X_{m}^{h}\right\|_{L^{2}(\Omega, H)} \\
& =: \operatorname{err}_{0}+e \operatorname{err}_{1} .
\end{aligned}
$$

Note that the space error $e r r_{0}$ is estimated by Lemma8. It remains to estimate the time error $\operatorname{err}_{1}$. We estimate the time error $\operatorname{err}_{1}$ for both $0 \leq \beta<1$ and $1 \leq \beta<2$ separately in the following two subsections.

### 3.2.1 Estimate of the time error for $0 \leq \beta<1$

We recall that the exact solution at $t_{m}$ of the semidiscrete problem is given by

$$
\begin{align*}
X^{h}\left(t_{m}\right) & =e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} X^{h}\left(t_{m-1}\right) \\
& +\int_{t_{m-1}}^{t_{m}} e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)} G_{m-1}^{h}\left(X^{h}(s)\right) d s \\
& +\int_{t_{m-1}}^{t_{m}} e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)} P_{h} B\left(X^{h}(s)\right) d W(s) . \tag{90}
\end{align*}
$$

We also recall that the numerical solution at $t_{m}$ given by 49) can be rewritten as

$$
\begin{align*}
X_{m}^{h} & =e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} X_{m-1}^{h} \\
& +\int_{t_{m-1}}^{t_{m}} e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)} G_{m-1}^{h}\left(X_{m-1}^{h}\right) d s \\
& +\int_{t_{m-1}}^{t_{m}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} P_{h} B\left(X_{m-1}^{h}\right) d W(s) . \tag{91}
\end{align*}
$$

If $m=1$ then it follows from (90) and (91) that

$$
\begin{align*}
& \left\|X\left(t_{1}\right)-X_{1}^{h}\right\|_{L^{2}(\Omega, H)} \\
\leq & \left\|\int_{0}^{\Delta t} e^{\left(A_{h}+J_{0}^{h}\right)(\Delta t-s)}\left[G_{0}^{h}\left(X^{h}(s)\right)-G_{0}^{h}\left(X_{0}^{h}\right)\right] d s\right\|_{L^{2}(\Omega, H)} \\
+ & \left\|\int_{0}^{\Delta t}\left[e^{\left(A_{h}+J_{0}^{h}\right)(\Delta t-s)} P_{h} B\left(X^{h}(s)\right)-e^{\left(A_{h}+J_{0}^{h}\right) \Delta t} P_{h} B\left(X_{0}^{h}\right)\right] d W(s)\right\|_{L^{2}(\Omega, H)} \\
= & I+I I . \tag{92}
\end{align*}
$$

Using Lemma 5, Lemma 6, 22, and the fact that $X_{0}^{h}=P_{h} X_{0}$ we obtain the following estimate

$$
\begin{equation*}
I \leq C \Delta t \tag{93}
\end{equation*}
$$

Using the Itô's isometry property, triangle inequality, Lemma 5. Assumption 4. (22) and the fact that $(a+b)^{2} \leq 2 a^{2}+2 b^{2}$ for all $a, b \in \mathbb{R}, \sqrt{u+v} \leq \sqrt{u}+\sqrt{v}$ for all positive real numbers $u$ and $v$, we obtain the following estimate

$$
\begin{equation*}
I I \leq C \Delta t^{1 / 2} \tag{94}
\end{equation*}
$$

Inserting (94) and (93) in (92) yields

$$
\begin{equation*}
\left\|X^{h}\left(t_{1}\right)-X_{1}^{h}\right\|_{L^{2}(\Omega, H)} \leq C \Delta t^{1 / 2} \tag{95}
\end{equation*}
$$

For $m \geq 2$, we iterate the mild solution at $t_{m}$ by substituting $X^{h}\left(t_{j}\right), j=$ $1,2, . ., m-1$ in 90 by their mild forms. We obtain

$$
\begin{align*}
& X^{h}\left(t_{m}\right) \\
& =e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{0}^{h}\right) \Delta t} X^{h}(0)+\int_{t_{m-1}}^{t_{m}} e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)} G_{m-1}^{h}\left(X^{h}(s)\right) d s \\
& +\int_{t_{m-1}}^{t_{m}} e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)} P_{h} B\left(X^{h}(s)\right) d W(s)  \tag{96}\\
& +\sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} G_{m-k-2}^{h}\left(X^{h}(s)\right) d s \\
& +\sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} P_{h} B\left(X^{h}(s)\right) d W(s) .
\end{align*}
$$

Similarly, for $m \geq 2$, we iterate the numerical solution (91) at $t_{m}$ by substituting $X_{j}^{h}, j=1,2, . ., m-1$ only in the first term of 91 by their expressions. We obtain
$X_{m}^{h}$

$$
\begin{align*}
& =e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{0}^{h}\right) \Delta t} X^{h}(0)+\int_{t_{m-1}}^{t_{m}} e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)} G_{m-1}^{h}\left(X_{m-1}^{h}\right) d s \\
& +\int_{t_{m-1}}^{t_{m}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} P_{h} B\left(X_{m-1}^{h}\right) d W(s) \\
& +\sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}  \tag{97}\\
& G_{m-k-2}^{h}\left(X_{m-k-2}^{h}\right) d s \\
& +\sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t} P_{h} B\left(X_{m-k-2}^{h}\right) d W(s) .
\end{align*}
$$

Therefore, it follows from (96) and (97) and the triangle inequality that

$$
\begin{equation*}
\frac{1}{4}\left\|X^{h}\left(t_{m}\right)-X_{m}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \leq I I I+I V+V+V I \tag{98}
\end{equation*}
$$

where

$$
\begin{aligned}
I I I & =\left\|\int_{t_{m-1}}^{t_{m}} e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)}\left[G_{m-1}^{h}\left(X^{h}(s)\right)-G_{m-1}^{h}\left(X_{m-1}^{h}\right)\right] d s\right\|_{L^{2}(\Omega, H)}^{2}, \\
I V & =\left\|\int_{t_{m-1}}^{t_{m}}\left(e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)} P_{h} B\left(X^{h}(s)\right)-e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} P_{h} B\left(X_{m-1}^{h}\right)\right) d W(s)\right\|_{L^{2}(\Omega, H)}^{2},
\end{aligned}
$$

$$
\begin{aligned}
V= & \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \left(G_{m-k-2}^{h}\left(X^{h}(s)\right)-G_{m-k-2}^{h}\left(X_{m-k-2}^{h}\right)\right) d s \|_{L^{2}(\Omega, H)}^{2}, \\
V I= & \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{n-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} \\
& \left(e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} P_{h} B\left(X^{h}(s)\right)-e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} P_{h} B\left(X_{m-k-2}^{h}\right)\right) d W(s) \|_{L^{2}(\Omega, H)}^{2} .
\end{aligned}
$$

Using Holder's inequality, the stability property of $S_{m}^{h}(t)$, Lemma 6, the triangle inequality and the fact that $(a+b)^{2} \leq 2 a^{2}+2 b^{2}$ yields

$$
\begin{aligned}
& I I I \\
\leq & \left(\int_{t_{m-1}}^{t_{m}}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)}\left(G_{m-1}^{h}\left(X^{h}(s)\right)-G_{m-1}^{h}\left(X_{m-1}^{h}\right)\right)\right\|_{L^{2}(\Omega, H)} d s\right)^{2} \\
\leq & \left(\int_{t_{m-1}}^{t_{m}}\left(\mathbb{E}\left[\left\|e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)}\right\|_{L(H)}^{2}\left\|\left(G_{m-1}^{h}\left(X^{h}(s)\right)-G_{m-1}^{h}\left(X_{m-1}^{h}\right)\right)\right\|^{2}\right]\right)^{1 / 2} d s\right)^{2} \\
\leq & C\left(\int_{t_{m-1}}^{t_{m}}\left(\mathbb{E}\left\|G_{m-1}^{h}\left(X^{h}(s)\right)-G_{m-1}^{h}\left(X_{m-1}^{h}\right)\right\|^{2}\right)^{1 / 2} d s\right)^{2} \\
\leq & \left.\left.C\left(\int_{t_{m-1}}^{t_{m}}\left(\mathbb{E} \| X^{h}(s)-X_{m-1}^{h}\right) \|^{2}\right)^{1 / 2} d s\right)^{2}=C\left(\int_{t_{m-1}}^{t_{m}} \| X^{h}(s)-X_{m-1}^{h}\right) \|_{L^{2}(\Omega, H)} d s\right)^{2} \\
\leq & C\left(\int_{t_{m-1}}^{t_{m}}\left\|X^{h}(s)-X^{h}\left(t_{m-1}\right)\right\|_{L^{2}(\Omega, H)} d s\right)^{2}+C \Delta t^{2}\left\|X^{h}\left(t_{m-1}\right)-X_{m-1}^{h}\right\|_{L^{2}(\Omega, H)}^{2} .(99)
\end{aligned}
$$

Using Lemma 4 it follows from (99) that

$$
\begin{align*}
I I I & \leq C\left(\int_{t_{m-1}}^{t_{m}}\left(s-t_{m-1}\right)^{\beta / 2} d s\right)^{2}+C \Delta t^{2}\left\|X^{h}\left(t_{m-1}\right)-X_{m-1}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \\
& \leq C \Delta t^{2+\beta}+C \Delta t^{2}\left\|X^{h}\left(t_{m-1}\right)-X_{m-1}^{h}\right\|_{L^{2}(\Omega, H)}^{2} . \tag{100}
\end{align*}
$$

Since the estimates of $I V$ and $V I$ are much more complicated, let us estimate $V$ first. We use inequality $(a+b)^{2} \leq 2 a^{2}+2 b^{2}$ to split $V$ into two terms. This yields

$$
\begin{align*}
V \leq & 2 \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \left(G_{m-k-2}^{h}\left(X^{h}(s)\right)-G_{m-k-2}^{h}\left(X^{h}\left(t_{m-k-2}\right)\right)\right) d s \|_{L^{2}(\Omega, H)}^{2} \\
+ & 2 \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \left(G_{m-k-2}^{h}\left(X^{h}\left(t_{m-k-2}\right)\right)-G_{m-k-2}^{h}\left(X_{m-k-2}^{h}\right)\right) d s \|_{L^{2}(\Omega, H)}^{2} \\
= & 2 V_{1}+2 V_{2} . \tag{101}
\end{align*}
$$

Using triangle inequality gives

$$
\begin{align*}
V_{1}= & \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \left(G_{m-k-2}^{h}\left(X^{h}(s)\right)-G_{m-k-2}^{h}\left(X^{h}\left(t_{m-k-2}\right)\right)\right) d s \|_{L^{2}(\Omega, H)}^{2} \\
\leq & m \sum_{k=0}^{m-2} \| \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \left(G_{m-k-2}^{h}\left(X^{h}(s)\right)-G_{m-k-2}^{h}\left(X_{m-k-2}^{h}\right)\right) d s \|_{L^{2}(\Omega, H)}^{2} \\
\leq & m \sum_{k=0}^{m-2}\left(\int _ { t _ { m - k - 2 } } ^ { t _ { m - k - 1 } } \left(\mathbb { E } \left[\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\right\|_{L(H)}^{2}\right.\right.\right. \\
& \| e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right) \|_{L(H)}^{2}} \\
& \left.\left.\left.\times\left\|\left(G_{m-k-2}^{h}\left(X^{h}(s)\right)-G_{m-k-2}^{h}\left(X^{h}\left(t_{m-k-2}\right)\right)\right)\right\|^{2}\right]\right)^{1 / 2} d s\right)^{2} \tag{102}
\end{align*}
$$

Using Lemma 10 with $\nu=0$ and Lemma 5 yields

$$
V_{1} \leq C m \sum_{k=0}^{m-2}\left(\int_{t_{m-k-2}}^{t_{m-k-1}}\left(\mathbb{E}\left\|G_{m-k-2}^{h}\left(X^{h}(s)\right)-G_{m-k-2}^{h}\left(X^{h}\left(t_{m-k-2}\right)\right)\right\|^{2}\right)^{1 / 2} d s\right)^{2}(1.03)
$$

Using Lemma 6 and Lemma 4 , it follows from (103) that

$$
\begin{align*}
V_{1} & \leq C m \sum_{k=0}^{m-2}\left(\int_{t_{m-k-2}}^{t_{m-k-1}}\left(\mathbb{E}\left\|X^{h}(s)-X^{h}\left(t_{m-k-2}\right)\right\|^{2}\right)^{1 / 2} d s\right)^{2} \\
& =C m \sum_{k=0}^{m-2}\left(\int_{t_{m-k-2}}^{t_{m-k-1}}\left\|X^{h}(s)-X^{h}\left(t_{m-k-2}\right)\right\|_{L^{2}(\Omega, H)} d s\right)^{2} \\
& \leq m C \sum_{k=0}^{m-2}\left(\int_{t_{m-k-2}}^{t_{m-k-1}}\left(s-t_{m-k-2}\right)^{\beta / 2} d s\right)^{2} \leq C \Delta t^{\beta} \tag{104}
\end{align*}
$$

Using triangle inequality, Lemma 5, Lemma 10 with $\nu=0$, Lemma 6 and Holder's inequality yields

$$
\begin{align*}
V_{2}= & \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \left(G_{m-k-2}\left(X^{h}\left(t_{m-k-2}\right)\right)-G_{m-k-2}\left(X_{m-k-2}^{h}\right)\right) d s \|_{L^{2}(\Omega, H)}^{2} \\
\leq & m \sum_{k=0}^{m-2} \| \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-2}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \left(G_{m-k-2}\left(X^{h}\left(t_{m-k-2}\right)\right)-G_{m-k-2}\left(X_{m-k-2}^{h}\right)\right) d s \|_{L^{2}(\Omega, H)}^{2} \\
\leq & C m \Delta t \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left\|X^{h}\left(t_{m-k-2}\right)-X_{m-k-2}^{h}\right\|_{L^{2}(\Omega, H)}^{2} d s \\
\leq & C \sum_{k=0}^{m-2} \Delta t\left\|X^{h}\left(t_{m-k-2}\right)-X_{m-k-2}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \\
= & C \Delta t \sum_{k=0}^{m-2}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} . \tag{105}
\end{align*}
$$

Substituting (105) and 104 in 101 yields

$$
\begin{equation*}
V \leq C \Delta t^{\beta}+C \sum_{k=0}^{m-2} \Delta t\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \tag{106}
\end{equation*}
$$

To estimate $V I$, we use the triangle inequality to split it in two terms

$$
\begin{align*}
& V I \\
\leq & 2 \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& {\left[P_{h} B\left(X^{h}(s)\right)-P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)\right] d W(s) \|_{L^{2}(\Omega, H)}^{2} } \\
+ & 2 \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} \\
& {\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t} P_{h} B\left(X_{m-k-2}^{h}\right)\right] d W(s) \|_{L^{2}(\Omega, H)}^{2} } \\
= & 2 V I_{1}+2 V I_{2} . \tag{107}
\end{align*}
$$

Since the expectation of the cross-product vanishes, Using Cauchy-Schwartz inequality, it follows that

$$
\begin{aligned}
& V I_{1} \\
= & \mathbb{E}\left[\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right. \\
& {\left.\left[P_{h} B\left(X^{h}(s)\right)-P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)\right] d W(s) \|^{2}\right] } \\
= & \sum_{k=0}^{m-2} \mathbb{E}\left[\| e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right. \\
& \left.\int_{t_{m-k-2}}^{t_{m-k-1}}\left[P_{h} B\left(X^{h}(s)\right)-P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)\right] d W(s) \|^{2}\right] \\
\leq & \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
\times & \left.\mathbb{E}\left\|\int_{t_{m-k-2}}^{t_{m-k-1}}\left[P_{h} B\left(X^{h}(s)\right)-P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)\right] d W(s)\right\|^{\frac{1}{2}}\right)^{\frac{1}{2}} .
\end{aligned}
$$

Using the Burkhölder-Davis-Gundy inequality ( 20, Lemma 5.1]), Lemma 10 with $\nu=0$, Assumption 4 and the fact that $S_{k}^{h}(\omega)$ is uniformly bounded (independently of $h, k$ and the sample $\omega$ ) yields

$$
\begin{align*}
V I_{1} & \leq C \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|\int_{t_{m-k-2}}^{t_{m-k-1}}\left[P_{h} B\left(X^{h}(s)\right)-P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)\right] d W(s)\right\|^{4}\right)^{\frac{1}{2}} \\
& \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left\|P_{h} B\left(X^{h}(s)\right)-P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)\right\|_{L_{2}^{0}}^{2} d s \\
& \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left\|X^{h}(s)-X^{h}\left(t_{m-k-2}\right)\right\|_{L^{2}(\Omega, H)}^{2} d s . \tag{108}
\end{align*}
$$

Applying Lemma 4 it follows from (108) that

$$
\begin{equation*}
V I_{1} \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(s-t_{m-k-2}\right)^{\beta} d s \leq C \Delta t^{\beta} \tag{109}
\end{equation*}
$$

Using the inequality $(a+b)^{2} \leq 2 a^{2}+2 b^{2}$, we split $V I_{2}$ in two terms

$$
\begin{align*}
& V I_{2} \\
= & \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right. \\
& \left.P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t} P_{h} B\left(X_{m-k-2}^{h}\right)\right] d W(s) \|_{L^{2}(\Omega, H)}^{2} \\
\leq & 2 \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} \\
& {\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right) d W(s) \|_{L^{2}(\Omega, H)}^{2} } \\
+ & 2 \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t} \\
& {\left[P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)-P_{h} B\left(X_{m-k-2}^{h}\right)\right] d W(s) \|_{L^{2}(\Omega, H)}^{2} } \\
= & 2 V I_{21}+2 V I_{22} \tag{110}
\end{align*}
$$

Since the expectation of the cross-product vanishes, inserting an appropriate power of $A_{h}$ and using Cauchy-Schwartz inequality yields

$$
\begin{aligned}
& V I_{21} \\
= & \mathbb{E}\left[\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\right. \\
& {\left.\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right) d W(s) \|^{2}\right] } \\
= & \sum_{k=0}^{m-2} \mathbb{E}\left[\| e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\right. \\
& \left.\int_{t_{m-k-2}}^{t_{m-k-1}}\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right) d W(s) \|^{2}\right] \\
= & \sum_{k=0}^{m-2} \mathbb{E}\left[\| e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{\beta}{2}}\right. \\
& \left.\int_{t_{m-k-2}}^{t_{m-k-1}}\left(-A_{h}\right)^{-\frac{\beta}{2}}\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right) d W(s) \|^{2}\right] \\
\leq & \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{\beta}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
\times & \left(\mathbb{E}\left\|\int_{t_{m-k-2}}^{t_{m-k-1}}\left(-A_{h}\right)^{-\frac{\beta}{2}}\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right) d W(s)\right\|^{4}\right)
\end{aligned}
$$

Using the Burkhölder-Davis-Gundy inequality yields

$$
\begin{align*}
& V I_{21}  \tag{111}\\
& \leq \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{\beta}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
& \times \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left\|\left(-A_{h}\right)^{-\frac{\beta}{2}}\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)\right\|_{L_{2}^{0}(H)}^{2} d s .
\end{align*}
$$

As $S_{k}^{h}(t)$ is a semigroup, we obviously have

$$
\begin{equation*}
S_{k}^{h}(t+s)=S_{k}^{h}(t) S_{k}^{h}(s), \quad t, s \geq 0 \tag{112}
\end{equation*}
$$

Using relation 112, Lemma 9 (i) with $\gamma_{1}=\frac{\beta}{2}$ and $\gamma_{2}=0$ and Lemma 10 with $\nu=\frac{\beta}{2}$ in 111 allows to have

$$
\begin{align*}
V I_{21} \leq & \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{\beta}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
& \times \mathbb{E}\left[\left\|\left(-A_{h}\right)^{\frac{-\beta}{2}}\left(\mathbf{I}-S_{m-k-2}^{h}\left(s-t_{m-k-2}\right)\right)\right\|_{L(H)}^{2}\left\|S_{m-k-2}^{h}\left(t_{m-k-1}-s\right)\right\|_{L(H)}^{2}\right. \\
& \left.\times\left\|P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)\right\|_{L_{2}^{0}}^{2}\right] d s  \tag{113}\\
\leq & C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} t_{k+1}^{-\beta}\left(s-t_{m-k-2}\right)^{\beta}\left\|S_{m-k-2}^{h}\left(t_{m-k-1}-s\right)\right\|_{L(H)}^{2}  \tag{114}\\
& \times\left\|B\left(X^{h}\left(t_{m-k-2}\right)\right)\right\|_{L_{2}^{0}}^{2} d s .
\end{align*}
$$

Using Assumption 4 and the fact that the random perturbed semigroup $S_{k}^{h}$ is uniformly bounded independently of $k, h$ and the sample $\omega$, it follows from (113) that

$$
\begin{align*}
V I_{21} & \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} t_{k+1}^{-\beta}\left(s-t_{m-k-2}\right)^{\beta} d s \\
& \leq C \Delta t^{\beta} \sum_{k=0}^{m-2} t_{k+1}^{-\beta} \Delta t \leq C \Delta t^{\beta} . \tag{115}
\end{align*}
$$

Let us estimate $V I_{22}$

$$
\begin{align*}
V I_{22}:= & \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t} \\
& {\left[P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)-P_{h} B\left(X_{m-k-2}^{h}\right)\right] d W(s) \|_{L^{2}(\Omega . H)}^{2} } \tag{116}
\end{align*}
$$

Following the same lines as in the estimate of $V I_{1}$, the following estimate holds for $V I_{22}$

$$
\begin{align*}
V I_{22} & \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left\|X^{h}\left(t_{m-k-2}\right)-X_{m-k-2}^{h}\right\|_{L^{2}(\Omega, H)}^{2} d s \\
& \leq C \sum_{k=0}^{m-2} \Delta t\left\|X^{h}\left(t_{m-k-2}\right)-X_{m-k-2}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \\
& =C \Delta t \sum_{k=0}^{m-2}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} . \tag{117}
\end{align*}
$$

Inserting (117) and 115 in 110 gives

$$
\begin{equation*}
V I_{2} \leq C \Delta t^{\beta}+C \Delta t \sum_{k=0}^{m-2}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} . \tag{118}
\end{equation*}
$$

Substituting 109 and 118 in 107 yields

$$
\begin{equation*}
V I \leq C \Delta t^{\beta}+C \Delta t \sum_{k=0}^{m-2}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \tag{119}
\end{equation*}
$$

Following the same lines as for the estimate of $V I$, we obtain

$$
\begin{align*}
I V= & \| \int_{t_{m-1}}^{t_{m}}\left[e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)} P_{h} B\left(X^{h}(s)\right)\right. \\
& \left.-e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} P_{h} B\left(X_{m-1}^{h}\right)\right] d W(s) \|_{L^{2}(\Omega, H)}^{2} \\
\leq & C \Delta t^{\beta}+C \Delta t\left\|X^{h}\left(t_{m-1}\right)-X_{m-1}^{h}\right\|_{L^{2}(\Omega, H)}^{2} . \tag{120}
\end{align*}
$$

Gathering estimates of $I I I, I V, V$ and $V I$ in yields

$$
\left\|X^{h}\left(t_{m}\right)-X_{m}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \leq C \Delta t^{\beta}+C \Delta t \sum_{k=0}^{m-1}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} .(121)
$$

Applying the discrete Gronwall lemma to (121) yields

$$
\begin{equation*}
\left\|X^{h}\left(t_{m}\right)-X_{m}^{h}\right\|_{L^{2}(\Omega, H)} \leq C \Delta t^{\beta / 2} \tag{122}
\end{equation*}
$$

Using Lemma 8 together with the estimate 122 completes the proof of Theorem 9 for $0 \leq \beta<1$.

### 3.2.2 Estimate of the time error for $1 \leq \beta \leq 2$

Note that the estimates of $I I I$ and $V$ in Section 3.2 .1 hold for $\beta \in[1,2]$ and due to the time regularity in Lemma 4, we obtain from (106) and (100)

$$
\begin{equation*}
I I I+V \leq C \Delta t+C \Delta t \sum_{k=0}^{m-1}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \tag{123}
\end{equation*}
$$

We only need to re-estimate $I V$ and $V I$. We will only estimate $V I$ in details since the the estimate of $I V$ is similar to that of $V I$. Let us recall that using triangle inequality we obtain

$$
\begin{equation*}
V I \leq 2 V I_{1}+2 V I_{2} \tag{124}
\end{equation*}
$$

where $V I_{1}$ and $V I_{2}$ are defined by 107 in Section 3.2.1. Applying Lemma 4 it follows from 108) that

$$
\begin{equation*}
V I_{1} \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(s-t_{m-k-2}\right) d s \leq C \Delta t \tag{125}
\end{equation*}
$$

Using the inequality $(a+b)^{2} \leq 2 a^{2}+2 b^{2}$, we split $V I_{2}$ in two terms

$$
\begin{equation*}
V I_{2} \leq V I_{21}+V I_{22} \tag{126}
\end{equation*}
$$

where $V I_{21}$ and $V I_{22}$ are given by 110 in Subsection 3.2.1. We recall that from (117) the following estimate holds for $V I_{22}$

$$
\begin{equation*}
V I_{22} \leq C \Delta t \sum_{k=0}^{m-2}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \tag{127}
\end{equation*}
$$

Since the expectation of the cross product vanishing, inserting an appropriate power of $A_{h}$ and using Cauchy-Schwartz inequality yields

$$
\begin{aligned}
& V I_{21} \\
= & \mathbb{E} \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} \\
& {\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right) d W(s) \|^{2} } \\
= & \sum_{k=0}^{m-2} \mathbb{E} \| e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} \\
& \int_{t_{m-k-2}}^{t_{m-k-1}}\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right) d W(s) \|^{2} \\
= & \sum_{k=0}^{m-2} \mathbb{E} \| e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{1-\gamma}{2}} \\
& \int_{t_{m-k-2}}^{t_{m-k-1}}\left(-A_{h}\right)^{\frac{-1+\gamma}{2}}\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right) d W(s) \|^{2} \\
\leq & \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{1-\gamma}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
\times & \left(\mathbb{E}\left\|\int_{t_{m-k-2}}^{t_{m-k-1}}\left(-A_{h}\right)^{\frac{-1+\gamma}{2}}\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right) d W(s)\right\|^{4}\right)^{\frac{1}{2}} .
\end{aligned}
$$

Using Burkhölder-Davis-Gundy inequality and the triangle inequality, we split $V I_{21}$ in two parts as

$$
\begin{align*}
& \leq \sum_{k=0}^{V I_{21}} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \ldots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{1-\gamma}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
& \times \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left\|\left(-A_{h}\right)^{\frac{-1+\gamma}{2}}\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)\right\|_{L_{2}^{0}}^{2} d s \\
& \leq 2 \sum_{k=0}^{m-1}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{1-\gamma}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
& \times \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left\|\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right]\left[P_{h} B\left(X^{h}\left(t_{m-k-2}\right)\right)-P_{h} B\left(X\left(t_{m-k-2}\right)\right)\right]\right\|_{L_{2}^{0}}^{2} d s \\
& \times 2 \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{1-\gamma}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
& \times \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left\|\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right] P_{h} B\left(X\left(t_{m-k-2}\right)\right)\right\|_{L_{2}^{0}}^{2} d s \\
& :=2 V I_{211}+2 V I_{212} . \tag{128}
\end{align*}
$$

Using Lemma 5 and Assumption 4 yields

$$
\begin{align*}
& V I_{211}  \tag{129}\\
\leq & C \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{1-\gamma}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
\times & \int_{t_{m-k-2}}^{t_{m-k-1}}\left[\left\|\left(-A_{h}\right)^{\frac{-1+\gamma}{2}}\left(e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right)\right\|_{L(H)}^{2}\left\|X\left(t_{m-k-2}\right)-X^{h}\left(t_{m-k-2}\right)\right\|^{2}\right] d s
\end{align*}
$$

Using Lemma 8 and Lemma 10 with $\nu=0$, we obtain

$$
\begin{align*}
V I_{211} & \leq C \sum_{k=0}^{m-2} t_{k+1}^{-1+\gamma} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left\|X\left(t_{m-k-2}\right)-X^{h}\left(t_{m-k-2}\right)\right\|^{2} d s \\
& \leq C h^{2 \beta} . \tag{130}
\end{align*}
$$

Inserting an appropriated power of $-A_{h}$ yields the following estimate

$$
\begin{align*}
& V I_{212} \\
\leq & \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{1-\gamma}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
\times & \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left[\left\|\left(-A_{h}\right)^{\frac{-1+\gamma}{2}}\left(e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right)\left(-A_{h}\right)^{\frac{-\gamma}{2}}\right\|_{L(H)}^{2}\right. \\
& \left.\times\left\|\left(-A_{h}\right)^{\frac{\gamma}{2}} P_{h} B\left(X\left(t_{m-k-2}\right)\right)\right\|_{L_{2}^{0}}^{2}\right] d s . \tag{131}
\end{align*}
$$

Using Lemma 10 with $\nu=\frac{1-\gamma}{2}$ in 131, we obatin

$$
\begin{align*}
& V I_{212} \\
\leq & C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} t_{k+1}^{-1+\gamma} \mathbb{E}\left[\|\left(-A_{h}\right)^{\frac{1+\gamma}{2}}\left(e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right)\right. \\
& \left.\times\left(-A_{h}\right)^{\frac{-\gamma}{2}}\left\|_{L(H)}^{2}\right\|\left(-A_{h}\right)^{\frac{\gamma}{2}} P_{h} B\left(X\left(t_{m-k-2}\right)\right) \|_{L_{2}^{0}}^{2}\right] d s \\
\leq & C \sum_{k=0}^{m-2} t_{k+1}^{-1+\gamma} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left[\|\left(-A_{h}\right)^{\frac{-1+\gamma}{2}}\left[e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right.\right. \\
& \left.\left.\quad-e^{\left(A_{h}+J_{m-k-2}^{h}\right) \Delta t}\right]\left(-A_{h}\right)^{-\frac{\gamma}{2}}\left\|_{L(H)}^{2}\right\|\left(-A_{h}\right)^{\frac{\gamma}{2}} P_{h} B\left(X\left(t_{m-k-2}\right)\right) \|_{L_{2}^{0}}^{2}\right] d s . \tag{132}
\end{align*}
$$

Using relation 112, Lemma (i) with $\gamma_{1}=\frac{1-\gamma}{2}$ and $\gamma_{2}=\frac{\gamma}{2}$ yields

$$
\begin{align*}
V I_{212} \leq & C \sum_{k=0}^{m-2} t_{k+1}^{-1+\gamma} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left[\|\left(-A_{h}\right)^{\frac{-1+\gamma}{2}} S_{m-k-2}^{h}\left(t_{m-k-1}-s\right)\right. \\
& \left.\left(S_{m-k-2}^{h}\left(s-t_{m-k-2}\right)-\mathbf{I}\right)\left(-A_{h}\right)^{-\frac{\gamma}{2}}\left\|_{L(H)}^{2}\right\|\left(-A_{h}\right)^{\frac{\gamma}{2}} P_{h} B\left(X\left(t_{m-k-2}\right)\right) \|_{L_{2}^{0}}^{2}\right] d s \\
\leq & C \sum_{k=0}^{m-2}\left(t _ { k + 1 } ^ { - 1 + \gamma } \int _ { t _ { m - k - 2 } } ^ { t _ { m - k - 1 } } \mathbb { E } \left[\left\|\left(-A_{h}\right)^{\frac{-1+\gamma}{2}} S_{m-k-2}^{h}\left(t_{m-k-1}-s\right)\left(-A_{h}\right)^{\frac{1-\gamma}{2}}\right\|_{L(H)}^{2}\right.\right. \\
\times & \left\|\left(-A_{h}\right)^{\frac{-1+\gamma}{2}}\left(S_{m-k-2}^{h}\left(s-t_{m-k-2}\right)-\mathbf{I}\right)\left(-A_{h}\right)^{-\frac{\gamma}{2}}\right\|_{L(H)}^{2} \\
& \left.\left.\left\|\left(-A_{h}\right)^{\frac{\gamma}{2}} P_{h} B\left(X\left(t_{m-k-2}\right)\right)\right\|_{L_{2}^{0}}^{2}\right] d s\right) \\
\leq & C \sum_{k=0}^{m-2} t_{k+1}^{-1+\gamma} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(s-t_{m-k-2}\right) \mathbb{E}\left\|\left(-A_{h}\right)^{\frac{\gamma}{2}} P_{h} B\left(X\left(t_{m-k-2}\right)\right)\right\|_{L_{2}^{0}}^{2} d s \\
\leq & C \Delta t^{2} \sum_{k=0}^{m-2} t_{k+1}^{-1+\gamma} \mathbb{E}\left\|\left(-A_{h}\right)^{\frac{\gamma}{2}} P_{h} B\left(X\left(t_{m-k-2}\right)\right)\right\|_{L_{2}^{0}}^{2} . \tag{133}
\end{align*}
$$

Using the definition of the $L_{2}^{0}$ norm, Assumption 5, Lemma 1 . Theorem 1 and estimate (9), we obtain

$$
\begin{align*}
\mathbb{E}\left\|\left(-A_{h}\right)^{\frac{\gamma}{2}} P_{h} B\left(X\left(t_{m-k-2}\right)\right)\right\|_{L_{2}^{0}}^{2} & =\mathbb{E}\left[\sum_{i=0}^{\infty}\left\|\left(-A_{h}\right)^{\frac{\gamma}{2}} P_{h} B\left(X\left(t_{m-k-2}\right)\right) Q^{1 / 2} e_{i}\right\|^{2}\right] \\
& \leq C \mathbb{E}\left[\sum_{i=0}^{\infty}\left\|(-A)^{\frac{\gamma}{2}} B\left(X\left(t_{m-k-2}\right)\right) Q^{1 / 2} e_{i}\right\|^{2}\right] \\
& =C \mathbb{E} \|(-A)^{\frac{\gamma}{2}} B\left(X\left(t_{m-k-2}\right) \|_{L_{2}^{0}}^{2}\right. \\
& \leq C \mathbb{E}\left(1+\left\|(-A)^{\frac{\gamma}{2}} X\left(t_{m-k-2}\right)\right\|^{2}\right) \\
& \leq C\left(1+\mathbb{E}\left(\left\|X_{0}\right\|_{\gamma}^{2}\right)\right)<\infty . \tag{134}
\end{align*}
$$

Substituting 134 in 133) yields

$$
\begin{equation*}
V I_{212} \leq C \Delta t^{2} \sum_{k=0}^{m-2} t_{k+1}^{-1+\gamma} \leq C \Delta t \tag{135}
\end{equation*}
$$

Inserting 135 and 130 in 128 gives

$$
\begin{equation*}
V I_{21} \leq C h^{2 \beta}+C \Delta t \tag{136}
\end{equation*}
$$

Inserting (136) and 127) in 126 gives

$$
\begin{equation*}
V I_{2} \leq C h^{2 \beta}+C \Delta t+C \Delta t \sum_{k=0}^{m-2}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \tag{137}
\end{equation*}
$$

Substituting estimates of $V I_{2} \quad 137$ and $V I_{1} 125$ in 124) yields

$$
\begin{equation*}
V I \leq C h^{2 \beta}+C \Delta t+C \Delta t \sum_{k=0}^{m-2}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \tag{138}
\end{equation*}
$$

Following the same lines as for $V I$, we obtain

$$
\begin{align*}
I V= & \| \int_{t_{m-1}}^{t_{m}}\left[e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)} P_{h} B\left(X^{h}(s)\right)\right. \\
& \left.-e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} P_{h} B\left(X_{m-1}^{h}\right)\right] d W(s) \|_{L^{2}(\Omega, H)}^{2} \\
\leq & C h^{2 \beta}+C \Delta t+C \Delta t\left\|X^{h}\left(t_{m-1}\right)-X_{m-1}^{h}\right\|_{L^{2}(\Omega, H)}^{2} . \tag{139}
\end{align*}
$$

Substituting (123), (138) and (139) in (98) yields

$$
\begin{align*}
& \left\|X^{h}\left(t_{m}\right)-X_{m}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \\
\leq & C h^{2 \beta}+C \Delta t+C \Delta t \sum_{k=0}^{m-1}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \tag{140}
\end{align*}
$$

Applying the discrete Gronwall lemma to yields

$$
\begin{equation*}
\left\|X^{h}\left(t_{m}\right)-X_{m}^{h}\right\|_{L^{2}(\Omega, H)} \leq C h^{\beta}+C \Delta t^{1 / 2} . \tag{141}
\end{equation*}
$$

### 3.3 Proof of Theorem 10

Recall that we only need to estimate the time error since the space error is estimated in Lemma 8. Recall also that the time error can be recast as follow

$$
\begin{equation*}
\frac{1}{4}\left\|X^{h}\left(t_{m}\right)-X_{m}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \leq I I I+I V+V+V I \tag{142}
\end{equation*}
$$

where $I I I$ and $V$ the same as in Section 3.2.1. The terms involving the noise $I V$ and $V I$ are in this case given by

$$
\begin{equation*}
I V=\left\|\int_{t_{m-1}}^{t_{m}}\left(e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)}-e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t}\right) P_{h} d W(s)\right\|_{L^{2}(\Omega, H)}^{2} \tag{143}
\end{equation*}
$$

and

$$
\begin{aligned}
V I= & \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} \\
& \left(e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}-e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\right) P_{h} d W(s) \|_{L^{2}(\Omega, H)}^{2}(144)
\end{aligned}
$$

Recall that from we have

$$
\begin{equation*}
I I I \leq C \Delta t^{2}+C \Delta t^{2}\left\|X^{h}\left(t_{m-1}\right)-X_{m-1}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \tag{145}
\end{equation*}
$$

It remains to estimate $V, I V$ and $V I$. Let us recall that from we have

$$
\begin{equation*}
V \leq 2 V_{1}+2 V_{2}, \tag{146}
\end{equation*}
$$

where from 105) we have

$$
\begin{equation*}
V_{2} \leq C \Delta t \sum_{k=0}^{m-2}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \tag{147}
\end{equation*}
$$

Let us recall that from (101) we have

$$
\begin{align*}
\sqrt{V_{1}}= & \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \cdot\left(G_{m-k-2}^{h}\left(X^{h}(s)\right)-G_{m-k-2}^{h}\left(X^{h}\left(t_{m-k-2}\right)\right)\right) d s \|_{L^{2}(\Omega, H)} . \tag{148}
\end{align*}
$$

Using the Taylor formula in Banach space yields

$$
\begin{align*}
\sqrt{V_{1} \leq} & \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \cdot\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right)\left(e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}-\mathbf{I}\right) X^{h}\left(t_{m-k-2}\right) d s \|_{L^{2}(\Omega, H)} \\
+ & \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right) \int_{t_{m-k-2}}^{s} e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)} G_{m-k-2}^{h}\left(X^{h}(\sigma)\right) d \sigma d s \|_{L^{2}(\Omega, H)} \\
+ & \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \cdot\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right) \int_{t_{m-k-2}}^{s} e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)} P_{h} d W(\sigma) d s \|_{L^{2}(\Omega, H)} \\
+ & \left\|\sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} R_{G_{m-k-2}^{h}}\right\|_{L^{2}(\Omega, H)} \\
:= & \sqrt{V_{11}}+\sqrt{V_{12}}+\sqrt{V_{13}}+\sqrt{V_{14}}, \tag{149}
\end{align*}
$$

where

$$
R_{G_{m-k-2}^{h}}:=\int_{0}^{1}\left(G_{m-k-2}^{h}\right)^{\prime \prime}\left(X^{h}\left(t_{m-k-2}\right)+\lambda\left(X^{h}(s)-X^{h}\left(t_{m-k-2}\right)\right)\right)\left(X^{h}(s)-X^{h}\left(t_{m-k-2}\right), X^{h}(s)-X^{h}\left(t_{m-k-2}\right)\right)(1-\lambda) d \lambda .
$$

Using triangle inequality yields

$$
\begin{align*}
\sqrt{V_{11}} \leq & \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \| e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}  \tag{150}\\
& \cdot\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right)\left(e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}-\mathbf{I}\right) X^{h}\left(t_{m-k-2}\right) \|_{L^{2}(\Omega, H)} d s \\
\leq & \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(\mathbf { E } \left[\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right\|_{L(H)}^{2}\right.\right. \\
& \left.\left.\times\left\|\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right)\left(e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}-\mathbf{I}\right) X^{h}\left(t_{m-k-2}\right)\right\|^{2}\right]\right)^{1 / 2} d s .
\end{align*}
$$

Using Lemma 10 with $\nu=0$ and Lemma 9 (ii) with $\gamma_{1}=0$, it holds that

$$
\begin{align*}
& \left\|e^{\left(A_{h}+J_{m-1}^{h}(\omega)\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}(\omega)\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}(\omega)\right)\left(t_{m-k-1}-s\right)}\right\|_{L(H)} \\
\leq & \left\|e^{\left(A_{h}+J_{m-1}^{h}(\omega)\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}(\omega)\right) \Delta t}\right\|_{L(H)}\left\|e^{\left(A_{h}+J_{m-k-2}^{h}(\omega)\right)\left(t_{m-k-1}-s\right)}\right\|_{L(H)} \\
\leq & C \tag{151}
\end{align*}
$$

for all $\omega \in \Omega$. Substituting (151) in (150), employing Lemma 12 , inserting an appropriate power of $-A_{h}$, using Lemma 9 (i) with $\gamma_{1}=0$ and $\gamma_{2}=\frac{\beta}{2}-\epsilon$, and Lemma 3 yields

$$
\begin{align*}
\sqrt{V_{11}} & \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left\|\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right)\left(e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}-\mathbf{I}\right) X^{h}\left(t_{m-k-2}\right)\right\|_{L^{2}(\Omega, H)} \\
& \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left\|\left(e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}-\mathbf{I}\right) X^{h}\left(t_{m-k-2}\right)\right\|_{L^{2}(\Omega, H)} \\
& \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left\|\left(e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}-\mathbf{I}\right)\left(-A_{h}\right)^{-\frac{\beta}{2}+\epsilon}\left(-A_{h}\right)^{\frac{\beta}{2}-\epsilon} X^{h}\left(t_{m-k-2}\right)\right\|_{L^{2}(\Omega, H)} \\
& \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(s-t_{m-k-2}\right)^{\frac{\beta}{2}-\epsilon}\left\|\left(-A_{h}\right)^{\frac{\beta}{2}-\epsilon} X^{h}\left(t_{m-k-2}\right)\right\|_{L^{2}(\Omega, H)} d s \\
& \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(s-t_{m-k-2}\right)^{\frac{\beta}{2}-\epsilon} d s \leq C \Delta t^{\frac{\beta}{2}-\epsilon} \tag{152}
\end{align*}
$$

Using the triangle inequality, Lemma 12, Lemma 6, Lemma 3, Lemma 10 and Lemma 9 , it holds that

$$
\begin{align*}
\sqrt{V_{12} \leq} \leq & \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \| e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right) \int_{t_{m-k-2}}^{s} e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)} G_{m-k-2}^{h}\left(X^{h}(\sigma)\right) d \sigma \|_{L^{2}(\Omega, H)} d s \\
\leq & \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(\mathbb { E } \left[\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right)\left(t_{m-k-1}-s\right)}\right\|_{L(H)}^{2}\right.\right. \\
& \left.\left.\times\left\|e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right\|_{L(H)}^{2}\left\|\int_{t_{m-k-2}}^{s} e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)} G_{m-k-2}^{h}\left(X^{h}(\sigma)\right) d \sigma\right\|_{L(H)}^{2}\right]\right)^{1 / 2} d s \\
\leq & C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(\mathbb{E}\left[\int_{t_{m-k-2}}^{s}\left\|e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)}\right\|_{L(H)}\left\|G_{m-k-2}^{h}\left(X^{h}(\sigma)\right)\right\| d \sigma\right]^{2}\right)^{1 / 2} d s \\
\leq & C \Delta t \tag{153}
\end{align*}
$$

Since the expectation of the cross-product terms vanishes, Cauchy-Schwartz inequality yields

$$
\begin{aligned}
& V_{13} \\
= & \mathbb{E}\left[\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right. \\
& \left.\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right) \int_{t_{m-k-2}}^{s} e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)} P_{h} d W(\sigma) d s \|^{2}\right] \\
= & \sum_{k=0}^{m-2} \mathbb{E}\left[\| \int_{t_{m-k-2}}^{t_{m-k-1}} \int_{t_{m-k-2}}^{s} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right. \\
& \left.\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right) e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)} P_{h} d W(\sigma) d s \|^{2}\right] \\
\leq & \Delta t \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left[\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right\|_{L(H)}^{2}\right. \\
\times & \left.\left\|\int_{t_{m-k-2}}^{s}\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right) e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)} P_{h} d W(\sigma)\right\|^{2}\right] d s .
\end{aligned}
$$

Using again Cauchy-Schwartz inequality, it follows that
$\quad V_{13}$
$\leq \Delta t \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}}$
$\times\left(\mathbb{E}\left\|\int_{t_{m-k-2}}^{s}\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right) e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)} P_{h} d W(\sigma)\right\|^{4}\right)^{\frac{1}{2}} d s$.

Using the Burkhölder-Davis-Gundy inequality ( 20 , Lemma 5.1]), Lemma 10 and Lemma 9, it holds that

$$
\begin{aligned}
& V_{13} \\
& \leq C \Delta t \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}}\left(\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
& \times \int_{t_{m-k-2}}^{s} \mathbb{E}\left\|\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right) e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)} P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2} d \sigma d s \\
& \leq C \Delta t \sum_{k=}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \int_{t_{m-k-2}}^{s} \mathbb{E}\left[\left\|\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right) e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)} P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2}\right] d \sigma d s .
\end{aligned}
$$

Using Lemma 12, inserting an appropriate power of $-A_{h}$, using Lemma 11
Lemma (ii) with $\gamma_{1}=0$ (if $\beta \geq 1$ ) and Lemma 9 (ii) with $\gamma_{1}=\frac{1-\beta}{2}$ (if
$\beta \leq 1)$, it holds that

$$
\begin{align*}
& \mathbb{E}\left[\left\|\left(G_{m-k-2}^{h}\right)^{\prime}\left(X^{h}\left(t_{m-k-2}\right)\right) e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)} P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2}\right] \\
\leq & \mathbb{E}\left[\left\|e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)}\left(-A_{h}\right)^{\frac{1-\beta}{2}}\left(-A_{h}\right)^{\frac{\beta-1}{2}} P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2}\right] \\
\leq & \mathbb{E}\left[\left\|e^{\left(A_{h}+J_{m-k-2}^{h}\right)(s-\sigma)}\left(-A_{h}\right)^{\frac{1-\beta}{2}}\right\|_{L(H)}^{2}\left\|\left(-A_{h}\right)^{\frac{\beta-1}{2}} P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2}\right] \\
\leq & C(s-\sigma)^{\min (-1+\beta, 0)} . \tag{154}
\end{align*}
$$

Substituting 154 in (154) yields

$$
\begin{align*}
V_{13} & \leq C \Delta t \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \int_{t_{m-k-2}}^{s}(s-\sigma)^{\min (-1+\beta, 0)} d \sigma d s \\
& \leq C \Delta t^{\min (1+\beta, 2)} . \tag{155}
\end{align*}
$$

To estimate $\sqrt{V_{14}}$, we note by using Lemma 12 and Lemma 4 that

$$
\begin{align*}
\left\|\left(-A_{h}\right)^{-\frac{\eta}{2}} R_{G_{m-k-2}^{h}}\right\|_{L^{2}(\Omega, H)} & \leq C\| \| X^{h}(s)-X^{h}\left(t_{m-k-2}\right)\left\|^{2}\right\|_{L^{2}(\Omega, H)} \\
& \leq C\left\|X^{h}(s)-X^{h}\left(t_{m-k-2}\right)\right\|_{L^{4}(\Omega, H)}^{2} \\
& \leq C \Delta t^{\min (\beta, 1)} . \tag{156}
\end{align*}
$$

Therefore the following estimate holds for $\sqrt{V_{14}}$

$$
\begin{equation*}
\sqrt{V_{14}} \leq C \Delta t^{\min (\beta, 1)} \tag{157}
\end{equation*}
$$

Substituting (157, 157, (153) and 152 in 149 yields

$$
\begin{equation*}
V_{1} \leq C \Delta t^{\beta-2 \epsilon} . \tag{158}
\end{equation*}
$$

Substituting 158 and 147) in 146 yields

$$
\begin{equation*}
V \leq C \Delta t^{\beta-2 \epsilon}+C \Delta t \sum_{k=0}^{m-1}\left\|X^{h}\left(t_{k}\right)-X_{k}^{h}\right\|_{L^{2}(\Omega, H)}^{2} \tag{159}
\end{equation*}
$$

Let us move to the estimate of $I V$. Applying the Itô-isometry to 143 yields

$$
\begin{align*}
I V & \leq \int_{t_{m-1}}^{t_{m}}\left\|\left(e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)}-e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t}\right) P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2} d s  \tag{160}\\
& =\int_{t_{m-1}}^{t_{m}}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)}\left(\mathbf{I}-e^{\left(A_{h}+J_{m-1}^{h}\right)\left(s-t_{m-1}\right)}\right) P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2} d s
\end{align*}
$$

Inserting $\left(-A_{h}\right)^{\frac{1-\beta}{2}}\left(-A_{h}\right)^{\frac{\beta-1}{2}}$ in 160 , using (5) and Lemma 11 yields

$$
\begin{aligned}
I V \leq & \int_{t_{m-1}}^{t_{m}}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)}\left(\mathbf{I}-e^{\left(A_{h}+J_{m-1}^{h}\right)\left(s-t_{m-1}\right)}\right)\left(-A_{h}\right)^{\frac{1-\beta}{2}}\right\|_{L(H)}^{2} \\
& \times\left\|\left(-A_{h}\right)^{\frac{\beta-1}{2}} P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2} d s \\
\leq & C \int_{t_{m-1}}^{t_{m}}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)}\left(\mathbf{I}-e^{\left(A_{h}+J_{m-1}^{h}\right)\left(s-t_{m-1}\right)}\right)\left(-A_{h}\right)^{\frac{1-\beta}{2}}\right\|_{L(H)}^{2} d(161)
\end{aligned}
$$

Inserting $\left(-A_{h}\right)^{\frac{1-\epsilon}{2}}\left(-A_{h}\right)^{\frac{\epsilon-1}{2}}$ in 161 and using Lemma 9 (ii) with $\gamma_{1}=\frac{1-\epsilon}{2}$, Lemma 9 (iv) with $\gamma_{1}=\frac{1-\epsilon}{2}$ and $\gamma_{2}=\frac{1-\beta}{2}$ (or Lemma 9 with $\gamma_{1}=\frac{1-\epsilon}{2}$ and $\gamma_{2}=\frac{\beta-\mathrm{T}}{2}$ if $\left.\beta \in[1,2]\right)$ yields

$$
\begin{align*}
I V \leq & C \int_{t_{m-1}}^{t_{m}}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right)\left(t_{m}-s\right)}\left(-A_{h}\right)^{\frac{1-\epsilon}{2}}\right\|_{L(H)}^{2} \\
& \times\left\|\left(-A_{h}\right)^{\frac{\epsilon-1}{2}}\left(\mathbf{I}-e^{\left(A_{h}+J_{m-1}^{h}\right)\left(s-t_{m-1}\right)}\right)\left(-A_{h}\right)^{\frac{1-\beta}{2}}\right\|_{L(H)}^{2} d s \\
\leq & C \int_{t_{m-1}}^{t_{m}}\left(t_{m}-s\right)^{-1+\epsilon}\left(s-t_{m-1}\right)^{\beta-\epsilon} d s \\
\leq & C \Delta t^{\beta-\epsilon} \int_{t_{m-1}}^{t_{m}}\left(t_{m}-s\right)^{-1+\epsilon} d s \leq C \Delta t^{\beta} . \tag{162}
\end{align*}
$$

Let us now turn our attention to the estimate of $V I$. Since the expectation of the cross-product vanishes, using Cauchy-Schwartz inequality, it follows from (144) that

$$
\begin{aligned}
V I= & \mathbb{E} \| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \left(\mathbf{I}-e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}\right) P_{h} d W(s) \|^{2} \\
= & \sum_{k=0}^{m-2} \mathbb{E} \| e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t} \\
& \int_{t_{m-k-2}}^{t_{m-k-1}} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\left(\mathbf{I}-e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}\right) P_{h} d W(s) \|^{2} \\
\leq & \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{1-\epsilon}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
\times & \left.\mathbb{E}\left\|\int_{t_{m-k-2}}^{t_{m-k-1}}\left(-A_{h}\right)^{\frac{-1+\epsilon}{2}} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\left(\mathbf{I}-e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}\right) P_{h} d W(s)\right\|^{4}\right)^{\frac{1}{2}} .
\end{aligned}
$$

Using the Burkhölder-Davis-Gundy inequality ( 20 , Lemma 5.1]), it follows that

$$
\begin{align*}
& \leq I  \tag{163}\\
& \leq C \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{1-\epsilon}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
& \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left\|\left(-A_{h}\right)^{\frac{-1+\epsilon}{2}} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\left(\mathbf{I}-e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}\right) P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2} d s .
\end{align*}
$$

Inserting $\left(-A_{h}\right)^{\frac{1-\beta}{2}}\left(-A_{h}\right)^{\frac{\beta-1}{2}}$ in 163 , using (5) and Lemma 11 yields

$$
\begin{align*}
& V I  \tag{164}\\
& \leq C \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{1-\epsilon}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
& \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left\|\left(-A_{h}\right)^{\frac{-1+\epsilon}{2}} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\left(\mathbf{I}-e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}\right)\left(-A_{h}\right)^{\frac{1-\beta}{2}}\right\|_{L(H)}^{2} \\
&\left\|\left(-A_{h}\right)^{\frac{\beta-1}{2}} P_{h} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_{2}(H)}^{2} d s \\
& \leq C \sum_{k=0}^{m-2}\left(\mathbb{E}\left\|e^{\left(A_{h}+J_{m-1}^{h}\right) \Delta t} \cdots e^{\left(A_{h}+J_{m-k-1}^{h}\right) \Delta t}\left(-A_{h}\right)^{\frac{1-\epsilon}{2}}\right\|_{L(H)}^{4}\right)^{\frac{1}{2}} \\
& \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E}\left\|\left(-A_{h}\right)^{\frac{-1+\epsilon}{2}} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\left(\mathbf{I}-e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}\right)\left(-A_{h}\right)^{\frac{1-\beta}{2}}\right\|_{L(H)}^{2} d s .
\end{align*}
$$

Using Lemma 10 with $\nu=\frac{1-\epsilon}{2}$ yields

$$
\begin{align*}
V I \leq & C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} t_{k+1}^{-1+\epsilon} \mathbb{E} \|\left(-A_{h}\right)^{\frac{\epsilon-1}{2}} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)} \\
& \left(\mathbf{I}-e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}\right)\left(-A_{h}\right)^{\frac{1-\beta}{2}} \|_{L(H)}^{2} d s . \tag{165}
\end{align*}
$$

Inserting $\left(-A_{h}\right)^{\frac{1-\epsilon}{2}}\left(-A_{h}\right)^{\frac{\epsilon-1}{2}}$ in 165 and using Lemma 9 (iii) with $\gamma_{1}=\gamma_{2}=$ $\frac{1-\epsilon}{2}$, Lemma 9 (iv) with $\gamma_{1}=\frac{1-\epsilon}{2}$ and $\gamma_{2}=\frac{1-\beta}{2}$ when $0 \leq \beta \leq 1$ and Lemma 9
(i) with $\gamma_{1}=\frac{1-\epsilon}{2}$ and $\gamma_{2}=\frac{1-\beta}{2}$ when $1 \leq \beta \leq 2$ yields

$$
\begin{align*}
& V I \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} t_{k+1}^{-1+\epsilon} \mathbb{E}\left[\left\|\left(-A_{h}\right)^{\frac{\epsilon-1}{2}} e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(t_{m-k-1}-s\right)}\left(-A_{h}\right)^{\frac{1-\epsilon}{2}}\right\|_{L(H)}^{2}\right. \\
&\left.\times\left\|\left(-A_{h}\right)^{\frac{-1+\epsilon}{2}}\left(\mathbf{I}-e^{\left(A_{h}+J_{m-k-2}^{h}\right)\left(s-t_{m-k-2}\right)}\right)\left(-A_{h}\right)^{\frac{1-\beta}{2}}\right\|_{L(H)}^{2}\right] d s \\
& \leq C \sum_{k=1}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} t_{k}^{-1+\epsilon}\left(s-t_{m-k-1}\right)^{\beta-\epsilon} d s \\
& \leq C \Delta t^{\beta-\epsilon} \sum_{k=1}^{m-2} t_{k}^{-1+\epsilon} \int_{t_{m-k-2}}^{t_{m-k-1}} d s=C \Delta t^{\beta-\epsilon} \sum_{k=1}^{m-2} t_{k}^{-1+\epsilon} \Delta t . \tag{166}
\end{align*}
$$

Let us recall the following estimate

$$
\begin{equation*}
\sum_{k=1}^{m-2} t_{k}^{-1+\epsilon} \Delta t \leq C \tag{167}
\end{equation*}
$$

Inserting (167) in 166) yields

$$
\begin{equation*}
V I \leq C \Delta t^{\beta-\epsilon} . \tag{168}
\end{equation*}
$$

Substituting (168), 162 , (159) and 145 in $\sqrt{142}$ and applying the discrete Gronwall lemma yields

$$
\begin{equation*}
\left\|X^{h}\left(t_{m}\right)-X_{m}^{h}\right\|_{L^{2}(\Omega, H)} \leq C \Delta t^{\beta / 2-\epsilon / 2} \leq C \Delta t^{\beta / 2-\epsilon} . \tag{169}
\end{equation*}
$$

This completes the proof of Theorem 10 .

## 4 Numerical simulations

Here we provide three examples to sustain our theoretical results. The first example has exact solution. The reference solution or "the exact solution" used in the errors computation for our second and third example are taken to be the numerical solution with small time step. In the legends of our graphs, we use the following notations

1. SERS denotes the strong errors from our SERS scheme.
2. SETD1 denotes the strong errors from the stochastic exponential scheme 23] given by (51).

The exponential matrix function $\varphi_{1}$ is computed by Krylov subspace technique with fixed dimension $m=10$ and tolerance $t o l=10^{-6} 9,33,35$. Note that we compute at every time step the action on the exponential matrix function on a vector and not the whole exponential matrix function. Our code was implemented in Matlab 8.1. Note that the initial solution is taken to be $X_{0}=0$ throughout our simulations, so optimal convergence order in time will depend only on the regularity of the noise.
4.1 Additive noise with exact solution

We first consider the following stochastic reaction diffusion equation with stiff reaction driven by additive noise in two dimensions with Neumann boundary conditions

$$
\begin{equation*}
d X(t)=[D \Delta X(t)-100 X(t)] d t+d W(t), \quad X(0)=X_{0}, \quad t \in[0, T] \tag{170}
\end{equation*}
$$

on the domain $\Lambda=\left[0, L_{1}\right] \times\left[0, L_{2}\right], D=10^{-1}$. A simple computation shows that the eigenfunctions $\left\{e_{i, j}\right\}_{i, j \geq 0}=\left\{e_{i}^{(1)} \otimes e_{j}^{(2)}\right\}_{i, j \geq 0}$ with the corresponding eigenvalues $\left\{\lambda_{i, j}\right\}_{i, j \geq 0}=\left\{\left(\lambda_{i}^{(1)}\right)^{2}+\left(\lambda_{j}^{(2)}\right)^{2}\right\}$ of $-\Delta$ are given by

$$
\begin{equation*}
e_{0}^{(l)}(x)=\sqrt{\frac{1}{L_{l}}}, \quad e_{i}^{(l)}(x)=\sqrt{\frac{2}{L_{l}}} \cos \left(\lambda_{i}^{(l)} x\right), \quad \lambda_{0}^{(l)}=0, \quad \lambda_{i}^{(l)}=\frac{i \pi}{L_{l}}, \tag{171}
\end{equation*}
$$

where $l=1,2, x \in \Lambda$ and $i \in \mathbb{N}$. In the abstract form (1) our linear operator $A$ is taken to be $A=D \Delta$ and $F(X)=-100 X$ which obviously satisfies Assumption 3 and Assumption 7. We take $L_{1}=L_{2}=1$ and the triangulation $\mathcal{T}$ has been constructed from uniform Cartesian grid of sizes $\Delta x=\Delta y=$ $1 / 100$.

We assume that the covariance operator $Q$ and $A$ have the same eigenfunctions. We take the following values for $\left\{q_{i, j}\right\}_{i+j>0}$ in the representation (2)

$$
\begin{equation*}
q_{i, j}=\frac{1}{\left(i^{2}+j^{2}\right)^{\beta+\delta}}, \quad 0 \leq \beta \leq 2, \text { and } \quad \delta>0 \text { small enough. } \tag{172}
\end{equation*}
$$

We can easily prove that Assumption 6 is fulfilled, since

$$
\sum_{(i, j) \in \mathbb{N}^{2}} \lambda_{i, j}^{\beta-1} q_{i, j}<\pi^{2} \sum_{(i, j) \in \mathbb{N}^{2}}\left(i^{2}+j^{2}\right)^{-(1+\delta)}<\infty, \quad 0 \leq \beta \leq 2
$$

To have trace class noise, it is enough to take $\beta+\delta>1$. We take $\beta=1$ and $\delta=0.001$. According to Theorem 10, the order of convergence in time should be close to 0.5 . The exact solution of 170 is constructed in 36 . Figure 1 shows the strong convergence of SERS and SETD1 schemes. This figure also shows that SETD1 is unstable for large time steps. We can observe the good stability property of the new SERS scheme even for large time steps. We can also observe that the two schemes have the same order of accuracy. Indeed although SETD1 seems to be more accurate, the two graphs become very close for small time step. The orders of convergence of the two methods are 0.4971 and 0.4980 for SERS and SETD1 schemes respectively, which are very close to theoretical results. Note that we only use the stable part of the data in the computation of the order of convergence of SETD1 scheme.


Fig. 1 Strong convergence of SERS and SETD1 scheme, we can also observe that SETD1 is unstable for large time steps. The orders of convergence of the two methods are 0.4971 and 0.4980 for SERS and SETD1 schemes respectively. The noise regularity parameter used is $\beta=1$ and 50 samples are used in the errors computation.
4.2 Additive noise without exact solution and with locally Lipschitz nonlinear function

We consider here the following stochastic reaction diffusion equation driven by additive noise in two dimensions with Neumann boundary conditions

$$
\begin{equation*}
d X(t)=\left[D \Delta X(t)+X(t)-X(t)^{3}\right] d t+d W(t), \quad X(0)=X_{0} \tag{173}
\end{equation*}
$$

on the domain $\Lambda=\left[0, L_{1}\right] \times\left[0, L_{2}\right], D=10^{-2}$ and $t \in[0, T]$. We take $L_{1}=$ $L_{2}=1$ and the triangulation $\mathcal{T}$ has been constructed from uniform Cartesian grid of sizes $\Delta x=\Delta y=1 / 100$. The reference solution or "the exact solution" using in the errors computation is the numerical solution with the time step $\Delta t=1 / 2048$. The goal of this example is to prove that our novel scheme can be stable and convergent for more complicated nonlinear function $F(X)=$ $X-X^{3}$. Although the existence and the uniqueness of the solution of 173 ) is well known 16, 26, the well-posedness of the numerical solution with our novel scheme is not yet understood since the nonlinear function is only locally Lipschitz $\sqrt[16,26]{26}$. In our simulation, the noise's representation $\sqrt{172}$ is used with $\beta=1.2$ and $\delta=0.001$. The orders of convergence of the two methods are 0.65 and 0.62 for SERS and SETD1 schemes respectively. If our convergence theorem (Theorem 10) was also valid for locally Lipschitz nonlinear function $F$, our convergence orders should be then close to the expected order 0.6 . We can also observe that the two schemes have the same order of accuracy. Indeed although SETD1 seems to be more accurate, the two graphs become very close for small time step. We can also observe the good stability property of the new SERS scheme even for large time step.


Fig. 2 Strong convergence of SERS and SETD1 scheme can be observed for nonlinear function $F(X)=X-X^{3}$. The orders of convergence of the two methods are 0.65 and 0.52 for SERS and SETD1 schemes respectively. The noise regularity parameter used is $\beta=1.2$ and 50 samples are used in the errors computation.
4.3 Multiplicative noise without exact solution

As a more challenging example, we consider the stochastic advection-diffusionreaction SPDE with multiplicative noise in two dimensions on the domain $\Lambda=[0,1] \times[0,1]$.

$$
\begin{align*}
d X & =\left[\nabla \cdot(\mathbf{D} \nabla X)-\nabla \cdot(\mathbf{q} X)-\frac{10 X}{X+1}\right] d t+X d W  \tag{174}\\
\mathbf{D} & =\left(\begin{array}{cc}
10^{-2} & 0 \\
0 & 10^{-3}
\end{array}\right) \tag{175}
\end{align*}
$$

with mixed Neumann-Dirichlet boundary conditions. The Dirichlet boundary condition is $X=1$ at $x=0$ and we use the homogeneous Neumann boundary conditions elsewhere. The Darcy velocity $\mathbf{q}$ is obtained as in 23 and to deal with high Péclet flows we discretize in space using finite volume method (viewed as the finite element method as in [34]) in rectangular grid of sizes $\Delta x=\Delta y=1 / 110$. The reference solution or "the exact solution" using in the errors computation is the numerical solution with the time step $\Delta t=1 / 2048$. Relatively small time steps are used to stabilize the scheme SETD1. The noise used is the same as in the first example with 172 and $\beta=1$ and $\delta=0.001$, corresponding to trace class noise. Our linear operator $A$ is given by

$$
\begin{equation*}
A=\nabla \cdot \mathbf{D} \nabla(.)-\nabla \cdot \mathbf{q}(.) . \tag{176}
\end{equation*}
$$

and the functions $f$ and $b$ are given by

$$
\begin{equation*}
f(x, u)=\frac{-10 u}{u+1}, \quad b(x, u)=u, \quad \forall x \in \Lambda, \quad u \in \mathbb{R} \tag{177}
\end{equation*}
$$

Therefore, from [15, Section 4] it follows that the operators $F$ and $B$ defined by (13) fulfil obviously Assumption 3 and Assumption 4


Fig. 3 Strong convergence of SERS and SETD1 scheme can be observed. The orders of convergence of the two methods are 0.5367 and 0.5337 for SERS and SETD1 schemes respectively. The noise regularity parameter used are $\beta=1$ and $\delta=0.001$. Note that 50 samples have been used in the errors computation.

Figure 3 shows the strong convergence of SERS scheme and SETD1 scheme presented in 23]. We can also observe that the two schemes have the same order of accuracy. Indeed although SERS seems to be more accurate, the difference between the two errors is small. The orders of convergence of the two methods are 0.5367 and 0.5337 for SERS and SETD1 schemes respectively, which are very close to 0.5 (from the theoretical results in Theorem 9 ).

## 5 Concluding remark

In this work, we have analyzed the strong convergence of the exponential Rosenbrock-Euler method for a semilinear parabolic SPDE. The method is based on a continuous linearization of the problem at each time step. The linearisation technique consists of adding the Jacobian of the nonlinear function to a linear operator while the nonlinear function is replaced by its reminder. The linear operator is assumed to be a generator of an analytic semigroup. By 5. Theorem 2.10, Page 176, Chapter 3] there exists a constant $a>0$ such that $A+L$ generates an analytic semigroup for every $A$-bounded operator $L$ having $A$-bound $a_{0}<a$ (see [5, Definition 2.1, Page 169, Chapter 3]). As the nonlinear function $F$ is assumed to be Fréchet differentiable with bounded derivative in our analysis, we can weaken that hypothesis on $F$ by replacing Assumption 3 by the following weaker assumption.

Assumption 11 The nonlinear function $F: H \longrightarrow H$ is assumed to be Fréchet differentiable with derivative relatively $A$-bounded with $A$-bound $a_{0}<$ $a$, i.e there exist constants $a_{0} \in[0, a)$ and $b \geq 0$ such that

$$
\begin{equation*}
\left\|F^{\prime}(u) v\right\| \leq a_{0}\|A v\|+b\|v\|, \quad u \in H, v \in \mathcal{D}(A) \tag{178}
\end{equation*}
$$

Under Assumption 11, for all $\omega \in \Omega, A_{h}+J_{m}^{h}(\omega)$ generates an analytic semigroup $S_{m}^{h}(\omega)(t):=e^{\left(A_{h}+J_{m}^{h}(\omega)\right) t}$ (see 5. Theorem 2.10, page 176, Chapter 3]) and therefore the numerical scheme (49) is well posed. Under Assumption 11 , the convergence analysis of the numerical method (49) is not straightforward. This is due to the presence of the linear operator $A$ in the right hand side of (178), which may produce some irregularities. This will be our interest for future work. Further investigations will be done also for locally Lipschitz nonlinear function $F$.

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## References

1. Baglama, J., Calvetti, D., and Reichel,L.: Fast Léja points. Electron. Trans. Num. Anal., 7, 124-140 (1998)
2. Carlos, M., M.: Weak exponential schemes for stochastic differential equations with additive noise. IMA Journal of Numerical Analysis. 25, 486-506 (2005)
3. Carlos, M., M.: Numerical solution of conservative finite-dimensional stochastic Schrödinger equations. Ann. Appl. Probab. 15, 2144-2171 (2005)
4. Elliot, C. M., Larsson S.: Error estimates with smooth and nonsmooth data for a finite element method for the Cahn-Hilliard equation. Math. Comput. 58, 603-630 (1992)
5. Engel, K. J., Nagel, R.: One-Parameter Semigroup for Linear Evolution Equations. Springer-Verlag, New York, Inc, 2000
6. Fujita, F., Suzuki, T.: Evolution problems (Part 1). Handbook of Numerical Analysis (P. G. Ciarlet and J.L. Lions eds). vol. 2. Amsterdam, The Netherlands: North-Holland, pp. 789-928 (1991)
7. Geiger S., Lord G., Tambue A.: Exponential time integrators for stochastic partial differential equations in 3D reservoir simulation. Computational Geosciences, 16(2), 323 - 334 (2012)
8. Henry, D.: Geometric Theory of semilinear parabolic Equations. Lecture notes in Mathematics, vol. 840. Berlin : Springer, 1981
9. Hochbruck, M., Lubich, C.: On Krylov subspace approximations to the matrix exponential operator. SIAM J. Numer. Anal., 34(5), 1911-1925 (1997)
10. Hochbruck, M., Ostermann, A.: Exponential integrators. Acta Numerica. 19, 209-286 (2010)
11. Hochbruck, M., Ostermann, A., Schweitzer, J.: Exponential Rosenbrock-type methods. SIAM J. Numer. Anal. 47(1), 786-803 (2009)
12. Hochbruck, M., Ostermann, A.: Explicit integrators of Rosenbrock-type. Oberwolfach Reports, 3(2), 1107-1110 (2006)
13. Jentzen, A., Kloeden, P. E.: Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 465(2102), 649-667 (2009)
14. Jentzen, A., Kloeden, P. E., Winkel, G.: Efficient simulation of nonlinear parabolic SPDEs with additive noise. Ann. Appl. Probab. 21(3), 908-950 (2011)
15. Jentzen, A., Röckner, M.: Regularity analysis for stochastic partial differential equations with nonlinear multiplicative trace class noise. J. Diff. Equat. 252(1), 114-136 (2012)
16. Jentzen, A.: Pathwise Numerical approximations of SPDEs with additive noise under non-global Lipschitz coefficients. Potential Analysis, 31(4), 375-404 (2009)
17. Kloeden, P. E., Lord, G. J., Neuenkirch, A., Shardlow, T.: The exponential integrator scheme for stochastic partial differential equations : Pathwise errors bounds J. Comput. Appl. Math. 235(5), 1245-1260 (2011)
18. Kloeden, P. E., Platen, E.: Numerical solution of stochastic Differential equations. Springer, Berlin (1992)
19. Kovács M., Larsson, S., and Lindgren, F.: Strong convergence of the finite element method with truncated noise for semilinear parabolic stochastic equations with additive noise. Numer. Algor. 53, 309-220 (2010)
20. Kruse, R.: Optimal error estimates of Galerkin finite element methods for stochastic partial differential equations with multiplicative noise. IMA J. Numer. Anal. 34(1), 217-251 (2014)
21. Kruse, R., Larsson, S.: Optimal regularity for semilinear stochastic partial differential equations with multiplicative noise. Electron. J. Probab. 17(65), 1-19 (2012)
22. Larsson, S.: Nonsmooth data error estimates with applications to the study of the long-time behavior of finite element solutions of semilinear parabolic problems Preprint 1992-36, Department of Mathematics, Chalmers University of Technology (1992).
23. Lord, G. J., Tambue, A.: Stochastic exponential integrators for the finite element discretization of SPDEs for multiplicative \& additive noise. IMA J. Numer. Anal. 2, 515-543 (2013)
24. Mukam, J. D., Tambue, A.: A note on exponential Rosenbrock-Euler method for the finite element discretization of a semilinear parabolic partial differential equations. arXiv:1610.05525, 2016
25. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences; Springer-Verlag, New York, v 44, 1983
26. Prato, D., Zabczyk, G. J.: Stochastic equations in infinite dimensions. Encyclopedia of Mathematics and its Applications, vol. 44. Cambridge: Cambridge University Press (1992)
27. Printems, J.: On the discretization in time of parabolic stochastic partial differential equations. Math. modelling and numer. Anal. 35(6), 1055-1078 (2001)
28. Prévôt, C., Röckner, M.: A Concise Course on Stochastic Partial Differential Equations. Lecture Notes in Mathematics, vol. 1905 (Springer, Berlin, 2007)
29. Ramos, J. I., García-López, C. M.: Piecewise-linearized methods for initial-value problems. Appl. Math. Comput. 82(2-3), 273-302 (1997)
30. Schweitzer, J.: The exponential Rosenbrock-Euler method for nonsmooth initial data Preprint, 2015
31. Schweitzer, J.: The exponential Rosenbrock-Euler method with variable time step sizes for nonsmooth initial data. Technical report, Karlsruhe Institute of Technology, 2014
32. Shardlow, T.: Weak convergence of a numerical method for a stochastic heat equation. BIT. 43(1), 179-193 (2003)
33. Tambue, A.: Efficient Numerical Schemes for Porous Media Flow. PhD Thesis, Department of Mathematics, Heriot-Watt University, 2010
34. Tambue, A.: An exponential integrator for finite volume discretization of a reaction-advection-diffusion equation. Comput. Math. Appl. 71(9), 1875-1897 (2016)
35. Tambue, A., Berre, I., Nordbotten, J., M.: Efficient simulation of geothermal processes in heterogeneous porous media based on the exponential Rosenbrock-Euler and Rosenbrock-type methods. Advances in Water Resources. 53, 250-262 (2013)
36. Tambue, A., Ngnotchouye, J. M. T.: Weak convergence for a stochastic exponential integrator and finite element discretization of stochastic partial differential equation with multiplicative \& additive noise. Appl. Num. Math. 108, 57-86 (2016)
37. Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems. Springer-Verlag, Berlin, (1997)
38. Wang, X.: Weak error estimates of the exponential Euler scheme for semi-linear SPDE without Malliavin calculus. Discret. Contin. Dyn. Sys. Ser. A, 36, 481-497 (2016)
39. Wang, X.: Strong convergence rates of the linear implicit Euler method for the finite element discretization of SPDEs with additive noise. IMA J. Numer. Anal. DOI: https://doi.org/10.1093/imanum/drw016, 2016
40. Wang, X., Ruisheng, Q.: A note on an accelerated exponential Euler method for parabolic SPDEs with additive noise. Appl. Math. Lett. 46, 31-37 (2015)
41. Yan, Y.: Galerkin finite element methods for stochastic parabolic partial differential equations. SIAM J. Num. Anal. 43(4), 1363-1384 (2005)
42. Yan, Y.: Semidiscrete Galerkin approximation for a linear stochastic parabolic partial differential equation driven by an additive noise. BIT Numer. Math. 44(4), 829-847 (2004)

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[^1]:    ${ }^{1}$ Full implicit or semi-implicit methods

[^2]:    ${ }^{2}$ The proposition indeed is general and provides some estimates for any semigroup and its generator.

[^3]:    ${ }^{3}$ Think about for example a multiple of Laplace operator $A=\alpha \Delta$, when $\alpha \rightarrow 0$

