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Abstract This article is concerned with *a posteriori* error estimates for a discrete-fracture, multidimensional, numerical model for flow in a fractured porous medium. Local residual error estimators are defined and upper and lower bounds in terms of these estimators for both the pressure and the Darcy velocity are derived. Numerical examples using these estimates for automatic grid refinement are given.

Keywords A posteriori error estimate · mixed finite element · reduced model · fractured porous media.

Mathematics Subject Classification (2010) 65N15 · 65N30 · 65N50 · 76S05

1 Introduction

The purpose of this paper is to obtain *a posteriori* error estimates for a numerical model for fluid flow in a porous medium with fractures. By a fracture in a porous medium we mean a part of the medium that has very small width in comparison with the other dimensions of the problem and that has a permeability either much higher than or much lower than that in the surrounding medium. Because they are very thin it is difficult to include fractures in a numerical model, but because of their large influence on the flow, providing natural channels for flow if they are highly permeable but blocking flow if they are of very low permeability, their presence needs to be taken into account in any reasonable numerical model. Many different ways of accounting for the fractures have been proposed. Homogenization methods are used to obtain double continuum models or double porosity models in which the fractures are assumed to have enough regularity in their distribution that their effect may be averaged. In the case of a network of highly permeable fractures sometimes flow is calculated only in the fractures, flow in the surrounding domain being considered negligible. In the type of model considered here the flow is calculated both in the fractures and in the surrounding domain, but

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the fractures are included individually in the model. However, because they are so thin they are treated as interfaces. Such models are called discrete models (since the fractures are taken into account individually and not averaged) or reduced models (since the fractures are modeled as surfaces of co-dimension 1). This type of model has been much studied in the mathematical and engineering literature for many types of flow problems and for many different types of numerical schemes. Using the techniques of domain decomposition [15], a first reduced model has been proposed for flow in fractured porous media in the case of very permeable fracture in [1]. Later on, a generalization of this model considering also the case of fracture with low permeability has been proposed in [21], where the fracture can be seen as geological barrier. This approach has been also used to modeling the two phase flow in [13] and [17], where Darcy's law is still applicable separately for each phase and coupling conditions were added. This model has been extended to the case of flow and transport in porous media in[26] and [13]. Finally porous media with a networks of fracture was also treated in [11] and [25].

Here we consider incompressible, single-phase flow ignoring gravity and assume that the flow respects Darcy's law. We use the model of [21] which was derived through a process of averaging across the fracture with the flow equation written in mixed form.

Let Ω be a convex domain in $\mathbb{R}^n$, $n = 2$ or 3, and $\Gamma = \partial\Omega$ the boundary of Ω. We suppose that a fracture domain γ separates Ω into two connected subdomains Ω_1 and Ω_2, ($\Omega\backslash\gamma = \Omega_1 \cup \Omega_2, \quad \Omega_1 \cap \Omega_2 = \emptyset$). The flow in each part Ω_i is governed by a conservation equation together with Darcy's law relating the gradient of the pressure p to the Darcy velocity $\mathbf{u}$. As the fracture γ is considered to be an $(n-1)-$dimensional porous medium, there too the fluid flow is governed by a conservation law and Darcy's law but in dimension $(n-1)$. There is also the possibility for exchange between the fracture and the domains Ω_i: both the pressure and the normal component of the Darcy velocity may be discontinuous across γ with the sum of the fluxes through γ from the two domains Ω_i being a source (or sink) term for the conservation equation in γ.

We use the notation ∇_τ, respectively div_τ, for the tangential gradient, respectively tangential divergence, operator along the fracture. We assume that the index i of the subdomains varies in $\mathbb{Z}/2\mathbb{Z}$ (so that 2 + 1 = 1).

For simplicity we suppose here that the fracture can be identified to a surface γ in a hyperplane of $\mathbb{R}^n$. We denote by Γ_i the part of the boundary of Ω_i shared with the boundary of Ω, for $i = 1,2$, and we denote by $\mathbf{n}_i$ the unit, normal, outward-pointing vector field on $\partial\Omega_i$. For $i = 1,2$, the permeability K_i of the subdomain Ω_i is a symmetric, uniformly continuous, uniformly positive-definite, $n\times n$ tensor field on Ω_i. We suppose that the permeability in the (n-dimensional) physical fracture, that is represented in the model by the (($n-1$)-dimensional) interface γ, can be split into a tangential part $K_{\gamma,\tau}$ and a normal part $K_{\gamma,\nu}$. Both of these parts are considered to be functions on γ (i.e. they are invariant over cross-sections of the physical fracture), and the tangential permeability $K_{\gamma,\tau}$ is supposed to satisfy the same properties as K_i except of course that it is an $(n-1)\times(n-1)$ tensor. The normal permeability $K_{\gamma,\nu}$ is a positive scalar field bounded above and away from 0. (In practice these are all assumed to be piecewise constant.) The approximate width of the physical fracture is represented in the model by the constant d. The model given in [21] is

$$\begin{aligned} \mathbf{u}_i &= -K_i\nabla p_i && \text{in } \Omega_i,\ \ i=1,2 \\ \mathrm{div}\mathbf{u}_i &= f_i && \text{in } \Omega_i,\ \ i=1,2 \\ \mathbf{u}_\gamma &= -dK_{\gamma,\tau}\nabla_\tau p_\gamma && \text{on } \gamma, \\ \mathrm{div}_\tau\,\mathbf{u}_\gamma &= f_\gamma + [\mathbf{u}\cdot\mathbf{n}] && \text{on } \gamma, \\ p_i &= p_\gamma + \frac{d}{2K_{\gamma,\nu}}\{\mathbf{u}\cdot\mathbf{n}\}_{\xi,i} && \text{on } \gamma,\ \ i\in\mathbb{Z}_{[2]} \\ p_i &= \bar{p}_i && \text{on } \Gamma_i,\ \ i=1,2 \\ p_\gamma &= \bar{p}_\gamma && \text{on } \partial\gamma, \end{aligned} \tag{1}$$

where, for $i = 1,2$, the scalar function p_i is the fluid pressure in Ω_i, the n-dimensional vector function $\mathbf{u}_i$ is the Darcy velocity in Ω_i, f_i is an external source term, and $\bar{p}_i$ is the given pressure on Γ_i. For $x\in\gamma$, $p_\gamma(x)$ is the average pressure across the cross-section of the physical fracture represented by x, and the $(n-1)$-dimensional vector function $\mathbf{v}_\gamma(x)$ is the integral across that same cross-section of the tangential component of the Darcy velocity. The given pressure on the boundary of γ is $\bar{p}_\gamma$, and f_γ is an eventual external source term in γ. The additional source term $[\mathbf{u}\cdot\mathbf{n}]$ in γ is defined by

$$[\mathbf{u}\cdot\mathbf{n}] = \mathbf{u}_1\cdot\mathbf{n}_1 + \mathbf{u}_2\cdot\mathbf{n}_2 \tag{2}$$

and is the jump across γ of the flux, the difference in what enters γ from one side and what leaves through the other side. While the normal component of the Darcy velocity in the fracture is not directly included in the model it is approximated by the weighted averages $\{\mathbf{u}\cdot\mathbf{n}\}_{\xi,i}$, $i=1,2$, where

$$\{\mathbf{u}\cdot\mathbf{n}\}_{\xi,i} = \xi\mathbf{u}_i\cdot\mathbf{n}_i - (1-\xi)\mathbf{u}_{i+1}\cdot\mathbf{n}_{i+1}, \tag{3}$$

with $\xi \in (1/2,1]$ a quadrature parameter used to approximate the integral across the half-cross-sections of the physical fracture of the normal component of the Darcy velocity. In this way the normal component of the Darcy law in the fracture is used to obtain nonstandard Robin type interface conditions on the interface γ.

The purpose of this paper is to derive *a posteriori* error estimates, for such a model in order to introduce an auto-adaptive mesh refinement technique.

Since the pioneering work of Babushka and Reinbold [3], much has been written about adaptive methods for finite element approximation with emphasis on both theoretical and computational aspects of the method. Several *a posteriori* error estimators for mixed finite element discretization of elliptic problems have been derived. For the residual-based estimators, which is the type of estimator that we use here, we can distinguish two types of estimation: the first of these was introduced by Braess and Verfürth [9] and gives bounds on the error in a mesh dependent norm which is close to the energy norm of the continuous problem in its primal form. In the presense of a saturation assumption (which is not always satisfied) this estimator is reliable and efficient in this norm, but somehow it is not efficient in the natural norm of the mixed formulation. This estimate was improved by Lovadina and Stenberg [20] and Larson and Malqvist [18] by introducing a postprocessing technique.

The second type of residual-based estimation for mixed finite element discretizations gives bounds on the error in the $H(\mathrm{div};\Omega)\times L^2(\Omega)$-norm. Such an estimate was first introduced by Alonso who in [2] obtained an upper bound of the error only on the dual variable in the $L^2(\Omega)$-norm. This estimate was generalized by Carstensen who in [10] obtained upper and lower bounds on the error in the natural norm for the primal and dual variables in the 2D case, by using a Helmholtz-like decomposition of vectors of $H(\mathrm{div};\Omega)$. Hoppe and Wohlmuth in [30] gave a comparison of such estimates with hierarchical ones and estimates using resolution of local problems. In Nicaise and Creusé [23] one can find a generalization to the anisotropic 2D and 3D cases of such estimates. The error estimation that we use here is based on that of [10] and of [23].

For mortar mixed finite element discretization, Wheeler and Yotov in [29] derived *a posteriori* error estimates that provide lower and upper bounds for the pressure, velocity, and mortar error in two and three dimensions for Darcy's equations in the $H(\mathrm{div};\Omega)\times L^2(\Omega)$ formulation under a saturation assumption. While we use spatial discretizations for the subdomains Ω_1 and Ω_2 that do not match-up at the interface γ, we have not relied on the work in [29] as the role of the mortar element in our model is played by the elements in the fracture γ.

In Bernardi, Hecht and Mghazli [6] optimal *a posteriori* error estimates are derived for the $(L^2(\Omega))^d\times H^1(\Omega)$ formulation and for a nonconforming approximation.

Vohralik in [28] gave a unified framework for a priori and a posteriori error analysis of mixed finite element discretization of second-order elliptic problems based on an H^1-conforming reconstruction of the pressure. The technique of [28] for obtaining *a posteriori* error estimates in the context of our fracture model was investigated in [22].

The remainder of this paper is organized as follows: in Sections 2 and 3 we recall, respectively the continuous mixed variational formulation of (1) with some of its properties and the discretization using the mixed finite element method of problem (1), and some technical lemmas related to the projection and interpolation operators that will be used in the *a posteriori* analysis . The error indicators and the statement of the main results are given in Section 4. The proof of upper and lower bounds for the error as a function of the indicators are carried out in Section 5 and Section 6 respectively. A numerical experiment and a summary conclude this paper.

Remark 1 : *When* $n=2$*, so that the dimension of* γ *is 1, both* ∇_τ *and* div_τ *are simply* $\dfrac{\partial}{\partial\tau}$ *but we will still usually write* ∇_τ *and* div_τ *in order not to have to distinguish the cases* $n=2$ *and* $n=3$*. For the same reason we will in this case write* $\mathbf{v}_\gamma$ *for the Darcy velocity in the fracture even though it is a scalar function.* □

2 The mixed weak formulation

2.1 Some notation and definitions

We begin this section by fixing some notation while recalling the definitions of some standard function spaces and operators. For D a domain in $\mathbb{R}^n$, n =1, 2 or 3, let $L^2(D)$ be the space of square integrable functions on D and let $H^1(D)$ be the subspace of functions in $L^2(D)$ whose gradient belongs to $(L^2(D))^n$:

$$H^1(D) := \{q \in L^2(D) \; : \; \nabla q \in (L^2(D))^n\}.$$

We will use the notation $\mathbf{L}^2(D)$ for $(L^2(D))^n$ and $\mathbf{H}^1(D)$ for $(H^1(D))^n$. We will also use the notation $\|\cdot\|_{0,D}$ or $\|\cdot\|_0$ for the norm on $L^2(D)$ as well as for the norm on $\mathbf{L}^2(D)$ and the notation $|\cdot|_{1,D}$ or $|\cdot|_1$, respectively $\|\cdot\|_{1,D}$ or $\|\cdot\|_1$, for the semi-norm, respectively norm, on both $H^1(D)$ and $\mathbf{H}^1(D)$. The trace map $\zeta \; : \; H^1(D) \longrightarrow L^2(\partial D)$ is continuous, and its kernel is denoted $H^1_0(D)$ while its image is the space $H^{\frac{1}{2}}(\partial D)$ with norm $\|\cdot\|_{\frac{1}{2},\partial D}$ or $\|\cdot\|_{\frac{1}{2}}$ defined by

$$\|\zeta q\|_{\frac{1}{2},\partial D} := \inf_{\substack{q' \in H^1(D) \\ \text{with } \zeta q' = \zeta q}} \|q'\|_{1,D}.$$

We will often use, for functions $q \in H^1(D)$, the notation $q|_{\partial D}$ instead of ζq.

To write the variational formulation of our problem we will need also the space $\mathbf{H}(\mathrm{div};D)$, the subspace of functions in $\mathbf{L}^2(D)$ whose divergence belongs to $L^2(D)$:

$$\mathbf{H}(\mathrm{div};D) := \{\mathbf{v} \in \mathbf{L}^2(D) \; : \; \mathrm{div}\mathbf{v} \in L^2(D)\}.$$

The norm on $\mathbf{H}(\mathrm{div};D)$ is defined by

$$\|\mathbf{v}\|^2_{\mathrm{div},D} \text{ or } \|\mathbf{v}\|^2_{\mathrm{div}} = \|\mathbf{v}\|^2_{0,D} + \|\mathrm{div}\mathbf{v}\|^2_{0,D}.$$

The normal trace map $\zeta_{\mathbf{n}} \; : \; \mathbf{H}(\mathrm{div};D) \longrightarrow \mathscr{D}'(\partial D)$ is continuous, and its kernel is denoted $\mathbf{H_0}(\mathrm{div};D)$ while its image is the space $H^{-\frac{1}{2}}(\partial D)$ with norm $\|\cdot\|_{-\frac{1}{2},\partial D}$ or $\|\cdot\|_{-\frac{1}{2}}$ defined by

$$\|\zeta_{\mathbf{n}}\mathbf{v}\|_{-\frac{1}{2},\partial D} := \inf_{\substack{\mathbf{v}' \in \mathbf{H}(\mathrm{div};D) \\ \text{with } \zeta_{\mathbf{n}}\mathbf{v}' = \zeta_{\mathbf{n}}\mathbf{v}}} \|\mathbf{v}'\|_{\mathrm{div},D}.$$

We will often use, for functions $\mathbf{v} \in \mathbf{H}(\mathrm{div};D)$, the notation $\mathbf{v}\cdot\mathbf{n}|_{\partial D}$ instead of $\zeta_{\mathbf{n}}\mathbf{v}$, with $\mathbf{n}$ denoting the outward-pointing, unit, normal vector field on ∂D. We have Green's formula

$$\int_D \mathrm{div}\mathbf{v}\, s + \int_D \nabla s \cdot \mathbf{v} = < \mathbf{v}\cdot\mathbf{n}\, , s >, \quad \forall s \in H^1(D) \text{ and } \mathbf{v} \in \mathbf{H}(\mathrm{div};D), \tag{4}$$

with $< \cdot, \cdot >$ denoting the pairing between $H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$.

To calculate the *a posteriori* error bounds we will use the space $\mathbf{H}(\mathbf{curl};D)$, the space of functions in $\mathbf{L}^2(D)$ if $n = 3$, or in $L^2(D)$ if $n = 2$, for which the **curl** belongs to $\mathbf{L}^2(D)$:

– for n=3,

$$\mathbf{H}(\mathbf{curl};D) := \left\{\mathbf{z} \in \mathbf{L}^2(D) \; : \; \mathbf{curl}\,\mathbf{z} \in \mathbf{L}^2(D)\right\}$$

equipped with the norm

$$\|\mathbf{z}\|^2_{\mathbf{curl},D} = \|\mathbf{z}\|^2_{0,D} + \|\mathbf{curl}\mathbf{z}\|^2_{0,D},$$

where the **curl** operator is defined as usual by

$$\mathbf{curl}\mathbf{z} = \nabla \times \mathbf{z} := \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ z_1 & z_2 & z_3 \end{pmatrix} = \left(\frac{\partial z_3}{\partial x_2} - \frac{\partial z_2}{\partial x_3}, \frac{\partial z_1}{\partial x_3} - \frac{\partial z_3}{\partial x_1}, \frac{\partial z_2}{\partial x_1} - \frac{\partial z_1}{\partial x_2}\right),$$

- for $n = 2$,
  - for scalar function
  $$H(\mathbf{curl}; D) := \left\{z \in L^2(D) \,:\, \mathbf{curl}\, z \in \mathbf{L}^2(D)\right\},$$
  equipped with the norm
  $$\|z\|^2_{\mathbf{curl},D} := \|z\|^2_{0,D} + \|\mathbf{curl}\, z\|^2_{0,D},$$
  where the **curl** operator is defined as usual by
  $$\mathbf{curl}\, z = \left(\frac{\partial z}{\partial x_2}, -\frac{\partial z}{\partial x_1}\right).$$
  - for a vector valued function the space $\mathbf{H}(\mathbf{curl}; D)$ is defined to be the space of all functions in $\mathbf{L}^2(D)$ whose curl is in $L^2(D)$:
  $$\mathbf{H}(\mathbf{curl}; D) := \{\mathbf{z} \in \mathbf{L}^2(D) \,:\, \mathrm{curl}\, \mathbf{z} \in L^2(D)\}.$$
  with
  $$\mathrm{curl}\mathbf{z} = \frac{\partial z_2}{\partial x_1} - \frac{\partial z_1}{\partial x_2}.$$

Of course when $n = 1$, $\mathbf{H}(\mathbf{curl}; D)$ is simply $\mathbf{H}^1(D)$.

The tangential trace map $\zeta_\tau : \mathbf{H}(\mathbf{curl}; D) \longrightarrow \mathscr{D}'(\partial D)$ is continuous, and its kernel is denoted $H_0(\mathbf{curl}; D)$ while its image is the space $(H^{-\frac{1}{2}}(\partial D))^3$ if $n = 3$ and is $H^{-\frac{1}{2}}(\partial D)$ if $n = 2$. We will often use, for functions $\mathbf{z} \in \mathbf{H}(\mathbf{curl}; D)$ the notation $\mathbf{z} \times \mathbf{n}|_{\partial D}$ instead of $\zeta_\tau \mathbf{z}$, if $n = 3$, and for functions $\mathbf{z} \in \mathbf{H}(\mathrm{curl}; D)$, the notation $\mathbf{z} \cdot \tau|_{\partial D}$ instead of $\zeta_\tau \mathbf{z}$, if $n = 2$ with $\mathbf{n}$ denoting the outward-pointing, unit, normal vector field on ∂D and τ denoting the unit tangential vector on ∂D defined by $\tau = (-n_2, n_1)$ if $n = 2$ and $\mathbf{n} = (n_1, n_2)$. We have the Green's formula ([14] page 34)

$$\begin{aligned} &\int_D \mathbf{z} \cdot \mathbf{curl}\, \mathbf{r} - \int_D \mathbf{curl}\, \mathbf{z} \cdot \mathbf{r} = \langle \zeta_\tau \mathbf{r}\,,\mathbf{z}\rangle, \quad \forall \mathbf{r} \in \mathbf{H}(\mathbf{curl}; D) \\ &\forall \mathbf{z} \in \mathbf{H}^1(D) \text{ if } n = 3 \text{ and } z \in H^1(D) \text{ if } n = 2, \end{aligned} \tag{5}$$

with $\langle \cdot\,,\cdot \rangle$ denoting the pairing between $(H^{-\frac{1}{2}}(\partial D))^3$ and $(H^{\frac{1}{2}}(\partial D))^3$ if $n = 3$ or between $H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$ if $n = 2$.

2.2 The weak formulation of the fracture problem

All of the operators and spaces in the previous subsection were defined for a domain D of dimension n. When Ω is of dimension 3, the fracture γ is of dimension 2, and the operators ∇, div, curl and **curl** on γ will be denoted ∇_τ, $\mathrm{div}_\tau, \mathrm{curl}_\tau$, and $\mathbf{curl}_\tau$, respectively. We will speak of the spaces $\mathbf{H}(\mathrm{div}_\tau; \gamma)$, and $H(\mathbf{curl}_\tau; \gamma)$. If $n = 2$ then the dimension of γ is 1 so that ∇_τ and div_τ are just $\frac{\partial}{\partial \tau}$, where τ is the unit vector on γ obtained by rotating $\mathbf{n}_1$ by $\frac{\pi}{2}$ degrees. In this case $H(\mathrm{div}_\tau; \gamma)$ is of course just $H^1(\gamma)$.

To define the variational formulation of (1) we need a few more spaces, operators, and some bilinear forms. As in the standard mixed formulation for Darcy flow the pressure p is sought in an L^2-space M and the Darcy velocity $\mathbf{u}$ is an $H(\mathrm{div})$-space $\mathbf{W}$ defined as follows:

$$\begin{aligned} M &= \{q = (q_1, q_2, q_\gamma) \in L^2(\Omega_1) \times L^2(\Omega_2) \times L^2(\gamma)\} \\ \mathbf{W} &= \{\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_\gamma) \in \mathbf{H}(\mathrm{div}; \Omega_1) \times \mathbf{H}(\mathrm{div}; \Omega_2) \times \mathbf{H}(\mathrm{div}_\tau; \gamma) : \\ &\qquad \mathbf{v}_i \cdot \mathbf{n}_i \in L^2(\gamma), i = 1, 2\}. \end{aligned}$$

These spaces are Hilbert spaces when equipped with the norms

$$\|q\|^2_M = \sum_{i=1}^{2} \|q_i\|^2_{0,\Omega_i} + \|q_\gamma\|^2_{0,\gamma},$$

$$\|\mathbf{v}\|_{\mathbf{W}}^2 = \sum_{i=1}^{2} \left(\|\mathbf{v}_i\|_{0,\Omega_i}^2 + \|\mathrm{div}\,\mathbf{v}_i\|_{0,\Omega_i}^2 \right) + \|\mathbf{v}_\gamma\|_{0,\gamma}^2 + \|\mathrm{div}_\tau\,\mathbf{v}_\gamma\|_{0,\gamma}^2 + \sum_{i=1}^{2} \|\mathbf{v}_i \cdot \mathbf{n}_i\|_{0,\gamma}^2.$$

If $\mathbf{v} \in \mathbf{W}$ we will often write $\mathrm{div}\,\mathbf{v}$ for $(\mathrm{div}\mathbf{v}_1, \mathrm{div}\mathbf{v}_2, \mathrm{div}_\tau \mathbf{v}_\gamma)$, but also we define the operator $\mathrm{Div} : \mathbf{W} \longrightarrow M$ by

$$\mathrm{Div}(\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_\gamma)) = (\mathrm{div}\mathbf{v}_1, \mathrm{div}\mathbf{v}_2, \mathrm{div}_\tau \mathbf{v}_\gamma - [\mathbf{v} \cdot \mathbf{n}]).$$

We denote the kernel of this latter operator by $\widetilde{\mathbf{W}}$:

$$\widetilde{\mathbf{W}} = \{\mathbf{v} \in \mathbf{W} \ : \ \mathrm{Div}\,\mathbf{v} = 0\}.$$

The weak formulation of problem (1) is given by (cf.[21]):

$$\begin{aligned} &\text{Find } (\mathbf{u}, p) \in \mathbf{W} \times M \text{ such that} \\ &a_\xi(\mathbf{u}, \mathbf{v}) - \beta(\mathbf{v}, p) = -L_d(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{W}, \\ &\beta(\mathbf{u}, q) = L_f(q), \qquad\qquad\qquad \forall q \in M, \end{aligned} \tag{6}$$

where the bilinear forms a_ξ and β and the linear forms L_f and L_d are defined as follows:

$$\begin{aligned} a_\xi(\mathbf{u}, \mathbf{v}) &= \sum_{i=1}^{2} \Big(K_i^{-1} \mathbf{u}_i, \mathbf{v}_i \Big)_{\Omega_i} + \Big\langle (dK_{\gamma,\tau})^{-1} \mathbf{u}_\gamma, \mathbf{v}_\gamma \Big\rangle_\gamma + \sum_{i=1}^{2} \Big\langle \frac{d}{2K_{f,\nu}} \{\mathbf{u} \cdot \mathbf{n}\}_{\xi,i}, \mathbf{v}_i \cdot \mathbf{n}_i \Big\rangle_\gamma, \\ \beta(\mathbf{u}, q) &= \sum_{i=1}^{2} \Big(\mathrm{div}\mathbf{u}_i, q_i \Big)_{\Omega_i} + \Big\langle \mathrm{div}_\tau \mathbf{u}_\gamma, q_\gamma \Big\rangle_\gamma - \Big\langle [\mathbf{u} \cdot \mathbf{n}], q_\gamma \Big\rangle_\gamma := < \mathrm{Div}\mathbf{u}, q >_{M',M}, \\ L_f(q) &= \sum_{i=1}^{2} \Big(f_i, q_i \Big)_{\Omega_i} + \Big\langle f_\gamma, q_\gamma \Big\rangle_\gamma, \\ L_d(\mathbf{v}) &= \sum_{i=1}^{2} \Big\langle \bar{p}_i, \mathbf{v}_i \cdot \mathbf{n}_i \Big\rangle_{\Gamma_i} + \Big\langle\!\Big\langle \bar{p}_\gamma, \mathbf{v}_\gamma \cdot \mathbf{n}_\gamma \Big\rangle\!\Big\rangle_{\partial\gamma}, \end{aligned}$$

with $\Big(\cdot,\cdot\Big)_{\Omega_i}$ denoting the scalar product in $L^2(\Omega_i),\ i = 1,2$, with $\Big\langle \cdot,\cdot \Big\rangle_{\Gamma_i}$, respectively $\Big\langle \cdot,\cdot \Big\rangle_\gamma$, denoting the scalar product in $L^2(\Gamma_i),\ i = 1,2$, respectively $L^2(\gamma)$, with $\Big\langle\!\Big\langle \cdot,\cdot \Big\rangle\!\Big\rangle_{\partial\gamma}$ denoting the scalar product on $L^2(\partial\gamma)$ and with $\{\mathbf{u} \cdot \mathbf{n}\}_{\xi,i}$ and $[\mathbf{u} \cdot \mathbf{n}]$ as defined in (3) and (2) respectively. When $n = 2$, γ is a line segment and the notation $\Big\langle\!\Big\langle \cdot,\cdot \Big\rangle\!\Big\rangle_{\partial\gamma}$ is understood to be simply the sum (over the two endpoints of γ) of the product of the two functions evaluated at the endpoints.

Problem (6) has a unique solution (see [24],[21]) since the bilinear form $a_\xi(\cdot,\cdot)$ is $\widetilde{\mathbf{W}}-$elliptic: i.e.

$$\exists C_\alpha > 0, \qquad \inf_{\mathbf{v} \in \widetilde{\mathbf{W}}} \frac{a_\xi(\mathbf{v}, \mathbf{v})}{\|\mathbf{v}\|_{\mathbf{W}}^2} \geq C_\alpha, \tag{7}$$

and the bilinear form $\beta(\cdot,\cdot)$ satisfies the following inf-sup condition:

$$\exists C_\beta > 0, \qquad \inf_{q \in M} \sup_{\mathbf{v} \in \mathbf{W}} \frac{\beta(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{W}} \|q\|_M} \geq C_\beta. \tag{8}$$

We now state a lemma concerning the space $\widetilde{\mathbf{W}}$ that we will use to obtain the *a posteriori* error estimates.

Lemma 1 *Suppose that* $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_\gamma) \in \widetilde{\mathbf{W}}$.

1. **If** *$n=3$, there exist $\mathbf{r_i}\in\mathbf{H}^1(\Omega_i)$, $r_\gamma\in H^1_0(\gamma)$ and $s_\gamma\in H^1(\gamma)/\mathbb{R}$ such that*

$$\mathbf{v}_i=\mathbf{curl}\,\mathbf{r_i}\qquad in\ \Omega_i\qquad for\ i=1,2, \tag{9}$$
$$\mathbf{v}_\gamma=\mathbf{curl}_\tau r_\gamma+\nabla_\tau s_\gamma,\qquad in\ \gamma. \tag{10}$$

Moreover there exists $\rho_\gamma\in H^1_0(\gamma)$ such that

$$\nabla_\tau s_\gamma-[\mathbf{r}\times\mathbf{n}]=\mathbf{curl}_\tau\rho_\gamma, \tag{11}$$
$$\mathbf{v}_\gamma=\mathbf{curl}_\tau(r_\gamma+\rho_\gamma)+[\mathbf{r}\times\mathbf{n}], \tag{12}$$

where $[\mathbf{r}\times\mathbf{n}]=\sum_{i=1}^{2}(\mathbf{r_i}\times\mathbf{n}_i)$.

2. **If** *$n=2$, there exist $r_i\in H^1(\Omega_i)$ and $r_\gamma\in H^1_0(\gamma)\cap H^2(\gamma)$ such that*

$$\mathbf{v}_i=\mathbf{curl}\,r_i\qquad in\ \Omega_i\qquad for\ i=1,2, \tag{13}$$
$$v_\gamma=\frac{\partial r_\gamma}{\partial\tau}\qquad in\ \gamma. \tag{14}$$

Proof The proofs of (9), (10) and (13) follow immediately from [14, Theorems 3.1, 3.2 and 3.4]. For (14) we recall that γ is a one-dimensional domain and $\mathrm{div}_\tau v_\gamma=\frac{\partial \mathbf{v}_\gamma}{\partial\tau}$. To prove (11), let $\mathbf{v}=(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_\gamma)\in\widetilde{\mathbf{W}}$ and let $\mathbf{r_i}$, $i=1,2$, r_γ and s_γ be as in (9) and (10). Then we have $\mathbf{v}_i\cdot\mathbf{n}_i=(\mathbf{curl}\ \mathbf{r_i})\cdot\mathbf{n}_i=\mathrm{div}_\tau(\mathbf{r_i}\times\mathbf{n}_i)$, so we deduce that

$$\mathrm{div}_\tau\mathbf{v}_\gamma-\sum_{i=1}^{2}\mathbf{v}_i\cdot\mathbf{n}_i=\mathrm{div}_\tau(\nabla_\tau s_\gamma-\sum_{i=1}^{2}(\mathbf{r_i}\times\mathbf{n}_i))=0,\qquad \text{on }\gamma,$$

where we have identified 3D vector fields on γ for which the component normal to γ vanishes, with their 2D projection on γ. By [14, Theorem 3.1] we conclude that

$$\exists\rho_\gamma\in H^1_0(\gamma)\text{ such that }\nabla_\tau s_\gamma-\sum_{i=1}^{2}(\mathbf{r_i}\times\mathbf{n}_i)=\mathbf{curl}_\tau\rho'_\gamma.$$

The proof is completed on noting that (12) is simply a combination of (10) and (11). ■

Remark 2 : For $n=3$, s_γ satisfies (see [14]) $\Delta_\tau s_\gamma=\mathrm{div}\mathbf{v}_\gamma$, so $\nabla_\tau s_\gamma$ is in $H(\mathrm{div};\gamma)$. □

Before turning to the discretization of problem (6), we define one more vector space that we will use only for the *a posteriori* error estimates:

$$\mathbf{N}\ =\ \{\mathbf{z}=(\mathbf{z}_1,\mathbf{z}_2,z_\gamma)\in\mathbf{H}(\mathbf{curl};\Omega_1)\times\mathbf{H}(\mathbf{curl};\Omega_2)\times H(\mathbf{curl}_\tau;\gamma)\}. \tag{15}$$

3 Discretization of the problem

In [21] problem (6) was discretized using a mixed finite element method with a conforming grid; i. e. a finite element mesh on all of Ω was chosen such that γ lay on the union of the edges of the elements in the mesh, and the mesh used for γ was that inherited from this mesh. In [12] it was shown that one could choose a mesh for each of the subdomains Ω_i and a mesh for γ all independently. Here, following [12], we will not assume that the meshes are conforming.

Let $\{\mathscr{T}_{h,i}\}_{h_i\in\mathscr{H}_i}$, for $i=1,2$, be a regular family of triangulations of Ω_i consisting of closed n-simplexes, and $\{\mathscr{T}_{h,\gamma}\}_{h_\gamma\in\mathscr{H}_\gamma}$ a regular family of triangulations of the hypersurface γ made up of closed $(n-1)$-simplexes. For $i=1,2$, let $\mathscr{E}_{h,i}$ be the set of $(n-1)$-dimensional faces of elements of $\mathscr{T}_{h,i}$; let $\mathscr{E}^{\Gamma_i}_{h,i}$, $\mathscr{E}^{\gamma}_{h,i}$ and $\mathscr{E}^{0}_{h,i}$ be the subsets of $\mathscr{E}_{h,i}$ consisting of the faces lying on the boundary Γ_i, on γ and in the interior of Ω_i, respectively. Let $\mathscr{G}_{h,i}$ be the set of $(n-2)$ dimensional faces of elements of $\mathscr{E}_{h,i}$, let $\mathscr{G}^{\Gamma_i}_{h,i}$ be the set of $(n-2)$ dimensional faces of elements of $\mathscr{E}^{\Gamma_i}_{h,i}$ and let $\mathscr{G}^{\partial\Gamma_i}_{h,i}$ denote the elements of $\mathscr{G}_{h,i}^{\Gamma_i}$ lying on $\partial\Gamma_i$. Similarly, let $\mathscr{E}_{h,\gamma}$ be the set of $(n-2)$-dimensional faces of elements of $\mathscr{T}_{h,\gamma}$ and $\mathscr{E}_{h,\gamma}^0$ the subset of $\mathscr{E}_{h,\gamma}$ consisting of faces lying in the interior of γ and $\mathscr{E}_{h,\gamma}^{\partial\gamma}$ those lying on the boundary of γ. In the case $d=2$, the $(n-2)$-dimensional faces are reduced to points in Γ_i or γ. Elements of $\mathscr{T}_{h,i}$ will be denoted by T, their $(n-1)$-dimensional faces by E and their $(n-2)$-dimensional faces by G, whereas elements of $\mathscr{T}_{h,\gamma}$ will be denoted t and their edges by e. Further, for $E \in \mathscr{E}_{h,i}$ let $\mathscr{T}_{h,E}$ denote the set of elements of $\mathscr{T}_{h,i}$ having E as a face, and for $e \in \mathscr{T}_{h,\gamma}$ let $\mathscr{T}_{h,e}$ denote the set of elements of $\mathscr{T}_{h,\gamma}$ having e as a face.

We will also introduce some notation for jumps of normal or tangential components of vector functions at interfaces between elements. If $E \in \mathscr{E}_{h,i}$ and $T \in \mathscr{T}_{h,E}$ then let $\mathbf{n}_{T,E}$ denote the unit normal vector on E pointing outward from T, and for $e \in \mathscr{E}_{h,\gamma}$ and $t \in \mathscr{T}_{h,\gamma}$, let $\mathbf{n}_{t,e}$ denote the unit normal vector (in the plane of γ) on e pointing outward from t. Now if $\mathbf{v}$ is a sufficiently regular, vector valued function on Ω_i, respectively on γ, then we define the jump in the normal component of $\mathbf{v}$ across an element $E \in \mathscr{E}_{h,i}^0$, across respectively an element $e \in \mathscr{E}_{h,\gamma}^0$, by

$$[\mathbf{v}]_{\mathbf{n},E} = \sum_{T \in \mathscr{T}_{h,E}} \left((\mathbf{v}_{|T})_{|E} \cdot \mathbf{n}_{T,E}\right) \qquad [\mathbf{v}]_{\mathbf{n},e} = \sum_{t \in \mathscr{T}_{h,e}} \left((\mathbf{v}_{|t})_{|e} \cdot \mathbf{n}_{t,e}\right).$$

Similarly we define the jump in the tangential component of a sufficiently regular $\mathbf{v}$ (for $n=3$) by

$$[\mathbf{v}]_{\tau,E} = \sum_{T \in \mathscr{T}_{h,E}} \left((\mathbf{v}_{|T})_{|E} \times \mathbf{n}_{T,E}\right) \qquad [\mathbf{v}]_{\tau,e} = \sum_{t \in \mathscr{T}_{h,e}} \left((\mathbf{v}_{|t})_{|e} \cdot \mathbf{t}_{t,e}\right).$$

For $n=2$ we have

$$[\mathbf{v}]_{\tau,E} = \sum_{T \in \mathscr{T}_{h,E}} \left((\mathbf{v}_{|T})_{|E} \cdot \mathbf{t}_{T,E}\right) \qquad [\mathbf{v}]_{\tau,e} = 0$$

as there is no tangential component for a jump across a vertex in a line segment.

3.1 Approximation spaces and projection operators

The finite element spaces used to approximate problem (6) are conforming: $M_h \subset M$ and $\mathbf{W_h} \subset \mathbf{W}$. The scalar approximation space $M_h = M_{h,1} \times M_{h,2} \times M_{h,\gamma} \subset M$ is a space of piecewise constant functions. For $i=1,2$, $M_{h,i} \subset L^2(\Omega_i)$ is the space of functions that are constant on each element of $\mathscr{T}_{h,i}$, and $M_{h,\gamma} \subset L^2(\gamma)$ is the space of functions that are constant on each element of $\mathscr{T}_{h,\gamma}$:

$$M_{h,i} = \mathscr{P}_{\mathbf{0}}(\mathscr{T}_{h,i}) := \{q \in L^2(\Omega_i) : q_{|T} \in P_0(T),\ \forall T \in \mathscr{T}_{h,i}\},\ \text{for } i=1,2,$$

$$M_{h,\gamma} = \mathscr{P}_{\mathbf{0}}(\mathscr{T}_{h,\gamma}) := \{q_\gamma \in L^2(\gamma) : q_{\gamma_{|t}} \in P_0(t),\ \forall t \in \mathscr{T}_{h,\gamma}\},$$

where $P_0(T)$, respectively $P_0(t)$, is the space of functions constant on T, respectively on t. With this discretization space is associated the L^2-projection operator defined by

$$P_h^0 = (P_{h,1}^0, P_{h,2}^0, P_{h,\gamma}^0) : M \longrightarrow M_h,$$

and we have the following estimates (see [7, Theorem 2.1 p. 164])

$$\begin{array}{ll} \|q_i - P_{h,i}^0 q_i\|_{0,T} \preceq \|q_i\|_{0,T},\ T \in \mathscr{T}_{h,i},\ i=1,2, & \|q_\gamma - P_{h,\gamma}^0 q_\gamma\|_{0,t} \preceq \|q_\gamma\|_{0,t},\ t \in \mathscr{T}_{h,\gamma}, \\ \|q_i - P_{h,i}^0 q_i\|_{0,T} \preceq h|q_i|_{1,T},\ T \in \mathscr{T}_{h,i},\ i=1,2, & \|q_\gamma - P_{h,\gamma}^0 q_\gamma\|_{0,t} \preceq h|q_\gamma|_{1,t},\ t \in \mathscr{T}_{h,\gamma}, \end{array} \tag{16}$$

where we use the shorthand notation

$$x \preceq y \tag{17}$$

for $x \leq Cy$ with the positive constant C independent of x, y, and the meshes involved.

The vector approximation space $\mathbf{W_h} = \mathbf{W}_{h,1} \times \mathbf{W}_{h,2} \times \mathbf{W}_{h,\gamma} \subset \mathbf{W}$ is a product of lowest order Raviart-Thomas (Nédelec) (RT(N)) spaces; see [8] or [24]. For $i=1,2$, $\mathbf{W}_{h,i} \subset \mathbf{H}(\mathrm{div};\Omega_i)$ is the space of lowest order RT(N) elements (in n-D) subordinate to the grid $\mathscr{T}_{h,i}$, $i=1,2$, and $\mathbf{W}_{h,\gamma}\subset\mathbf{H}(\mathrm{div}_\tau;\gamma)$ is the space of lowest order RT elements (in $(n-1)$-D) subordinate to the grid $\mathscr{T}_{h,\gamma}$:

$$\mathbf{W}_{h,i}=\mathbf{RT_0}(\mathscr{T}_{h,i}):=\{\mathbf{v}\in\mathbf{H}(\mathrm{div},\Omega_i)\,:\,\mathbf{v}|_T\in\mathbf{P}_0(T)\oplus\mathbf{x}P_0(T)\,\forall T\in\mathscr{T}_{h,i}\},\ \text{for } i=1,2,$$

$$\mathbf{W}_{h,\gamma}=\mathbf{RT_0}(\mathscr{T}_{h,\gamma}):=\{\mathbf{v}\in\mathbf{H}(\mathrm{div}_\tau,\gamma)\,:\,\mathbf{v}|_t\in\mathbf{P}_0(t)\oplus\mathbf{x}P_0(t)\,\forall t\in\mathscr{T}_{h,\gamma}\},$$

where $\mathbf{P}_0(T)=P_0(T)^n$ and $\mathbf{P}_0(t)=P_0(t)^{n-1}$. Of course when $n=2$, $\mathbf{W}_{h,\gamma}=\mathscr{P}_\mathbf{1}(\mathscr{T}_{h,\gamma}):=\{v_\gamma\in H^1(\gamma)\,:\,v_{\gamma|_t}\in P_1(t),\forall t\in\mathscr{T}_{h,\gamma}\}$ where $P_1(t)$ is the space of polynomial functions of degree ≤ 1.

The interpolation operator associated with $\mathbf{W}$ is defined using degrees of freedom that require extra regularity: if $s>2$, then one may define (see [8, (2.5.1)])

$$\Pi_h^D=(\Pi_{h,1}^D,\Pi_{h,2}^D,\Pi_{h,\gamma}^D):\mathbf{W}^{(s)}:=\mathbf{W}\cap(\mathbf{L}^s(\Omega_1)\times\mathbf{L}^s(\Omega_2)\times\mathbf{L}^s(\gamma))\longrightarrow\mathbf{W_h}$$

using the following degrees of freedom to define $\Pi_{h,i}^D$, $i=1,2$:

$$\int_E(\mathbf{v}_i-\Pi_{h,i}^D(\mathbf{v}_i))\cdot\mathbf{n}_E ds=0,\quad \forall E\in\mathscr{E}_{h,i}, \tag{18}$$

and the following degrees of freedom to define $\Pi_{h,\gamma}^D$ in the case $n=3$:

$$\int_e(\mathbf{v}_\gamma-\Pi_{h,\gamma}^D(\mathbf{v}_\gamma))\cdot\mathbf{n}_e ds=0,\quad \forall e\in\mathscr{E}_{h,\gamma}, \tag{19}$$

where $\mathbf{n}_E$, respectively $\mathbf{n}_e$, is a unit normal vector on E, respectively a unit normal vector on e in the plane of γ. In the case $n=2$ (so γ is of dimensional one), $\Pi_{h,\gamma}^D$ is just the Lagrange interpolation operator .

With these projection operators we have the following commutative diagram (see [8, Prop. 2.5.2]):

$$\begin{array}{ccc}\mathbf{W}^{(s)} & \xrightarrow{\mathrm{div}} & M\\ \Big\downarrow \Pi_h^D & & \Big\downarrow P_h^0\\ \mathbf{W_h} & \xrightarrow{\mathrm{div}} & M_h\end{array}\ . \tag{20}$$

Note that the commuting diagram property also implies that if $\mathbf{v}\in\mathbf{W}^{(s)}$ then

$$\mathrm{div}\,\Pi_{h,i}^D\mathbf{v}_i=P_{h,i}^0\mathrm{div}\,\mathbf{v}_i \qquad \text{and} \qquad \mathrm{div}_\tau\,\Pi_{h,\gamma}^D\mathbf{v}_i=P_{h,\gamma}^0\mathrm{div}_\tau\,\mathbf{v}_\gamma, \tag{21}$$

$$\left(\mathrm{div}\,(\mathbf{v}_i-\Pi_{h,i}^D\mathbf{v}_i)\,,q_0\right)_T=0, \qquad \forall q_0\in P_0(T),\ \forall T\in\mathscr{T}_{h,i},\ i=1,2, \tag{22}$$

$$\left\langle\mathrm{div}_\tau(\mathbf{v}_\gamma-\Pi_{h,\gamma}^D\mathbf{v}_\gamma),q_0\right\rangle_t=0, \qquad \forall q_0\in P_0(t),\ \forall t\in\mathscr{T}_{h,\gamma}. \tag{23}$$

Remark 3 : *In the case in which the grids match up along γ, we also have the following commutative diagram:*

$$\begin{array}{ccc}\mathbf{W}^{(s)} & \xrightarrow{\mathrm{Div}} & M\\ \Big\downarrow \Pi_h^D & & \Big\downarrow P_h^0\\ \mathbf{W_h} & \xrightarrow{\mathrm{Div}} & M_h\end{array}\ . \tag{24}$$

□

We have the following approximation properties (see [8, Prop. 2.5.1]): for $\mathbf{v}\in\mathbf{W}^{(s)}$

$$\begin{aligned}&\|\mathbf{v}-\Pi_h^D\mathbf{v}\|_\mathbf{W}\preceq\|\mathbf{v}\|_\mathbf{W},\\ &\|\mathbf{v}-\Pi_h^D\mathbf{v}\|_{0,T}\preceq h_T|\mathbf{v}|_{1,T},\forall T\in\mathscr{T}_{h,i},\ i=1,2;\qquad \|\mathbf{v}_\gamma-\Pi_{h,\gamma}^D\mathbf{v}_\gamma\|_{0,t}\preceq h_t|\mathbf{v}_\gamma|_{1,t},\forall t\in\mathscr{T}_{h,\gamma},\\ &\|\mathbf{v}-\Pi_h^D\mathbf{v}\|_{0,E}\preceq h_E^{1/2}|\mathbf{v}|_{1,\widetilde{T}_E},\forall E\in\mathscr{E}_{h,i},\ i=1,2;\quad \|\mathbf{v}_\gamma-\Pi_{h,\gamma}^D\mathbf{v}_\gamma\|_{0,e}\preceq h_e^{1/2}|\mathbf{v}_\gamma|_{1,\widetilde{t}_e},\forall e\in\mathscr{E}_{h,\gamma},\end{aligned} \tag{25}$$

where $\widetilde{T}_E$, respectively $\tilde{t}_e$, denotes the union of all elements $T' \in \mathscr{T}_{h,i}$, respectively $t' \in \mathscr{T}_{h,\gamma}$, such that $E \subset \partial T'$, respectively $e \subset \partial t'$.

We will also make use of approximation spaces $\mathbf{N}_h = \mathbf{N}_{h,1} \times \mathbf{N}_{h,2} \times \mathbf{N}_{h,\gamma} \subset \mathbf{N}$, (with $\mathbf{N}$ defined in (15)), that will be used only for the *a posteriori* estimates. These are the lowest order Nédélec spaces of the first kind:

$$\text{for } n=3, \qquad \begin{array}{l} \mathbf{N}_{h,i} = \mathbf{N}_0(\mathscr{T}_{h,i}) \;:= \{\mathbf{z}_i \in \mathbf{H}(\mathbf{curl};\Omega_i) \,:\, \mathbf{z}|_T \in \mathbf{P}_0(T) \oplus (\mathbf{x} \times \mathbf{P}_0(T)), \forall T \in \mathscr{T}_{h,i}\}, \\ \mathbf{N}_{h,\gamma} = \mathbf{N}_0(\mathscr{T}_{h,\gamma}) := \{\mathbf{z}_\gamma \in \mathbf{H}(\mathrm{curl}_\tau;\gamma) \,:\, \mathbf{z}|_t \in \mathbf{P}_0(t) \oplus \mathbf{x}^\perp P_0(t), \forall t \in \mathscr{T}_{h,\gamma}\}, \end{array}$$

$$\text{for } n=2, \qquad \begin{array}{l} \mathbf{N}_{h,i} = \mathbf{N}_0(\mathscr{T}_{h,i}) \;:= \{\mathbf{z}_i \in \mathbf{H}(\mathbf{curl};\Omega_i) \,:\, \mathbf{z}|_T \in \mathbf{P}_0(T) \oplus \mathbf{x}^\perp P_0(T), \forall T \in \mathscr{T}_{h,i}\}, \\ \mathbf{N}_{h,\gamma} = \mathbf{N}_0(\mathscr{T}_{h,\gamma}) := \mathscr{P}_\mathbf{1}(\mathscr{T}_{h,\gamma}) := \{v_\gamma \in H^1(\gamma) \,:\, v_{\gamma|_t} \in P_1(t), \forall t \in \mathscr{T}_{h,\gamma}\}, \end{array}$$

where $\mathbf{x}^\perp = (-x_2, x_1)$ whenever $\mathbf{x} = (x_1, x_2)$.

The definition of the interpolation operator associated with the **curl** operator also requires added regularity. Define the space $\mathbf{N}^{(1)}$ by

$$\mathbf{N}^{(1)} = H(\mathbf{curl},\Omega_1) \cap (H^1(\Omega_1))^3 \times H(\mathbf{curl},\Omega_2) \cap (H^1(\Omega_2))^3 \times H(\mathbf{curl},\gamma) \cap (H^1(\gamma))^2$$

and define the interpolation operator

$$\Pi_h^C = (\Pi_{h,1}^C, \Pi_{h,2}^C, \Pi_{h,\gamma}^C) : \mathbf{N}^{(1)} \longrightarrow \mathbf{N}_h$$

componentwise and according to whether $n=2$ or $n=3$. As in the remark 2.1.5 page 51 in [8], it turns out that in two dimensions the space $\mathbf{H}(\mathbf{curl})$ is isomorphic to $\mathbf{H}(\mathrm{div})$. As a consequence the approximation of $\mathbf{H}(\mathbf{curl})$ can be obtained from $\mathbf{H}(\mathrm{div})$. For $n=3$, we use the Clément type regularization operator introduced in [5]. To every element G of $\mathscr{G}_{h,i}$, for $i=1,2$, we associate an element T_G of $\mathscr{T}_{h,i}$, and we define the operator π_G as the orthogonal projection operator of $\mathbf{L}^2(T_G)$ on $\mathbf{P}_0(T_G)$. For $\mathbf{r_i} \in H(\mathbf{curl},\Omega_i) \cap H^1(\Omega_i)$ $i=1,2$, let

$$\Pi_{h,i}^C(\mathbf{r}_i) = \sum_{G \in \mathscr{G}_{h,i}} (\int_G ((\pi_G \mathbf{r_i})(s) \cdot \mathbf{t}_G ds)\, \xi_G \tag{26}$$

where ξ_G are the basic functions associated to G in $\mathbf{N}_{h,i}$, and $\mathbf{t}_G$ the tangent of G .
Similarly, we defined on γ the operator $\Pi_{h,\gamma}^C$ with (26) by taking $e \in \mathscr{E}_{h,\gamma}$ instead of $G \in \mathscr{G}_{h,i}$. This operator verify the commuting diagram, i.e

$$\mathbf{curl}(\Pi_h^C(\mathbf{z})) = \Pi_h^D(\mathbf{curl}\,\mathbf{z}). \tag{27}$$

For that it suffices to note that $\mathbf{curl}(\Pi_{h,i}^C \mathbf{z})$ is in $RT_0(\mathscr{T}_{h,i})$ and satisfies for all $E \in \mathscr{E}_{h,i}$

$$\int_E \mathbf{curl}(\Pi_{h,i}^C \mathbf{z}) \cdot \mathbf{n} = \int_E \mathbf{curl}(\mathbf{z}) \cdot \mathbf{n}.$$

Then

$$\begin{tikzcd} \mathbf{N}^{(s)} \arrow[r, "\mathbf{curl}"] \arrow[d, "\Pi_h^C"] & \mathbf{W} \arrow[d, "\Pi_h^D"] \\ \mathbf{N}_h \arrow[r, "\mathbf{curl}"] & \mathbf{W_h} \end{tikzcd}$$

which may be resumed in the following equations: or otherwise written, if $\mathbf{z} \in \mathbf{N}^{(s)}$ then

$$\mathbf{curl}\, \Pi_{h,i}^C \mathbf{z}_i = \Pi_{h,i}^D \mathbf{curl}\, \mathbf{z}_i \qquad \text{and} \qquad \mathbf{curl}_\tau\, \Pi_{h,\gamma}^C z_\gamma = \Pi_{h,\gamma}^D \mathbf{curl}_\tau\, z_\gamma \tag{28}$$

The following estimates follow from the estimates in the component spaces which can be found in ([5]):

$$\begin{array}{l} \|\mathbf{r}_i - \Pi_{h,i}^C(\mathbf{r}_i)\|_{0,T} \preceq h_T \|\mathbf{r}_i\|_{1,(\mathscr{U}_T)}, \quad \forall T \in \mathscr{T}_{h,i} \\ \|\mathbf{r}_\gamma - \Pi_{h,\gamma}^C(\mathbf{r}_\gamma)\|_{0,T} \preceq h_t \|\mathbf{r}_\gamma\|_{1,(\mathscr{U}_t)}, \quad \forall t \in \mathscr{T}_{h,\gamma} \\ \|\mathbf{r}_i - \Pi_{h,i}^C(\mathbf{r}_i)\|_{0,E} \preceq h_E^{1/2} \|\mathbf{r}\|_{1,\widetilde{T}_E}, \quad \forall E \in \mathscr{E}_{h,i}. \end{array} \tag{29}$$

where $\mathscr{U}_T$, (resp. $\mathscr{U}_t$) is the union of elements which have an intersection with T (resp.t). We will also make use of the L^2-projection operators

$$\widetilde{P}_{h,i}^{\Gamma_i}: L^2(\Gamma_i) \to H_{h,i}^{\Gamma_i}, \qquad \widetilde{P}_{h,i}^{\gamma}: L^2(\partial\gamma) \to M_{h,i}^{\gamma}, \qquad \widetilde{P}_{h,\gamma}^{\partial\gamma}: L^2(\gamma) \to M_{h,\gamma}^{\partial\gamma}, \text{ and } P_{\mathscr{E}_{h,i}^{\gamma}}: \mathscr{P}_0(\mathscr{E}_{h,i+1}^{\gamma}) \to \mathscr{P}_0(\mathscr{E}_{h,i}^{\gamma}) \tag{30}$$

where $H_{h,i}^{\Gamma_i}$, $M_{h,i}^{\gamma}$ and $M_{h,\gamma}^{\partial\gamma}$ are the spaces of piecewise constant functions on Γ_i, γ and $\partial\gamma$, respectively (constant on each element of $\mathscr{E}_{h,i}^{\Gamma_i}$, $\mathscr{E}_{h,i}^{\gamma}$ and $\mathscr{E}_{h,\gamma}^{\partial\gamma}$, respectively).

We conclude this subsection with a lemma concerning elements of $\widetilde{\mathbf{W}}$ and the projection operator $\Pi_{h,i}^D$.

Lemma 2 *Let $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_\gamma) \in \widetilde{\mathbf{W}}$, and let $\mathbf{r_i} \in \mathbf{H}^1(\Omega_i)$, $r_\gamma \in H_0^1(\gamma)$ and $\rho_\gamma \in H_0^1(\gamma)$ be as defined in (9), (10), and (12), respectively. Then*

(i) we have, for all $\mathbf{w}_{h,i} \in \mathbf{W}_{h,i}$, $i = 1,2$,

$$\sum_{T\in\mathscr{T}_{h,i}} (K_i^{-1}\mathbf{w}_{h,i}, \mathbf{v}_i - \Pi_{h,i}^D \mathbf{v}_i)_T = -\sum_{E\in\mathscr{E}_{h,i}} \left\langle [K_i^{-1}\mathbf{w}_{h,i}]_{\tau,E}\,, \mathbf{r_i} - \Pi_{h,i}^C \mathbf{r_i} \right\rangle_E, \tag{31}$$

where $[\cdot]_{\tau,E}$ is defined in the beginning of Section 3.
(ii) In the case $n = 3$ we also have, for all $\mathbf{w}_{h,\gamma} \in \mathbf{W}_{h,\gamma}$,

$$\begin{aligned} &\sum_{t\in\mathscr{T}_{h,\gamma}} \left\langle \left(dK_{\gamma,\tau}\right)^{-1} \mathbf{w}_{h,\gamma}, \mathbf{v}_\gamma - \Pi_{h,\gamma}^D \mathbf{v}_\gamma \right\rangle_t \\ &\quad = -\sum_{e\in\mathscr{E}_{h,\gamma}} \left\langle\!\!\left\langle \left[\left(dK_{\gamma,\tau}\right)^{-1} \mathbf{w}_{h,\gamma}\cdot\tau_t\right]_{\tau,e}, \left((r_\gamma+\rho_\gamma) - \Pi_{h,\gamma}^C(r_\gamma+\rho_\gamma)\right) \right\rangle\!\!\right\rangle_e \\ &\qquad + \sum_{t\in\mathscr{T}_{h,\gamma}} \left\langle \left(dK_{\gamma,\tau}\right)^{-1} \mathbf{w}_{h,\gamma}, \sum_{i=1}^2 (\mathbf{r_i} - \Pi_{h,i}^C \mathbf{r_i}) \times \mathbf{n}_i \right\rangle_t. \end{aligned} \tag{32}$$

Proof To show (31) we use the fact that $\mathbf{v}_i - \Pi_{h,i}^D \mathbf{v}_i = \mathbf{curl}(\mathbf{r_i} - \Pi_{h,i}^C \mathbf{r_i})$, Green's formula as given in (5) in each $T \in \mathscr{T}_{h,i}$, and the facts that K_i is constant on each $T \in \mathscr{T}_{h,i}$, and that the **curl** operator vanishes on lowest order Raviart-Thomas-Nédélec elements:

$$\sum_{T\in\mathscr{T}_{h,i}} (K_i^{-1}\mathbf{u}_{h,i}\,, \mathbf{v}_i - \Pi_{h,i}^D \mathbf{v}_i)_T \;= -\sum_{T\in\mathscr{T}_{h,i}} \left\langle \zeta_\tau\left(K_i^{-1}\mathbf{u}_{h,i}\right), \mathbf{r_i} - \Pi_{h,i}^C \mathbf{r_i} \right\rangle_{\partial T}.$$

Since the tangential component of $(K_i^{-1}\mathbf{u}_{h,i})$ is not continuous through $E \in \mathscr{E}_{h,i}$, we introduce the jump across the interior elements $E \in \mathscr{E}_{h,i}^0$ and obtain (31).

To show (32), we recall that by (12) we have $\mathbf{v}_\gamma = \mathbf{curl}_\tau(r_\gamma + \rho_\gamma) + \sum_{i=1}^2 (\mathbf{r_i} \times \mathbf{n}_i)$, and apply the projection operator $\Pi_{h,\gamma}^D$ to obtain

$$\Pi_{h,\gamma}^D \mathbf{v}_\gamma = \mathbf{curl}_\tau \Pi_{h,\gamma}^C (r_\gamma + \rho_\gamma) + \sum_{i=1}^2 (\Pi_{h,i}^C \mathbf{r_i} \times \mathbf{n}_i), \tag{33}$$

where we have used the second equation of (28). Now (32) follows by (33) and using Green's formula (5). ■

Remark 4 : *In the case $n = 3$, if we identify $\mathbf{w}_{h,\gamma}$ with the 3 dimensional vector whose component in the direction normal to γ is 0 and whose projection on γ is $\mathbf{w}_{h,\gamma}$, we can rewrite the last term of* (32) *as follows:*

$$\sum_{t\in\mathscr{T}_{h,\gamma}} \left\langle \left(dK_{\gamma,\tau}\right)^{-1} \mathbf{w}_{h,\gamma}, \sum_{i=1}^2 (\mathbf{r_i} - \Pi_{h,i}^C \mathbf{r_i}) \times \mathbf{n}_i \right\rangle_t = -\sum_{t\in\mathscr{T}_{h,\gamma}} \sum_{i=1}^2 \left\langle \left(dK_{\gamma,\tau}\right)^{-1} \mathbf{w}_{h,\gamma} \times \mathbf{n}_i, \mathbf{r_i} - \Pi_{h,i}^C \mathbf{r_i} \right\rangle_t.$$

□

3.2 The discretized problem

The discrete mixed formulation of problem (1) is given by

$$\begin{aligned}
&\text{Find } (\mathbf{u}_h, p_h) \in \mathbf{W_h} \times M_h \text{ such that} \\
&a_\xi(\mathbf{u}_h, \mathbf{v}_h) - \beta(\mathbf{v}_h, p_h) = -L_d(\mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{W_h} \\
&\beta(\mathbf{u}_h, q_h) = L_f(q_h), \qquad \forall q_h \in M_h.
\end{aligned} \tag{34}$$

Problem (34) is well posed (cf.[21], [12]) and we have an optimal a priori error estimate; i. e.

$$\|\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{W_h}} + \|p-p_h\|^2_M \le Ch\left(\|p\|_H + \|\mathbf{u}\|_{\mathbf{H}} + \|\mathrm{div}\mathbf{u}\|_H + \sum_{i=1}^{2} \|\mathbf{u}\cdot\mathbf{n}\|_{H^1(\gamma)}\right), \tag{35}$$

where $H = H^1(\Omega_1) \times H^1(\Omega_2) \times H^1(\gamma)$ and $\mathbf{H} = \mathbf{H}^1(\Omega_1) \times \mathbf{H}^1(\Omega_2) \times \mathbf{H}^1(\gamma)$.

Furthermore the solution satisfies

$$\begin{aligned}
&P^0_{h,i} f_i = \mathrm{div}\mathbf{u}_{h,i},\ i = 1,2, \\
&P^0_{h,\gamma} f_\gamma = \mathrm{div}_\tau \mathbf{u}_{h,\gamma} - P^0_{h,\gamma}([\mathbf{u}_h\cdot\mathbf{n}]).
\end{aligned} \tag{36}$$

4 Definition of the error estimators and main results

Our goal now is to derive an *a posteriori* error estimate of the errors $\mathbf{e_u} = \mathbf{u} - \mathbf{u}_h$, and $\varepsilon_p = p - p_h$. We have the residual equations

$$\begin{aligned}
&a_\xi(\mathbf{e_u}, \mathbf{v}) - \beta(\mathbf{v}, \varepsilon_p) = -L_d(\mathbf{v}) - a_\xi(\mathbf{u}_h, \mathbf{v}) + \beta(\mathbf{v}, p_h), \qquad \forall \mathbf{v} \in \mathbf{W}, \\
&\beta(\mathbf{e_u}, q) = L_f(q) - \beta(\mathbf{u}_h, q), \qquad \forall q \in M,
\end{aligned} \tag{37}$$

and the orthogonality conditions

$$\begin{aligned}
&a_\xi(\mathbf{e_u}, \mathbf{v}_h) - \beta(\mathbf{v}_h, \varepsilon_p) = 0, \qquad \forall \mathbf{v}_h \in \mathbf{W_h}, \\
&\beta(\mathbf{e_u}, q_h) = 0, \qquad \forall q_h \in M_h.
\end{aligned} \tag{38}$$

Following [10] and [23], we will use error indicators associated with lack of regularity in the discrete solution and failure of the discrete solution to satisfy the equations (1) locally. Let $J_{\tau,E}(.)$ defined by

$$J_{\tau,E}(\mathbf{v}_{h,i}) := \begin{cases} [K_i^{-1}\mathbf{v}_{h,i}]_{\tau,E}; & \text{if } E \in \mathscr{E}^0_{h,i} \cup \mathscr{E}^{\Gamma_i}_{h,i} \\ \left(K_i^{-1}\mathbf{v}_{h,i} - \left(dK_{\gamma,\tau}\right)^{-1}\mathbf{v}_{h,\gamma}\right)_{|E} \times \mathbf{n}_i & \text{if } E \in \mathscr{E}^\gamma_{h,i}, \end{cases} \tag{39}$$

where $\left[\cdot\right]_{\tau,E}$ is defined in section 3.

Similarly, for any $\mathbf{v}_{h,\gamma} \in \mathbf{W}_{h,\gamma}$, $e \in \mathscr{E}_{h,\gamma}$ and $t \in \mathscr{T}_{h,\gamma}$ having e as a face, we define

$$J_{\tau,e}(\mathbf{v}_{h,\gamma}) := [dK^{-1}_{\gamma,\tau}\mathbf{v}_{h,\gamma}]_{\tau,e} \quad \text{if } e \in \mathscr{E}^0_{h,\gamma}, \tag{40}$$

which, by the definition of $[\cdot]_{\tau,e}$, vanishes in the case $n = 2$ (cf. Section 3).

The local residual error estimators are defined as follows:

1. **Indicators associated with the faces of the elements** (These are related to the lack of regularity in the discrete solution). The jumps of tangential traces give rise to error indicators defined as follows: Define, $\forall T \in \mathscr{T}_{h,i}$, $i = 1,2$, and $\forall t \in \mathscr{T}_{h,\gamma}$, and

$$\begin{aligned}
&\sigma^{(i)}_E := h_E^{1/2}\|J_{\tau,E}(\mathbf{u}_{h,i})\|_{0,E}, \\
&\sigma_e := h_e^{1/2}\|J_{\tau,e}(\mathbf{u}_{h,\gamma})\|_{0,e} \qquad \text{if } n = 3.
\end{aligned} \tag{41}$$

2. **Indicators related to the local residual equations** (These are related to the failure of the discrete solution to satisfy Darcy's law locally in the elements or the continuity equation (see (34)). Define, $\forall T \in \mathscr{T}_{h,i}$, $i = 1,2$, and $\forall t \in \mathscr{T}_{h,\gamma}$,

$$\eta_T^{(i)} := h_T \|K_i^{-1}\mathbf{u}_{h,i}\|_{0,T} \quad \text{and} \quad \eta_t^{(\gamma)} := h_t \|(dK_{\gamma,\tau})^{-1}\mathbf{u}_{h,\gamma}\|_{0,t}, \tag{42}$$

$$\omega_T^{(i)} := \|f_i - \mathrm{div}\mathbf{u}_{h,i}\|_{0,T} \quad \text{and} \quad \omega_t := \|f_\gamma - \mathrm{div}_\tau \mathbf{u}_{h,\gamma} - P^0_{h,\gamma}([\mathbf{u}_h \cdot \mathbf{n}])\|_{0,t} \tag{43}$$

3. **Indicators related to the interface condition.** Define, $\forall E \in \mathscr{E}^{\gamma}_{h,i}$, $i = 1,2$,

$$\delta_E^{(i)} := \|\mathscr{A}_E^{(i)}(\mathbf{u}_h, p_h)\|_{0,E}, \tag{44}$$

$$\text{where} \qquad \mathscr{A}_E^{(i)}(\mathbf{u}_h, p_h) := \left(p_{h,i} - p_{h,\gamma} - \frac{d}{2K_{\gamma,\nu}}\{\mathbf{u}_h \cdot \mathbf{n}\}_{\xi,i} \right)\Big|_E, \tag{45}$$

4. **Indicators related to the non conformity of the meshes.** Define, $\forall E \in \mathscr{E}^{\gamma}_{h,i}$, $i = 1,2$, and $\forall t \in \mathscr{T}_{h,\gamma}$,

$$\bar{\delta}_E^{(i)} := h_E^{1/2} \|P^{\gamma}_{h,i}(p_{h,\gamma}) - p_{h,\gamma}\|_{0,E}, \tag{46}$$

$$\bar{\delta}_t := \left\|P^0_{h,\gamma}([\mathbf{u}_h \cdot \mathbf{n}]) - [\mathbf{u}_h \cdot \mathbf{n}]\right\|_{0,t} \tag{47}$$

$$\Delta_E^{(i)} := h_E^{1/2} \left\| \frac{d}{2K_{\gamma,\nu}}(1-\xi)(\mathbf{u}_{h,i+1} \cdot \mathbf{n}_{i+1} - P_{\mathscr{E}^{\gamma}_{h,i}}(\mathbf{u}_{h,i+1} \cdot \mathbf{n}_{i+1})) \right\|_{0,E} \tag{48}$$

5. **Indicators related to the boundary data.** Define, $\forall E \in \mathscr{E}^{\Gamma_i}_{h,i}$, $i = 1,2$, and $\forall e \in \mathscr{E}^{\partial\gamma}_{h,\gamma}$,

$$\begin{aligned} &\overline{\omega}_E^{(i)} := \|\bar{p}_i - \widetilde{P}^{\Gamma_i}_{h,i}\bar{p}_i\|_{0,E}, , \qquad &&\overline{\omega}_{1E}^{(i)} := \|\bar{p}_i - \widetilde{P}^{\Gamma_i}_{h,i}\bar{p}_i\|_{1,E}, \ , \\ &\overline{\omega}_e := \|\bar{p}_\gamma - \widetilde{P}^{\partial\gamma}_{h,\gamma}\bar{p}_\gamma\|_{0,e} \ , \qquad &&\overline{\omega}_{1e} := \|\bar{p}_\gamma - \widetilde{P}^{\partial\gamma}_{h,\gamma}\bar{p}_\gamma\|_{1,E}, \quad \text{if } n = 3. \end{aligned} \tag{49}$$

Remark 5 : In light of (36)*, the indicators* $\omega_T^{(i)}$ *and* ω_t *can be viewed as indicators related to source-term data.* □

The following three propositions state the main results of this work. We have adopted the notation of (17).

Proposition 1 *Let* $\mathbf{u}$ *be the solution of (6) and* $\mathbf{u}_h$ *the solution of (34). Then the following a posteriori error estimate holds in the case* $n = 3$*:*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{W}} \preceq \sum_{i=1}^{2} \Bigg\{ &\Bigg(\sum_{E \in \mathscr{E}_{h,i}} (\sigma_E^{(i)})^2 \Bigg)^{1/2} + \Bigg(\sum_{E \in \mathscr{E}^{\gamma}_{h,i}} (\delta_E^{(i)})^2 \Bigg)^{1/2} + \Bigg(\sum_{E \in \mathscr{E}_{h,i}} (\overline{\omega}_{1E}^{(i)})^2 \Bigg)^{1/2} + \Bigg(\sum_{T \in \mathscr{T}_{h,i}} (\omega_T^{(i)})^2 \Bigg)^{1/2} \Bigg\} \\ &+ \Bigg(\sum_{e \in \mathscr{E}_{h,\gamma}} (\sigma_e)^2 \Bigg)^{1/2} + \Bigg(\sum_{t \in \mathscr{T}_{h,\gamma}} (\bar{\delta}_t)^2 \Bigg)^{1/2} + \Bigg(\sum_{e \in \mathscr{E}^{\partial\gamma}_{h,\gamma}} (\overline{\omega}_{1e})^2 \Bigg)^{1/2} + \Bigg(\sum_{t \in \mathscr{T}_{h,\gamma}} (\omega_t)^2 \Bigg)^{1/2}. \end{aligned} \tag{50}$$

In the case $n = 2$*, the estimate differs only in that the term* $\sum_{e \in \mathscr{E}_{h,\gamma}} (\sigma_e)^2$ *in the right-hand-side of* (50) *should be replaced by* $\sum_{t \in \mathscr{T}_{h,\gamma}} (\eta_t^{(\gamma)})^2$.

Proposition 2 *Let* $\mathbf{u}$ *be the solution of (6) and* $\mathbf{u}_h$ *the solution of (34) then the following a posteriori error estimates hold:*

$$\begin{aligned}
\|p-p_h\|_M \preceq \sum_{i=1}^{2}\left\{\left(\sum_{E\in\mathscr{E}_{h,i}^{\Gamma_i}}(h_E^{\frac{1}{2}}\overline{\varpi}_E^{(i)})^2\right)^{1/2}+\left(\sum_{T\in\mathscr{T}_{h,i}}(\eta_T^{(i)})^2\right)^{1/2}+\left(\sum_{T\in\mathscr{T}_{h,i}}(h_T\,\omega_T^{(i)})^2\right)^{1/2}\right\}\\
+\sum_{i=1}^{2}\left\{\left(\sum_{E\in\mathscr{E}_{h,i}^{\gamma}}(\bar{\delta}_E^{(i)})^2\right)^{1/2}+\left(\sum_{E\in\mathscr{E}_{h,i}^{\gamma}}(\Delta_E^{(i)})^2\right)^{1/2}\right\}\\
+\left(\sum_{e\in\mathscr{E}_{h,\gamma}^{\partial\gamma}}(h_e^{\frac{1}{2}}\overline{\varpi}_e)^2\right)^{1/2}+\left(\sum_{t\in\mathscr{T}_{h,\gamma}}(\eta_t^{(\gamma)})^2\right)^{1/2}+\left(\sum_{t\in\mathscr{T}_{h,\gamma}}(h_t\,\omega_t)^2\right)^{1/2}+\left(\sum_{t\in\mathscr{T}_{h,\gamma}}\left(h_t\,\bar{\delta}_t\right)^2\right)^{1/2}.
\end{aligned}$$

The following proposition concerns a lower bound of the error by the indicators. Let us define some notation. For $D\subset\Omega_i$ we put $\mathbf{W}_i(D):=H(\mathrm{div},D)$, and for $D\subset\gamma$ we put $\mathbf{W}_f(D):=H(\mathrm{div}_\tau,D)$.

Proposition 3 *For* $i=1,2$, *and for all* $T\in\mathscr{T}_{h,i}$, $t\in\mathscr{T}_{h,\gamma}$, $E\in\mathscr{E}_{h,i}$ *and* $e\in\mathscr{E}_{h,\gamma}$, *we have the following estimates:*

$$\begin{aligned}
\sigma_E^{(i)} &\preceq \|K_i^{-1}(\mathbf{u}_i-\mathbf{u}_{h,i})\|_{0,\widetilde{T}_E}+\|\left(K_i^{-1}\mathbf{u}_i-(dK_{\gamma,\tau})^{-1}\mathbf{u}_\gamma\right)_{\tau,E}\|_{0,E},\\
\sigma_e &\preceq \|\mathbf{u}_\gamma-\mathbf{u}_{h,\gamma}\|_{0,\widetilde{t}_e}
\end{aligned}\tag{51}$$

$$\begin{aligned}
\eta_T^{(i)} &\preceq \|\mathbf{u}_i-\mathbf{u}_{h,i}\|_{0,T}+\|p_i-p_{h,i}\|_{0,T},\\
\eta_t^{(\gamma)} &\preceq \|\mathbf{u}_\gamma-\mathbf{u}_{h,\gamma}\|_{0,t}+\|p_\gamma-p_{h,\gamma}\|_{0,t},
\end{aligned}\tag{52}$$

$$\delta_E^{(i)} \preceq \|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{W}}+\|p_\gamma-p_{h,\gamma}\|_M,\tag{53}$$

$$\begin{aligned}
\bar{\delta}_E^{(i)} &\preceq \|p_\gamma-p_{h,\gamma}\|_{0,E}+\|p_\gamma-\widetilde{P}_{h,i}^{\gamma}(p_\gamma)\|_{0,E},\\
\bar{\delta}_t &\preceq \|\mathrm{div}_\tau(\mathbf{u}_{h,\gamma}-\mathbf{u}_\gamma)\|_{0,t}+\|(P_{h,\gamma}^0(f_\gamma)-f_\gamma)\|_{0,t}+\sum_{i=1}^{2}\|(\mathbf{u}_{h,i}-\mathbf{u}_i)\cdot\mathbf{n}_i\|_{0,t}
\end{aligned}\tag{54}$$

$$\Delta_E^{(i)}\preceq \|\mathbf{u}_{i+1}\cdot\mathbf{n}_{i+1}-\mathbf{u}_{h,i+1}\cdot\mathbf{n}_{i+1}\|_{0,E}+\|P_{\mathscr{E}_{h,i}^{\gamma}}(\mathbf{u}_{i+1}\cdot\mathbf{n}_{i+1})-\mathbf{u}_{i+1}\cdot\mathbf{n}_{i+1}\|_{0,E}.\tag{55}$$

5 Upper bounds for the errors in terms of the indicators

In this section we will prove Propositions 1 and 2, which concern the reliability of the estimators, by deriving upper bounds, in terms of the indicators, for the error in the velocity, $\mathbf{e_u}=\mathbf{u}-\mathbf{u}_h$, and for the error in the pressure, $\varepsilon_p=p-p_h$, in the natural norms, $\|\mathbf{e_u}\|_{\mathbf{W}}$ and $\|\varepsilon_p\|_M$.

5.1 An upper bound for the velocity error

To derive an upper bound for $\|\mathbf{e_u}\|_{\mathbf{W}}$ we will use the following lemma:

Lemma 3 *If* $\mathbf{u}$ *is the solution of (6) and* $\mathbf{u}_h$ *the solution of (34), the following estimate holds:*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{W}} \preceq \sup_{\substack{\mathbf{v}\in\widetilde{\mathbf{W}}\\ \mathbf{v}\neq 0}} \frac{a_\xi(\mathbf{u}-\mathbf{u}_h,\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{W}}} + \sum_{i=1}^{2}\left(\sum_{T\in\mathscr{T}_{h,i}} (\omega_T^{(i)})^2\right)^{1/2} + \left(\sum_{t\in\mathscr{T}_{h,\gamma}} (\omega_t)^2\right)^{1/2} + \left(\sum_{t\in\mathscr{T}_{h,\gamma}} (\bar{\delta}_t)^2\right)^{1/2}. \tag{56}$$

Proof Let $\mathbf{u}$ be the solution of (6) and $\mathbf{u}_h$ the solution of (34). Since $a_\xi(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ satisfy (7) and (8) respectively (see [12]), it follows (see [8,24]) that for any $g\in M'$ the following auxiliary problem has a unique solution:

$$\begin{aligned} &\text{Find } (\mathbf{w},y)\in\mathbf{W}\times M \text{ such that}\\ &a_\xi(\mathbf{w},\mathbf{v})-\beta(\mathbf{v},y)=0, \qquad \forall\mathbf{v}\in\mathbf{W},\\ &\qquad\qquad\beta(\mathbf{w},q)=g(q), \qquad \forall q\in M, \end{aligned} \tag{57}$$

and that there is a constant C independent of g such that

$$\|\mathbf{w}\|_W \le C\|g\|_{M'}. \tag{58}$$

Now let $\mathbf{e_u}=\mathbf{u}-\mathbf{u}_h$, and let $(\mathbf{w},y)\in\mathbf{W}\times M$ be the solution of problem (57) for $g\in M'$ defined by $g(q)=\beta(\mathbf{e_u},q)$, $\forall q\in M$. We have $(\mathbf{e_u}-\mathbf{w})\in\widetilde{\mathbf{W}}$ since for all $q\in M$, $\beta(\mathbf{e_u}-\mathbf{w},q)=\beta(\mathbf{e_u},q)-\beta(\mathbf{w},q)=\beta(\mathbf{e_u},q)-g(q)=0$.

The ellipticity of $a_\xi(\cdot,\cdot)$ in $\widetilde{\mathbf{W}}$ gives (see (7))

$$\begin{aligned} C_\alpha\|\mathbf{e_u}-\mathbf{w}\|_{\mathbf{W}}^2 &\le a_\xi(\mathbf{e_u}-\mathbf{w},\mathbf{e_u}-\mathbf{w}) = a_\xi(\mathbf{e_u},\mathbf{e_u}-\mathbf{w})-a_\xi(\mathbf{w},\mathbf{e_u}-\mathbf{w})\\ &\le a_\xi(\mathbf{e_u},\mathbf{e_u}-\mathbf{w})+\|a_\xi\|\|\mathbf{w}\|_{\mathbf{W}}\|\mathbf{e_u}-\mathbf{w}\|_{\mathbf{W}}. \end{aligned}$$

Then $C_\alpha\|\mathbf{e_u}-\mathbf{w}\|_{\mathbf{W}} \le \dfrac{a_\xi(\mathbf{e_u},\mathbf{e_u}-\mathbf{w})}{\|\mathbf{e_u}-\mathbf{w}\|_{\mathbf{W}}}+\|a_\xi\|\|\mathbf{w}\|_{\mathbf{W}}$, and taking the supremum on $\widetilde{\mathbf{W}}$ we deduce that

$$C_\alpha\|\mathbf{e_u}-\mathbf{w}\|_{\mathbf{W}} \le \sup_{\mathbf{v}\in\widetilde{\mathbf{W}},\mathbf{v}\neq 0}\frac{a_\xi(\mathbf{e_u},\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{W}}}+\|a_\xi\|\|\mathbf{w}\|_{\mathbf{W}}. \tag{59}$$

From the second equation of (37) we have that for any $q\in M$,

$$g(q)=\sum_{i=1}^{2}(f_i-\operatorname{div}\mathbf{u}_{h,i}\,,q_i)_{\Omega_i}+(f_\gamma-\operatorname{div}_\tau\mathbf{u}_{h,\gamma}+[\mathbf{u}_h\cdot\mathbf{n}]\,,q_\gamma)_\gamma. \tag{60}$$

Thus

$$\begin{aligned} \|g\|_{M'} &= \sup_{\substack{q\in M\\ q\neq 0}}\frac{1}{\|q\|_M}g(q)\\ &\le \sum_{i=1}^{2}\left(\sum_{T\in\mathscr{T}_{h,i}}\|f_i-\operatorname{div}\mathbf{u}_{h,i}\|_{0,T}^2\right)^{1/2} + \left(\sum_{t\in\mathscr{T}_{h,\gamma}}\left\|f_\gamma-\operatorname{div}_\tau\mathbf{u}_{h,\gamma}+P_{h,\gamma}^0([\mathbf{u}_h\cdot\mathbf{n}])\right\|_{0,t}^2\right)^{1/2}\\ &\quad+\left(\sum_{t\in\mathscr{T}_{h,\gamma}}\|[\mathbf{u}_h\cdot\mathbf{n}]-P_{h,\gamma}^0([\mathbf{u}_h\cdot\mathbf{n}])\|_{0,t}^2\right)^{1/2}\\ &\le \sum_{i=1}^{2}\left(\sum_{T\in\mathscr{T}_{h,i}}(\omega_T^{(i)})^2\right)^{1/2}+\left(\sum_{t\in\mathscr{T}_{h,\gamma}}(\omega_t)^2\right)^{1/2}+\left(\sum_{t\in\mathscr{T}_{h,\gamma}}(\bar{\delta}_t)^2\right)^{1/2}. \end{aligned}$$

So by (58)

$$\|\mathbf{w}\|_{\mathbf{W}} \le C\sum_{i=1}^{2}\left(\sum_{T\in\mathscr{T}_{h,i}}(\omega_T^{(i)})^2\right)^{1/2}+\left(\sum_{t\in\mathscr{T}_{h,\gamma}}(\omega_t)^2\right)^{1/2}+\left(\sum_{t\in\mathscr{T}_{h,\gamma}}(\bar{\delta}_t)^2\right)^{1/2}. \tag{61}$$

The lemma follows from this last inequality together with (59) and the triangle inequality. ∎

We now observe that in Lemma 3 we could replace the supremum over $\mathbf{v} \in \widetilde{\mathbf{W}}, \mathbf{v} \neq 0$ by the supremum over $\mathbf{v} \in \widetilde{\mathbf{W}}_\star, \mathbf{v} \neq 0$, where $\widetilde{\mathbf{W}}_\star$ is the intersection of $\widetilde{\mathbf{W}}$ with $\mathbf{W}_\star := C^\infty(\Omega_1) \times C^\infty(\Omega_2) \times C^\infty(\gamma)$ since $\widetilde{\mathbf{W}}_\star$ is dense in $\widetilde{\mathbf{W}}$. So to derive estimate (50) and prove Proposition 1, there remains to estimate $\dfrac{a_\xi(\mathbf{e_u}, \mathbf{v})}{\|\mathbf{v}\|_{\mathbf{W}}}$ for an arbitrary vector $\mathbf{v} \neq 0$ in $\widetilde{\mathbf{W}}_\star$.

Proof of Proposition 1 Subtracting (34) from (37) we see that for all $\mathbf{v} \in \mathbf{W}_\star$ and an arbitrary $\mathbf{v}_h \in \mathbf{W_h}$ we have

$$\begin{aligned} a_\xi(\mathbf{u}-\mathbf{u}_h, \mathbf{v}) - \beta(\mathbf{v}, p - p_h) &= -L_d(\mathbf{v}-\mathbf{v}_h) - a_\xi(\mathbf{u}_h, \mathbf{v}-\mathbf{v}_h) + \beta(\mathbf{v}-\mathbf{v}_h, p_h), \\ &= -\sum_{i=1}^{2} \left\langle \bar{p}_i, (\mathbf{v}_i - \mathbf{v}_{h,i}) \cdot \mathbf{n}_i \right\rangle_{\Gamma_i} - \left\langle\!\!\left\langle \bar{p}_\gamma, (\mathbf{v}_\gamma - \mathbf{v}_{h,\gamma}) \cdot \mathbf{n}_\gamma \right\rangle\!\!\right\rangle_{\partial\gamma} \\ &\quad -\sum_{i=1}^{2} \left(K_i^{-1} \mathbf{u}_{h,i}, \mathbf{v}_i - \mathbf{v}_{h,i} \right)_{\Omega_i} - \left\langle (dK_{\gamma,\tau})^{-1} \mathbf{u}_{h,\gamma}, \mathbf{v}_\gamma - \mathbf{v}_{h,\gamma} \right\rangle_\gamma \\ &\quad -\sum_{i=1}^{2} \left\langle \frac{d}{2K_{\gamma,\nu}} \{\mathbf{u}_h \cdot \mathbf{n}\}_{\xi,i}, (\mathbf{v}_i - \mathbf{v}_{h,i}) \cdot \mathbf{n}_i \right\rangle_\gamma - \left\langle p_{h,\gamma}, [(\mathbf{v}-\mathbf{v}_h)\cdot \mathbf{n}] \right\rangle_\gamma \\ &\quad +\sum_{i=1}^{2} \left(\operatorname{div}(\mathbf{v}_i - \mathbf{v}_{h,i}), p_{h,i} \right)_{\Omega_i} + \left\langle (\operatorname{div}_\tau(\mathbf{v}_\gamma - \mathbf{v}_{h,\gamma}), p_{h,\gamma} \right\rangle_\gamma . \end{aligned}$$

Now if $\mathbf{v} \in \widetilde{\mathbf{W}}_\star$ and if $\mathbf{v}_h$ is taken to be $\mathbf{v}_h = \Pi_h^D \mathbf{v}$, the last two terms on the righthand side vanish, by (18) and (19), leaving us with the following equality for all $\mathbf{v} \in \widetilde{\mathbf{W}}_\star$:

$$\begin{aligned} a_\xi(\mathbf{u}-\mathbf{u}_h, \mathbf{v}) = &-\sum_{i=1}^{2} \left\langle \bar{p}_i, (\mathbf{v}_i - \Pi_{h,i}^D \mathbf{v}_i) \cdot \mathbf{n}_i \right\rangle_{\Gamma_i} - \left\langle\!\!\left\langle \bar{p}_\gamma, (\mathbf{v}_\gamma - \Pi_{h,\gamma}^D \mathbf{v}_\gamma) \cdot \mathbf{n}_\gamma \right\rangle\!\!\right\rangle_{\partial\gamma} \\ &-\sum_{i=1}^{2} \left(K_i^{-1} \mathbf{u}_{h,i}, \mathbf{v}_i - \Pi_{h,i}^D \mathbf{v}_i \right)_{\Omega_i} - \left\langle (dK_{\gamma,\tau})^{-1} \mathbf{u}_{h,\gamma}, \mathbf{v}_\gamma - \Pi_{h,\gamma}^D \mathbf{v}_\gamma \right\rangle_\gamma \\ &-\sum_{i=1}^{2} \left\langle \frac{d}{2K_{\gamma,\nu}} \{\mathbf{u}_h \cdot \mathbf{n}\}_{\xi,i}, (\mathbf{v}_i - \Pi_{h,i}^D \mathbf{v}_i) \cdot \mathbf{n}_i \right\rangle_\gamma - \left\langle p_{h,\gamma}, [(\mathbf{v} - \Pi_{h,i}^D \mathbf{v}) \cdot \mathbf{n}] \right\rangle_\gamma . \end{aligned} \tag{62}$$

For $n = 2$, using the fact that in the fracture $H(div, \gamma)$ is simply $H^1(\gamma)$, then $\Pi_{h,\gamma}^D$ is simply the Lagrange interpolation operator, so the second term in the right hand side of (62) vanishes, and by (25) the fourth term is estimated as following

$$-\left\langle (dK_{\gamma,\tau})^{-1} \mathbf{u}_{h,\gamma}, \mathbf{v}_\gamma - \Pi_{h,\gamma}^D \mathbf{v}_\gamma \right\rangle_\gamma \preceq \sum_{t \in \mathscr{T}_{h,\gamma}} h_t \|(dK_{\gamma,\tau})^{-1} \mathbf{u}_{h,\gamma}\|_{0,t} |\mathbf{v}_\gamma|_{1,t}.$$

We come back to the case $n = 3$, now with $\mathbf{r_i}$, r_γ, s_γ and ρ_γ given by Lemma 1, using (31) with $\mathbf{w}_{h,i} = \mathbf{u}_{h,i}$ and (32) with $\mathbf{w}_{h,\gamma} = \mathbf{u}_{h,\gamma}$, we may write the sum of the third and fourth terms of the right hand side of (62) as follows:

$$\begin{aligned} &\sum_{i=1}^{2} \sum_{E \in \mathscr{E}_{h,i}^0} \left\langle J_{\tau,E}(\mathbf{u}_{h,i}), \mathbf{r_i} - \Pi_{h,i}^C \mathbf{r_i} \right\rangle_E - \sum_{e \in \mathscr{E}_{h,\gamma}^0} \left\langle\!\!\left\langle J_{\tau,e}(\mathbf{u}_{h,\gamma}), (r_\gamma - \Pi_{h,\gamma}^C r_\gamma) + (\rho_\gamma - \Pi_{h,\gamma}^C(\rho_\gamma) \right\rangle\!\!\right\rangle_e \\ &+\sum_{i=1}^{2} \sum_{E \in \mathscr{E}_{h,i}^{\Gamma_i}} \left\langle (K_i^{-1} \mathbf{u}_{h,i}) \times \mathbf{n}_i, \mathbf{r_i} - \Pi_{h,i}^C \mathbf{r_i} \right\rangle_E - \sum_{i=1}^{2} \sum_{E \in \mathscr{E}_{h,i}^\gamma} \left\langle (K_i^{-1} \mathbf{u}_{h,i}) \times \mathbf{n}_i, (\mathbf{r_i} - \Pi_{h,i}^C \mathbf{r_i}) \right\rangle_E \\ &+\sum_{i=1}^{2} \sum_{t \in \mathscr{T}_{h,\gamma}} \left\langle \left(dK_{\gamma,\tau} \right)^{-1} \mathbf{u}_{h,\gamma} \times \mathbf{n}_i, \mathbf{r_i} - \Pi_{h,i}^C \mathbf{r_i} \right\rangle_t , \end{aligned} \tag{63}$$

where we have also used the equality $< \mathbf{w}_i \times \mathbf{n}_i, \mathbf{z}_i >_\gamma = - < \mathbf{w}_i, \mathbf{z}_i \times \mathbf{n}_i >_\gamma$, for all $\mathbf{w}_i \in \mathbf{W}_i$ and $\mathbf{z}_i \in \mathbf{H}^1(\Omega_i)$ and Remark 4. The last two terms of the above equation can then be combined to yield (with the notation of (39)) the term

$$\sum_{i=1}^{2} \sum_{E \in \mathscr{E}_{h,i}^\gamma} \left\langle J_{\tau,E}(\mathbf{u}_{h,i}), \mathbf{r_i} - \Pi_{h,i}^C \mathbf{r_i} \right\rangle_E .$$

Remark 6 : This term is well defined despite the non-conformity of the meshes because all the functions used are in $L^2(\gamma)$. We can write this term over all $t \in \mathscr{T}_{h,\gamma}$ or over all $E \in \mathscr{E}^{\gamma}_{h,i}$. We can also use a projection operator $\mathbf{P}^{\gamma,D}_{h,i}$ on the space of restriction on γ of functions in $\mathbf{W}_{h,i}$, and define $J_{\tau,E}$ in (39) by

$$J_{\tau,E}(\mathbf{v}_{h,i}) = \left(K_i^{-1}\mathbf{v}_{h,i} - \mathbf{P}^{\gamma,D}_{h,i}\left(\left(dK_{\gamma,\tau}\right)^{-1}\mathbf{v}_{h,\gamma}\right)\right)|_E \times \mathbf{n}_i \quad \text{if } E \in \mathscr{E}^{\gamma}_{h,i}.$$

This will introduce another indicator related to the non-conformity of the meshes.

$$\bar{\delta}^{\gamma,i}_E = h_E^{1/2}\|\mathbf{P}^{\gamma,D}_{h,i}\left(\left(dK_{\gamma,\tau}\right)^{-1}\mathbf{v}_{h,\gamma}\right) - \left(dK_{\gamma,\tau}\right)^{-1}\mathbf{v}_{h,\gamma}\|_{0,E}.$$

In this case, we add the term $\left(\sum_{E\in\mathscr{E}^{\gamma}_{h,i}}(\bar{\delta}^{\gamma,i}_E)^2\right)^{1/2}$ in the estimate (50). □

The terms in the last line of (62) can be assembled to obtain

$$\sum_{i=1}^{2}\sum_{t\in\mathscr{T}_{h,\gamma}}\left\langle -p_{h,\gamma} - \frac{d}{2K_{\gamma,\nu}}\{\mathbf{u}_h\cdot\mathbf{n}\}_{\xi,i}\,,\,(\mathbf{v}_i - \Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\right\rangle_t,$$

and by (18), the above term may be rewritten as

$$\sum_{i=1}^{2}\sum_{E\in\mathscr{E}^{\gamma}_{h,i}}\left\langle p_{h,i} - p_{h,\gamma} - \frac{d}{2K_{\gamma,\nu}}\{\mathbf{u}_h\cdot\mathbf{n}\}_{\xi,i}\,,\,(\mathbf{v}_i - \Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\right\rangle_E.$$

This term may now be written in terms of the indicator $\mathscr{A}^{(i)}_E(\mathbf{u}_h, p_h)$ defined in (45):

$$\sum_{i=1}^{2}\sum_{E\in\mathscr{E}^{\gamma}_{h,i}}\left\langle \mathscr{A}^{(i)}_E(\mathbf{u}_h,p_h)\,,\,(\mathbf{v}_i - \Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\right\rangle_E.$$

To treat the first term of (62), we add and subtract $\sum_{i=1}^{2}\sum_{E\in\mathscr{E}^{\Gamma_i}_{h,i}}\left\langle P^{\Gamma_i}_{h,i}(\bar{p}_i)\,,\,(\mathbf{v}_i - \Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\right\rangle_E$ and use (18), to obtain

$$\sum_{i=1}^{2}\left\langle \bar{p}_i\,,\,(\mathbf{v}_i - \Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\right\rangle_{\Gamma_i} = \sum_{i=1}^{2}\sum_{E\in\mathscr{E}^{\Gamma_i}_{h,i}}\left\langle \bar{p}_i - P^{\Gamma_i}_{h,i}(\bar{p}_i)\,,\,(\mathbf{v}_i - \Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\right\rangle_E. \tag{64}$$

The second term of the right-hand side of (62) is treated similarly:

$$\sum_{e\in\mathscr{E}^{\partial\gamma}_{h,\gamma}}\left\langle\!\!\left\langle \bar{p}_\gamma, (\mathbf{v}_\gamma - \Pi^D_{h,\gamma}\mathbf{v}_\gamma)\cdot\mathbf{n}_\gamma\right\rangle\!\!\right\rangle_e = \sum_{e\in\mathscr{E}^{\partial\gamma}_{h,\gamma}}\left\langle\!\!\left\langle \bar{p}_\gamma - P^{\partial\gamma}_{h,\gamma}(\bar{p}_\gamma), (\mathbf{v}_\gamma - \Pi^D_{h,\gamma}\mathbf{v}_\gamma)\cdot\mathbf{n}_\gamma\right\rangle\!\!\right\rangle_e. \tag{65}$$

We obtain

$$\begin{aligned}
a_\xi(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) = &-\sum_{i=1}^{2}\sum_{E\in\mathscr{E}_{h,i}}\left\langle J_{\tau,E}(\mathbf{u}_{h,i}),(\mathbf{r_i} - \Pi^C_{h,i}\mathbf{r_i})\right\rangle_E - \sum_{e\in\mathscr{E}_{h,\gamma}}\left\langle\!\!\left\langle J_{\tau,e}(\mathbf{u}_{h,\gamma}),(r_\gamma - \Pi^C_{h,\gamma}r_\gamma) - (\rho_\gamma - \Pi^C_{h,\gamma}(\rho_\gamma))\right\rangle\!\!\right\rangle_e \\
&+\sum_{i=1}^{2}\sum_{E\in\mathscr{E}^{\gamma}_{h,i}}\left\langle \mathscr{A}^{(i)}_E(\mathbf{u}_h,p_h)\,,\,(\mathbf{v}_i - \Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\right\rangle_E \\
&-\sum_{i=1}^{2}\sum_{E\in\mathscr{E}^{\Gamma_i}_{h,i}}\left\langle \bar{p}_i - P^{\Gamma_i}_{h,i}(\bar{p}_i)\,,\,(\mathbf{v}_i - \Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\right\rangle_E - \sum_{e\in\mathscr{E}^{\partial\gamma}_{h,\gamma}}\left\langle\!\!\left\langle \bar{p}_\gamma - P^{\partial\gamma}_{h,\gamma}(\bar{p}_\gamma), (\mathbf{v}_\gamma - \Pi^D_{h,\gamma}\mathbf{v}_\gamma)\cdot\mathbf{n}_\gamma\right\rangle\!\!\right\rangle_e.
\end{aligned} \tag{66}$$

To obtain (50), we use lemma 3 and the Cauchy-Schwarz inequality in the last equality. For the two first terms in the right-hand side of (66), the second estimation of (29) and the decomposition (12) give the indicators related to (41).

$$\left|\left\langle J_{\tau,E}(\mathbf{u}_{h,i}),(\mathbf{r_i}-\Pi^C_{h,i}\mathbf{r_i})\right\rangle_E\right| \preceq h_E^{\frac{1}{2}}\sigma_E^{(i)}\|\mathbf{v}\|_{\mathbf{W}}. \quad (67)$$

$$\left|\left\langle\!\!\left\langle J_{\tau,e}(\mathbf{u}_{h,\gamma}),(r_\gamma-\Pi^C_{h,\gamma}r_\gamma)-(\rho_\gamma-\Pi^C_{h,\gamma}(\rho_\gamma))\right\rangle\!\!\right\rangle_e\right| \preceq h_e^{\frac{1}{2}}\sigma_e\|\mathbf{v}\|_{\mathbf{W}}. \quad (68)$$

The three last term contain the normal component of vectors and we have to be careful about it. First, we notice that for $E\in\mathscr{E}^{\gamma}_{h,i}$, we have $(\mathbf{v}_i-\Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\in L^2(E)$ and we can write

$$\left|\left\langle \mathscr{A}_E^{(i)}(\mathbf{u}_h,p_h)\,,\,(\mathbf{v}_i-\Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\right\rangle_E\right| \le \|\mathscr{A}_E^{(i)}(\mathbf{u}_h,p_h)\|_{0,E}\|(\mathbf{v}_i-\Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\|_{0,E}\preceq \delta_E^{(i)}\|\mathbf{v}\|_{\mathbf{W}}. \quad (69)$$

To estimate the last terms of (66), we require that $\bar{p}_i\in H^1(\Gamma_i)$ and $\bar{p}_\gamma\in H^1(\partial\gamma)$. We use the interpolation inequality given in [19] page 49. Indeed $H^{1/2}$ is the interpolation space of index $1/2$ between L^2 and H^1: $H^{1/2}=[H^1,L^2]$ and for all $z\in H^1$ we have $\|z\|_{\frac{1}{2}}\le C\|z\|_1^{1/2}\|z\|_0^{1/2}$. (a similar argument was used for elasticity problem in [4]). Using that and the fact that the $H^1(\Gamma_i)$ norm is higher than the $L^2(\Gamma_i)$ norm, and by the continuity of normal trace operator, we obtain

$$\begin{aligned}\sum_{i=1}^{2}\left\langle \bar{p}_i\,,\,(\mathbf{v}_i-\Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\right\rangle_{\Gamma_i} &\le \sum_{i=1}^{2}\sum_{E\in\mathscr{E}^{\Gamma_i}_{h,i}}\|\bar{p}_i-P^{\Gamma_i}_{h,i}(\bar{p}_i)\|_{1/2,E}\|(\mathbf{v}_i-\Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\|_{-1/2,E}\\ &\preceq \sum_{i=1}^{2}\sum_{E\in\mathscr{E}^{\Gamma_i}_{h,i}}\|\bar{p}_i-P^{\Gamma_i}_{h,i}(\bar{p}_i)\|_{1,E}^{1/2}\|\bar{p}_i-P^{\Gamma_i}_{h,i}(\bar{p}_i)\|_{0,E}^{1/2}\|(\mathbf{v}_i-\Pi^D_{h,i}\mathbf{v}_i)\|_W\\ &\preceq \sum_{i=1}^{2}\sum_{E\in\mathscr{E}^{\Gamma_i}_{h,i}}\|\bar{p}_i-P^{\Gamma_i}_{h,i}(\bar{p}_i)\|_{1,E}\|\mathbf{v}\|_W.\end{aligned} \quad (70)$$

With a similar argument we obtain

$$\sum_{e\in\mathscr{E}^{\partial\gamma}_{h,\gamma}}\left\langle\!\!\left\langle \bar{p}_\gamma-P^{\partial\gamma}_{h,\gamma}(\bar{p}_\gamma),(\mathbf{v}_\gamma-\Pi^D_{h,\gamma}\mathbf{v}_\gamma)\cdot\mathbf{n}_\gamma\right\rangle\!\!\right\rangle_e \le \sum_{e\in\mathscr{E}^{\partial\gamma}_{h,\gamma}}\|\bar{p}_\gamma-P^{\partial\gamma}_{h,\gamma}(\bar{p}_\gamma)\|_{1,E}\|\mathbf{v}\|_W.$$

Dividing by $\|\mathbf{v}\|_W$ and taking the supremum over $\widetilde{\mathbf{W}}$ we obtain (50). ■

Remark 7 : In the development of the indicators for velocity we can use an integration by part in the first term of (62) on each $E\in\mathscr{E}^{\Gamma_i}_{h,i}$ *and obtain*

$$\begin{aligned}\sum_{i=1}^{2}\left\langle \bar{p}_i\,,\,(\mathbf{v}_i-\Pi^D_{h,i}\mathbf{v}_i)\cdot\mathbf{n}_i\right\rangle_{\Gamma_i} &= -\sum_{i=1}^{2}\sum_{E\in\mathscr{E}^{\Gamma_i}_{h,i}}\left\langle \nabla_{\tau_i}(\bar{p}_i)\,,\,(\mathbf{r_i}-\Pi^C_{h,i}\mathbf{r_i})\times\mathbf{n}_i\right\rangle_E\\ &\quad+\sum_{i=1}^{2}\sum_{G\in\mathscr{G}^{\Gamma_i}_{h,i}}\left\langle\!\!\left\langle \bar{p}_i-P^{\Gamma_i}_{h,i}(\bar{p}_i)\,,\,(\mathbf{r_i}-\Pi^C_{h,i}\mathbf{r_i})\cdot\mathbf{t}_G\right\rangle\!\!\right\rangle_G.\end{aligned}$$

The first term in the right-hand side of the last equality can be assembled with the term related to the tangential trace of the velocity and define $J_{\tau,E}$ *with another expression for* $E\in\mathscr{E}^{\Gamma_i}_{h,i}$*. The problem of this way is the difficulty to make a sense to the trace of the trace of* $\mathbf{r_i}$*. Indeed,* $\mathbf{r_i}|_{\Gamma_i}$ *is in* $H^{1/2}(\Gamma_i)$ *which do not have a trace.* □

5.2 An upper bound for the pressure error

In this section, we prove Proposition 2. To derive an upper bound for the pressure error we again use the auxiliary problem (57) with

$$g(q) = \sum_{i=1}^{2} \int_{\Omega_i} \varepsilon_{p,i}\, q_i \mathrm{dx} + \int_{\gamma} \varepsilon_{p,\gamma} q_\gamma \mathrm{d}\sigma, \quad \forall q \in M,$$

where $\varepsilon_p = (\varepsilon_{p,1}, \varepsilon_{p,2}, \varepsilon_{p,\gamma})$. Let $(\mathbf{w}, y) \in \mathbf{W} \times M$ be the solution of

$$\begin{aligned} a_\xi(\mathbf{w}\,,\, \mathbf{v}) - \beta(\mathbf{v}\,,\, y) &= 0, \quad \forall \mathbf{v} \in \mathbf{W}, \\ -\beta(\mathbf{w}\,,\, q) &= g(q), \ \forall q \in M. \end{aligned} \tag{71}$$

We suppose that we have the elliptic regularity assumption

$$\exists C_s > 0 \quad \text{such that} \quad \|\mathbf{w}\|_1 + \|y\|_1 \le C_s \|\varepsilon_p\|_M. \tag{72}$$

Proof of Proposition 2. Let $(\mathbf{w}, y) \in \mathbf{W} \times M$ be the solution of (71), and let $\mathbf{w}_h = \Pi_h^D \mathbf{w}$ and $y_h = P_h^0 y$, where Π_h^D and P_h^0 are the quasi-interpolation operators defined in Section 3.

Taking $\mathbf{e_u} \in \mathbf{W}$ and $\varepsilon_p \in M$ as test functions in (71), adding the two equations, and using the symmetry of $a_\xi(\,,\,)$, then the orthogonality conditions (38) and then the residual equations (37), we have that

$$\begin{aligned} \|\varepsilon_p\|^2 &= -\beta(\mathbf{w}\,,\, \varepsilon_p) + a_\xi(\mathbf{w}\,,\, \mathbf{e_u}) - \beta(\mathbf{e_u}\,,\, y) \\ &= a_\xi(\mathbf{e_u}\,,\, \mathbf{w} - \mathbf{w}_h) - \beta(\mathbf{w} - \mathbf{w}_h\,,\, \varepsilon_p) - \beta(\mathbf{e_u}\,,\, y - y_h) \\ &= -L_d(\mathbf{w} - \mathbf{w}_h) - L_f(y - y_h) - a_\xi(\mathbf{u}_h\,,\, \mathbf{w} - \mathbf{w}_h) + \beta(\mathbf{w} - \mathbf{w}_h\,,\, p_h) + \beta(\mathbf{u}_h\,,\, y - y_h). \end{aligned} \tag{73}$$

Taking $\mathbf{w}_h = \Pi_h^D \mathbf{w}$ and $y_h = P_h^0 y$ and using (18), (19) and (36), we obtain

$$\begin{aligned} \|\varepsilon_p\|^2 = &-\sum_{i=1}^{2} \Big\langle (\mathbf{w}_i - \mathbf{w}_{h,i}) \cdot \mathbf{n}_i, \bar{p}_i \Big\rangle_{\Gamma_i} - \Big\langle\!\Big\langle (\mathbf{w}_\gamma - \mathbf{w}_{h,\gamma}) \cdot \mathbf{n}_\gamma, \bar{p}_\gamma \Big\rangle\!\Big\rangle_{\partial\gamma} \\ &- \sum_{i=1}^{2} \Big(f_i - P_{h,i}^0(f_i), y_i - y_{h,i} \Big)_{\Omega_i} - \Big\langle f_\gamma - P_{h,\gamma}^0(f_\gamma), y_\gamma - y_{h,\gamma} \Big\rangle_\gamma \\ &- \sum_{i=1}^{2} \Big(K_i^{-1} \mathbf{u}_{h,i}, \mathbf{w}_i - \mathbf{w}_{h,i} \Big)_{\Omega_i} - \Big\langle (dK_{\gamma,\tau})^{-1} \mathbf{u}_{h,\gamma}, \mathbf{w}_\gamma - \mathbf{w}_{h,\gamma} \Big\rangle_\gamma \\ &- \sum_{i=1}^{2} \Big\langle \frac{d}{2K_{\gamma,\nu}} \{\mathbf{u}_h \cdot \mathbf{n}_i\}_{\xi,i}\,,\, (\mathbf{w}_i - \mathbf{w}_{h,i}) \cdot \mathbf{n}_i \Big\rangle_\gamma - \Big\langle [(\mathbf{w} - \mathbf{w}_h) \cdot \mathbf{n}_i]\,,\, p_{h,\gamma} \Big\rangle_\gamma - \Big\langle [\mathbf{u}_h \cdot \mathbf{n}] - P_{h,\gamma}^0([\mathbf{u}_h \cdot \mathbf{n}])\,,\, y_\gamma - y_{h,\gamma} \Big\rangle_\gamma. \end{aligned}$$

The term $\sum_{i=1}^{2} \Big\langle \frac{d}{2K_{\gamma,\nu}} \{\mathbf{u}_h \cdot \mathbf{n}_i\}_{\xi,i}\,,\, (\mathbf{w}_i - \mathbf{w}_{h,i}) \cdot \mathbf{n}_i \Big\rangle_\gamma$ vanishes if the meshes $\mathscr{T}_{h_1} \cup \mathscr{T}_{h_2}$ are conforming, otherwise it will be equal to,

$$\sum_{i=1}^{2} \Big\langle \frac{d}{2K_{\gamma,\nu}} (1-\xi)(\mathbf{u}_{h,i+1} \cdot \mathbf{n}_{i+1} - P_{\mathscr{E}_{h,i}^\gamma}(\mathbf{u}_{h,i+1} \cdot \mathbf{n}_{i+1})), (\mathbf{w}_i - \mathbf{w}_{h,i}) \cdot \mathbf{n}_i \Big\rangle_\gamma$$

where $P_{\mathscr{E}_{h,i}^\gamma}(\mathbf{u}_{h,i+1})$ is the L^2-projection of $\mathbf{u}_{h,i+1} \cdot \mathbf{n}_{i+1}$ on $\mathscr{P}_0(\mathscr{E}_{h,i}^\gamma)$.

Now, we introduce $\widetilde{P}_{h,i}^{\Gamma_i} \bar{p}_i$, $\widetilde{P}_{h,\gamma}^{\partial\gamma} \bar{p}_\gamma$, and $\widetilde{P}_{h,i}^{\gamma} p_{h,\gamma}$ for $i = 1,2$. Using a Cauchy Scwartz inequality, then the estimates (25) and (16), and the regularity assumption (72) we have

$$
\begin{aligned}
\|\varepsilon_p\| \preceq \Bigg\{ & \sum_{i=1}^{2} \sum_{E\in\mathscr{E}_{h,i}^{\Gamma_i}} h_E^{\frac{1}{2}} \|\bar{p}_i - \widetilde{P}_{h,i}^{\Gamma_i} \bar{p}_i\|_{0,E} + \sum_{e\in\mathscr{E}_{h,\gamma}^{\partial\gamma}} h_e^{\frac{1}{2}} \|\bar{p}_\gamma - \widetilde{P}_{h,\gamma}^{\partial\gamma} \bar{p}_\gamma\|_e \\
& + \sum_{i=1}^{2} \sum_{T\in\mathscr{T}_{h,i}} h_T \|f_i - P_{h,i}^0 f_i\|_{0,T} + \sum_{t\in\mathscr{T}_{h,\gamma}} h_t \|f_\gamma - P_{h,\gamma}^0 f_\gamma\|_{0,t} \\
& + \sum_{i=1}^{2} \sum_{T\in\mathscr{T}_{h,i}} h_T \|K_i^{-1}\mathbf{u}_{h,i}\|_{0,T} + \sum_{t\in\mathscr{T}_{h,\gamma}} h_t \|(dK_{\gamma,\tau})^{-1}\mathbf{u}_{h,\gamma}\|_{0,t} \\
& + \sum_{t\in\mathscr{T}_{h,\gamma}} h_t \left\| P_{h,\gamma}^0([\mathbf{u}_h\cdot\mathbf{n}]) - [\mathbf{u}_h\cdot\mathbf{n}] \right\|_{0,t} \\
& + \sum_{i=1}^{2} \sum_{E\in\mathscr{E}_{h,i}^{\gamma}} \Bigg(h_E^{\frac{1}{2}} \|p_{h,\gamma} - P_{h,i}^{\gamma} p_{h,\gamma}\|_{0,E} \\
& + h_E^{\frac{1}{2}} \left\| \tfrac{d}{2K_{\gamma,\nu}}(1-\xi)(\mathbf{u}_{h,i+1}\cdot\mathbf{n}_{i+1} - P_{\mathscr{E}_{h,i}^{\gamma}}(\mathbf{u}_{h,i+1}\cdot\mathbf{n}_{i+1}) \right\|_E \Bigg) \Bigg\}
\end{aligned}
\tag{74}
$$

This yields the desired result ■

Remark 8 : *Another possibility to work with the term* $-\sum_{i=1}^{2}\left\langle \dfrac{d}{2K_{\gamma,\nu}}\{\mathbf{u}_h\cdot\mathbf{n}_i\}_{\xi,i}\,,\,(\mathbf{w}_i-\mathbf{w}_{h,i})\cdot\mathbf{n}_i \right\rangle_\gamma$ *is to add* $\left\langle (\mathbf{w}_i - \mathbf{w}_{h,i})\cdot\mathbf{n}_i\,,\,p_{h,i} \right\rangle_\gamma$ *as in the estimation for the velocity, since vanishes for* $\mathbf{w}_h = \Pi_h^D\mathbf{w}$ *and we obtain the estimation*

$$
\begin{aligned}
\|\varepsilon_p\| \preceq \Bigg\{ & \sum_{i=1}^{2} \sum_{E\in\mathscr{E}_{h,i}^{\Gamma_i}} h_E^{\frac{1}{2}} \|\bar{p}_i - \widetilde{P}_{h,i}^{\Gamma_i} \bar{p}_i\|_{0,E} + \sum_{e\in\mathscr{E}_{h,\gamma}^{\partial\gamma}} h_e^{\frac{1}{2}} \|\bar{p}_\gamma - \widetilde{P}_{h,\gamma}^{\partial\gamma} \bar{p}_\gamma\|_e \\
& + \sum_{i=1}^{2} \sum_{T\in\mathscr{T}_{h,i}} h_T \|f_i - P_{h,i}^0 f_i\|_{0,T} + \sum_{t\in\mathscr{T}_{h,\gamma}} h_t \|f_\gamma - P_{h,\gamma}^0 f_\gamma\|_{0,t} \\
& + \sum_{i=1}^{2} \sum_{T\in\mathscr{T}_{h,i}} h_T \|K_i^{-1}\mathbf{u}_{h,i}\|_{0,T} + \sum_{t\in\mathscr{T}_{h,\gamma}} h_t \|(dK_{\gamma,\tau})^{-1}\mathbf{u}_{h,\gamma}\|_{0,t} \\
& + \sum_{t\in\mathscr{T}_{h,\gamma}} h_t \left\| P_{h,\gamma}^0([\mathbf{u}_h\cdot\mathbf{n}]) - [\mathbf{u}_h\cdot\mathbf{n}] \right\|_{0,t} + \sum_{E\in\mathscr{E}_{h,i}^{\gamma}} \|\mathscr{A}_E^{(i)}(\mathbf{u}_h,p_h)\|_{0,E} \Bigg\}
\end{aligned}
\tag{75}
$$

□

6 The lower bound of the error

In order to prove the lower bound of the error, we use the usual localization technique based on triangle-bubble and edge-bubble functions, the inverse inequalities and integrating by parts. We recall some notations and preliminary results, collected in the following lemma. For all $T \in \mathscr{T}_{h,i}$ let b_T be the bubble-fonction on T defined as the product of $n+1$ barycentric co-ordinates of T associated to vertices of T. For all $E \in \mathscr{E}_{h,i}$ let b_E be the bubble-fonction on $\widetilde{T}_E = T_1 \cup T_2$ (where $T_1 \cap T_2 = E$), defined as the product of n barycentric co-ordinates of each T_i associated to vertices of T_i, continuous on $\widetilde{T}_E$ and vanishing on $\partial\widetilde{T}_E$. With a similar notation, for $t \in \mathscr{T}_{h,\gamma}$ and $e \in \mathscr{E}_{h,\gamma}$ we consider also the bubble functions b_t and b_e.

Lemma 4 *(Verfüth [27])*

For $T \in \mathscr{T}_{h,i}$, $E \in \mathscr{E}_{h,i}$, $\varphi \in \mathscr{P}_m(T)$ $z \in P_m(E)$, and $\psi \in \mathbf{P}_m(T)$ we have

$$\|\varphi\|_{0,T} \preceq \|b_T^{1/2}\varphi\|_{0,T} \qquad \|b_T\varphi\|_{0,T} \preceq \|\varphi\|_{0,T} \tag{76}$$

$$\|z\|_{0,E} \preceq \|b_E^{1/2} z\|_{0,E} \qquad \|\nabla(b_E z)\|_{0,\widetilde{T}_E} \preceq h_E^{-1/2}\|z\|_{0,E} \tag{77}$$

$$\|\nabla(b_T\varphi)\|_{0,T} \preceq h_T^{-1}\|\varphi\|_{0,T} \qquad \|\mathrm{div}(b_T\psi)\|_{0,T} \preceq h_T^{-1}\|\psi\|_{0,T}. \tag{78}$$

We can now give the proof of Proposition3.

Proof of Proposition 3.

1. The indicators $\sigma_E^{(i)}$ and σ_e, defined by (41), are related to the jump of the velocities through $E \in \mathscr{E}_{h,i}$ and $e \in \mathscr{E}_{h,\gamma}$ respectively. We prove the first estimate of (51), the proof of the second is similar.
   There are three cases. The proof of the two first cases is analogous to proof in [10], [30], and [23],
   (a) Let $E \in \mathscr{E}_{ih}^0$ and b_E the bubble function defined on $\widetilde{T}_E$ and such that $b_E|_{\partial \widetilde{T}_E} = 0$.
   Let
   $$\phi_E = \begin{cases} \mathscr{R}_E(J_{\tau,E}(\mathbf{u}_h))b_E & \text{in } \widetilde{T}_E \\ 0 & \text{in } \Omega_i \setminus \widetilde{T}_E, \end{cases}$$
   where $J_{\tau,E}(\mathbf{u}_h)$ is defined by (39) and $\mathscr{R}_E$ is is a lifting operator of trace, from $L^2(E)$ to $H^1(\widetilde{T}_E)$ in 2D and to $(H^1(\widetilde{T}_E))^3$ in 3D, satisfying
   $$|\mathscr{R}_E(J_{\tau,E}(\mathbf{u}_h))|_{1,\widetilde{T}_E} \preceq h_E^{-1/2}\|J_{\tau,E}(\mathbf{u}_h)\|_{0,E}$$
   By (77), and the definition of ϕ_E we have
   $$\begin{aligned}
   \|J_{\tau,E}(u_{i,h})\|_{0,E}^2 &\preceq \int_E |J_{\tau,E}(u_{i,h})|^2 b_E \\
   &= \int_E J_{\tau,E}(\mathbf{u}_{h,i})\cdot\phi_E = -\int_E J_{\tau,E}(\mathbf{u}_i - \mathbf{u}_{h,i})\cdot\phi_E \\
   &= \sum_{T\subset\widetilde{T}_E}\int_T \left(K_i^{-1}(\mathbf{u}_i-\mathbf{u}_{h,i})\cdot\mathrm{curl}\,\phi_E - \mathrm{curl}\,(K_i^{-1}(\mathbf{u}_i-\mathbf{u}_{h,i}))\cdot\phi_E\right) \\
   &\qquad \text{where we suppose } \mathbf{u}_i \in \mathbf{H}^1(\Omega_i) \\
   &= \sum_{T\subset\widetilde{T}_E}\int_T K_i^{-1}(\mathbf{u}_i-\mathbf{u}_{h,i})\cdot\mathrm{curl}\phi_E \\
   &\qquad \text{since } K_i^{-1}\mathbf{u}_i = \nabla p_i, \text{ and the operator} \\
   &\qquad \text{Curl vanishes on the lowest order space } RT_0 \\
   &\preceq \|K_i^{-1}(\mathbf{u}_i-\mathbf{u}_{h,i})\|_{0,\widetilde{T}_E}\|\mathrm{curl}\,\phi_E\|_{0,\widetilde{T}_E} \\
   &\preceq \|K_i^{-1}(\mathbf{u}_i-\mathbf{u}_{h,i})\|_{0,\widetilde{T}_E} h_E^{-1/2}\|J_{\tau,E}(u_h)\|_{0,E}, \text{ by (77)}.
   \end{aligned}$$
   We obtain
   $$h_E^{1/2}\|J_{\tau,E}(\mathbf{u}_{h,i})\|_{0,E} \preceq \|K_i^{-1}(\mathbf{u}_i-\mathbf{u}_{h,i})\|_{0,\widetilde{T}_E}.$$
   (b) In the case where $E \in \mathscr{E}_{ih}^{\Gamma_i}$, $\widetilde{T}_E$ is reduced to one element T, and the same arguments as above give the estimate.
   (c) For the case $E \in \mathscr{E}_{ih}^{\gamma}$, by (39) and the triangle inequality we have
   $$\|J_{\tau,E}(\mathbf{u}_{h,i})\|_{0,E} \le \|\left(K_i^{-1}(\mathbf{u}_{h,i}-\mathbf{u}_i)\right)_{\tau,E}\|_{0,E} + \|(dK_{\gamma,\tau})^{-1}(\mathbf{u}_{h,\gamma}-\mathbf{u}_\gamma)\|_{0,E}$$
   $$+\|\left((K_i)^{-1}\mathbf{u}_i - (dK_{\gamma,\tau})^{-1}\mathbf{u}_\gamma\right)_{\tau,E}\|_{0,E}.$$
   the last one is a model error on γ.

2. Let us examine now the indicators (42), related to local residual equations, and prove (52).
Let b_T the bubble function on T. By (76), then (78) and the fact that K_i^{-1} is piecewise constant , $K_i^{-1}\nabla p_i = \mathbf{u}_i$ and $\nabla p_{h,i} = 0$ on T, we have

$$\begin{aligned}\left(\eta_T^{(i)}\right)^2 &= h_T^2 \|K_i^{-1}\mathbf{u}_{h,i}\|_{0,T}^2 \\ &\preceq h_T^2 \|b_T^{1/2} K_i^{-1}\mathbf{u}_{h,i}\|_{0,T}^2 \\ &= h_T^2 \int_T b_T K_i^{-1}(\mathbf{u}_{h,i} - \mathbf{u}_i)\cdot K_i^{-1}\mathbf{u}_{h,i} - h_T^2 \int_T b_T \nabla(p_i - p_{h,i})\cdot K_i^{-1}\mathbf{u}_{h,i}, \\ &= h_T^2 \int_T b_T K_i^{-1}(\mathbf{u}_{h,i} - \mathbf{u}_i)\cdot K_i^{-1}\mathbf{u}_{h,i} + h_T^2 \int_T (p_i - p_{h,i})\mathrm{div}(b_T K_i^{-1}\mathbf{u}_{h,i}).\end{aligned}$$

Using Cauchy-Schwarz's inequality and the inverse inequality (78) we get

$$\eta_T^{(i)} \preceq \|\mathbf{u}_{h,i} - \mathbf{u}_i\|_{0,T} + \|p_i - p_{h,i}\|_{0,T}$$

The same argument gives the second estimate of (42).

3. We consider now the indicators defined by (45) and $\mathscr{A}_E^{(i)}(\mathbf{u}_h, p_h)$ defined by (45) for $i = 1,2$. Let $E \in \mathscr{E}_{h,i}^{\gamma}$, and $T \in \mathscr{T}_{h,i}$ such that $E = \partial T \cap \gamma$. Let g defined by $g|_E = b_E \mathscr{A}_E^{(i)}(\mathbf{u}_h, p_h)|_E$ and $g|_{\partial T\setminus E} = 0$ where b_E is the bubble function in E. There exists $\Psi_i \in H^1(T)$ solution, for $i = 1,2$, of the problem

$$\begin{cases} -\Delta\Psi_i + \Psi_i = 0 & \text{in } T \\ \frac{\partial \psi_i}{\partial n_{\partial T}} = g & \text{on } E \\ \psi_i = 0 & \text{on } \partial T \setminus E \end{cases}$$

which admits a solution satisfying

$$\|\Psi_i\|_{1,T} \leq \|b_E \mathscr{A}_E^{(i)}(\mathbf{u}_h, p_h)\|_{0,E}.$$

So the function $\mathbf{w}_i = \nabla\Psi_i$ is in $H(\mathrm{div}, T)$ and satisfies

$$\|\mathbf{w}_i\|_{\mathrm{div},T} \leq \|b_E \mathscr{A}_E^{(i)}(\mathbf{u}_h, p_h)\|_{0,E}.$$

We denote the extension by 0 of $\mathbf{w}_i$ by $\widetilde{\mathbf{w}}_i$ which is in $H(\mathrm{div};\Omega_i)$. We take now $\bar{\mathbf{v}} = (\widetilde{\mathbf{w}}_1, 0, 0)$ for $i = 1$, or $\bar{\mathbf{v}} = (0, \widetilde{\mathbf{w}}_2, 0)$ for $i = 2$.

$$\begin{aligned} a_\xi(\mathbf{u} - \mathbf{u}_h, \bar{\mathbf{v}}) - \beta(\bar{\mathbf{v}}, p - p_h) &= \int_T K_i^{-1}(\mathbf{u}_i - \mathbf{u}_{h,i})\cdot\widetilde{\mathbf{w}}_i + \int_E \frac{d}{2K_{\gamma,\nu}}\{\mathbf{u}\cdot\mathbf{n} - \mathbf{u}_h\cdot\mathbf{n}\}_{\xi_i}\widetilde{\mathbf{w}}_i\cdot\mathbf{n}_i \\ &\quad + \int_T \mathrm{div}\widetilde{\mathbf{w}}_i(p_i - p_{h,i}) + \int_E (p_\gamma - p_{h,\gamma})\widetilde{\mathbf{w}}_i\cdot\mathbf{n}_i \\ &= \int_T K_i^{-1}(\mathbf{u}_i - \mathbf{u}_{h,i})\cdot\widetilde{\mathbf{w}}_i + \int_T \mathrm{div}\widetilde{\mathbf{w}}_i(p_i - p_{h,i}) \\ &\quad + \int_E (p_\gamma + \frac{d}{2K_{\gamma,\nu}}\{\mathbf{u}\cdot\mathbf{n}\}_{\xi_i})\widetilde{\mathbf{w}}_i\cdot\mathbf{n}_i \\ &\quad - \int_E (p_{h,\gamma} + \frac{d}{2K_{\gamma,\nu}}\{\mathbf{u}_h\cdot\mathbf{n}\}_{\xi_i})\widetilde{\mathbf{w}}_i\cdot\mathbf{n}_i. \end{aligned}$$

Since on γ we have $p_i = p_\gamma + \frac{d}{2K_{\gamma,\nu}}\{\mathbf{u}\cdot\mathbf{n}\}_{\xi_i}$ we can write

$$\int_E (p_\gamma + \frac{d}{2K_{\gamma,\nu}}\{\mathbf{u}\cdot\mathbf{n}\}_{\xi_i})\mathbf{w}_i\cdot\mathbf{n}_i - \int_E (p_{h,\gamma} + \frac{d}{2K_{\gamma,\nu}}\{\mathbf{u}_h\cdot\mathbf{n}\}_{\xi_i})\mathbf{w}_i\cdot\mathbf{n}_i =$$

$$\int_E (p_i - p_{h,i})\mathbf{w}_i\cdot\mathbf{n}_i + \int_E (p_{h,i} - p_{h,\gamma} - \frac{d}{2K_{\gamma,\nu}}\{\mathbf{u}_h\cdot\mathbf{n}\}_{\xi_i})\mathbf{w}_i\cdot\mathbf{n}_i.$$

By taking into account the value of $\mathbf{w}_i \cdot \mathbf{n}_i$ on E we obtain

$$\int_E b_E \mathscr{A}_E^{(i)}(\mathbf{u}_h, p_h)^2 = a_\xi(\mathbf{u} - \mathbf{u}_h, \bar{\mathbf{v}}) - \beta(\bar{\mathbf{v}}, p - p_h) - \int_T K_i^{-1}(\mathbf{u}_i - \mathbf{u}_{h,i}) \cdot \\ - \int_T \operatorname{div}\mathbf{w}_i(p_i - p_{h,i}) - \int_E (p_i - p_{h,i})\mathbf{w}_i \cdot \mathbf{n}_i.$$

By (76) and the estimate on $\mathbf{w}_i$ we have

$$\begin{aligned} \|\mathscr{A}_E^{(i)}(\mathbf{u}_h, p_h)\|_{0,E}^2 &\preceq \|b_E^{1/2}\mathscr{A}_E^{(i)}(\mathbf{u}_h, p_h)\|_{0,E}^2 \\ &\preceq (\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{W}} + \|p - p_h\|_M)\, \|\mathbf{w}_i\|_{\mathbb{H}(\operatorname{div},T)} \\ &\preceq (\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{W}} + \|p - p_h\|_M)\, \|\mathscr{A}_E^{(i)}(\mathbf{u}_h, p_h)\|_{0,E} \end{aligned}$$

4. The inequalities (54) and (55) are concerned by the indicators related to the non conformity of the meshes. We firstly remark that even in the context of quasi-conformity $\bar{\delta}_E^{(i)}$ and $\bar{\delta}_t$ may be no vanishing. Indeed, in the situation where $h_t < h_E$ and $E = \cup_k t_k$, only $\bar{\delta}_t$ vanishes since $[\mathbf{u}_h \cdot \mathbf{n}]$ is a constant on t, and in the situation where $h_E < h_t$ and $t = \cup_k E_k$ only $\bar{\delta}_E^{(i)}$ vanishes. Otherwise we have

$$\begin{aligned} \bar{\delta}_E^{(i)} &:= h_E^{1/2}\|p_{h,\gamma} - P_{h,i}^\gamma(p_{h,\gamma})\|_{0,E} \\ &\le h_E^{1/2}\|p_\gamma - p_{h,\gamma}\|_{0,E} + h_E^{1/2}\|p_\gamma - P_{h,i}^\gamma(p_\gamma)\|_{0,E} + h_E^{1/2}\|P_{h,i}^\gamma(p_\gamma) - P_{h,i}^\gamma(p_{h,\gamma})\|_{0,E} \\ &\preceq \left(\|p_{h,\gamma} - p_\gamma\|_{0,E} + \|p_\gamma - P_{h,i}^\gamma(p_\gamma)\|_{0,E}\right) \text{ since the operator } P_{h,i}^\gamma \text{ is continuous} \end{aligned}$$

$$\begin{aligned} \bar{\delta}_t^{\,2} &= \|P_{h,\gamma}^0(\sum_{i=1}^2 \mathbf{u}_{h,i} \cdot \mathbf{n}_i) - \sum_{i=1}^2 \mathbf{u}_{h,i} \cdot \mathbf{n}_i\|_{0,t} \\ &= \|\operatorname{div}_\tau \mathbf{u}_{h,\gamma} - P_{h,\gamma}^0(f_\gamma) - \sum_{i=1}^2 \mathbf{u}_{h,i} \cdot \mathbf{n}_i\|_{0,t} \qquad \text{by (36)} \\ &= \|\operatorname{div}_\tau(\mathbf{u}_{h,\gamma} - \mathbf{u}_\gamma) - (P_{h,\gamma}^0(f_\gamma) - f_\gamma) - \sum_{i=1}^2 (\mathbf{u}_{h,i} - \mathbf{u}_i) \cdot \mathbf{n}_i\|_{0,t} \end{aligned}$$

To obtain (55), we add and subtract $\mathbf{u}_{i+1} \cdot \mathbf{n}_{i+1}$ and $P_{\mathscr{E}_{h,i}^\gamma}(\mathbf{u}_{i+1} \cdot \mathbf{n}_{i+1})$. ■

7 Numerical experiments

In this section we give some results related to the performance of the indicators defined in Section 4. We restrict our attention to the 2D case. All computations are carried out using FreeFem++ [16]. We consider the reduced model defined by (6) and the test problem proposed in [21], as indicated in Fig.1, where the domain Ω is composed of the three parts $\Omega_1 =]-1,0[\times]0,3[$, $\Omega_2 =]0,1[\times]0,3[$ and $\gamma = \{0\}\times]0,3[$. The analysis of the problem was given (see section 2) only with the boundary condition on the pressure, but this can be generalized to other conditions easily. Here, in the numerical test, we consider a boundary condition on velocity as presented in the figure Fig.1. We solve the discrete problem (34) (with $\xi = 1$) using Raviart-Thomas mixed finite elements of lowest order to obtain the couple $(\mathbf{u}_h, p_h)$.

The aim now is to illustrate numerically the convergence and the performance of the following indicators:

- $\sigma_E^{(i)}$, $E \in \mathscr{E}_{h,i}$, $i = 1,2$, given in (41) which are related to the lack of regularity the discrete solution,
- $\eta_T^{(i)}$, $T \in \mathscr{T}_{h,i}, i = 1,2$, and $\eta_t^{(\gamma)}$, $t \in \mathscr{T}_{h,\gamma}$, given in (42) which are related to the residual equation for Darcy's law,
- $\omega_T^{(i)}$, $T \in \mathscr{T}_{h,i}, i = 1,2$, and ω_t, $t \in \mathscr{T}_{h,\gamma}$, given in (43) which are related to the residual equation for the continuity equation,

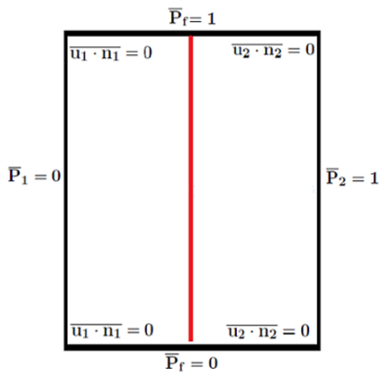


Fig. 1: The Data

- $\delta_E^{(i)}$, $E \in \mathscr{E}_{h,i}^{\gamma}$, $i = 1,2$ given in (45) which are related to the interface condition,
- $\bar{\delta}_E^{(i)}$, $E \in \mathscr{E}_{h,i}^{\gamma}$, $i = 1,2$, and $\bar{\delta}_t$, $t \in \mathscr{T}_{h,\gamma}$ given in (46) and (47), respectively, which are related to the non-conformity of the meshes.

Note that the indicators $\Delta_E^{(i)}$ vanish when $\xi = 1$. In our example all of the boundary data are constant so the related indicators $\overline{\omega}_E^{(i)}$, $\overline{\omega}_{1E}^{(i)}$, $\overline{\omega}_e$ and $\overline{\omega}_{1e}$ vanish as well. Note that the indicators $\omega_T^{(i)}$ and ω_t given in (43) are related to the source-term data and vanish in all of our experiments due to the property (36) and the fact that the source terms f_1, f_2 and f_γ are taken to be zero. Before continuing we introduce some more notation: Let $S_i(T)$ and S_i for $i = 1,2$ and $S_3(t)$ and S_3 be defined by

$$S_i(T) = (\eta_T^{(i)})^2 + (\omega_T^{(i)})^2 + \sum_{E \subset \partial T} (\sigma_E^{(i)})^2 + (\delta_E^{(i)})^2 + (\bar{\delta}_E^{(i)})^2, \quad T \in \mathscr{T}_{h,i}, \qquad S_i = \left(\sum_{T \in \mathscr{T}_{h,i}} S_i(T) \right)^{1/2},$$

$$S_3(t) = (\eta_t^{(\gamma)})^2 + (\omega_t)^2 + (\bar{\delta}_t)^2, \quad t \in \mathscr{T}_{h,\gamma}, \qquad S_3 = \left(\sum_{t \in Thg} S_3(t) \right)^{1/2}.$$

Let $\overline{S}$ and S_{TOL} be defined by

$$\overline{S} = \frac{1}{m} \left(\sum_{i=1}^{2} \sum_{T \in \mathscr{T}_{h,i}} S_i(T) + \sum_{t \in \mathscr{T}_{h,\gamma}} S_3(t) \right)^{1/2}, \quad \text{where } m = \sharp \mathscr{T}_{h,1} + \sharp \mathscr{T}_{h,2} + \sharp \mathscr{T}_{h,\gamma}$$

$$S_{TOL} = cc * \overline{S} \qquad \text{where } cc \in]0,1].$$

We start by showing that all of the indicators converge to zero when the mesh size goes to zero. This can be done by refining meshes uniformly iteration by iteration denoted in the tables by ite. For this we consider two cases: the case in which the meshes are quasi-conforming (Table 1) and the case in which they are not (Table 2). In the case of quasi-conforming meshes we have taken $K_1 = K_2 = 1$, $K_{\gamma,\tau} = 100$ and $d = 0.05$; then $dK_{\gamma,\tau} = 5$; whereas, in the case of non-conforming meshes we have taken $K_1 = 1$, $K_2 = 10$, $K_{\gamma,\tau} = 100$ and $d = 0.05$. The Tables are composed of three sub-tables, the first corresponds to Ω_1 the second to Ω_2 and the third to the fracture, where NT_i and h_i represent respectively the number of elements and the mesh size of the mesh $\mathscr{T}_{h,i}$, $i = 1,2,\gamma$. The others columns represent the global indicators. It can be seen that all the indicators, as well as S_i, for $i = 1,2,3$ converge to zero when the mesh size goes to zero. Nevertheless, as expected $\bar{\delta}_t$ almost vanishes in the "conforming" case, but $\delta_E^{(i)}$ could vanish only in the case where $\mathscr{T}_{h,\gamma} = \mathscr{E}_{h,i}^{\gamma}$. The second step of our numerical experiments is to use the indicators for a self-adaptive mesh-refinement. The strategy adopted here is to mark for refinement the elements $T \in \mathscr{T}_{h,i}$, $i = 1,2$, for which S_i is greater than tolerance S_{Tol}, and elements $t \in \mathscr{T}_{h,\gamma}$ for which S_3 is greater than S_{Tol}, where S_{Tol} is defined as above with $cc = 0.7$. In the case of nonconforming meshes we have taken $K_1 = 1$, $K_2 = 10$, $K_{\gamma,\tau} = 100$ and $d = 0.05$. The results are presented in Table 3. First of all, we remark that the indicators still converge with mesh refinement. Secondly, the self-adaptive strategy gives us "optimal" meshes to reach a fixed accuracy. Indeed we notice, in Table3, that in the iteration 4, for example, the sum S_1 is about 0.2 for only 1085 elements, contrary to uniform refinement, where it required 7200 elements to achieve this accuracy (see Table 2, $ite = 6$).

Fig. 2a represents the mesh used in the initial iteration for both Tables 2 and 3, and Fig. 2b represents the adapted meshes of iteration 6 of Table 3. Fig. 3 represents the solution with the adapted meshes of iteration 6. In Fig. 4 we plot S_i as a function of the number of elements NT_i for $i = 1,2,3$, in both the case of uniform refinement and that of refinement by adaptation. We can see that the strategy of refinement by adaptation is less expensive for the same accuracy since the meshes are "optimal".

Table 1: uniform refinement of meshes (quasi-conform).

ite	S_1	h_1	$\sigma_E^{(1)}$	$\eta_T^{(1)}$	$\delta_E^{(1)}$	$\omega^{(1)}$	$\overline{\delta_E^1}$	NT_1
1	0.581267	0.316228	0.476654	0.323672	0.0730908	1.19443e-015	0.0238092	200
2	0.333684	0.158114	0.288979	0.162408	0.0372705	2.42753e-015	0.00841595	800
3	0.238461	0.105409	0.210891	0.108362	0.0250112	3.82392e-015	0.00458058	1800
4	0.187077	0.0790569	0.167408	0.0812983	0.0188203	4.93277e-015	0.00297502	3200
5	0.154637	0.0632456	0.13946	0.0650495	0.015086	6.01288e-015	0.00212868	5000
6	0.132179	0.0527046	0.11988	0.0542131	0.0125882	7.78573e-015	0.00161932	7200

ité	S_2	h_2	$\sigma_E^{(2)}$	$\eta_T^{(2)}$	$\delta_E^{(2)}$	$\omega^{(2)}$	$\overline{\delta_E^2}$	NT_2
1	0.578976	0.316228	0.476654	0.323672	0.0517983	1.26172e-015	0.0238092	200
2	0.332575	0.158114	0.288979	0.162408	0.0254945	2.56606e-015	0.00841595	800
3	0.237746	0.105409	0.210891	0.108362	0.0168765	4.12932e-015	0.00458058	1800
4	0.186555	0.0790569	0.167408	0.0812983	0.0126051	5.53824e-015	0.00297502	3200
5	0.154227	0.0632456	0.13946	0.0650495	0.0100563	6.56781e-015	0.00212868	5000
6	0.131844	0.0527046	0.11988	0.0542131	0.00836364	8.15367e-015	0.00161932	7200

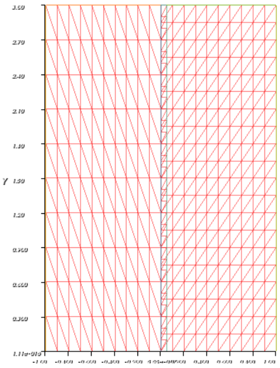
ité	S_3	h_t	$\eta_t^{(\gamma)}$	ω_t	$\overline{\delta_t}$	NT_3
1	0.058358	0.111803	0.058358	3.25657e-010	3.27901e-010	60
2	0.0291587	0.0707107	0.0291587	3.36494e-010	3.38721e-010	120
3	0.0194353	0.0600925	0.0194353	3.40223e-010	3.42438e-010	180
4	0.0145752	0.0559017	0.0145752	3.42115e-010	3.44323e-010	240
5	0.0116596	0.0538516	0.0116596	3.43259e-010	3.45462e-010	300
6	0.00971609	0.0527046	0.00971609	3.44025e-010	3.46225e-010	360

Table 2: uniform refinement of meshes (non-conform).

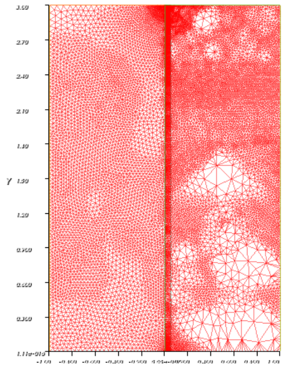
ité	S_1	h_1	$\sigma_E^{(1)}$	$\eta_T^{(1)}$	$\delta_E^{(1)}$	$\omega^{(1)}$	$\overline{\delta_E^1}$	NT_1
1	0.861483	0.316228	0.746871	0.418933	0.088564	1.71925e-015	0.0314416	200
2	0.551991	0.158114	0.508454	0.209718	0.0454171	3.66243e-015	0.0111463	800
3	0.41659	0.105409	0.391177	0.139862	0.030485	4.97107e-015	0.00606459	1800
4	0.33817	0.0790569	0.320643	0.104912	0.0229232	7.18826e-015	0.00393726	3200
5	0.286324	0.0632456	0.273114	0.0839357	0.0183588	7.53843e-015	0.00281633	5000
6	0.249221	0.0527046	0.238703	0.0699494	0.015306	1.00737e-014	0.00214193	7200

ité	S_2	h_2	$\sigma_E^{(2)}$	$\eta_T^{(2)}$	$\delta_E^{(2)}$	$\omega^{(2)}$	$\overline{\delta_E^2}$	NT_2
1	0.609914	0.180278	0.448429	0.410621	0.0425241	9.28686e-015	0.0221223	400
2	0.348657	0.0901388	0.278125	0.208744	0.0210254	1.82835e-014	0.0138592	1600
3	0.249309	0.0600925	0.205675	0.139777	0.013871	2.93787e-014	0.0110732	3600
4	0.195769	0.0450694	0.164611	0.105029	0.0103213	4.04206e-014	0.0095282	6400
5	0.161933	0.0360555	0.137872	0.0841061	0.00820677	4.83638e-014	0.00850186	10000
6	0.138472	0.0300463	0.118953	0.0701296	0.00680585	6.2314e-014	0.00775341	14400

ité	S_3	h_t	$\eta_t^{(\gamma)}$	ω_t	$\overline{\delta_t}$	NT_3
1	1.40731	0.111803	0.0790741	2.24381e-009	1.40509	60
2	1.07286	0.0707107	0.0389219	2.36445e-009	1.07215	120
3	0.893232	0.0600925	0.0258283	2.40452e-009	0.892859	180
4	0.776756	0.0559017	0.0193315	2.42353e-009	0.776516	240
5	0.693207	0.0538516	0.0154481	2.43418e-009	0.693035	300
6	0.629414	0.0527046	0.0128647	2.44076e-009	0.629283	360

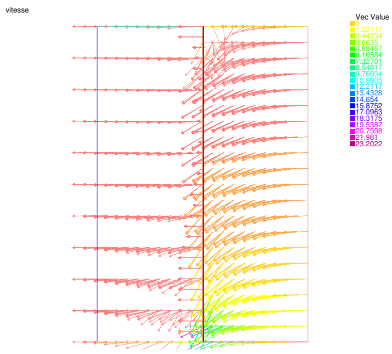


(a) initial mesh

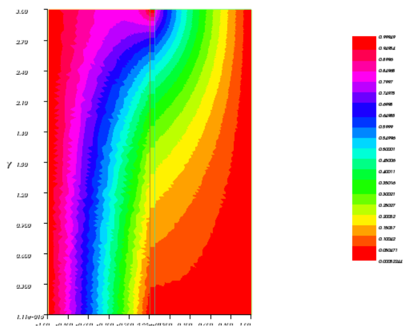


(b) mesh after adaptation

Fig. 2: Meshes



(a) The velocity



(b) The pressure

Fig. 3: Solution

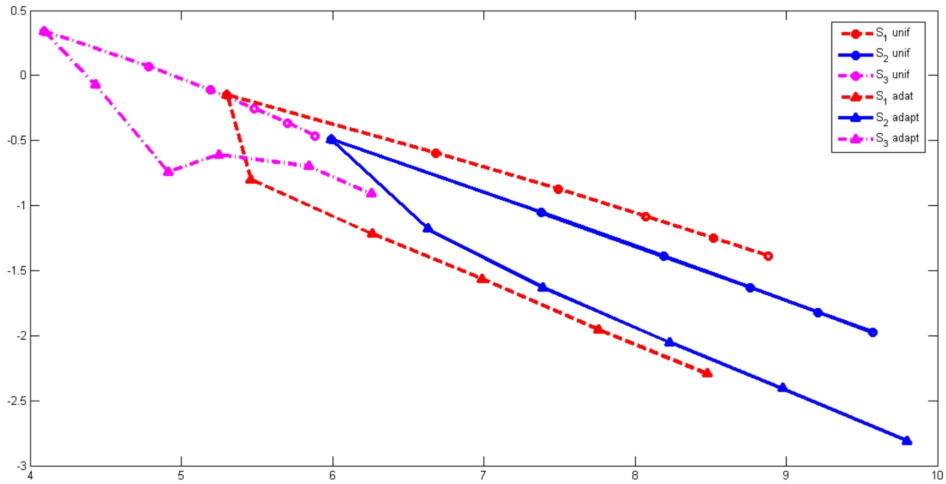


Fig. 4: Comparison between adaptative and uniform refinement

Table 3: Adaptation using $S_{Tol}\ cc = 0.7$.

ité	S_1	h_1	$\sigma_E^{(1)}$	$\eta_T^{(1)}$	$\delta_E^{(1)}$	$\omega^{(1)}$	$\overline{\delta_E^1}$	NT_1
1	0.861483	0.316228	0.746871	0.418933	0.088564	1.71925e-015	0.0314416	200
2	0.448564	0.369846	0.309643	0.315292	0.0763111	1.07442e-015	0.0099058	235
3	0.295902	0.25018	0.212977	0.199245	0.0493663	1.7474e-015	0.00794304	524
4	0.209265	0.166055	0.14535	0.14633	0.0346715	2.44545e-015	0.0071341	1085
5	0.141536	0.147124	0.105082	0.0923014	0.0213347	3.66268e-015	0.00393538	2342
6	0.100893	0.0807819	0.0719949	0.0691567	0.0141477	5.35043e-015	0.00363949	4804

ité	S_2	h_2	$\sigma_E^{(2)}$	$\eta_T^{(2)}$	$\delta_E^{(2)}$	$\omega^{(2)}$	$\overline{\delta_E^2}$	NT_2
1	0.609914	0.180278	0.448429	0.410621	0.0425241	9.28686e-015	0.0221223	400
2	0.307273	0.470264	0.200601	0.231325	0.0233625	1.71791e-014	0.0109015	757
3	0.195976	0.179074	0.117551	0.156133	0.0126563	2.90434e-014	0.0071024	1620
4	0.128155	0.176654	0.0823051	0.0976151	0.00840333	4.50863e-014	0.00709591	3752
5	0.0901892	0.0748915	0.0547407	0.0713185	0.00508682	7.01125e-014	0.00503452	7918
6	0.0602512	0.0851743	0.0400974	0.0446159	0.00368594	1.03643e-013	0.00427117	18036

ité	S_3	h_t	$\eta_t^{(\gamma)}$	ω_t	$\overline{\delta_t}$	NT_3
1	1.40731	0.111803	0.0790741	2.24381e-009	1.40509	60
2	0.932386	0.195662	0.0429223	2.36683e-009	0.931397	84
3	0.475513	0.0987598	0.0270506	2.44432e-009	0.474743	136
4	0.54336	0.0996846	0.0192478	2.45243e-009	0.543019	190
5	0.497737	0.0634958	0.0102959	2.45395e-009	0.497631	346
6	0.402759	0.0597944	0.00724793	2.45321e-009	0.402694	522

8 Conclusion

In this work, we have generalized the *a posteriori* error analysis for mixed finite elements ([10], [30], [23]) to a reduced coupled $nD/(n-1)D$ problem (n =2 or 3) for calculating flow in a fractured porous medium. In this analysis no conforming assumption between the three meshes invollved is required. Indeed we could choose a mesh for each of the subdomains Ω_i and a mesh for γ all independently. We obtained upper and lower bounds for the aproximation error for both the pressure and the velocity in terms of the indicators. The reduced model considered here does not require mortar finite elements, and the new indicators found are used to control the non-conformity between the meshes. Some of the indicators are standard; others related to interface conditions or to non-conformity of the meshes are new. The numerical tests show the convergence of the indicator and provide tool for a self-adaptive refinement.

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