# Hybridized Discontinuous Galerkin Method for Elliptic Interface Problems 

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New hybridized discontinuous Galerkin (HDG) methods for the interface problem for elliptic equations are proposed. Unknown functions of our schemes are $u_{h}$ in elements and $\hat{u}_{h}$ on inter-element edges. That is, we formulate our schemes without introducing the flux variable. Our schemes naturally satisfy the Galerkin orthogonality. The solution $u$ of the interface problem under consideration may not have a sufficient regularity, say $\left.u\right|_{\Omega_{1}} \in H^{2}\left(\Omega_{1}\right)$ and $\left.u\right|_{\Omega_{2}} \in H^{2}\left(\Omega_{2}\right)$, where $\Omega_{1}$ and $\Omega_{2}$ are subdomains of the whole domain $\Omega$ and $\Gamma=\partial \Omega_{1} \cap \partial \Omega_{2}$ implies the interface. We study the convergence, assuming $\left.u\right|_{\Omega_{1}} \in H^{1+s}\left(\Omega_{1}\right)$ and $\left.u\right|_{\Omega_{2}} \in H^{1+s}\left(\Omega_{2}\right)$ for some $s \in(1 / 2,1]$, where $H^{1+s}$ denotes the fractional order Sobolev space. Consequently, we succeed in deriving optimal order error estimates in an HDG norm and the $L^{2}$ norm. Numerical examples to validate our results are also presented.

Key words: discontinuous Galerkin method, elliptic interface problem
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## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^{d}, d=2,3$, with the boundary $\partial \Omega$. We suppose that $\Omega$ is divided into two disjoint subdomains $\Omega_{1}$ and $\Omega_{2}$. Then, $\Gamma=\partial \Omega_{1} \cap \partial \Omega_{2}$ implies the interface. See Fig. 1 for example.

[^0]

Figure 1: Examples of $\Omega_{1}, \Omega_{2}$ and $\Gamma$.

Suppose that we are given a matrix-valued function $A=A(x)$ of $\Omega \rightarrow \mathbb{R}^{d \times d}$ such that:

| (smoothness) | $\left.A\right\|_{\Omega_{i}}$ is a $C^{1}$ function in $\Omega_{i}$ | $(i=1,2) ;$ |
| :--- | :--- | :--- |
| (symmetry) | $\xi \cdot A(x) \eta=(A(x) \xi) \cdot \eta$, | $\left(\xi, \eta \in \mathbb{R}^{d}, x \in \Omega\right) ;$ |
| (elliptic condition) | $\lambda_{\min }\|\xi\|^{2} \leq \xi \cdot A(x) \xi \leq \lambda_{\max }\|\xi\|^{2}$ | $\left(\xi \in \mathbb{R}^{d}, x \in \Omega\right)$ |

with some positive constants $\lambda_{\min }$ and $\lambda_{\max }$. Hereinafter, $|\cdot|=|\cdot|_{\mathbb{R}^{d}}$ denotes the Euclidean norm in $\mathbb{R}^{d}$ and $\xi \cdot \eta$ the standard scalar product in $\mathbb{R}^{d}$.

We consider the following interface problem for second-order elliptic equations for the function $u=u(x), x \in \bar{\Omega}$,

$$
\begin{align*}
-\nabla \cdot A \nabla u & =f & & \text { in } \Omega \backslash \Gamma,  \tag{1a}\\
u & =0 & & \text { on } \partial \Omega,  \tag{1b}\\
\left.u\right|_{\Omega_{1}}-\left.u\right|_{\Omega_{2}} & =g_{D} & & \text { on } \Gamma,  \tag{1c}\\
\left.(A \nabla u)\right|_{\Omega_{1}} \cdot n_{1}+\left.(A \nabla u)\right|_{\Omega_{2}} \cdot n_{2} & =g_{N} & & \text { on } \Gamma, \tag{1d}
\end{align*}
$$

where $f, g_{D}, g_{N}$ are given functions, and $n_{1}, n_{2}$ are the unit normal vectors to $\Gamma$ outgoing from $\Omega_{1}, \Omega_{2}$, respectively. Moreover, $\left.u\right|_{\Omega_{1}}$ stands for the restriction of $u$ to $\Omega_{1}$ for example. We note that the gradient $\nabla u$ of the solution may be discontinuous across $\Gamma$, since $A$ may be discontinuous, even if $g_{D}=0$ and $g_{N}=0$.

Elliptic interface problem of the form (1) arises in many fields of applications such as fluid dynamics and solid mechanics. For instance, the first author has proposed (1) as a convenient model for computing sheath voltage wave form in the radio frequency plasma source within reasonable computational time (see [18]). The model involves the interface where the electronic potential and flux have nontrivial gaps; see also [9]. The case $g_{D}=0$, which is sometimes referred to as elliptic problem with discontinuous (diffusion) coefficients, is formulated as the standard elliptic variational problem in $H^{1}(\Omega)$ and numerical methods are studied by many authors; see [6, 2, 23, 5] for instance. On the other hand, the case $g_{D} \neq 0$ has further difficulties and a lot of numerical methods have been proposed (see [3, 17, 19] for example).
The present paper has dual purpose. The first one is to propose new schemes for solving (1) based on the hybridized discontinuous Galerkin (HDG) method. The HDG
method is a class of the discontinuous Galerkin (DG) method that is proposed by Cockburn et al. (see [7]; see also [14, 15, 21] for other pioneering works). In the HDG method, we introduce a new unknown function $\hat{u}_{h}$ on inter-element edges in addition to the usual unknown function $u_{h}$ in elements. We can eliminate $u_{h}$ from the resulting linear system and obtain the system only for $\hat{u}_{h}$; consequently, the size of the system becomes smaller than that of the DG method. In this paper, we present another advantage of the HDG method. That is, elliptic interface problem (1) is readily discretized by the HDG method and the resulting schemes (10) and (11) described below naturally satisfy the consistency (see Lemma 5) together with the Galerkin orthogonality (see (21)). It should be kept in mind that Huynh et al. [12] proposed an HDG scheme for (11). They introduced further unknown function $q=A \nabla u$ and rewrote (1) into the system for $(u, q, \hat{u})$ based on the idea of [7], while our unknowns are only ( $u, \hat{u}$ ) by following the idea of [21, 20]. Herein, $\hat{u}$ denotes the trace of $u$ into inter-element edges. Moreover, results of numerical experiments were well discussed and no theoretical consideration was undertaken in [12].

The second purpose of this paper is to establish error estimates for the HDG method when a sufficient regularity of solution, say $u \in H^{2}(\Omega)$, could not be assumed. Actually, if $g_{D} \neq 0$, the solution cannot be continuous across $\Gamma$. Moreover, we do not always have partial regularities $\left.u\right|_{\Omega_{i}} \in H^{2}\left(\Omega_{i}\right), i=1,2$. As a matter of fact, if $\partial \Omega \cap \Gamma \neq \emptyset$, then we know that $\left.u\right|_{\Omega_{i}}$ may not belong to $H^{2}\left(\Omega_{i}\right)$, even when $\Gamma$ and $\partial \Omega$ are sufficiently smooth; see Remark 2. To surmount of this obstacle, we employ the fractional order Sobolev space $H^{1+s}\left(\Omega_{i}\right), s \in(1 / 2,1], i=1,2$, and are going to attempt to derive an error estimate in an HDG norm $\|\cdot\|_{1+s, h}$ defined in terms of the $H^{1+s}\left(\Omega_{i}\right)$-seminorms (see (18)). One of our final error estimate reads (see Theorem 13)

$$
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{1+s, h} \leq C h^{s}\left(\|u\|_{H^{1+s}\left(\Omega_{1}\right)}+\|u\|_{H^{1+s}\left(\Omega_{2}\right)}\right),
$$

where $\boldsymbol{u}=(u, \hat{u})$ and $\boldsymbol{u}_{h}=\left(u_{h}, \hat{u}_{h}\right)$. Moreover, we also derive (see Theorem 14)

$$
\left\|u-u_{h}\right\|_{L^{2}(\Omega)} \leq C h^{2 s}\left(\|u\|_{H^{1+s}\left(\Omega_{1}\right)}+\|u\|_{H^{1+s}\left(\Omega_{2}\right)}\right)
$$

following the Aubin-Nitsche duality argument. To derive those inequalities, we improve the standard boundness inequality for the bilinear form (see Lemma 11) and inverse inequality (see Lemma 10) to fit our purpose. We note that those results are actually optimal order estimates, since we assume only $\left.u\right|_{\Omega_{1}} \in H^{1+s}\left(\Omega_{1}\right)$ and $\left.u\right|_{\Omega_{2}} \in H^{1+s}\left(\Omega_{2}\right)$.

In this paper, we concentrate our consideration on the case where $\Omega_{1}$ and $\Omega_{2}$ are polyhedral domains in order to avoid unessential complications about approximation of smooth surfaces/curves. The case of a smooth $\Gamma$ is of great interest; we postpone it for future study. On the other hand, we only consider the case $\partial \Omega \cap \Gamma \neq \emptyset$, since the modification to the case $\partial \Omega \cap \Gamma=\emptyset$ is readily and straightforward.

This paper is composed of five sections with an appendix. In Section 2, we recall the variational formulation of (1) and state our HDG schemes. The consistency is also proved there. The well-posedness of the schemes is verified in Section 3. Section 4 is devoted to error analysis using the fractional order Sobolev space. Finally, we conclude this paper by reporting numerical examples to confirm our error estimates in Section 5. In the appendix, we state the proof of a modification of inverse inequality (Lemma 10).

## 2. Variational formulation and HDG schemes

For the geometry of $\Omega \subset \mathbb{R}^{d}, d=2,3$, we assume the following:

$$
\begin{equation*}
\Omega, \Omega_{1}, \Omega_{2} \text { are all polyhedral domains and } \partial \Omega \cap \Gamma \neq \emptyset . \tag{H1}
\end{equation*}
$$

That is, we consider only Case (I) in Fig. 1.
To state a variational formulation, we need several function spaces. Namely, we use $L^{2}(\Omega), H^{m}(\Omega), m$ being a positive integer, $H_{0}^{1}(\Omega), L^{2}(\Gamma), H^{1 / 2}(\Gamma), H_{0}^{3 / 2}(\Gamma)$ and so on. We follow the notation of [16] for those Lebesgue and Sobolev spaces and their norms. The standard seminorm of $H^{m}(\Omega)$ is denoted by $|v|_{H^{m}(\Omega)}$. Supposing that $S$ is a part of $\partial \Omega$ or $\Gamma$, we let $\gamma(\Omega, S)$ be the trace operator from $H^{1}(\Omega)$ into $L^{2}(S)$. Set

$$
H_{\Gamma}^{1}\left(\Omega_{i}\right)=\left\{v \in H^{1}\left(\Omega_{i}\right) \mid \gamma\left(\Omega_{i}, \partial \Omega \cap \partial \Omega_{i}\right) v=0\right\}, \quad i=1,2 .
$$

Further set $\gamma_{i}=\gamma\left(\Omega_{i}, \Gamma\right), i=1,2$. We introduce

$$
V=\left\{v \in L^{2}(\Omega)|v|_{\Omega_{i}} \in H_{\Gamma}^{1}\left(\Omega_{i}\right), i=1,2\right\}
$$

and write $v_{i}=\left.v\right|_{\Omega_{i}}, i=1,2$, for $v \in V$.
Variational formulation of (11) is given as follows: Find $u \in V$ such that

$$
\begin{gather*}
\gamma_{1} u_{1}-\gamma_{2} u_{2}=g_{D} \quad \text { on } \Gamma,  \tag{2a}\\
a(u, v)=\int_{\Omega} f v d x+\int_{\Gamma} g_{N} v d S \quad\left(\forall v \in H_{0}^{1}(\Omega)\right), \tag{2b}
\end{gather*}
$$

where

$$
\begin{equation*}
a(u, v)=\int_{\Omega_{1}} A \nabla u_{1} \cdot \nabla v_{1} d x+\int_{\Omega_{2}} A \nabla u_{2} \cdot \nabla v_{2} d x \tag{2c}
\end{equation*}
$$

To state the well-posedness of Problem (2), we have to recall the so-called LionsMagenes space (see [16, §1.11.5])

$$
H_{00}^{1 / 2}(\Gamma)=\left\{\mu \in H^{1 / 2}(\Gamma) \mid \varrho^{-1 / 2} \mu \in L^{2}(\Gamma)\right\}
$$

which is a Hilbert space equipped with the norm $\|\mu\|_{H_{00}^{1 / 2}(\Gamma)}^{2}=\|\mu\|_{H^{1 / 2}(\Gamma)}^{2}+\left\|\varrho^{-1 / 2} \mu\right\|_{L^{2}(\Gamma)}^{2}$. Herein, $\varrho \in C^{\infty}(\bar{\Gamma})$ denotes any positive function satisfying $\left.\varrho\right|_{\partial \Gamma}=0$ and, for $x_{0} \in \partial \Gamma$, $\lim _{x \rightarrow x_{0}} \varrho(x) /$ dist $(x, \partial \Gamma)=\varrho_{0}>0$ with some $\varrho_{0}>0$. In particular, $H_{00}^{1 / 2}(\Gamma)$ is strictly included in $H^{1 / 2}(\Gamma)$. The following result follows directly from [10, Theorem 2.5] and [11, Theorem 1.5.2.3]. (A partial result is also reported in [24, Theorems 1.1 and 5.1].)

Lemma 1. The trace operator $v \mapsto \mu=\gamma_{1} v$ is a linear and continuous operator of $H_{\Gamma}^{1}\left(\Omega_{1}\right) \rightarrow H_{00}^{1 / 2}(\Gamma)$. Conversely, there exists a linear and continuous operator $\mathcal{E}_{1}$ of $H_{00}^{1 / 2}(\Gamma) \rightarrow H_{\Gamma}^{1}\left(\Omega_{1}\right)$, which is called a lifting operator, such that $\gamma_{1}\left(\mathcal{E}_{1} \mu\right)=\mu$ for all $\mu \in H_{00}^{1 / 2}(\Gamma)$. The same propositions remain true if $\gamma_{1}$ and $\Omega_{1}$ are replaced by $\gamma_{2}$ and $\Omega_{2}$, respectively.

Suppose that

$$
\begin{equation*}
f \in L^{2}(\Omega), \quad g_{D} \in H_{00}^{1 / 2}(\Gamma) \quad \text { and } \quad g_{N} \in L^{2}(\Gamma) \tag{H2}
\end{equation*}
$$

In view of Lemma 1 , there is $\tilde{g}_{D} \in V$ such that $\gamma_{1} \tilde{g}_{D}=\gamma_{2} \tilde{g}_{D}=g_{D}$ on $\Gamma$ and $\left\|\tilde{g}_{D}\right\|_{H^{1}(\Omega)} \leq$ $C\left\|g_{D}\right\|_{H_{00}^{1 / 2}(\Gamma)}$.

Hereinafter, the symbol $C$ denotes various generic positive constants depending on $\Omega$. In particular, it is independent of the discretization parameter $h$ introduced below. If it is necessary to specify the dependence on other parameters, say $\mu_{1}, \mu_{2}, \ldots$, then we write them as $C\left(\mu_{1}, \mu_{2}, \ldots\right)$.

Therefore, we can apply the Lax-Milgram theory to conclude that the problem (2) admits a unique solution $u \in V$ satisfying

$$
\left\|u_{1}\right\|_{H^{1}\left(\Omega_{1}\right)}+\left\|u_{2}\right\|_{H^{1}\left(\Omega_{2}\right)} \leq C\left(\|f\|_{L^{2}(\Omega)}+\left\|g_{D}\right\|_{H_{00}^{1 / 2}(\Gamma)}+\left\|g_{N}\right\|_{L^{2}(\Gamma)}\right),
$$

where $C=C(A)$.
Next we review the regularity property of the solution $u$. Suppose further that

$$
g_{D} \in H_{0}^{3 / 2}(\Gamma) \quad \text { and } \quad g_{N} \in H^{1 / 2}(\Gamma)
$$

However, in general, we do not expect that $u_{1} \in H^{2}\left(\Omega_{1}\right)$ and $u_{2} \in H^{2}\left(\Omega_{2}\right)$, because of the presence of intersection points $\Gamma \cap \partial \Omega$. (Even if we consider the case $\Gamma \cap \partial \Omega=\emptyset$, we may have $u_{1} \notin H^{2}\left(\Omega_{1}\right)$ and $u_{2} \notin H^{2}\left(\Omega_{2}\right)$.) To state regularity properties of $u_{1}$ and $u_{2}$, it is useful to introduce fractional order Sobolev spaces. We set

$$
\begin{equation*}
|v|_{H^{1+\theta}(\omega)}^{2}=\sum_{i=1}^{d} \iint_{\omega \times \omega} \frac{\left|\partial_{i} v(x)-\partial_{i} v(y)\right|^{2}}{|x-y|^{d+2 \theta}} d x d y \tag{3a}
\end{equation*}
$$

where $\omega \subset \mathbb{R}^{d}, \theta \in(0,1)$, and $\partial_{i}=\partial / \partial x_{i}$. Then, fractional order Sobolev spaces $H^{1+\theta}\left(\Omega_{i}\right), i=1,2$, are defined as

$$
\begin{equation*}
H^{1+\theta}\left(\Omega_{i}\right)=\left\{v \in H^{1}\left(\Omega_{i}\right)\left|\|v\|_{H^{1+\theta}\left(\Omega_{i}\right)}^{2}=\|v\|_{H^{1}\left(\Omega_{i}\right)}^{2}+|v|_{H^{1+\theta}\left(\Omega_{i}\right)}^{2}<\infty\right\} .\right. \tag{3b}
\end{equation*}
$$

We assume that

$$
g_{D} \in H_{0}^{s+1 / 2}(\Gamma) \quad \text { and } \quad g_{N} \in H^{s-1 / 2}(\Gamma)
$$

and that the solution $u \in V$ of (2) has the following regularity property,

$$
\left\{\begin{array}{l}
u_{1} \in H^{1+s}\left(\Omega_{1}\right), \quad u_{2} \in H^{1+s}\left(\Omega_{2}\right) \quad \text { and }  \tag{4}\\
N_{s}(u) \leq C\left(\|f\|_{L^{2}(\Omega)}+\left\|g_{D}\right\|_{H_{0}^{s+1 / 2}(\Gamma)}+\left\|g_{N}\right\|_{H^{s-1 / 2}(\Gamma)}\right)
\end{array}\right.
$$

for some $s \in(1 / 2,1]$, where $N_{s}(u)=\left\|u_{1}\right\|_{H^{1+s}\left(\Omega_{1}\right)}+\left\|u_{2}\right\|_{H^{1+s}\left(\Omega_{2}\right)}$ and $C=C(A)$.
Remark 2. We can find no explicit reference to (4). Nevertheless, we consider the problem under (4) on the analogy of Poisson interface problem. As an illustration, we consider the case $d=2$. Suppose that $x_{0}$ is an intersection point of $\partial \Omega$ and $\bar{\Gamma}$. We then
set $U=\mathscr{O} \cap \Omega$ and $U_{i}=U \cap \Omega_{i}, i=1,2$, where $\mathscr{O}$ is a neighbourhood of $x_{0}$. Assume that $U$ contains no corners of $\partial \Omega \cup \Gamma$ and no other intersection points except for $x_{0}$. Consider the unique solution $w \in H_{0}^{1}(\Omega)$ of

$$
\kappa_{1} \int_{\Omega_{1}} \nabla w \cdot \nabla v d x+\kappa_{2} \int_{\Omega_{2}} \nabla w \cdot \nabla v d x=\int_{\Omega} f v d x \quad\left(\forall v \in H_{0}^{1}(\Omega)\right)
$$

where $f \in L^{2}(\Omega)$ and $\kappa_{1}, \kappa_{2}$ are positive constants with $\kappa_{1} \neq \kappa_{2}$. Then, we have (see [22, Theorem 6.2])

$$
\left.w\right|_{\Omega_{i}} \in H^{1+\beta}\left(U_{i}\right), \quad i=1,2, \quad \beta=\min \left\{1, \frac{\pi}{2 \theta}\right\} \in(1 / 2,1]
$$

where $\theta$ denotes the maximum interior angle of $\partial \Omega_{1}$ and $\partial \Omega_{2}$ at $x_{0}$.
We proceed to the presentation of our HDG schemes. We introduce a family of quasi-uniform triangulations $\left\{\mathcal{T}_{h}\right\}_{h}$ of $\Omega$. That is, $\left\{\mathcal{T}_{h}\right\}_{h}$ is a family of shape-regular triangulations that satisfies the inverse assumptions (see [4, (4.4.15)]). Hereinafter, we set $h=\max \left\{h_{K} \mid K \in \mathcal{T}_{h}\right\}$, where $h_{K}$ denotes the diameter of $K$. Let $\mathcal{E}_{h}=\{e \subset$ $\left.\partial K \mid K \in \mathcal{T}_{h}\right\}$ be the set of all faces $(d=3) /$ edges $(d=2)$ of elements, and set $S_{h}=\cup_{K \in \mathcal{T}_{h}} \partial K=\cup_{e \in \mathcal{E}_{h}} e$. We assume that there is a positive constant $\nu_{1}$ which is independent of $h$ such that

$$
\begin{equation*}
\max \left\{\frac{h_{e}}{\rho_{K}}, \frac{h_{K}}{h_{e}}\right\} \leq \nu_{1} \quad\left(\forall e \subset \partial K, \forall K \in \forall \mathcal{T}_{h} \in\left\{\mathcal{T}_{h}\right\}_{h}\right) \tag{H3}
\end{equation*}
$$

where $h_{e}$ denotes the diameter of $e$ and $\rho_{K}$ the diameter of the inscribed ball of $K$.
We use the following function spaces:

$$
\begin{aligned}
H^{1+s}\left(\mathcal{T}_{h}\right) & =\left\{v \in L^{2}(\Omega)|v|_{K} \in H^{1+s}(K), K \in \mathcal{T}_{h}\right\} \\
L_{\partial \Omega}^{2}\left(S_{h}\right) & =\left\{\hat{v} \in L^{2}\left(S_{h}\right)|\hat{v}|_{e}=0, e \in \mathcal{E}_{h, \partial \Omega}\right\} \\
H_{\partial \Omega}^{1 / 2}\left(S_{h}\right) & =\left\{\hat{v} \in H^{1 / 2}\left(S_{h}\right)|\hat{v}|_{e}=0, e \in \mathcal{E}_{h, \partial \Omega}\right\} \\
\boldsymbol{V}^{1+s}(h) & =H^{1+s}\left(\mathcal{T}_{h}\right) \times H_{\partial \Omega}^{1 / 2}\left(S_{h}\right)
\end{aligned}
$$

for $s \in(1 / 2,1]$.
Further, we assume that

$$
\begin{equation*}
\text { there exists a subset } \mathcal{E}_{h, \Gamma} \text { of } \mathcal{E}_{h} \text { such that } \Gamma=\bigcup_{e \in \mathcal{E}_{h, \Gamma}} e \tag{H4}
\end{equation*}
$$

as shown for illustration in Fig. 2,
We then set $\mathcal{E}_{h, \partial \Omega}=\left\{e \in \mathcal{E}_{h} \mid e \subset \partial \Omega\right\}$ and $\mathcal{E}_{h, 0}=\mathcal{E}_{h} \backslash\left(\mathcal{E}_{h, \Gamma} \cup \mathcal{E}_{h, \partial \Omega}\right)$. Assumption (H4) implies that $\mathcal{T}_{h, i}=\left\{K \in \mathcal{T}_{h} \mid K \subset \overline{\Omega_{i}}\right\}$ is a triangulation of $\Omega_{i}$ for $i=1,2$ and we can write

$$
\begin{equation*}
a(u, v)=\sum_{K \in \mathcal{T}_{h}} \int_{K} A \nabla u \cdot \nabla v d x \tag{5}
\end{equation*}
$$



Figure 2: Triangulation satisfying (H4).

Throughout this paper, we always assume that (H1), (H2), (H2'), (H3) and (H4) are satisfied.
For derivation of our HDG schemes, we examine a local conservation property of the flux of the solution $u$. Let $K \in \mathcal{T}_{h}$. Recall that, if $u$ is suitably regular, we have by (1a) and Gauss-Green's formula

$$
\int_{\Omega}\left(A \nabla u \cdot n_{K}\right) w d S=\int_{K} A \nabla u \cdot \nabla w d x-\int_{K} f w d x
$$

for any $w \in H^{1}(K)$, where $n_{K}$ denotes the outer normal vector to $\partial K$. As mentioned above, the left-hand side of this identity is well-defined, since (4) is assumed for some $s \in(1 / 2,1]$. However, we derive local conservation properties (Lemmas 3 and 4 below) without using the further regularity property (4). That is, based on the identity above, we introduce a functional $\left\langle A \nabla u \cdot n_{K}, \cdot\right\rangle_{\partial K}$ on $H^{1 / 2}(\partial K)$ by

$$
\begin{equation*}
\left\langle A \nabla u \cdot n_{K}, \phi\right\rangle_{\partial K}=\int_{K} A \nabla u \cdot \nabla(Z \phi) d x-\int_{K} f(Z \phi) d x \tag{6}
\end{equation*}
$$

for any $\phi \in H^{1 / 2}(\partial K)$, where $Z \phi \in H^{1}(K)$ denotes a suitable extension of $\phi$ such that $\|Z \phi\|_{H^{1}(K)} \leq C\|\phi\|_{H^{1 / 2}(\partial K)}$. Actually, the definition of $\left\langle A \nabla u \cdot n_{K}, \cdot\right\rangle_{\partial K}$ above does not depend on the way of extension of $\phi$. Below, for the solution $u$ of (1), we simply write

$$
\begin{equation*}
\int_{\partial K}\left(A \nabla u \cdot n_{K}\right) \phi d S=\int_{K} A \nabla u \cdot \nabla(Z \phi) d x-\int_{K} f(Z \phi) d x \tag{7}
\end{equation*}
$$

to express (6).
The following lemmas are readily obtainable consequences of (5) and (7).
Lemma 3. For the solution $u$ of (2), we have

$$
\begin{equation*}
\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left(A \nabla u \cdot n_{K}\right) \hat{v} d S=\int_{\Gamma} g_{N} \hat{v} d S \quad\left(\hat{v} \in H_{\partial \Omega}^{1 / 2}\left(S_{h}\right)\right) \tag{8}
\end{equation*}
$$

Lemma 4. For the solution $u$ of (2), we have

$$
\begin{align*}
\sum_{K \in \mathcal{T}_{h}} \int_{K} A \nabla u \cdot \nabla v d x+\sum_{K \in \mathcal{T}_{h}} & \int_{\partial K}\left(A \nabla u \cdot n_{K}\right)(\hat{v}-v) d S \\
& =\sum_{K \in \mathcal{T}_{h}} \int_{K} f v d x+\int_{\Gamma} g_{N} \hat{v} d S \quad\left((v, \hat{v}) \in \boldsymbol{V}^{1}(h)\right) . \tag{9}
\end{align*}
$$

We discretize the expression (9) by the idea of HDG. We use the following finite element spaces:

$$
\begin{aligned}
& \boldsymbol{V}_{h}=V_{h} \times \hat{V}_{h} \\
& V_{h}=V_{h, k}=\left\{v \in H^{1}\left(\mathcal{T}_{h}\right)|v|_{K} \in P_{k}(K), K \in \mathcal{T}_{h}\right\}, \quad k \geq 1: \text { integer; } \\
& \hat{V}_{h}=\hat{V}_{h, l}=\left\{\hat{v} \in L_{\partial \Omega}^{2}\left(S_{h}\right)|\hat{v}|_{e} \in P_{l}(e), e \in \mathcal{E}_{h, 0} \cup \mathcal{E}_{h, \Gamma}\right\}, \quad l \geq 1: \text { integer, }
\end{aligned}
$$

where $P_{k}(K)$ denotes the set of all polynomials defined in $K$ with degree $\leq k$.
At this stage, we can state our scheme: Find $\boldsymbol{u}_{h}=\left(u_{h}, \hat{u}_{h}\right) \in \boldsymbol{V}_{h}$ such that

$$
\begin{equation*}
B_{h}\left(\boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)=L_{h}\left(\boldsymbol{v}_{h}\right) \quad\left(\forall \boldsymbol{v}_{h}=\left(v_{h}, \hat{v}_{h}\right) \in \boldsymbol{V}_{h}\right), \tag{10a}
\end{equation*}
$$

where

$$
\begin{align*}
& B_{h}\left(\boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)=\underbrace{\sum_{K \in \mathcal{T}_{h}} \int_{K} A \nabla u_{h} \cdot \nabla v_{h} d x}_{=B_{1}} \underbrace{-\underbrace{\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left(A \nabla u_{h} \cdot n_{K}\right)\left(v_{h}-\hat{v}_{h}\right) d S}_{=B_{2}}}_{=B_{3}} \\
& \underbrace{\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left(A \nabla v_{h} \cdot n_{K}\right)\left(u_{h}-\hat{u}_{h}\right) d S}_{=B_{4}}+\underbrace{}_{\sum_{K \in \mathcal{T}_{h}} \sum_{e \subset \partial K} \int_{e} \frac{\eta_{e}}{h_{e}}\left(u_{h}-\hat{u}_{h}\right)\left(v_{h}-\hat{v}_{h}\right) d S} \tag{10b}
\end{align*}
$$

and

$$
\begin{align*}
L_{h}\left(\boldsymbol{v}_{h}\right)=\underbrace{\sum_{K \in \mathcal{T}_{h}} \int_{K} f v_{h} d x+\int_{\Gamma} g_{N} \hat{v} d S}_{=L_{1}} & \underbrace{-\int_{\Gamma} g_{D}\left(A \nabla v_{h} \cdot n_{1}\right) d S}_{=L_{2}} \\
& \underbrace{\sum_{e \in \mathcal{E}_{h, \Gamma}} \int_{e} \frac{\sigma_{K, e}}{2} \frac{\eta_{e}}{h_{e}} g_{D}\left(v_{h}-\hat{v}_{h}\right) d S}_{=L_{3}} . \tag{10c}
\end{align*}
$$

Therein, $\sigma_{K, e}$ is defined by

$$
\sigma_{K, e}= \begin{cases}1 & \left(K \in \mathcal{T}_{h, 1}\right)  \tag{10d}\\ -1 & \left(K \in \mathcal{T}_{h, 2}\right)\end{cases}
$$

and $\eta_{e}$ denotes the penalty parameter such that

$$
\begin{equation*}
0<\eta_{\min }=\inf _{\mathcal{T}_{h} \in\left\{\mathcal{T}_{h}\right\}_{h}} \min _{e \in \mathcal{E}_{h}} \eta_{e}, \quad \eta_{\max }=\sup _{\mathcal{T}_{h} \in\left\{\mathcal{T}_{h}\right\}_{h}} \max _{e \in \mathcal{E}_{h}} \eta_{e}<\infty . \tag{10e}
\end{equation*}
$$

The main advantage of the scheme 10 is stated as the following lemma.
Lemma 5 (Consistency). Let $u \in V$ be the solution of (2) and introduce $\hat{u} \in H_{\partial \Omega}^{1 / 2}\left(S_{h}\right)$ by

$$
\hat{u}= \begin{cases}\frac{1}{2}\left[\gamma\left(K_{1}, e\right) u+\gamma\left(K_{2}, e\right) u\right] & \left(e \in \mathcal{E}_{h, 0} \cup \mathcal{E}_{h, \Gamma}, e=\partial K_{1} \cap \partial K_{2}\right) \\ \gamma(K, e) u & \left(e \in \mathcal{E}_{h, \partial \Omega, e \subset \partial K)}\right.\end{cases}
$$

Then, $\boldsymbol{u}=(u, \hat{u}) \in \boldsymbol{V}^{1}(h)$ solves

$$
B_{h}\left(\boldsymbol{u}, \boldsymbol{v}_{h}\right)=L_{h}\left(\boldsymbol{v}_{h}\right) \quad\left(\forall \boldsymbol{v}_{h} \in \boldsymbol{V}_{h}\right)
$$

Proof. In view of Lemma 4, we know that $B_{1}+B_{2}=L_{1}$. We show that $B_{3}=L_{2}$ and $B_{4}=L_{3}$. For $e \in \mathcal{E}_{h, 0} \cap \mathcal{E}_{h, \partial \Omega}$, we have $u-\hat{u}=0$ on $e$, since $\hat{u}=\gamma(\Omega, \Gamma) u$ on $e$. Hence,

$$
\begin{aligned}
B_{3} & =\sum_{e \in \mathcal{E}_{h, \Gamma}}\left[\int_{e}\left(A \nabla v_{h} \cdot n_{1}\right)\left(u_{h}-\hat{u}_{h}\right) d S+\int_{e}\left(A \nabla v_{h} \cdot n_{2}\right)\left(u_{h}-\hat{u}_{h}\right) d S\right] \\
& =\sum_{e \in \mathcal{E}_{h, \Gamma}}\left[\int_{e}\left(A \nabla v_{h} \cdot n_{1}\right) \frac{u_{1}-u_{2}}{2} d S-\int_{e}\left(A \nabla v_{h} \cdot n_{1}\right) \frac{u_{2}-u_{1}}{2} d S\right] \\
& =\int_{\Gamma}\left(A \nabla v_{h} \cdot n_{1}\right)\left(u_{1}-u_{2}\right) d S,
\end{aligned}
$$

where $e=\partial K_{1} \cap \partial K_{2}$ with $K_{1} \in \mathcal{T}_{h, 1}$ and $K_{2} \in \mathcal{T}_{h, 2}$. This, together with (2a), gives $B_{3}=L_{2}$. Using the same notion, we have

$$
\begin{aligned}
B_{4} & =\sum_{e \in \mathcal{E}_{h, \Gamma}}\left[\int_{e} \frac{\eta_{e}}{h_{e}}\left(u_{1}-\hat{u}\right)\left(v_{h, 1}-\hat{v}_{h}\right) d S+\int_{e} \frac{\eta_{e}}{h_{e}}\left(u_{2}-\hat{u}\right)\left(v_{h, 2}-\hat{v}_{h}\right) d S\right] \\
& =\sum_{e \in \mathcal{E}_{h, \Gamma}}\left[\int_{e} \frac{\eta_{e}}{h_{e}} \frac{g_{D}}{2}\left(v_{h, 1}-\hat{v}_{h}\right) d S-\int_{e} \frac{\eta_{e}}{h_{e}} \frac{g_{D}}{2}\left(v_{h, 2}-\hat{v}_{h}\right) d S\right]=L_{3},
\end{aligned}
$$

which completes the proof.
An alternative scheme is given as

$$
\begin{equation*}
B_{h}\left(\boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)=L_{h}^{\prime}\left(\boldsymbol{v}_{h}\right) \quad\left(\forall \boldsymbol{v}_{h} \in \boldsymbol{V}_{h}\right) \tag{11a}
\end{equation*}
$$

where

$$
\begin{equation*}
L_{h}^{\prime}\left(\boldsymbol{v}_{h}\right)=L_{1}+L_{2}+\sum_{e \in \mathcal{E}_{h, \Gamma}} \int_{e} \sigma_{K, e}^{\prime} \frac{\eta_{e}}{h_{e}} g_{D}\left(v_{h}-\hat{v}_{h}\right) d S \tag{11b}
\end{equation*}
$$

and

$$
\sigma_{K, e}^{\prime}= \begin{cases}1 & \left(K \in \mathcal{T}_{h, 1}\right)  \tag{11c}\\ 0 & \left(K \in \mathcal{T}_{h, 2}\right)\end{cases}
$$

Lemma 5 remains valid for with an obvious modification of the definition of $\hat{u}$. Therefore, all the following results also remain true for (11). Hence, we explicitly study only (10) below.
Remark 6. We restrict ourselves to simplicial triangulations; that is, we are assuming that each $K \in \mathcal{T}_{h} \in\left\{\mathcal{T}_{h}\right\}_{h}$ is a $d$-simplex. However, we are able to consider more general shape of elements. For example, for $d=2$, each $K$ could be an $m$-polygonal domain, where $m$ is an integer and can differ with $K$. We assume that $m$ is bounded from above independently of a family of triangulations and $\partial K$ does not intersect with itself. In particular, we can consider rectangular meshes as well. Moreover, $\boldsymbol{V}_{h}$ could be replaced by any finite dimensional subspace $\boldsymbol{V}_{h}^{\prime}$ of $\boldsymbol{V}^{1}(h)$. See [20, 21] for the detail of modifications.

## 3. Well-posedness

In this section, we establish the well-posedness of the scheme 10). First, we recall the following standard results; (12) is the standard inverse inequality (see [4, Lemma 4.5.3]) and (13) follows from the standard trace inequalities (see also Appendix A).

Lemma 7. For $K \in \mathcal{T}_{h}$, we have following inequalities.
(Inverse inequality)

$$
\begin{equation*}
\left|v_{h}\right|_{H^{1}(K)} \leq C_{\mathrm{IV}} h_{K}^{-1}\left\|v_{h}\right\|_{L^{2}(K)} \quad\left(v_{h} \in V_{h}\right) \tag{12}
\end{equation*}
$$

(Trace inequalities)

$$
\begin{array}{rlr}
\|v\|_{L^{2}(e)}^{2} \leq C_{0, \mathrm{~T}} h_{e}^{-1}\left(\|v\|_{L^{2}(K)}^{2}+h_{K}^{2}|v|_{H^{1}(K)}^{2}\right) & \left(v \in H^{1}(K)\right), \\
\|\nabla v\|_{L^{2}(e)}^{2} \leq C_{1, \mathrm{~T}} h_{e}^{-1}\left(\|v\|_{H^{1}(K)}^{2}+h_{K}^{2}|v|_{H^{2}(K)}^{2}\right) & & \left(v \in H^{2}(K)\right) \tag{13b}
\end{array}
$$

Those $C_{\mathrm{IV}}, C_{0, \mathrm{~T}}$ and $C_{1, \mathrm{~T}}$ are absolute positive constants.
We use the following HDG norms:

$$
\begin{align*}
& \|\boldsymbol{v}\|_{1, h}^{2}=\sum_{K \in \mathcal{T}_{h}}|v|_{H^{1}(K)}^{2}+\sum_{K \in \mathcal{T}_{h}} \sum_{e \subset \partial K} \frac{\eta_{e}}{h_{e}}\|\hat{v}-v\|_{L^{2}(e)}^{2} ;  \tag{14a}\\
& \|\boldsymbol{v}\|_{2, h}^{2}=\sum_{K \in \mathcal{T}_{h}}|v|_{H^{1}(K)}^{2}+\sum_{K \in \mathcal{T}_{h}} h_{K}^{2}|v|_{H^{2}(K)}^{2}+\sum_{K \in \mathcal{T}_{h}} \sum_{e \subset \partial K} \frac{\eta_{e}}{h_{e}}\|\hat{v}-v\|_{L^{2}(e)}^{2} . \tag{14b}
\end{align*}
$$

Moreover, set

$$
\alpha=\max \left\{\sup _{x \in \Omega_{1}} \sup _{\xi \in \mathbb{R}^{d}} \frac{|A(x) \xi|}{|\xi|}, \sup _{x \in \Omega_{2}} \sup _{\xi \in \mathbb{R}^{d}} \frac{|A(x) \xi|}{|\xi|}\right\} .
$$

Remark 8. In view of 12 , two norms $\|\boldsymbol{v}\|_{1, h}$ and $\|\boldsymbol{v}\|_{2, h}$ are equivalent norms in the finite dimensional space $\boldsymbol{V}_{h}$. That is, there exists a positive constant $C_{0}$ that depends only on $C_{\text {IV }}$ such that

$$
\begin{equation*}
\|\boldsymbol{v}\|_{1, h} \leq\|\boldsymbol{v}\|_{2, h} \leq C_{0}\|\boldsymbol{v}\|_{1, h} \quad\left(\boldsymbol{v}_{h} \in \boldsymbol{V}_{h}\right) \tag{15}
\end{equation*}
$$

Lemma 9. (Boundness) For any $\eta_{\min }>0$, there exists a positive constant $C_{\mathrm{b}}=$ $C_{\mathrm{b}}\left(\alpha, \eta_{\text {min }}, d, C_{1, \mathrm{~T}}\right)$ such that

$$
\begin{equation*}
B_{h}(\boldsymbol{w}, \boldsymbol{v}) \leq C_{\mathbf{b}}\|\boldsymbol{w}\|_{2, h}\|\boldsymbol{v}\|_{2, h} \quad\left(\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{V}^{2}(h)\right) . \tag{16}
\end{equation*}
$$

(Coercivity) There exist positive constants $\eta^{*}=\eta^{*}\left(\alpha, \lambda_{\min }, d, C_{1, \mathrm{~T}}, C_{\mathrm{IV}}\right)$ and $C_{\mathrm{c}}=$ $C_{\mathrm{c}}\left(\lambda_{\text {min }}, C_{\mathrm{IV}}\right)$ such that, if $\eta_{\text {min }} \geq \eta^{*}$, we have

$$
\begin{equation*}
B_{h}\left(\boldsymbol{v}_{h}, \boldsymbol{v}_{h}\right) \geq C_{\mathrm{c}}\left\|\boldsymbol{v}_{h}\right\|_{2, h}^{2} \quad\left(\boldsymbol{v}_{h} \in \boldsymbol{V}_{h}\right) \tag{17}
\end{equation*}
$$

Both inequalities are essentially well-known; however, we briefly state their proofs, since the contribution of parameters on $C_{\mathrm{c}}$ and $C_{\mathrm{b}}$ should be clarified. Moreover, we shall state the extension of (16) below (see Lemma 11) so it is useful to recall the proof of (16) at this stage.

Proof of Lemma 9. (Boundness) Let $\boldsymbol{w}=(w, \hat{w}), \boldsymbol{v}=(v, \hat{v}) \in \boldsymbol{V}^{2}(h)$. For $e \subset \partial K$, $K \in \mathcal{T}_{h}$, we have by Schwarz' inequality

$$
\left|\int_{e}\left(A \nabla w \cdot n_{K}\right)(v-\hat{v}) d S\right| \leq \alpha\left(\frac{h_{e}}{\eta_{e}}\right)^{1 / 2}\|\nabla w\|_{L^{2}(e)} \cdot\left(\frac{\eta_{e}}{h_{e}}\right)^{1 / 2}\|v-\hat{v}\|_{L^{2}(e)}
$$

Hence, using Schwarz' inequality again,

$$
\begin{aligned}
B_{h}(\boldsymbol{w}, \boldsymbol{v}) & \leq \sum_{K \in \mathcal{T}} \alpha|w|_{H^{1}(K)}|v|_{H^{1}(K)} \\
& +\sum_{K \in \mathcal{T}_{h}} \sum_{e \subset \partial K} \frac{\alpha}{\eta_{\min }^{1 / 2}} h_{e}^{1 / 2}\|\nabla w\|_{L^{2}(e)} \cdot\left(\frac{\eta_{e}}{h_{e}}\right)^{1 / 2}\|v-\hat{v}\|_{L^{2}(e)} \\
& +\sum_{K \in \mathcal{T}_{h}} \sum_{e \subset \partial K} \frac{\alpha}{\eta_{\min }^{1 / 2}} h_{e}^{1 / 2}\|\nabla v\|_{L^{2}(e)} \cdot\left(\frac{\eta_{e}}{h_{e}}\right)^{1 / 2}\|w-\hat{w}\|_{L^{2}(e)} \\
& +\sum_{K \in \mathcal{T}_{h}} \sum_{e \subset \partial K} \frac{\eta_{e}}{h_{e}}\|w-\hat{w}\|_{L^{2}(e)} \cdot \frac{\eta_{e}}{h_{e}}\|v-\hat{v}\|_{L^{2}(e)} \\
\leq & C\left[\sum_{K \in \mathcal{T}_{h}}|w|_{H^{1}(K)}^{2}+\sum_{e \subset \partial K}\left(h_{e}^{-1}\|\nabla w\|_{L^{2}(e)}^{2}+\frac{\eta_{e}}{h_{e}}\|w-\hat{w}\|_{L^{2}(e)}^{2}\right)\right]^{1 / 2} \\
& \cdot\left[\sum_{K \in \mathcal{T}_{h}}|v|_{H^{1}(K)}^{2}+\sum_{e \subset \partial K}\left(h_{e}^{-1}\|\nabla v\|_{L^{2}(e)}^{2}+\frac{\eta_{e}}{h_{e}}\|v-\hat{v}\|_{L^{2}(e)}^{2}\right)\right]^{1 / 2}
\end{aligned}
$$

Therefore, using (13b), we obtain 16).
(Coercivity) Let $\boldsymbol{v}_{h}=\left(v_{h}, \hat{v}_{h}\right) \in \boldsymbol{V}_{h}$. Then,

$$
\begin{aligned}
B_{h}\left(\boldsymbol{v}_{h}, \boldsymbol{v}_{h}\right) \geq \lambda_{\min } & \sum_{K \in \mathcal{T}_{h}}\left|v_{h}\right|_{H^{1}(K)}^{2} \\
& +\sum_{K \in \mathcal{T}_{h}} \sum_{e \subset \partial K} \frac{\eta_{e}}{h_{e}}\|\hat{v}-v\|_{L^{2}(e)}^{2}-2 \sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left(A \nabla v_{h} \cdot n_{K}\right)\left(v_{h}-\hat{v}_{h}\right) d S .
\end{aligned}
$$

Letting $e \subset \partial K, K \in \mathcal{T}_{h}$, we have by (13b), (12), Schwarz' and Young's inequalities

$$
\begin{aligned}
& \left|\int_{e}\left(A \nabla v_{h} \cdot n_{K}\right)\left(v_{h}-\hat{v}_{h}\right) d S\right| \\
& \leq \alpha\left\|\nabla v_{h}\right\|_{L^{2}(e)}\left\|v_{h}-\hat{v}_{h}\right\|_{L^{2}(e)} \\
& \leq \alpha C_{1, \mathrm{~T}} h_{e}^{-1 / 2}\left(\left|v_{h}\right|_{H^{1}(K)}^{2}+h_{K}^{2}\left|v_{h}\right|_{H^{2}(K)}^{2}\right)^{1 / 2} \cdot\left\|v_{h}-\hat{v}_{h}\right\|_{L^{2}(e)} \\
& \leq C_{*}\left(\delta \eta_{e}\right)^{-1 / 2}\left|v_{h}\right|_{H^{1}(K)} \cdot\left(\frac{\eta_{e} \delta}{h_{e}}\right)^{1 / 2}\left\|v_{h}-\hat{v}_{h}\right\|_{L^{2}(e)} \\
& \leq \frac{C_{*}^{2}}{\delta \eta_{e}}\left|v_{h}\right|_{H^{1}(K)}^{2}+\delta \frac{\eta_{e}}{h_{e}}\left\|v_{h}-\hat{v}_{h}\right\|_{L^{2}(e)}^{2},
\end{aligned}
$$

where $C_{*}=C_{*}\left(\alpha, d, C_{1, \mathrm{~T}}, C_{\mathrm{IV}}\right)$ and $\delta$ is a positive constant specified later. Using this, we deduce

$$
\begin{aligned}
\left.B_{h}\left(\boldsymbol{v}_{h}, \boldsymbol{v}_{h}\right) \geq\left[\lambda_{\min }-2(d+1) \frac{C_{*}^{2}}{\delta \eta_{\min }}\right] \sum_{K \in \mathcal{T}_{h}} \right\rvert\, & \left|v_{h}\right|_{H^{1}(K)}^{2} \\
& +(1-2 \delta) \sum_{K \in \mathcal{T}_{h}} \sum_{e \subset \partial K} \frac{\eta_{e}}{h_{e}}\|\hat{v}-v\|_{L^{2}(e)}^{2} .
\end{aligned}
$$

At this stage, choosing $\delta$ and $\eta_{\text {min }}$ such that

$$
0<\delta \leq \frac{1}{4}, \quad \eta_{\min } \geq 4(d+1) \frac{C_{*}^{2}}{\lambda_{\min } \delta}
$$

we obtain

$$
B_{h}\left(\boldsymbol{v}_{h}, \boldsymbol{v}_{h}\right) \geq \frac{1}{2} \min \left\{1, \lambda_{\min }\right\}\left\|\boldsymbol{v}_{h}\right\|_{1, h}^{2},
$$

which, together with (15), implies (17).

## 4. Error analysis

This section is devoted to error analysis of our HDG scheme. We use a new HDG norm:

$$
\begin{equation*}
\|\boldsymbol{v}\|_{1+s, h}^{2}=\sum_{K \in \mathcal{T}_{h}}|v|_{H^{1}(K)}^{2}+\sum_{K \in \mathcal{T}_{h}} h_{K}^{2 s}|v|_{H^{1+s}(K)}^{2}+\sum_{K \in \mathcal{T}_{h}} \sum_{e \subset \partial K} \frac{\eta_{e}}{h_{e}}\|\hat{v}-v\|_{L^{2}(e)}^{2} \tag{18}
\end{equation*}
$$

for $s \in(1 / 2,1)$.
We have to improve Lemmas 7 and 9 for our purpose. First, the trace inequality for functions of $H^{1+s}(K)$ is given as follows; the proof will be stated in Appendix A.

Lemma 10. (Trace inequality) Let $s \in(1 / 2,1)$. For $K \in \mathcal{T}_{h}$, we have

$$
\begin{equation*}
\|\nabla v\|_{L^{2}(e)}^{2} \leq C_{1+s, \mathrm{~T}} h_{e}^{-1}\left(|v|_{H^{1}(K)}^{2}+h_{K}^{2 s}|v|_{H^{1+s}(K)}^{2}\right) \quad\left(v \in H^{1+s}(K)\right) . \tag{19}
\end{equation*}
$$

Moreover, we deduce the following lemma in exactly the same way as the proof of Lemma 9 using (19) instead of 13b).

Lemma 11. Let $s, t \in(1 / 2,1]$. For any $\eta_{\min }>0$, there exists a positive constant $C_{\mathrm{b}, s, t}=C_{\mathrm{b}, \mathrm{s}, t}\left(\alpha, \eta_{\min }, d, C_{1+s, \mathrm{~T}}, C_{1+t, \mathrm{~T}}, s, t\right)$ such that

$$
\begin{equation*}
B_{h}(\boldsymbol{w}, \boldsymbol{v}) \leq C_{\mathrm{b}, s, t}\|\boldsymbol{w}\|_{1+s, h}\|\boldsymbol{v}\|_{1+t, h} \quad\left(\boldsymbol{w} \in \boldsymbol{V}^{1+s}(h), \boldsymbol{v} \in \boldsymbol{V}^{1+t}(h)\right) . \tag{20}
\end{equation*}
$$

Theorem 12. Let $u \in V$ be the solution of (2) and assume that (4) for some $s \in$ $(1 / 2,1]$. Set $\boldsymbol{u} \in \boldsymbol{V}^{1+s}(h)$ as in Lemma 5. Moreover, let $\boldsymbol{u}_{h}=\left(u_{h}, \hat{u}_{h}\right) \in \boldsymbol{V}_{h}$ be the solution of (10). Then, we have the Galerkin orthogonality

$$
\begin{equation*}
B_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)=0 \quad\left(\forall \boldsymbol{v}_{h} \in \boldsymbol{V}_{h}\right) . \tag{21}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{1+s, h} \leq C \inf _{\boldsymbol{v}_{h} \in \boldsymbol{V}_{h}}\left\|\boldsymbol{u}-\boldsymbol{v}_{h}\right\|_{1+s, h} \tag{22}
\end{equation*}
$$

Proof. Let $\boldsymbol{v}_{h} \in V_{h}$ be arbitrarily. Then, (21) is a consequence of 10 and Lemma 5 . On the other hand,

$$
\begin{aligned}
C_{\mathrm{c}}\left\|\boldsymbol{v}_{h}-\boldsymbol{u}_{h}\right\|_{2, h}^{2} & \leq B_{h}\left(\boldsymbol{v}_{h}-\boldsymbol{u}_{h}, \boldsymbol{v}_{h}-\boldsymbol{u}_{h}\right) \\
& \leq B_{h}\left(\boldsymbol{v}_{h}-\boldsymbol{u}, \boldsymbol{v}_{h}-\boldsymbol{u}_{h}\right)+B_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}, \boldsymbol{v}_{h}-\boldsymbol{u}_{h}\right) \\
& \leq B_{h}\left(\boldsymbol{v}_{h}-\boldsymbol{u}, \boldsymbol{v}_{h}-\boldsymbol{u}_{h}\right) \\
& \leq C_{\mathrm{b}, s, 1}\left\|\boldsymbol{v}_{h}-\boldsymbol{u}\right\|_{1+s, h}\left\|\boldsymbol{v}_{h}-\boldsymbol{u}_{h}\right\|_{2, h}
\end{aligned}
$$

This implies

$$
\left\|\boldsymbol{v}_{h}-\boldsymbol{u}_{h}\right\|_{2, h} \leq \frac{C_{\mathrm{b}, s, 1}}{C_{\mathrm{c}}}\left\|\boldsymbol{v}_{h}-\boldsymbol{u}\right\|_{1+s, h}
$$

We apply the triangle inequality to obtain

$$
\begin{aligned}
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{1+s, h} & \leq\left\|\boldsymbol{u}-\boldsymbol{v}_{h}\right\|_{1+s, h}+C\left\|\boldsymbol{v}_{h}-\boldsymbol{u}_{h}\right\|_{2, h} \\
& \leq\left\|\boldsymbol{u}-\boldsymbol{v}_{h}\right\|_{1+s, h}+C\left\|\boldsymbol{v}_{h}-\boldsymbol{u}\right\|_{1+s, h},
\end{aligned}
$$

which gives (22).
Theorem 13. Under the same assumptions of Theorem 12, we have

$$
\begin{equation*}
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{1+s, h} \leq C h^{s}\left(\|u\|_{H^{1+s}\left(\Omega_{1}\right)}+\|u\|_{H^{1+s}\left(\Omega_{2}\right)}\right) \tag{23}
\end{equation*}
$$

Proof. It is done by the standard method; see [1, Paragraph 4.3] for example. However, we state the proof, since it is not apparent how to estimate the third term of the left-hand side of 14 b$)$. First, we introduce $u_{I} \in V_{h}$ as follows. Let $K \in \mathcal{T}_{h}$ and let $u_{I, K}=\left.\left(u_{I}\right)\right|_{K} \in \overline{P_{k}(K)}$ be the Lagrange interpolation of $\left.u\right|_{K}$. We remark here that $u_{I}$ is well-defined, since $\left.u\right|_{K} \in H^{1+s}(K)$. Further, we introduce $\hat{u}_{I} \in \hat{V}_{h}$ by setting $\left.\hat{u}_{I}\right|_{e}=\left(\left.u_{I, K_{1}}\right|_{e}+\left.u_{I, K_{2}}\right|_{e}\right) / 2$ for $e \in \mathcal{E}_{h, 0} \cup \mathcal{E}_{h, \Gamma}, e=\partial K_{1} \cap \partial K_{2}$ and $\left.\hat{u}_{I}\right|_{e}=\left.u_{I, K}\right|_{e}$ for
$e \in \mathcal{E}_{h, \partial \Omega}, e \subset \partial K$. Then, letting $\boldsymbol{w}_{h}=\left(u_{I}, \hat{u}_{I}\right) \in \boldsymbol{V}_{h}$, we derive an estimation for $\left\|\boldsymbol{u}-\boldsymbol{w}_{h}\right\|_{1+s, h}$.

For $e \in \mathcal{E}_{h, 0} \cup \mathcal{E}_{h, \Gamma}, e \subset \partial K$, we have by 13a

$$
\frac{\eta_{e}}{h_{e}}\left\|u-u_{I}\right\|_{L^{2}(e)}^{2} \leq C h_{e}^{-2}\left(\left\|u-u_{I}\right\|_{L^{2}(K)}^{2}+h_{K}^{2}\left|u-u_{I}\right|_{H^{1}(K)}^{2}\right)
$$

Hence, using (H3),

$$
\sum_{K \in \mathcal{T}_{h}} \sum_{e \subset \partial K} \frac{\eta_{e}}{h_{e}}\left\|u-u_{I}\right\|_{L^{2}(e)}^{2} \leq C \sum_{K \in \mathcal{T}_{h}}\left(h_{K}^{-2}\left\|u-u_{I}\right\|_{L^{2}(K)}^{2}+\left|u-u_{I}\right|_{H^{1}(K)}^{2}\right)
$$

On the other hand, for $e \in \mathcal{E}_{h, 0} \cup \mathcal{E}_{h, \Gamma}, e=\partial K_{1} \cap \partial K_{2}$,

$$
\left\|\hat{u}-\hat{u}_{I}\right\|_{L^{2}(e)}^{2} \leq C\left(\left\|\left.u\right|_{K_{1}}-u_{I, K_{1}}\right\|_{L^{2}(e)}^{2}+\left\|\left.u\right|_{K_{2}}-u_{I, K_{2}}\right\|_{L^{2}(e)}^{2}\right)
$$

Therefore, as above, we have

$$
\sum_{K \in \mathcal{T}_{h}} \sum_{e \subset \partial K} \frac{\eta_{e}}{h_{e}}\left\|\hat{u}-\hat{u}_{I}\right\|_{L^{2}(e)}^{2} \leq C \sum_{K \in \mathcal{T}_{h}}\left(h_{K}^{-2}\left\|u-u_{I}\right\|_{L^{2}(K)}^{2}+\left|u-u_{I}\right|_{H^{1}(K)}^{2}\right)
$$

Consequently, we obtain

$$
\begin{aligned}
& \left\|\boldsymbol{u}-\boldsymbol{w}_{h}\right\|_{1+s, h}^{2} \\
& \qquad C \sum_{K \in \mathcal{T}_{h}}\left(h_{K}^{-2}\left\|u-u_{I}\right\|_{L^{2}(K)}^{2}+\left|u-u_{I}\right|_{H^{1}(K)}^{2}+h_{K}^{2 s}\left|u-u_{I}\right|_{H^{1+s}(K)}^{2}\right) .
\end{aligned}
$$

At this stage, we recall

$$
\left|u-u_{I}\right|_{H^{t}(K)} \leq C h_{K}^{s+1-t}|u|_{H^{1+s}(K)} \quad(0 \leq t \leq 2)
$$

where $|\cdot|_{H^{0}(K)}$ is understood as $\|\cdot\|_{L^{2}(K)}$. See, for example, [8, Theorems 2.19, 2.22] where the case of integer $t$ is explicitly mentioned. However, the extension to the case of non-integer $t \in[0,1+s]$ is straightforward, since the imbedding $H^{t}(K) \subset H^{1+s}(K)$ is continuous. Combining those inequalities, we deduce

$$
\left\|\boldsymbol{u}-\boldsymbol{w}_{h}\right\|_{1+s, h} \leq C h^{s}\left(\|u\|_{H^{1+s}\left(\Omega_{1}\right)}+\|u\|_{H^{1+s}\left(\Omega_{2}\right)}\right)
$$

which completes the proof.
Theorem 14. Under the same assumptions of Theorem 12, we have

$$
\left\|u-u_{h}\right\|_{L^{2}(\Omega)} \leq C h^{2 s}\left(\|u\|_{H^{1+s}\left(\Omega_{1}\right)}+\|u\|_{H^{1+s}\left(\Omega_{2}\right)}\right)
$$

Proof. We follow the Aubin-Nitsche duality argument. Set $\boldsymbol{e}_{h}=\boldsymbol{u}-\boldsymbol{u}_{h} \in \boldsymbol{V}_{h}$ with $e_{h}=u-u_{h} \in V_{h}, \hat{e}_{h}=\hat{u}-\hat{u}_{h} \in \hat{V}_{h}$ and consider the adjoint problem: Find $\psi \in V$ such that

$$
\begin{equation*}
\gamma_{1} \psi_{1}-\gamma_{2} \psi_{2}=0 \text { on } \Gamma, \quad a(v, \psi)=\int_{\Omega} v e_{h} d x \quad(\forall v \in V) \tag{24}
\end{equation*}
$$

(Note that we have taken $f=e_{h}, g_{D}=0, g_{N}=0$ and used the symmetry of $a$.) In view of (4), we have $\psi_{1} \in H^{1+s}\left(\Omega_{1}\right), \psi_{2} \in H^{1+s}\left(\Omega_{2}\right)$ and

$$
\begin{equation*}
N_{s}(\psi) \leq C\left\|e_{h}\right\|_{L^{2}(\Omega)} . \tag{25}
\end{equation*}
$$

As is verified in Lemma $4 \boldsymbol{\psi}=(\psi, \hat{\psi}) \in \boldsymbol{V}^{1+s}(h)$ satisfies

$$
B_{h}(\boldsymbol{v}, \boldsymbol{\psi})=\int_{\Omega} v e_{h} d x \quad(\forall \boldsymbol{v} \in \boldsymbol{V}(h)) .
$$

HDG scheme for (24) reads as follows: Find $\boldsymbol{\psi}_{h} \in \boldsymbol{V}_{\boldsymbol{h}}$ such that

$$
B_{h}\left(\boldsymbol{v}_{h}, \boldsymbol{\psi}_{h}\right)=\int_{\Omega} v e_{h} d x \quad\left(\forall \boldsymbol{v}_{h} \in \boldsymbol{V}_{h}\right)
$$

Then, we have

$$
\begin{array}{rlrl}
\left\|e_{h}\right\|_{L^{2}(\Omega)}^{2} & =B_{h}\left(\boldsymbol{e}_{h}, \boldsymbol{\psi}\right)=B_{h}\left(\boldsymbol{e}_{h}, \boldsymbol{\psi}-\boldsymbol{\psi}_{h}\right) \\
& \leq C\left\|\boldsymbol{e}_{h}\right\|_{1+s, h}\left\|\boldsymbol{\psi}-\boldsymbol{\psi}_{h}\right\|_{1+s, h} & & (\text { by } 21)) \\
& \leq C h^{s} N_{s}(u) \cdot h^{s} N_{s}(\psi) & & (\text { by } 23) \\
& \leq C h^{2 s} N_{s}(u) \cdot\left\|e_{h}\right\|_{L^{2}(\Omega)}, & & (\text { by } 220)
\end{array}
$$

which completes the proof.

## 5. Numerical examples

In this section, we confirm the validity of error estimates described in Theorems 13 and 14 using simple numerical examples.

Example 15. Set $\Omega_{1}=(0,1) \times(0,1 / 2), \Omega_{2}=(0,1) \times(1 / 2,1)$ and consider

$$
\begin{gather*}
A=\lambda I, \quad \lambda=\left\{\begin{array}{ll}
4 & \text { in } \Omega_{1} \\
1 & \text { in } \Omega_{2},
\end{array} \quad(I: \text { the identity matrix }),\right.  \tag{26a}\\
f= \begin{cases}8 \pi^{2} \sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right) & \text { in } \Omega_{1} \\
-2 \pi^{2} \sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right) & \text { in } \Omega_{2} .\end{cases} \tag{26b}
\end{gather*}
$$

The exact solution is given as

$$
u= \begin{cases}\sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right) & \text { in } \Omega_{1} \\ -\sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right) & \text { in } \Omega_{2}\end{cases}
$$

(Functions $g_{D}$ and $g_{N}$ are computed by u.) It is apparent that $u_{1} \in H^{2}\left(\Omega_{1}\right)$ and $u_{2} \in H^{2}\left(\Omega_{2}\right)$ so that we are able to apply Theorems 13 and 14 for $s=1$.


Figure 3: Examples of $\Omega_{1}, \Omega_{2}$ and $\Gamma$.

Example 16. $\Omega, \Omega_{1}$ and $\Omega_{2}$ are given as shown for illustration in Fig. 3. $\Gamma$ is set as $\Gamma=\Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{3}$. We use $A$ and $f$ defined as (26). Functions $g_{D}$ and $g_{N}$ are given as

$$
g_{D}=\left\{\begin{array}{ll}
2 \sin \left(\pi x_{1}\right) & \text { on } \Gamma_{1} \\
2 & \text { on } \Gamma_{2} \\
2 \sin \left(\pi x_{1}\right) & \text { on } \Gamma_{3}
\end{array} \quad g_{N}=0 \text { on } \Gamma .\right.
$$

In this case, we have $u_{1} \in H^{1+s}\left(\Omega_{1}\right)$ and $u_{2} \in H^{1+s}\left(\Omega_{2}\right)$ for some $s \in(1 / 2,1)$, since $\Omega_{1}$ and $\Omega_{2}$ have concave corners.

We use the $Q_{1}$ element for $V_{h}$ on uniform rectangular meshes and $P_{1}$ for $\hat{V}_{h}$ (see Remark 6). Set

$$
\begin{equation*}
E_{h}=\left|u-u_{h}\right|_{H^{1}\left(\mathcal{T}_{h}\right)}, \quad e_{h}=\left\|u-u_{h}\right\|_{L^{2}(\Omega)} \tag{27}
\end{equation*}
$$

For Example 16, we use numerical solutions $u_{h^{\prime}}$ with extra fine mesh $h^{\prime}$ instead of the exact solution $u$. We examine $E_{h}$ and $e_{h}$ together with

$$
R_{h}=\frac{\log E_{h}-\log E_{h / 2}}{\log 2}, \quad r_{h}=\frac{\log e_{h}-\log e_{h / 2}}{\log 2}
$$

for several $h$ 's.
Results are reported in Tab. 1 and 2 for Examples 15 and 16, respectively. We observe from those tables theoretical convergences with $s=1$ and $s \in(1 / 2,1)$, respectively, for Examples 15 and 16 actually take place.

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| $h$ | $E_{h}$ | $R_{h}$ | $e_{h}$ | $r_{h}$ |
| :---: | :---: | :---: | :---: | :---: |
| 0.06250 | $1.62 \cdot 10^{-1}$ |  | $4.14 \cdot 10^{-3}$ |  |
| 0.03125 | $8.13 \cdot 10^{-2}$ | 1.00 | $1.04 \cdot 10^{-3}$ | 1.99 |
| 0.01563 | $4.06 \cdot 10^{-2}$ | 1.00 | $2.60 \cdot 10^{-4}$ | 2.00 |
| 0.00781 | $2.04 \cdot 10^{-2}$ | 0.99 | $6.50 \cdot 10^{-5}$ | 2.00 |
| 0.00391 | $1.02 \cdot 10^{-2}$ | 1.00 | $1.63 \cdot 10^{-5}$ | 2.00 |
| 0.00195 | $5.08 \cdot 10^{-3}$ | 1.01 | $4.09 \cdot 10^{-6}$ | 1.99 |

Table 1: Errors and convergence rates for Example 15.

| $h$ | $E_{h}$ | $R_{h}$ | $e_{h}$ | $r_{h}$ |
| :---: | :---: | :---: | :---: | :---: |
| 0.06250 | $1.49 \cdot 10^{-1}$ |  | $1.37 \cdot 10^{-3}$ |  |
| 0.03125 | $7.60 \cdot 10^{-2}$ | 0.98 | $3.48 \cdot 10^{-4}$ | 1.98 |
| 0.01563 | $3.88 \cdot 10^{-2}$ | 0.97 | $8.83 \cdot 10^{-5}$ | 1.98 |
| 0.007813 | $1.99 \cdot 10^{-2}$ | 0.97 | $2.26 \cdot 10^{-5}$ | 1.97 |
| 0.003906 | $1.02 \cdot 10^{-2}$ | 0.96 | $5.86 \cdot 10^{-6}$ | 1.95 |

Table 2: Errors and convergence rates for Example 16.

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## A. Proof of Lemma 10

Let $s \in(1 / 2,1)$. Let $K \in \mathcal{T}_{h}$ and $e \subset \partial K$.
The fractional order Sobolev space $H^{s}(K)$ is defined as

$$
H^{s}(K)=\left\{v \in L^{2}(K)\left|\|v\|_{H^{s}(K)}^{2}=\|v\|_{L^{2}(K)}^{2}+|v|_{H^{s}(K)}^{2}<\infty\right\},\right.
$$

where

$$
|v|_{H^{s}(K)}^{2}=\iint_{K \times K} \frac{|v(x)-v(y)|^{2}}{|x-y|^{d+2 s}} d x d y .
$$

It suffices to prove

$$
\begin{equation*}
\|v\|_{L^{2}(e)}^{2} \leq C_{s, \mathrm{~T}} h_{e}^{-1}\left(\|v\|_{L^{2}(K)}^{2}+h_{K}^{2 s}|v|_{H^{s}(K)}^{2}\right) \quad\left(v \in H^{s}(K)\right), \tag{28}
\end{equation*}
$$

since the desired inequality (19) is a direct consequence of (28).
Suppose that $\tilde{K}$ is the reference element in $\mathbb{R}^{d}$ with $\operatorname{diam}(\tilde{K})=1$. Moreover, let $\tilde{e} \subset \partial \tilde{K}$ be a face $(d=3) /$ edge $(d=2)$ of $\tilde{K}$. Trace theorem implies

$$
\|\tilde{v}\|_{L^{2}(e)}^{2} \leq \tilde{C}\left(\|\tilde{v}\|_{L^{2}(\tilde{K})}^{2}+|\tilde{v}|_{H^{s}(\tilde{K})}^{2}\right) \quad\left(\tilde{v} \in H^{1}(\tilde{K})\right)
$$

where $\tilde{C}$ denotes an absolute positive constant. See [13, Theorem 1, $\S$ V.1.1] for example.
Suppose that $\Phi(\xi)=B \xi+c, B \in \mathbb{R}^{d \times d}, c \in \mathbb{R}^{d}$, is the affine mapping which maps $\tilde{K}$ onto $K ; K=\Phi(\tilde{K})$. We know

$$
\|B\|=\sup _{|\xi|=1}|B \xi| \leq \frac{h_{K}}{\tilde{\rho}}, \quad\left\|B^{-1}\right\| \leq \frac{\tilde{h}}{\rho_{K}}, \quad d \xi=\frac{\operatorname{meas}_{d}(\tilde{K})}{\operatorname{meas}_{d}(K)} d x
$$

where $\tilde{h}=h_{\tilde{K}}, \tilde{\rho}=\rho_{\tilde{K}}$ and $\operatorname{meas}_{d}(K)$ denotes the $\mathbb{R}^{d}$-Lebesgue measure of $K$. Moreover,

$$
\frac{|x|}{\left|B^{-1} x\right|} \leq \sup _{\xi \in \mathbb{R}^{d}} \frac{|B \xi|}{|\xi|}=\|B\| \quad\left(x \in \mathbb{R}^{d}, x \neq 0\right) .
$$

We recall that there exists a positive constant $\nu_{2}$ that independent of $h$ such that $h_{K} / \rho_{K} \leq \nu_{2}\left(\forall K \in \forall \mathcal{T}_{h} \in\left\{\mathcal{T}_{h}\right\}_{h}\right)$ by the shape-regularity of the family of triangulations.

Now we can state the proof of (28). By the density, it suffices to consider (28) for $v \in C^{1}(K)$. Set $\tilde{v}=v \circ \Phi \in C^{1}(\tilde{K})$. Then,

$$
\int_{\tilde{K}} \tilde{v}^{2} d \xi=\frac{\operatorname{meas}_{d}(\tilde{K})}{\operatorname{meas}_{d}(K)} \int_{K} v^{2} d x \leq C \rho_{K}^{-d}\|v\|_{L^{2}(K)}^{2}
$$

and

$$
\begin{aligned}
\iint_{\tilde{K} \times \tilde{K}} \frac{|\tilde{v}(\xi)-\tilde{v}(\eta)|^{2}}{|\xi-\eta|^{d+2 s}} d \xi d \eta & \leq\left(\frac{\operatorname{meas}_{d}(\tilde{K})}{\operatorname{meas}_{d}(K)}\right)^{2} \iint_{K \times K} \frac{|v(x)-v(y)|^{2}}{\left|B^{-1} x-B^{-1} y\right|^{d+2 s}} d x d y \\
& \leq C \rho_{K}^{-2 d} \cdot\|B\|^{d+2 s} \iint_{K \times K} \frac{|v(x)-v(y)|^{2}}{|x-y|^{d+2 s}} d x d y \\
& \leq C h_{K}^{2 s} \nu_{2}^{d} \rho_{K}^{-d} \iint_{K \times K} \frac{|v(x)-v(y)|^{2}}{|x-y|^{d+2 s}} d x d y .
\end{aligned}
$$

Using those inequalities, we have

$$
\begin{aligned}
\|\tilde{v}\|_{L^{2}(e)}^{2} & =\frac{\operatorname{meas}_{d-1}(e)}{\operatorname{meas}_{d-1}(\tilde{e})} \int_{\tilde{e}} \tilde{v}(\xi)^{2} d \xi \\
& \leq C h_{e}^{d-1} \cdot \tilde{C}\left(\int_{\tilde{K}} \tilde{v}^{2} d \xi+\iint_{\tilde{K} \times \tilde{K}} \frac{|v(\xi)-v(\eta)|^{2}}{|\xi-\eta|^{d+2 s}} d \xi d \eta\right) \\
& \leq C \nu_{1}^{d} h_{e}^{-1}\left(\|v\|_{L^{2}(K)}^{2}+h_{K}^{2 s}|v|_{H^{s}(K)}^{2}\right)
\end{aligned}
$$

which completes the proof.


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