# A SPACE-TIME HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR LINEAR FREE-SURFACE WAVES 

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#### Abstract

We present and analyze a novel space-time hybridizable discontinuous Galerkin (HDG) method for the linear free-surface problem on prismatic space-time meshes. We consider a mixed formulation which immediately allows us to compute the velocity of the fluid. In order to show well-posedness, our space-time HDG formulation makes use of weighted inner products. We perform an a priori error analysis in which the dependence on the time step and spatial mesh size is explicit. We provide two numerical examples: one that verifies our analysis and a wave maker simulation.


## 1. Introduction

The study of water waves is crucial when designing, for example, ships, offshore structures, levees, and seawalls. For this reason free-surface problems are of great interest in many fields of engineering, such as naval and maritime engineering. In order to analyze the interaction between waves and prototypes of ships and other structures, experiments in water tanks using wave makers may be used to simulate realistic maritime situations. However, many complex cases can be difficult to reproduce in water tanks. Developing efficient and accurate numerical models is therefore important to help simulate water-wave phenomena.

In general, free-surface problems are mathematically described by a set of partial differential equations that model the movement of the fluid, and a set of boundary conditions that describe and determine the free-surface. These problems are particularly hard to solve because the free-surface that defines the shape of the domain is part of the solution to the problem. In order to simplify the problem, certain assumptions on the flow can be made. For example, many applications can be modeled by considering the fluid to be incompressible, inviscid, and irrotational. Moreover, assuming that the wave displacement is small, the free-surface boundary conditions may be linearized.

In order to effectively model water waves, we require a stable and higher-order accurate numerical method to minimize numerical diffusion and dispersion. These properties may be achieved, for example, by using a discontinuous Galerkin (DG) method for the spatial discretization combined with a higher-order accurate time stepping scheme, as done, for example, for linear water waves in [31]. In [31] they combined a second order accurate time stepping scheme with a higher-order accurate DG method for the spatial discretization. They proved stability and provided an a priori error analysis of their method.

[^0]Alternatively, one may use a DG method to discretize partial differential equations in space and time simultaneously. These space-time DG methods achieve higher-order accuracy in both space and time simply by increasing the polynomial approximation in space-time. Space-time DG methods have successfully been applied to many different problems, such as compressible flows [18, 33], incompressible flows [24, 28, 29, 30], depth-averaged flows [2, 21] and non-linear free-surface problems [12, 32]. However, there is a major drawback to space-time DG methods: since the dimension of the problem is increased by one, the number of degrees-of-freedom is much higher compared to standard DG methods. A consequence is that spacetime DG methods are generally computationally more expensive than traditional approaches.

Hybridizable discontinuous Galerkin (HDG) methods were first introduced in [9] as a computationally more efficient alternative to discontinuous Galerkin methods. HDG methods maintain the local conservation properties of DG methods, but have been shown to be as computationally efficient in certain cases as continuous Galerkin finite element methods [16, 36]. This is achieved by introducing a variable that exists only on the facets. Communication between two neighbouring elements is through this facet variable. By coupling degrees-of-freedom on elements only through the degrees-of-freedom on facets, the element degrees-of-freedom can be eliminated. This static condensation results in a global linear system for the facet variable only of which the size is significantly smaller than the global linear system obtained by a DG discretization. Once the facet variable is known the element unknowns can be reconstructed element-wise.

Space-time HDG methods apply the HDG method in space-time [14, 17, 22, 23, 34] and result in an efficient alternative to space-time DG methods. In this paper we present a novel space-time HDG method for the linear free-surface problem. We consider a mixed formulation based on the splitting introduced originally for the wave equation in [25]. This is different from previous works on DG methods for freesurface problems in which the primal form of the problem is considered [31, 32]. The reason to consider the mixed formulation is that it allows us to immediately obtain the velocity of the fluid without post-processing. To the best of our knowledge, such a splitting has not been considered for inviscid, incompressible and irrotational freesurface flow problems. Furthermore, as opposed to standard discontinuous Galerkin discretizations, our space-time HDG formulation uses weighted inner-products. The idea of using weighted inner-products was previously applied in [11] to the wave equation. Here, we use weighted inner-products to prove well-posedness of the space-time HDG formulation of the linear free-surface waves problem.

The a priori error analysis in this paper is projection based [10], but modified for weighted inner-products and space-time prismatic elements. Additionally, we derive scaling identities between the reference and physical space-time prisms that separate the spatial dimension from the temporal dimension. Such anisotropic scaling identities were previously derived in the context of anisotropic meshes in [13] and space-time meshes in [17, 27]. Furthermore, these scaling identities allow us to obtain a priori error estimates in which the dependence on the time step and the spatial mesh size is explicit. This is in contrast to error bounds that depend on the space-time mesh size as derived, for example, for parabolic problems in [8].

The outline of this paper is as follows. In section 2 we formulate the linear free-surface problem in a space-time setting. Next, in section 3, we present the


Figure 1. Depiction of the flow domain $\Omega \subset \mathbb{R}^{2}$.
space-time HDG method and show well-posedness of the discrete formulation. Section 4 provides the tools required for the a priori error analysis that we present in section 5 . Our theoretical results are verified by numerical examples in section 6 and conclusions are drawn in section 7 .

## 2. The linear free-surface problem

Let $b\left(x_{1}\right)>0$ be a piecewise linear polynomial representing the bottom topography and let $H$ be the average water depth. We define the flow domain as $\Omega:=\left\{\boldsymbol{x}=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}:-H+b\left(x_{1}\right)<x_{2}<0, L<x_{1}<R\right\}$ where $L$ and $R$ are given constants. The boundary of the domain, $\partial \Omega$, is partitioned into a free-surface boundary $\Gamma_{S}:=\left\{\boldsymbol{x} \in \mathbb{R}^{2}: x_{1} \in[L, R], x_{2}=0\right\}$, periodic boundaries $\Gamma_{P}$, and a solid boundary $\Gamma_{N}:=\left\{\boldsymbol{x} \in \mathbb{R}^{2}: x_{1} \in[L, R], x_{2}=-H+b\left(x_{1}\right)\right\}$. These boundaries do not overlap and are such that $\bar{\Gamma}_{S} \cup \bar{\Gamma}_{N} \cup \bar{\Gamma}_{P}=\partial \Omega$. The boundary outward unit normal vector is denoted by $\boldsymbol{n}$. See fig. 1 for an illustration of the notation.

We consider an incompressible, inviscid fluid, with an irrotational velocity field $\boldsymbol{u}$. Let $\zeta:[L, R] \times[0, T] \rightarrow \mathbb{R}$ denote the wave height and let $\phi: \Omega \times[0, T] \rightarrow \mathbb{R}$ denote the velocity potential so that $\boldsymbol{u}=\nabla \phi$. The linear free-surface problem for irrotational flow is given by

$$
\begin{align*}
-\nabla^{2} \phi & =0 & & \text { in } \Omega  \tag{2.1a}\\
\nabla \phi \cdot \boldsymbol{n} & =\partial_{t} \zeta & & \text { on } \Gamma_{S}  \tag{2.1b}\\
\partial_{t} \phi+\zeta & =0 & & \text { on } \Gamma_{S}  \tag{2.1c}\\
\nabla \phi \cdot \boldsymbol{n} & =0 & & \text { on } \Gamma_{N} \tag{2.1d}
\end{align*}
$$

where eq. (2.1b) and eq. (2.1c) are, respectively, the kinematic and dynamic linear free-surface boundary conditions, and where eq. (2.1d) imposes a no-normal flow condition on the solid boundary. Periodic boundary conditions are applied on $\Gamma_{P}$. To close the problem we require the initial conditions $\phi(\boldsymbol{x}, 0)=\phi^{0}(\boldsymbol{x})$ and $\zeta\left(x_{1}, 0\right)=\zeta^{0}\left(x_{1}\right)$.

In [31] the kinematic eq. (2.1b) and dynamic eq. (2.1c) boundary conditions were combined into a single equation for $\phi$. The resulting linear free-surface model
becomes:

$$
\begin{align*}
-\nabla^{2} \phi & =0 & & \text { in } \Omega, \\
\partial_{t t} \phi+\nabla \phi \cdot \boldsymbol{n} & =0 & & \text { on } \Gamma_{S},  \tag{2.2a}\\
\nabla \phi \cdot \boldsymbol{n} & =0 & & \text { on } \Gamma_{N} .
\end{align*}
$$

As discussed in the next section, we instead introduce a mixed formulation of the linear free-surface problem eq. (2.1) which is more suitable for the space-time HDG method.

## 3. The space-time hybridizable discontinuous Galerkin method

To introduce the space-time HDG method we first formulate the mixed spacetime formulation of the linear free-surface problem eq. (2.1).

Space-time methods do not distinguish between temporal and spatial variables. A point at time $t=x_{0}$ with position vector $\boldsymbol{x}=\left(x_{1}, x_{2}\right)$ has Cartesian coordinates $\left(x_{0}, \boldsymbol{x}\right)$. We therefore pose the linear free-surface problem eq. (2.1) on a spacetime domain defined as $\mathcal{E}:=\left\{\left(x_{0}, \boldsymbol{x}\right) \in \mathbb{R}^{3}: \boldsymbol{x} \in \Omega, 0<x_{0}<T\right\}$. The boundary $\partial \mathcal{E}$ of the space-time domain $\mathcal{E}$ consists of $\Omega_{0}:=\left\{\left(x_{0}, \boldsymbol{x}\right) \in \partial \mathcal{E}: x_{0}=0\right\}, \Omega_{N}:=$ $\left\{\left(x_{0}, \boldsymbol{x}\right) \in \partial \mathcal{E}: x_{0}=T\right\}$ and $\mathcal{Q}_{\mathcal{E}}:=\left\{\left(x_{0}, \boldsymbol{x}\right) \in \partial \mathcal{E}: 0<x_{0}<T\right\}$. Furthermore, $\mathcal{Q}_{\mathcal{E}}$ is subdivided as $\mathcal{Q}_{\mathcal{E}}=\partial \mathcal{E}_{\mathcal{S}} \cup \partial \mathcal{E}_{\mathcal{N}} \cup \partial \mathcal{E}_{\mathcal{P}}$, where $\partial \mathcal{E}_{\mathcal{S}}:=\left\{\left(x_{0}, \boldsymbol{x}\right) \in \partial \mathcal{E}: \boldsymbol{x} \in \Gamma_{S}, 0<x_{0}<T\right\}$, $\partial \mathcal{E}_{\mathcal{N}}:=\left\{\left(x_{0}, \boldsymbol{x}\right) \in \partial \mathcal{E}: \boldsymbol{x} \in \Gamma_{N}, 0<x_{0}<T\right\}$, and $\partial \mathcal{E}_{\mathcal{P}}:=\left\{\left(x_{0}, \boldsymbol{x}\right) \in \partial \mathcal{E}: \boldsymbol{x} \in \Gamma_{P}, 0<x_{0}<T\right\}$.

We next introduce two new variables, namely $\boldsymbol{q}=-\nabla \phi$ and $v=-\partial_{t} \phi$. A similar choice of variables was introduced in [19] for the wave equation. The linear free-surface problem eq. (2.1) on the space-time domain $\mathcal{E}$ can then be written as a mixed space-time formulation which is given by

$$
\begin{align*}
\partial_{t} \boldsymbol{q}-\nabla v & =0 & & \text { in } \mathcal{E},  \tag{3.1a}\\
\nabla \cdot \boldsymbol{q} & =0 & & \text { in } \mathcal{E},  \tag{3.1b}\\
-\boldsymbol{q} \cdot \boldsymbol{n} & =\partial_{t} v & & \text { on } \partial \mathcal{E}_{\mathcal{S}},  \tag{3.1c}\\
-\boldsymbol{q} \cdot \boldsymbol{n} & =0 & & \text { on } \partial \mathcal{E}_{\mathcal{N}},  \tag{3.1d}\\
\boldsymbol{q} & =-\nabla \phi_{0} & & \text { on } \Omega_{0},  \tag{3.1e}\\
v & =-\partial_{t} \phi(0, \boldsymbol{x}) & & \text { on } \Omega_{0} . \tag{3.1f}
\end{align*}
$$

Note that the wave height $\zeta$ will be the restriction of $v$ to the free-surface $\Gamma_{S}$.
3.1. Notation. Let $\mathcal{I}=[0, T]$ denote the time interval. We partition the time interval into time levels $0=t_{0}<t_{1}<t_{2}<\cdots<t_{N}=T$ and denote the $n^{\text {th }}$ time interval by $I_{n}=\left(t_{n}, t_{n+1}\right)$. The length of each time interval is constant and is denoted by $\Delta t$. The $n^{\text {th }}$ space-time slab is then defined as $\mathcal{E}^{n}:=\mathcal{E} \cap\left(I_{n} \times \mathbb{R}^{2}\right)$. Define $\Omega_{n}:=\left\{\left(x_{0}, \boldsymbol{x}\right) \in \mathcal{E}: x_{0}=t_{n}\right\}$. The boundary of a space-time slab, $\partial \mathcal{E}^{n}$, can then be divided into $\Omega_{n}, \Omega_{n+1}$ and $\mathcal{Q}_{\mathcal{E}}^{n}:=\partial \mathcal{E}^{n} \backslash\left(\Omega_{n+1} \cup \Omega_{n}\right)$. We can further subdivide $\mathcal{Q}_{\mathcal{E}}^{n}$ as $\mathcal{Q}_{\mathcal{E}}^{n}=\partial \mathcal{E}_{\mathcal{S}}^{n} \cup \partial \mathcal{E}_{\mathcal{N}}^{n} \cup \partial \mathcal{E}_{\mathcal{P}}^{n}$, where $\partial \mathcal{E}_{\mathcal{S}}^{n}:=\left\{\left(x_{0}, \boldsymbol{x}\right) \in \partial \mathcal{E}: \boldsymbol{x} \in \Gamma_{S}, t_{n}<x_{0}<t_{n+1}\right\}$, $\partial \mathcal{E}_{\mathcal{N}}^{n}:=\left\{\left(x_{0}, \boldsymbol{x}\right) \in \partial \mathcal{E}: \boldsymbol{x} \in \Gamma_{N}, t_{n}<x_{0}<t_{n+1}\right\}$, and $\partial \mathcal{E}_{\mathcal{P}}^{n}:=\left\{\left(x_{0}, \boldsymbol{x}\right) \in \partial \mathcal{E}: \boldsymbol{x} \in \Gamma_{P}, t_{n}<x_{0}<t_{n+1}\right\}$.

For linear free-surface waves the spatial domain $\Omega$ does not change with time. We therefore introduce a space-time mesh as follows. We first introduce a triangulation $\mathcal{T}:=\{K\}$ of the domain $\Omega_{n}$, which is the same for all $n$. Each space-time element $\mathcal{K} \subset \mathcal{E}^{n}$ is constructed as $\mathcal{K}=K \times I_{n}$. The set of all space-time elements, $\mathcal{T}^{n}:=$ $\left\{\mathcal{K}: \mathcal{K} \subset \mathcal{E}^{n}\right\}$ is then a triangulation of the space-time slab $\mathcal{E}^{n}$. This is repeated for all $n$. See fig. 2 for an illustration of $\mathcal{E}^{n}$.


Figure 2. Depiction of a space-time slab $\mathcal{E}^{n} \subset \mathbb{R}^{3}$.

Consider a space-time element $\mathcal{K}_{j}^{n} \in \mathcal{T}^{n}$. Let $K_{j}^{n}:=\left\{\left(x_{0}, \boldsymbol{x}\right) \in \partial \mathcal{K}_{j}^{n}: x_{0}=t_{n}\right\} \in$ $\mathcal{T}$. The boundary of a space-time element $\mathcal{K}_{j}^{n} \in \mathcal{T}^{n}$ is then composed of $K_{j}^{n}, K_{j}^{n+1}$ and $\mathcal{Q}_{\mathcal{K}_{j}^{n}}:=\partial \mathcal{K}_{j}^{n} \backslash \partial \mathcal{K}_{0}^{n}$ where $\partial \mathcal{K}_{0}^{n}:=K_{j}^{n} \cup K_{j}^{n+1}$.

The outward unit space-time normal vector field on $\partial \mathcal{K}_{j}^{n}$ is denoted by $\hat{\boldsymbol{n}}_{j}^{n}=$ $\left(\left(n_{t}\right)_{j}^{n}, \boldsymbol{n}_{j}^{n}\right)$, where $\left(n_{t}\right)_{j}^{n}$ and $\boldsymbol{n}_{j}^{n}$ are, respectively, the temporal and spatial parts of the space-time normal vector. Since the mesh does not change with time, $\hat{\boldsymbol{n}}_{j}^{n}=$ $(1, \mathbf{0})$ on $K_{j}^{n+1}, \hat{\boldsymbol{n}}_{j}^{n}=(-1, \mathbf{0})$ on $K_{j}^{n}$, and $\hat{\boldsymbol{n}}_{j}^{n}=\left(0, \boldsymbol{n}_{j}^{n}\right)$ on $\mathcal{Q}_{\mathcal{K}_{j}^{n}}$.

In the remainder of this paper we will omit the subscripts and superscripts when referring to space-time elements, their boundaries, and the normal vector wherever no confusion will occur.

In a space-time slab $\mathcal{E}^{n}$, the set and union of all faces in $\partial \mathcal{E}_{\mathcal{S}}$ are denoted by $\mathcal{F}_{\mathcal{S}}^{n}$ and $\Gamma_{\mathcal{S}}^{n}$, respectively. Furthermore, the set and union of all interior and boundary faces in $\mathcal{E}^{n}$ that are not on $\Omega_{n} \cup \Omega_{n+1}$ are denoted by $\mathcal{F}_{\mathcal{Q}}^{n}$ and $\Gamma_{\mathcal{Q}}^{n}$, respectively. On the other hand, faces on $\Omega_{n}$ and $\Omega_{n+1}$ are denoted by $\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)$ and $\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)$, respectively. Finally, $\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)$ denotes the set of edges $e \in \partial \mathcal{E}_{\mathcal{S}}^{n}$ at $t=t_{n}$.

For triangular prismatic space-time elements we introduce the following local spaces:

$$
\begin{align*}
W_{h}(\mathcal{K}) & :=P_{p}(K) \otimes P_{p}\left(I_{n}\right)  \tag{3.2a}\\
\boldsymbol{V}_{h}(\mathcal{K}) & :=\left[P_{p}(K) \otimes P_{p}\left(I_{n}\right)\right]^{2}  \tag{3.2b}\\
M_{h}(\mathcal{F}) & :=Q_{p}(\mathcal{F}) \quad \forall \mathcal{F} \subset \mathcal{Q}_{\mathcal{K}} \tag{3.2c}
\end{align*}
$$

where $P_{p}(D)$ is the space of polynomials of degree at most $p$ on a domain $D$ and $Q_{p}(D)$ denotes the tensor-product polynomials of degree $p$. The global finite element spaces are then defined as:

$$
\begin{align*}
W_{h} & :=\left\{w \in L^{2}\left(\mathcal{E}^{n}\right):\left.w\right|_{\mathcal{K}} \in W_{h}(\mathcal{K}), \forall \mathcal{K} \in \mathcal{T}^{n}\right\}  \tag{3.3a}\\
\boldsymbol{V}_{h} & :=\left\{\boldsymbol{v} \in\left[L^{2}\left(\mathcal{E}^{n}\right)\right]^{2}:\left.\boldsymbol{v}\right|_{\mathcal{K}} \in \boldsymbol{V}_{h}(\mathcal{K}), \forall \mathcal{K} \in \mathcal{T}^{n}\right\},  \tag{3.3b}\\
M_{h} & :=\left\{\mu \in L^{2}\left(\Gamma_{\mathcal{Q}}^{n}\right):\left.\mu\right|_{\mathcal{F}} \in M_{h}(\mathcal{F}), \forall \mathcal{F} \in \mathcal{F}_{\mathcal{Q}}^{n}\right\} . \tag{3.3c}
\end{align*}
$$

For scalar functions we introduce the following inner-products:

$$
\begin{align*}
(v, w)_{\mathcal{K}} & =\int_{\mathcal{K}} v w \mathrm{~d} \mathcal{K}, \quad\langle v, w\rangle_{\mathcal{Q}}=\int_{\mathcal{Q}} v w \mathrm{~d} \mathcal{Q}, \quad\langle v, w\rangle_{K_{j}^{n}}=\int_{K_{j}^{n}} v w \mathrm{~d} K \\
\langle\langle v, w\rangle\rangle_{e} & =\int_{e} v w \mathrm{~d} e, \quad\langle v, w\rangle_{\mathcal{F}} \tag{3.4}
\end{align*}
$$

and

$$
\begin{align*}
(v, w)_{\mathcal{T}^{n}} & =\sum_{\mathcal{K} \in \mathcal{T}^{n}}(v, w)_{\mathcal{K}}, \quad\langle v, w\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}=\sum_{\mathcal{K} \in \mathcal{T}^{n}}\langle v, w\rangle_{\mathcal{Q}} \\
\langle v, w\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)} & =\sum_{\mathcal{K} \in \mathcal{T}^{n}}\langle v, w\rangle_{K_{j}^{n}}, \quad\langle\langle v, w\rangle\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}=\sum_{e \in \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}\left\langle\langle v, w\rangle_{e},\right.  \tag{3.5}\\
\langle v, w\rangle_{\mathcal{F}_{\mathcal{S}}^{n}} & =\sum_{\mathcal{F} \in \mathcal{F}_{\mathcal{S}}^{n}}\langle v, w\rangle_{\mathcal{F}} .
\end{align*}
$$

Similar notation is used for vector functions, for example,

$$
(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{K}}=\int_{\mathcal{K}} \boldsymbol{v} \cdot \boldsymbol{w} \mathrm{d} \mathcal{K} \quad \text { and } \quad(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{T}^{n}}=\sum_{\mathcal{K} \in \mathcal{T}^{n}}(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{K}}
$$

The $L^{2}$-norm of a function $v$ on a domain $D$ will be denoted by $\|v\|_{D}$.
3.2. Discretization. In this section we present the space-time HDG discretization for eq. (2.1) on a space-time slab $\mathcal{E}^{n}$. To be able to show existence and uniqueness of a solution to our discretization we use weighted inner products, as done originally for the wave equation in [11].

Let $f_{n}$ be a weight function depending only on time and which is defined as $f_{n}(t)=e^{-\alpha\left(t-t_{n}\right)}$ with $\alpha>0$. The space-time HDG discretization for the linear free-surface problem eq. (3.1) is: Find $\left(\boldsymbol{q}_{h}, v_{h}, \lambda_{h}\right) \in \boldsymbol{V}_{h} \times W_{h} \times M_{h}$ such that for all $\left(\boldsymbol{r}_{h}, w_{h}, \mu_{h}\right) \in \boldsymbol{V}_{h} \times W_{h} \times M_{h}$ the following relations are satisfied:

$$
\begin{align*}
& -\left(\boldsymbol{q}_{h}, f_{n} \partial_{t} \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left(\boldsymbol{q}_{h}, \boldsymbol{r}_{h} f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left\langle\boldsymbol{q}_{h}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}  \tag{3.6a}\\
& \quad+\left(v_{h}, f_{n} \nabla \cdot \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left\langle\lambda_{h}, \boldsymbol{r}_{h} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}=\left\langle\boldsymbol{q}_{h}^{-}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}, \\
& -\left(w_{h}, f_{n} \nabla \cdot \boldsymbol{q}_{h}\right)_{\mathcal{T}^{n}}+\left\langle\tau\left(v_{h}-\lambda_{h}\right), w_{h} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}=0  \tag{3.6b}\\
& \left\langle\boldsymbol{q}_{h} \cdot \boldsymbol{n}-\tau\left(v_{h}-\lambda_{h}\right), \mu_{h} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}-\left\langle\lambda_{h}, f_{n} \partial_{t} \mu_{h}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}  \tag{3.6c}\\
& -\left\langle\lambda_{h}, \mu_{h} f_{n}^{\prime}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}+\left\langle\left\langle\lambda_{h}, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}\left(t_{n+1}\right)}=\left\langle\left\langle\lambda_{h}^{-}, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}\left(t_{n}\right)}, \quad \forall \mu_{h} \in M_{h}
\end{align*}
$$

where $\tau>0$ is a stabilization parameter, and $\boldsymbol{q}_{h}^{-}$and $\lambda_{h}^{-}$denote the known values of $\boldsymbol{q}_{h}$ and $\lambda_{h}$, respectively, at $x_{0}=t_{n}$ from the previous space-time slab $\mathcal{E}^{n-1}$, or the initial condition if $n=0$.
3.3. Well posedness. We now show the existence of a unique solution to the space-time HDG method eq. (3.6).
Theorem 3.1 (Existence and uniqueness). A unique solution $\left(\boldsymbol{q}_{h}, v_{h}, \lambda_{h}\right) \in \boldsymbol{V}_{h} \times$ $W_{h} \times M_{h}$ to eq. (3.6) exists if the stabilization parameter $\tau$ is positive.

Proof. It is sufficient to show that if the data is equal to zero, the only solution to eq. (3.6) is the trivial one. We only need to show this for an arbitrary space-time slab $\mathcal{E}^{n}$ assuming $\lambda_{h}^{-}=0$ and $\boldsymbol{q}_{h}^{-}=0$.

Take $\boldsymbol{r}_{h}=\boldsymbol{q}_{h}$ in eq. (3.6a), $w_{h}=v_{h}$ in eq. (3.6b) and $\mu_{h}=\lambda_{h}$ in eq. (3.6c), and add the three equations together:

$$
\begin{align*}
&\left.-\left(\boldsymbol{q}_{h}, f_{n} \partial_{t} \boldsymbol{q}_{h}\right)_{\mathcal{T}^{n}}-\left(\left|\boldsymbol{q}_{h}\right|^{2}, f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left.\langle | \boldsymbol{q}_{h}\right|^{2}, f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}  \tag{3.7}\\
&+\left\langle\tau\left(v_{h}-\lambda_{h}\right)^{2}, f_{n}\right\rangle_{\mathcal{F}_{\mathscr{Q}}^{n}}-\left\langle\lambda_{h} f_{n}, \partial_{t} \lambda_{h}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}-\left\langle\lambda_{h}^{2}, f_{n}^{\prime}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}} \\
&+\left\langle\left\langle\lambda_{h}^{2}, f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}\left(t_{n+1}\right)}=0
\end{align*}
$$

Note that since $\boldsymbol{q}_{h} \cdot \partial_{t} \boldsymbol{q}_{h}=\frac{1}{2} \partial_{t}\left(\left|\boldsymbol{q}_{h}\right|^{2}\right)$, we may write the first term on the left hand side, after integration by parts in time, as

$$
\begin{align*}
&\left.-\left(\boldsymbol{q}_{h}, f_{n} \partial_{t} \boldsymbol{q}_{h}\right)_{\mathcal{T}^{n}}=\frac{1}{2}\left(\left|\boldsymbol{q}_{h}\right|^{2}, f_{n}^{\prime}\right)_{\mathcal{T}^{n}}-\left.\frac{1}{2}\langle | \boldsymbol{q}_{h}\right|^{2}, f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}  \tag{3.8}\\
&\left.+\left.\frac{1}{2}\langle | \boldsymbol{q}_{h}\right|^{2}, f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}
\end{align*}
$$

Similarly, the fifth term on the left hand side of eq. (3.7) may be written as

$$
\begin{align*}
-\left\langle\lambda_{h} f_{n}, \partial_{t} \lambda_{h}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}=\frac{1}{2}\left\langle\lambda_{h}^{2}, f_{n}^{\prime}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}-\frac{1}{2}\left\langle\left\langle\lambda_{h}^{2}, f_{n}\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}\right. & +\frac{1}{2}\left\langle\left\langle\lambda_{h}^{2}, f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)} . \tag{3.9}
\end{align*}
$$

Combining eq. (3.7)-eq. (3.9), we obtain

$$
\begin{align*}
0=-\frac{1}{2}\left(\left|\boldsymbol{q}_{h}\right|^{2}\right. & \left.\left., f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left.\frac{1}{2}\langle | \boldsymbol{q}_{h}\right|^{2}, f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}  \tag{3.10}\\
+ & \left.\left.\frac{1}{2}\langle | \boldsymbol{q}_{h}\right|^{2}, f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}+\left\langle\tau\left(v_{h}-\lambda_{h}\right)^{2}, f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}} \\
& \quad-\frac{1}{2}\left\langle\lambda_{h}^{2}, f_{n}^{\prime}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}+\frac{1}{2}\left\langle\left\langle\lambda_{h}^{2}, f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}+\frac{1}{2}\left\langle\left\langle\lambda_{h}^{2}, f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}}\left(t_{n}\right)
\end{align*}
$$

Since $\tau>0, f_{n}>0, \forall t$, and $f_{n}^{\prime}<0, \forall t$, we conclude that $\boldsymbol{q}_{h}=0$ in $\mathcal{E}^{n}$ and on $K_{j}^{n} \cup K_{j}^{n+1}, v_{h}=\lambda_{h}$ on $\Gamma_{\mathcal{Q}}^{n}$, and $\lambda_{h}=0$ on $\Gamma_{\mathcal{S}}^{n}$ and on $\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right) \cup \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)$. Substituting into eq. (3.6a),

$$
\begin{align*}
0 & =\left(v_{h}, f_{n} \nabla \cdot \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left\langle\lambda_{h}, \boldsymbol{r}_{h} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}} \\
& =-\left(\nabla v_{h}, f_{n} \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}+\left\langle\left(v_{h}-\lambda_{h}\right), \boldsymbol{r}_{h} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}  \tag{3.11}\\
& =-\left(\nabla v_{h}, f_{n} \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}},
\end{align*}
$$

which holds for all $\boldsymbol{r}_{h} \in \boldsymbol{V}_{h}$. Since $f_{n}>0$ we conclude that $\nabla v_{h}=0$ in $\mathcal{E}^{n}$, implying that $v_{h}$ depends only on time. To show that $v_{h}=0$ in $\mathcal{E}^{n}$, consider the following. Since $v_{h}$ depends only on time, then $v_{h}$ is constant on $\Omega \times\{t\}$, for all $t \in\left(t_{n}, t_{n+1}\right)$. In addition, $v_{h}=\lambda_{h}=0$ on $\Gamma_{\mathcal{S}}^{n}$, so $v_{h}=0$ on $\mathcal{E}^{n}$.

## 4. Analysis tools

In this section, we develop some of the tools needed to perform the error analysis in section 5 .
4.1. Notation and anisotropic Sobolev spaces. Let the multi-index $\alpha$ be a vector of non-negative integers $\alpha_{i}$ and let $|\alpha|$ be defined as $|\alpha|=\sum_{i} \alpha_{i}$. By $D^{\alpha} v$ we denote the partial derivative of order $|\alpha|$ of $v$, i.e.,

$$
\begin{equation*}
D^{\alpha} v=\partial_{x_{0}}^{\alpha_{0}} \partial_{x_{1}}^{\alpha_{1}} \partial_{x_{2}}^{\alpha_{2}} v \tag{4.1}
\end{equation*}
$$

We define the Sobolev space $H^{s}(\Omega)=\left\{v \in L^{2}(\Omega): D^{\alpha} v \in L^{2}(\Omega)\right.$ for $\left.|\alpha| \leq s\right\}$. This space is equipped with the following norm and seminorm:

$$
\begin{equation*}
\|v\|_{H^{s}(\Omega)}^{2}=\sum_{|\alpha| \leq s}\left\|D^{\alpha} v\right\|_{L^{2}(\Omega)}^{2} \quad \text { and } \quad|v|_{H^{s}(\Omega)}^{2}=\sum_{|\alpha|=s}\left\|D^{\alpha} v\right\|_{L^{2}(\Omega)}^{2} \tag{4.2}
\end{equation*}
$$

For $\alpha_{i} \geq 0, i=0,1,2$, we introduce the anisotropic Sobolev space of order $\left(s_{t}, s_{s}\right)$ on $\mathcal{E} \subset \mathbb{R}^{3}$ by

$$
\begin{equation*}
H^{\left(s_{t}, s_{s}\right)}(\mathcal{E})=\left\{v \in L^{2}(\mathcal{E}): D^{\left(\alpha_{t}, \alpha_{s}\right)} v \in L^{2}(\mathcal{E}) \text { for } \alpha_{t} \leq s_{t},\left|\alpha_{s}\right| \leq s_{s}\right\} \tag{4.3}
\end{equation*}
$$

where $\alpha_{t}=\alpha_{0}$ and $\alpha_{s}=\left(\alpha_{1}, \alpha_{2}\right)$ and $D^{\left(\alpha_{t}, \alpha_{s}\right)} v=\partial_{x_{0}}^{\alpha_{t}} \partial_{x_{1}}^{\alpha_{1}} \partial_{x_{2}}^{\alpha_{2}} v$. The anisotropic Sobolev norm and seminorm are given by, respectively,
$\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{E})}^{2}=\sum_{\substack{\alpha_{t} \leq s_{t} \\\left|\alpha_{s}\right| \leq s_{s}}}\left\|D^{\left(\alpha_{t}, \alpha_{s}\right)} v\right\|_{L^{2}(\mathcal{E})}^{2}$ and $|v|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{E})}^{2}=\sum_{\substack{\alpha_{t}=s_{t} \\\left|\alpha_{s}\right|=s_{s}}}\left\|D^{\left(\alpha_{t}, \alpha_{s}\right)} v\right\|_{L^{2}(\mathcal{E})}^{2}$.
Let $\mathcal{K}$ denote any space-time prism that is constructed as $\mathcal{K}=K \times I$, where $K$ is a spatial triangular element and $I$ is an interval. Let $h_{K}$ and $\rho_{K}$ denote, respectively, the radii of the 2 -dimensional circumcircle and inscribed circle of $K$. We will assume spatial shape regularity, i.e., we assume there exists a constant $c_{r}>0$ such that

$$
\begin{equation*}
\frac{h_{K}}{\rho_{K}} \leq c_{r}, \quad \forall \mathcal{K} \in \mathcal{T}^{n} \tag{4.5}
\end{equation*}
$$

Additionally, we assume that $\mathcal{T}^{n}$ does not have any hanging nodes. Throughout the remainder of this paper we will denote by $C>0$ a generic constant that is independent of $h_{K}$ and $\Delta t$.
4.2. Space-time mappings. Let $\widehat{K}$ denote the reference triangle defined by the vertices $(0,0),(1,0),(0,1)$, and with reference coordinates $\widehat{\boldsymbol{x}}=\left(\widehat{x}_{1}, \widehat{x}_{2}\right)$. Furthermore, let $H_{K}$ denote the affine mapping $H_{K}: \widehat{K} \rightarrow K$ defined as $H_{K}(\widehat{\boldsymbol{x}})=B_{K} \widehat{\boldsymbol{x}}+\boldsymbol{c}$, where $B_{K}$ is a matrix and $\boldsymbol{c}$ is a vector.

To construct each space-time prism we follow a similar approach as [27]. Consider a reference prism $\widehat{\mathcal{K}}$ defined by the vertices $(-1,0,0),(-1,1,0),(-1,0,1),(1,0,0),(1,1,0),(1,0,1)$. The reference coordinates in $\widehat{\mathcal{K}}$ are denoted by $\left(\widehat{x}_{0}, \widehat{\boldsymbol{x}}\right)$. The space-time prism $\mathcal{K}$ is obtained as follows. First, we construct an intermediate element $\check{\mathcal{K}}$ from an affine mapping $F_{\mathcal{K}}: \widehat{\mathcal{K}} \mapsto \check{\mathcal{K}}$ defined as $F_{\mathcal{K}}\left(\widehat{x}_{0}, \widehat{\boldsymbol{x}}\right)=A_{\mathcal{K}}\left[\widehat{x}_{0}, \widehat{\boldsymbol{x}}\right]^{T}+\boldsymbol{b}$, where $A_{\mathcal{K}}=\operatorname{diag}\left(\frac{\Delta t}{2}, 1,1\right)$ and $\boldsymbol{b}$ is a vector of the form $\left[b_{0}, 0,0\right]^{T}$. The coordinates on $\check{\mathcal{K}}$ are $\left(\check{x}_{0}, \check{\boldsymbol{x}}\right)$. Then, $\mathcal{K}$ is obtained via the affine mapping, $G_{\mathcal{K}}: \check{\mathcal{K}} \mapsto \mathcal{K}$ defined as:

$$
G_{\mathcal{K}}\left(\check{x}_{0}, \check{\boldsymbol{x}}\right)=\left[\begin{array}{cc}
1 & \mathbf{0}  \tag{4.6}\\
\mathbf{0}^{T} & B_{K}
\end{array}\right]\left[\begin{array}{c}
\check{x}_{0} \\
\check{\boldsymbol{x}}
\end{array}\right]+\left[\begin{array}{l}
0 \\
\boldsymbol{c}
\end{array}\right]
$$

where $\mathbf{0}=[0,0]$ and $B_{K}$ denotes the matrix associated with the mapping $H_{K}$ defined above. See fig. 3.


Figure 3. An illustration of the different space-time mappings.

We denote by $\partial \widehat{\mathcal{K}}_{1}$ the boundary face of $\widehat{\mathcal{K}}$ with $\widehat{x}_{1}=0$. Similarly, $\partial \widehat{\mathcal{K}}_{2}$ and $\partial \widehat{\mathcal{K}}_{3}$ are the boundary faces of $\widehat{\mathcal{K}}$ with, respectively, $\widehat{x}_{2}=0$ and $\widehat{x}_{1}+\widehat{x}_{2}=1$. By $\partial \widehat{\mathcal{K}}_{0}$ we denote the boundary faces of $\widehat{\mathcal{K}}$ with $\widehat{x}_{0}=-1$ and $\widehat{x}_{0}=1$. Furthermore, $\partial \check{\mathcal{K}}_{i}$, $i=0,1,2,3$, will denote the boundary faces of $\check{\mathcal{K}}$ which are obtained by applying the transformation $F_{\mathcal{K}}$ to $\widehat{\mathcal{K}}$.
4.3. Trace and inverse trace identities. In this section we prove anisotropic trace and inverse trace identities. This is achieved by first considering different scaling identities. Similar identities were shown on hexahedra in [13, 27], but are modified here for prisms.

Lemma 4.1. Let $\check{u} \in H^{\left(s_{t}, s_{s}\right)}(\check{\mathcal{K}}), \alpha_{t}=\alpha_{0}, \alpha_{s}=\left(\alpha_{1}, \alpha_{2}\right)$, and $\alpha=\left(\alpha_{t}, \alpha_{s}\right)$. Then, the following identities hold for $\alpha=\left(\alpha_{t}, \alpha_{s}\right), \alpha_{i} \geq 0, i=0,1,2$,

$$
\begin{align*}
\left\|\check{D}^{\alpha} \check{u}\right\|_{\check{\mathcal{K}}}^{2} & =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}-1}\left\|\widehat{D}^{\alpha} \widehat{u}\right\|_{\widehat{\mathcal{K}}}^{2}  \tag{4.7a}\\
\left\|\check{D}^{\alpha} \check{u}\right\|_{\partial \check{\mathcal{K}}_{0}}^{2} & =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}}\left\|\widehat{D}^{\alpha} \widehat{u}\right\|_{\partial \widehat{\mathcal{K}}_{0}}^{2}  \tag{4.7b}\\
\left\|\check{D}^{\alpha} \check{u}\right\|_{\partial \check{\mathcal{K}}_{j}}^{2} & =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}-1}\left\|\widehat{D}^{\alpha} \widehat{u}\right\|_{\partial \widehat{\mathcal{K}}_{j}}^{2}, \quad j=1,2,3 \tag{4.7c}
\end{align*}
$$

where $\widehat{u}=\check{u} \circ F_{\mathcal{K}}$.
Proof. Note that by the chain rule,

$$
\begin{equation*}
\check{D}^{\alpha} \check{u}=\left(\frac{2}{\Delta t}\right)^{\alpha_{0}} \widehat{D}^{\alpha}\left(\check{u} \circ F_{\mathcal{K}}\right) \circ F_{\mathcal{K}}^{-1} \tag{4.8}
\end{equation*}
$$

We first show eq. (4.7a). By eq. (4.8)

$$
\begin{equation*}
\left\|\check{D}^{\alpha} \check{u}\right\|_{\check{\mathcal{K}}}^{2}=\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}} \int_{\check{\mathcal{K}}}\left(\widehat{D}^{\alpha}\left(\check{u} \circ F_{\mathcal{K}}\right) \circ F_{\mathcal{K}}^{-1}\right)^{2} \mathrm{~d} \check{x}_{0} \mathrm{~d} \check{\boldsymbol{x}} . \tag{4.9}
\end{equation*}
$$

Changing variables, we obtain

$$
\begin{align*}
\left\|\check{D}^{\alpha} \check{u}\right\|_{\check{\mathcal{K}}}^{2} & =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}} \int_{\widehat{\mathcal{K}}}\left(\widehat{D}^{\alpha}\left(\check{u} \circ F_{\mathcal{K}}\right) \circ F_{\mathcal{K}}^{-1} \circ F_{\mathcal{K}}\right)^{2}\left|\operatorname{det} A_{\mathcal{K}}\right| \mathrm{d} \widehat{x}_{0} \mathrm{~d} \widehat{\boldsymbol{x}} \\
& =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}} \int_{\widehat{\mathcal{K}}}\left(\widehat{D}^{\alpha}\left(\check{u} \circ F_{\mathcal{K}}\right)\right)^{2}\left|\operatorname{det} A_{\mathcal{K}}\right| \mathrm{d} \widehat{x}_{0} \mathrm{~d} \widehat{\boldsymbol{x}} \\
& =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}} \frac{\Delta t}{2} \int_{\widehat{\mathcal{K}}}\left(\widehat{D}^{\alpha} \widehat{u}\right)^{2} \mathrm{~d} \widehat{x}_{0} \mathrm{~d} \widehat{\boldsymbol{x}}  \tag{4.10}\\
& =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}-1}\left\|\widehat{D}^{\alpha} \widehat{u}\right\|_{\widehat{\mathcal{K}}}^{2}
\end{align*}
$$

To show eq. (4.7b) we note that time is fixed on $\partial \check{\mathcal{K}}_{0}$. Furthermore, since $\check{\boldsymbol{x}}=\widehat{\boldsymbol{x}}$ we note that for any function $\check{v}$ on $\check{\mathcal{K}}$ we have $\check{v}\left(t_{n}, \check{\boldsymbol{x}}\right)=\widehat{v}(1, \widehat{\boldsymbol{x}})$, i.e., $\left.\check{v}\right|_{\partial \check{\mathcal{K}}_{0}}=\left.\widehat{v}\right|_{\partial \widehat{\mathcal{K}}_{0}}$. In particular, if $\check{v}=\check{D}^{\alpha} \check{u}$,

$$
\begin{equation*}
\left.\left(\check{D}^{\alpha} \check{u}\right)\right|_{\partial \check{\mathcal{K}}_{0}}=\left.\left(\frac{2}{\Delta t}\right)^{\alpha_{0}}\left(\widehat{D}^{\alpha}\left(\check{u} \circ F_{\mathcal{K}}\right)\right)\right|_{\partial \widehat{\mathcal{K}}_{0}} \tag{4.11}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
\left\|\check{D}^{\alpha} \check{u}\right\|_{\partial \check{\mathcal{K}}_{0}}^{2} & =\int_{\partial \check{\mathcal{K}}_{0}}\left(\check{D}^{\alpha} \check{u}\right)^{2} \mathrm{~d} \check{\boldsymbol{x}} \\
& =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}} \int_{\partial \widehat{\mathcal{K}}_{0}}\left(\widehat{D}^{\alpha} \widehat{u}\right)^{2} \mathrm{~d} \widehat{\boldsymbol{x}}  \tag{4.12}\\
& =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}}\left\|\widehat{D}^{\alpha} \widehat{u}\right\|_{\partial \widehat{\mathcal{K}}_{0}}^{2}
\end{align*}
$$

which concludes the proof for eq. (4.7b).
Finally, we prove eq. (4.7c) for $j=1$. The proofs for $j=2,3$ are analogous. First let us define the mapping $J_{\partial \mathcal{K}}$ that maps $\partial \widehat{\mathcal{K}}_{1}$ onto $\partial \check{\mathcal{K}}_{1}$. This mapping is given by

$$
J_{\partial \mathcal{K}}\left(\widehat{x}_{0}, \widehat{x}_{2}\right)=\left[\begin{array}{cc}
\frac{\Delta t}{2} & 0  \tag{4.13}\\
0 & 1
\end{array}\right]\left[\begin{array}{l}
\widehat{x}_{0} \\
\widehat{x}_{2}
\end{array}\right]+\boldsymbol{d}
$$

where $\boldsymbol{d}$ is a constant two dimensional vector. Then, by the chain rule,

$$
\begin{align*}
\left.\left(\check{D}^{\alpha} \check{u}\right)\right|_{\partial \check{\mathcal{K}}_{1}} & =\left(\check{D}^{\alpha} \check{u}\right)\left(\check{x}_{0}, 0, \check{x}_{2}\right) \\
& =\left.\left(\frac{2}{\Delta t}\right)^{\alpha_{0}}\left(\widehat{D}^{\alpha}\left(\check{u} \circ F_{\mathcal{K}}\right)\right)\right|_{\partial \widehat{\mathcal{K}}_{1}} \circ J_{\partial \mathcal{K}}^{-1}  \tag{4.14}\\
& =\left(\frac{2}{\Delta t}\right)^{\alpha_{0}}\left[\left(\widehat{D}^{\alpha}\left(\check{u} \circ F_{\mathcal{K}}\right)\right)\left(\widehat{x}_{0}, 0, \widehat{x}_{2}\right)\right] \circ J_{\partial \mathcal{K}}^{-1}
\end{align*}
$$

We now find:

$$
\begin{align*}
\left\|\check{D}^{\alpha} \check{u}\right\|_{\partial \check{K}_{1}}^{2} & =\int_{\partial \check{\mathcal{K}}_{1}}\left(\check{D}^{\alpha} \check{u}\right)^{2} \mathrm{~d} \check{x}_{0} \mathrm{~d} \check{x}_{2} \\
& =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}} \int_{\partial \check{\mathcal{K}}_{1}}\left(\left[\left(\widehat{D}^{\alpha}\left(\check{u} \circ F_{\mathcal{K}}\right)\right)\left(\widehat{x}_{0}, 0, \widehat{x}_{2}\right)\right] \circ J_{\partial \mathcal{K}}^{-1}\right)^{2} \mathrm{~d} \check{x}_{0} \mathrm{~d} \check{x}_{2} \tag{4.15}
\end{align*}
$$

Note that the determinant of the Jacobian of $J_{\partial \mathcal{K}}$ is $\Delta t / 2$. Changing variables,

$$
\begin{align*}
\left\|\check{D}^{\alpha} \check{u}\right\|_{\partial \check{\mathcal{K}}_{1}}^{2} & =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}} \frac{\Delta t}{2} \int_{\partial \widehat{\mathcal{K}}_{1}}\left(\left(\widehat{D}^{\alpha} \widehat{u}\right)\left(\widehat{x}_{0}, 0, \widehat{x}_{2}\right)\right)^{2} \mathrm{~d} \widehat{x}_{0} \mathrm{~d} \widehat{x}_{2} \\
& =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}-1} \int_{\partial \widehat{\mathcal{K}}_{1}}\left(\left(\widehat{D}^{\alpha} \widehat{u}\right)\left(\widehat{x}_{0}, 0, \widehat{x}_{2}\right)\right)^{2} \mathrm{~d} \widehat{x}_{0} \mathrm{~d} \widehat{x}_{2}  \tag{4.16}\\
& =\left(\frac{2}{\Delta t}\right)^{2 \alpha_{0}-1}\left\|\widehat{D}^{\alpha} \widehat{u}\right\|_{\partial \widehat{\mathcal{K}}_{1}}^{2} .
\end{align*}
$$

The result follows.
In what follows we use the following inequalities that can be shown by standard scaling arguments:

$$
\begin{align*}
\left|\operatorname{det} B_{K}\right| & \leq C h_{K}^{2}  \tag{4.17a}\\
\|\tilde{u}\|_{\tilde{F}}^{2} & \leq C h_{K}^{-1}\|u\|_{F_{K}}^{2}  \tag{4.17b}\\
\|u\|_{F_{K}}^{2} & \leq C h_{K}\|\tilde{u}\|_{\tilde{F}}^{2} \tag{4.17c}
\end{align*}
$$

where $F_{K} \in \partial K$, and where $\tilde{u}=\widehat{u}$ and $\tilde{F}=F_{\widehat{K}} \in \partial \widehat{K}$, or $\tilde{u}=\check{u}$ and $\tilde{F}=F_{\breve{K}} \in \partial \check{K}$.
Lemma 4.2. Let $u \in H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})$. The following inequalities hold

$$
\begin{align*}
\|u\|_{\mathcal{K}}^{2} & \leq C h_{K}^{2}\|\check{u}\|_{\tilde{\mathcal{K}}}^{2},  \tag{4.18a}\\
\|\check{u}\|_{\check{\mathcal{K}}}^{2} & \leq C h_{K}^{-2}\|u\|_{\mathcal{K}}^{2},  \tag{4.18~b}\\
\|\check{u}\|_{\mathcal{F}_{\check{\mathcal{K}}}}^{2} & \leq C h_{K}^{-1}\|u\|_{\mathcal{F}_{\mathcal{K}}}^{2}, \tag{4.18c}
\end{align*}
$$

where $\check{u}=u \circ G_{\mathcal{K}}, \mathcal{F}_{\check{\mathcal{K}}} \in \mathcal{Q}_{\check{\mathcal{K}}}$, and $\mathcal{F}_{\mathcal{K}} \in \mathcal{Q}_{\mathcal{K}}$.
Proof. The results follow eq. (4.17).
We end this section by stating a trace inequality and two inverse trace inequalities.
Lemma 4.3. Let $X_{h}(\mathcal{K}) \subset W_{h}(\mathcal{K})$ be a finite dimensional subspace such that the trace map $\gamma_{\mathcal{F}_{\mathcal{K}}}: X_{h}(\mathcal{K}) \mapsto M_{h}(\mathcal{F})$ defined by $\gamma_{\mathcal{F}_{\mathcal{K}}}\left(v_{h}\right)=\left.v_{h}\right|_{\mathcal{F}_{\mathcal{K}}}$, for a face $\mathcal{F}_{\mathcal{K}} \subset \mathcal{Q}_{\mathcal{K}}$ is injective, then

$$
\begin{equation*}
\left\|v_{h}\right\|_{\mathcal{K}}^{2} \leq C h_{K}\left\|v_{h}\right\|_{\mathcal{F}_{\mathcal{K}}}^{2}, \quad \forall v_{h} \in X_{h}(\mathcal{K}) \tag{4.19}
\end{equation*}
$$

Proof. By eq. (4.18a), we obtain

$$
\begin{equation*}
\left\|v_{h}\right\|_{\mathcal{K}}^{2} \leq C h_{K}^{2}\left\|\check{v}_{h}\right\|_{\tilde{\mathcal{K}}}^{2} \tag{4.20}
\end{equation*}
$$

Since $v_{h}$ is a polynomial and $\gamma_{\mathcal{F}_{\mathcal{K}}}$ is injective, we have $\left\|\check{v}_{h}\right\|_{\check{\mathcal{K}}}^{2} \leq C\left\|\check{v}_{h}\right\|_{\mathcal{F}_{\check{\mathcal{K}}}}^{2}$. Combining this with eq. (4.20), the result follows after using eq. (4.18c).
Lemma 4.4. Let $\mathcal{K}=K \times I_{n}$ a space-time element in $\mathcal{T}^{n}, \mathcal{F} \subset \partial \mathcal{E}_{\mathcal{S}}^{n}$ a face on the free-surface boundary and $\partial \mathcal{F}_{0}$ the two edges of the face $\mathcal{F}$ that are on the time levels. For $v_{h} \in W_{h}(\mathcal{K})$ and $\lambda_{h} \in M_{h}(\mathcal{F})$, the following inverse trace inequalities hold

$$
\begin{align*}
\left\|\lambda_{h}\right\|_{\partial \mathcal{F}_{0}}^{2} & \leq C \Delta t^{-1}\left\|\lambda_{h}\right\|_{\mathcal{F}}^{2}  \tag{4.21a}\\
\left\|v_{h}\right\|_{\partial \mathcal{K}_{0}}^{2} & \leq C \Delta t^{-1}\left\|v_{h}\right\|_{\mathcal{K}}^{2}  \tag{4.21b}\\
\left\|v_{h}\right\|_{\mathcal{Q K}_{\mathcal{K}}}^{2} & \leq C h_{K}^{-1}\left\|v_{h}\right\|_{\mathcal{K}}^{2} \tag{4.21c}
\end{align*}
$$

Proof. Since the face $\mathcal{F}$ is a quadrilateral, the proof for eq. (4.21a) can be found, e.g., in [13, Corollary 3.49]. Equation (4.21b) and eq. (4.21c) can be obtained by the results in Lemma 4.1 and standard scaling arguments in space.

## 5. ERror analysis

In this section we present an a priori error analysis for the space-time HDG method eq. (3.6). For this we require the following spaces:

$$
\begin{equation*}
\widetilde{W}_{h}(\mathcal{K}):=P_{p-1}(K) \otimes P_{p}\left(I_{n}\right), \quad \widetilde{\boldsymbol{V}}_{h}(\mathcal{K}):=\left[P_{p-1}(K) \otimes P_{p}\left(I_{n}\right)\right]^{2} \tag{5.1}
\end{equation*}
$$

We require also the $f_{n}$-weighted $L^{2}$-norm defined on a domain $D$. For any $v \in$ $L^{2}(D)$ this norm is defined as $\|v\|_{f_{n}, D}^{2}:=\left(f_{n} v, v\right)_{D}$ while for $\boldsymbol{q} \in\left[L^{2}(D)\right]^{2}$ it is defined as $\|\boldsymbol{q}\|_{f_{n}, D}^{2}:=\left(f_{n} \boldsymbol{q}, \boldsymbol{q}\right)_{D}$.
5.1. The projection. The projection $\Pi_{h}$ onto $\boldsymbol{V}_{h} \times W_{h}$ used here is based on the projection defined in [10], but tailored to the spaces used in this work. The projected function is denoted by $\Pi_{h}(\boldsymbol{q}, v)$ or $\left(\Pi_{\boldsymbol{V}} \boldsymbol{q}, \Pi_{W} v\right)$, and is defined by requiring that the following equations are satisfied on each space-time element $\mathcal{K} \in \mathcal{T}^{n}$ :

$$
\begin{align*}
\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}, \boldsymbol{s}_{h} f_{n}\right)_{\mathcal{K}}=\left(\boldsymbol{q}, \boldsymbol{s}_{h} f_{n}\right)_{\mathcal{K}} & \forall s_{h} \in \widetilde{\boldsymbol{V}}_{h}(\mathcal{K})  \tag{5.2a}\\
\left(\Pi_{W} v, z_{h} f_{n}\right)_{\mathcal{K}}=\left(v, z_{h} f_{n}\right)_{\mathcal{K}} & \forall z_{h} \in \widetilde{W}_{h}(\mathcal{K}) \tag{5.2~b}
\end{align*}
$$

$\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}-\tau \Pi_{W} v, \sigma_{h} f_{n}\right\rangle_{\mathcal{F}}=\left\langle\boldsymbol{q} \cdot \boldsymbol{n}-\tau v, \sigma_{h} f_{n}\right\rangle_{\mathcal{F}} \quad \forall \sigma_{h} \in M_{h}(\mathcal{F}), \mathcal{F} \subset \mathcal{Q}_{\mathcal{K}}$.

Notice that $\Pi_{h}$ is well defined for functions $\boldsymbol{q}$ and $v$ such that their traces are in $L^{2}\left(\mathcal{Q}_{\mathcal{K}}\right)$. Therefore, the domain of $\Pi_{h}$ is in $\left[H^{1}\left(\mathcal{T}^{n}\right)\right]^{2} \times H^{1}\left(\mathcal{T}^{n}\right)$, where $H^{1}\left(\mathcal{T}^{n}\right):=$ $\prod_{\mathcal{K} \in \mathcal{T}^{n}} H^{1}(\mathcal{K})$.

In order to show existence and uniqueness of the projection and its approximation properties, it will be useful to define the following spaces:

$$
\begin{align*}
W_{h}^{\perp}(\mathcal{K}) & :=\left\{w \in W_{h}(\mathcal{K}):\left(w, \widetilde{w} f_{n}\right)_{\mathcal{K}}=0, \forall \widetilde{w} \in \widetilde{W}_{h}(\mathcal{K})\right\},  \tag{5.3a}\\
\boldsymbol{V}_{h}^{\perp}(\mathcal{K}) & :=\left\{\boldsymbol{v} \in \boldsymbol{V}_{h}(\mathcal{K}):\left(\boldsymbol{v}, \widetilde{\boldsymbol{v}} f_{n}\right)_{\mathcal{K}}=0, \forall \widetilde{\boldsymbol{v}} \in \widetilde{\boldsymbol{V}}_{h}(\mathcal{K})\right\} . \tag{5.3b}
\end{align*}
$$

The following lemma will be useful when showing the existence and uniqueness of the projection and its approximation properties.

Lemma 5.1. For any space-time element $\mathcal{K}$, the following is satisfied for any face $\mathcal{F}_{\mathcal{K}} \in \mathcal{Q}_{\mathcal{K}}:$

$$
\begin{align*}
w_{h} \in W_{h}^{\perp}(\mathcal{K}) \text { and }\left.w_{h}\right|_{\mathcal{F}_{\mathcal{K}}}=0, & \text { implies } w_{h}=0 \text { on } \mathcal{K},  \tag{5.4a}\\
\boldsymbol{v}_{h} \in \boldsymbol{V}_{h}^{\perp}(\mathcal{K}) \text { and }\left.\boldsymbol{v}_{h} \cdot \boldsymbol{n}\right|_{\mathcal{Q}_{\mathcal{K}} \backslash \mathcal{F}_{\mathcal{K}}}=0 & \text { implies } \boldsymbol{v}_{h}=0 \text { on } \mathcal{K} . \tag{5.4b}
\end{align*}
$$

Moreover, the following estimates are satisfied

$$
\begin{align*}
\left\|w_{h}\right\|_{f_{n}, \mathcal{K}} \leq C h_{K}^{1 / 2}\left\|w_{h}\right\|_{f_{n}, \mathcal{F}_{\mathcal{K}}} & \forall w_{h} \in W_{h}^{\perp}(\mathcal{K}),  \tag{5.5a}\\
\left\|\boldsymbol{v}_{h}\right\|_{f_{n}, \mathcal{K}} \leq C h_{K}^{1 / 2}\left\|\boldsymbol{v}_{h} \cdot \boldsymbol{n}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}} & \forall \boldsymbol{v}_{h} \in \boldsymbol{V}_{h}^{\perp}(\mathcal{K}) . \tag{5.5b}
\end{align*}
$$

Proof. We first show eq. (5.4a). Take $w_{h} \in W_{h}^{\perp}(\mathcal{K})$ such that $\left.w_{h}\right|_{\mathcal{F}_{\mathcal{K}}}=0$ and let $L$ be a nonzero linear function that vanishes on $\mathcal{F}_{\mathcal{K}}$. Then, $w_{h}$ can be written as $w_{h}=L \widetilde{p}$, where $\widetilde{p} \in \widetilde{W}_{h}(\mathcal{K})$ [7, Lemma 3.1.10]. Since $w_{h} \in W_{h}^{\perp}(\mathcal{K})$, we have that $\left(L \widetilde{p}, f_{n} \widetilde{p}\right)_{\mathcal{K}}=0$. Since $L$ cannot be zero on $\mathcal{K}$, we conclude that $\widetilde{p}$ must be zero and therefore, $w_{h}$ is zero on $\mathcal{K}$.

We next show eq. (5.4b). Let $p_{h}=\boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}}$, for any face $\mathcal{F}$ different than $\mathcal{F}_{\mathcal{K}}$. Notice that $p_{h} \in W_{h}^{\perp}(\mathcal{K})$ and $\left.p_{h}\right|_{\mathcal{F}}=0$. Using a similar argument as above, we can conclude that $p_{h}=0$ on $\mathcal{K}$. Therefore, $\boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}}=0$ on $\mathcal{K}$. Since the set $\left\{\boldsymbol{n}_{\mathcal{F}}: \mathcal{F} \in \mathcal{Q}_{\mathcal{K}} \backslash \mathcal{F}_{\mathcal{K}}\right\}$ is a basis of $\mathbb{R}^{2}$, we conclude that $\boldsymbol{v}_{h}$ must be zero on $\mathcal{K}$.

To show eq. (5.5a), we use the scaling identities from section 4.3. Since $\|\cdot\|_{f_{n}, \mathcal{K}}$ is a weighted norm and $w_{h} f_{n}$, for $w_{h} \in W_{h}^{\perp}(\mathcal{K})$, is not a broken polynomial, as required in the proof of Lemma 4.3, we cannot use Lemma 4.3 directly. However, since $f_{n}$ is uniformly bounded, there exists a constant $C_{f_{n}}>0$ such that $\sup _{t} f_{n}(t) \leq C_{f_{n}}$ for all $n$. Therefore,

$$
\begin{equation*}
\left\|w_{h}\right\|_{f_{n}, \mathcal{K}}^{2} \leq C_{f_{n}}\left\|w_{h}\right\|_{\mathcal{K}}^{2} \tag{5.6}
\end{equation*}
$$

Notice that eq. (5.4a) implies that the trace map $\gamma_{\mathcal{F}_{\mathcal{K}}}: W_{h}^{\perp}(\mathcal{K}) \mapsto M_{h}\left(\mathcal{F}_{\mathcal{K}}\right)$ defined by $\gamma_{\mathcal{F}_{\mathcal{K}}}\left(w_{h}\right)=\left.w_{h}\right|_{\mathcal{F}_{\mathcal{K}}}$ is injective. By Lemma 4.3 we then obtain

$$
\begin{equation*}
\left\|w_{h}\right\|_{f_{n}, \mathcal{K}}^{2} \leq C h_{K}\left\|w_{h}\right\|_{\mathcal{F}_{\mathcal{K}}}^{2} \tag{5.7}
\end{equation*}
$$

Equation (5.5a) now follows by equivalence of norms on finite-dimensional spaces.
Finally, we show eq. (5.5b). Let $\mathcal{F}_{1}, \mathcal{F}_{2} \in \mathcal{Q}_{\mathcal{K}}$ with $\mathcal{F}_{1} \neq \mathcal{F}_{\mathcal{K}}$ and $\mathcal{F}_{2} \neq \mathcal{F}_{\mathcal{K}}$. Notice that $\left\{\boldsymbol{n}_{\mathcal{F}_{1}}, \boldsymbol{n}_{\mathcal{F}_{2}}\right\}$ is a basis of $\mathbb{R}^{2}$, therefore, we can write $\boldsymbol{v}_{h}$ as $\boldsymbol{v}_{h}=\boldsymbol{v}_{1} \boldsymbol{v}_{h}$. $\boldsymbol{n}_{\mathcal{F}_{1}}+\boldsymbol{v}_{2} \boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}_{2}}$, where $\boldsymbol{v}_{1}, \boldsymbol{v}_{2} \in \mathbb{R}^{2}$ are constant vectors. Thus,

$$
\begin{align*}
\left\|\boldsymbol{v}_{h}\right\|_{f_{n}, \mathcal{K}} & =\left\|\boldsymbol{v}_{1} \boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}_{1}}+\boldsymbol{v}_{2} \boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}_{2}}\right\|_{f_{n}, \mathcal{K}} \\
& \leq\left\|\boldsymbol{v}_{1}\right\|_{f_{n}, \mathcal{K}}\left\|\boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}_{1}}\right\|_{f_{n}, \mathcal{K}}+\left\|\boldsymbol{v}_{2}\right\|_{f_{n}, \mathcal{K}}\left\|\boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}_{2}}\right\|_{f_{n}, \mathcal{K}}  \tag{5.8}\\
& \leq C\left(\left\|\boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}_{1}}\right\|_{f_{n}, \mathcal{K}}+\left\|\boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}_{2}}\right\|_{f_{n}, \mathcal{K}}\right) .
\end{align*}
$$

Applying eq. (5.5a) to the scalar functions $\boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}_{1}}$ and $\boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}_{2}}$, we have

$$
\begin{align*}
\left\|\boldsymbol{v}_{h}\right\|_{f_{n}, \mathcal{K}} & \leq C h_{K}^{1 / 2}\left(\left\|\boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}_{1}}\right\|_{f_{n}, \mathcal{F}_{1}}+\left\|\boldsymbol{v}_{h} \cdot \boldsymbol{n}_{\mathcal{F}_{2}}\right\|_{f_{n}, \mathcal{F}_{2}}\right)  \tag{5.9}\\
& \leq C h_{K}^{1 / 2}\left\|\boldsymbol{v}_{h} \cdot \boldsymbol{n}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}
\end{align*}
$$

proving the result.
We next prove existence and uniqueness of $\Pi_{h}$.
Lemma 5.2. The projection $\Pi_{h}$ defined by eq. (5.2) exists and is unique.
Proof. To see that $\Pi_{h}$ exists and is unique, we first verify that eq. (5.2) is a square system. First, recall that in two dimensions,

$$
\begin{equation*}
\operatorname{dim} P_{p}\left(K_{j}^{n}\right)=\frac{1}{2}(p+1)(p+2), \quad \operatorname{dim} Q_{p}(\mathcal{F})=(p+1)^{2} \tag{5.10}
\end{equation*}
$$

Thus,

$$
\begin{align*}
\operatorname{dim} W_{h}(\mathcal{K}) & =\frac{1}{2}(p+1)^{2}(p+2), \\
\operatorname{dim} \boldsymbol{V}_{h}(\mathcal{K}) & =(p+1)^{2}(p+2),  \tag{5.11}\\
\operatorname{dim} M_{h}(\mathcal{F}) & =(p+1)^{2}
\end{align*}
$$

Moreover,

$$
\begin{align*}
\operatorname{dim} \widetilde{W}_{h}(\mathcal{K}) & =\frac{1}{2} p(p+1)^{2} \\
\operatorname{dim} \widetilde{\boldsymbol{V}}_{h}(\mathcal{K}) & =p(p+1)^{2} \tag{5.12}
\end{align*}
$$

It follows that the number of unknowns in eq. (5.2) is

$$
\begin{equation*}
\operatorname{dim} W_{h}(\mathcal{K})+\operatorname{dim} \boldsymbol{V}_{h}(\mathcal{K})=\frac{3}{2}(p+1)^{2}(p+2) \tag{5.13}
\end{equation*}
$$

Since any space-time prism element $\mathcal{K}$ has only three faces in $\mathcal{Q}_{\mathcal{K}}$, the number of equations in eq. (5.2) is

$$
\begin{align*}
& \operatorname{dim} \widetilde{W}_{h}(\mathcal{K})+\operatorname{dim} \widetilde{\boldsymbol{V}}_{h}(\mathcal{K})+3 \operatorname{dim} M_{h}(\mathcal{F})  \tag{5.14}\\
& \quad=\frac{1}{2} p(p+1)^{2}+p(p+1)^{2}+3(p+1)^{2}=\frac{3}{2}(p+1)^{2}(p+2)
\end{align*}
$$

Since the number of equations and unknowns in eq. (5.2) coincide, eq. (5.2) is a square system.

Now, taking $\boldsymbol{q}=0$ and $v=0$ in eq. (5.2), we see that

$$
\begin{align*}
\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}, \boldsymbol{s}_{h} f_{n}\right)_{\mathcal{K}}=0 & \forall s_{h} \in \widetilde{\boldsymbol{V}}_{h}(\mathcal{K}),  \tag{5.15a}\\
\left(\Pi_{W} v, z_{h} f_{n}\right)_{\mathcal{K}}=0 & \forall z_{h} \in \widetilde{W}_{h}(\mathcal{K}),  \tag{5.15b}\\
\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}-\tau \Pi_{W} v, \sigma_{h} f_{n}\right\rangle_{\mathcal{F}}=0 & \forall \sigma_{h} \in M_{h}(\mathcal{F}), \mathcal{F} \subset \mathcal{Q}_{\mathcal{K}} \tag{5.15c}
\end{align*}
$$

Let $w_{h} \in W_{h}^{\perp}(\mathcal{K})$. By eq. (5.15a), since $\nabla w_{h} \in \tilde{\boldsymbol{V}}_{h}(\mathcal{K})$, we have
$\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}, \nabla w_{h} f_{n}\right)_{\mathcal{K}}=0$.
Applying integration by parts in space,

$$
\begin{equation*}
-\left(\nabla \cdot \boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}, w_{h} f_{n}\right)_{\mathcal{K}}+\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}, w_{h} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}=0 \tag{5.17}
\end{equation*}
$$

Since $\nabla \cdot \boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q} \in \widetilde{W}_{h}(\mathcal{K})$, then $\left(\nabla \cdot \boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}, w_{h} f_{n}\right)_{\mathcal{K}}=0$, thus

$$
\begin{equation*}
\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}, w_{h} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}=0 \quad \forall w_{h} \in W_{h}^{\perp}(\mathcal{K}) \tag{5.18}
\end{equation*}
$$

By eq. (5.15c) and recalling that $\tau>0$, we have

$$
\begin{equation*}
\left\langle\Pi_{W} v, w_{h} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}=0 \quad \forall w_{h} \in W_{h}^{\perp}(\mathcal{K}) \tag{5.19}
\end{equation*}
$$

Note that by eq. (5.15b), $\Pi_{W} v \in W_{h}^{\perp}(\mathcal{K})$. Thus, taking $w_{h}=\Pi_{W} v$ in eq. (5.19), we see that $\Pi_{W} v=0$ on $\mathcal{Q}_{\mathcal{K}}$. Then, using eq. (5.4a) we conclude that $\Pi_{W} v=0$ in $\mathcal{K}$. Taking $\sigma_{h}=\Pi_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}$ in eq. (5.15c), since $\Pi_{W} v=0$, we see that $\Pi_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}=0$ on $\mathcal{Q}_{\mathcal{K}}$. Using eq. (5.4b), we conclude that $\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}=0$ in $\mathcal{K}$. The result follows.

In addition to the projection $\Pi_{h}$, we define also $P_{M}^{f_{n}}$ as the $f_{n}$-weighted $L^{2}$ projection onto $M_{h}$, so that

$$
\begin{equation*}
\left\langle P_{M}^{f_{n}} v, \sigma_{h} f_{n}\right\rangle_{\mathcal{F}}=\left\langle v, \sigma_{h} f_{n}\right\rangle_{\mathcal{F}} \quad \forall \sigma_{h} \in M_{h}(\mathcal{F}), \mathcal{F} \subset \mathcal{Q}_{\mathcal{K}} \tag{5.20}
\end{equation*}
$$

Note that the domain of $P_{M}^{f_{n}}$ is $L^{2}\left(\Gamma_{\mathcal{Q}}^{n}\right)$.
We next show approximation properties of $\Pi_{h}$. For this, let $P_{W}^{f_{n}}, P_{\widetilde{W}}^{f_{n}}$, and $\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}}$ denote, respectively, the $f_{n}$-weighted $L^{2}$-projections onto $W_{h}, \widetilde{W}_{h}$, and $\boldsymbol{V}_{h}$.

Theorem 5.3 (Approximation properties of the projection). Assume that $\tau$ is uniformly bounded above and below by constants $C_{\tau}^{\max }$ and $C_{\tau}^{\min }$, respectively. The projection $\Pi_{h}$ satisfies the following bounds

$$
\begin{align*}
\left\|v-\Pi_{W} v\right\|_{f_{n}, \mathcal{K}} \leq & \left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \mathcal{K}}+C C_{\tau} h_{K}^{1 / 2}\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}} \\
& +\frac{C}{C_{\tau}^{\min }} h_{K}\left\|\nabla \cdot \boldsymbol{q}-P_{\widetilde{W}}^{f_{n}} \nabla \cdot \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}}  \tag{5.21a}\\
\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}} \leq & \left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}}+C h_{K}^{1 / 2}\left\|\left(\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}\right) \cdot \boldsymbol{n}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}  \tag{5.21b}\\
& +C C_{\tau}^{\max } h_{K}^{1 / 2}\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}} \\
\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \partial \mathcal{K}_{0}} \leq & \left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}\right\|_{f_{n}, \partial \mathcal{K}_{0}}+C \Delta t^{-1 / 2}\left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}}  \tag{5.21c}\\
& +C \Delta t^{-1 / 2}\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}}
\end{align*}
$$

where $C_{\tau}=C_{\tau}^{\max } / C_{\tau}^{\min }$.
Proof. Let $\boldsymbol{\delta}_{\boldsymbol{q}}:=\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}$ and $\delta_{v}:=\Pi_{W} v-P_{W}^{f_{n}} v$. Note that $\boldsymbol{\delta}_{\boldsymbol{q}}$ and $\delta_{v}$ satisfy the following equations

$$
\begin{align*}
\left(\boldsymbol{\delta}_{\boldsymbol{q}}, \boldsymbol{s}_{h} f_{n}\right)_{\mathcal{K}}=0 & \forall s_{h} \in \widetilde{\boldsymbol{V}}_{h}(\mathcal{K})  \tag{5.22a}\\
\left(\delta_{v}, z_{h} f_{n}\right)_{\mathcal{K}}=0 & \forall z_{h} \in \widetilde{W}_{h}(\mathcal{K}) \tag{5.22b}
\end{align*}
$$

$$
\begin{equation*}
\left\langle\left(\boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n}-\tau \delta_{v}\right), \sigma_{h} f_{n}\right\rangle_{\mathcal{F}}=\left\langle\left(\boldsymbol{I}_{\boldsymbol{q}} \cdot \boldsymbol{n}-\tau I_{v}\right), \sigma_{h} f_{n}\right\rangle_{\mathcal{F}} \quad \forall \sigma_{h} \in M_{h}(\mathcal{F}), \mathcal{F} \subset \mathcal{Q}_{\mathcal{K}}, \tag{5.22c}
\end{equation*}
$$

where $\boldsymbol{I}_{\boldsymbol{q}}=\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}$ and $I_{v}=v-P_{W}^{f_{n}} v$.
We first prove eq. (5.21a). Notice that for any $w_{h} \in W_{h}^{\perp}(\mathcal{K}),\left.w_{h}\right|_{\mathcal{F}} \in M_{h}(\mathcal{F})$, for any $\mathcal{F} \subset \mathcal{Q}_{\mathcal{K}}$. Therefore, by eq. (5.22c),

$$
\begin{equation*}
\left\langle\left(\boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n}-\tau \delta_{v}\right), w_{h} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}=\left\langle\left(\boldsymbol{I}_{\boldsymbol{q}} \cdot \boldsymbol{n}-\tau I_{v}\right), w_{h} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}} \quad \forall w_{h} \in W_{h}^{\perp}(\mathcal{K}) \tag{5.23}
\end{equation*}
$$

Using integration by parts in space, note that

$$
\begin{equation*}
\left\langle\boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n}, w_{h} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}=\left(\nabla \cdot \boldsymbol{\delta}_{\boldsymbol{q}}, w_{h} f_{n}\right)_{\mathcal{K}}+\left(\boldsymbol{\delta}_{\boldsymbol{q}}, f_{n} \nabla w_{h}\right)_{\mathcal{K}} \tag{5.24}
\end{equation*}
$$

Since $\nabla \cdot \boldsymbol{\delta}_{\boldsymbol{q}} \in \widetilde{W}_{h}(\mathcal{K})$ and $w_{h} \in W_{h}^{\perp}(\mathcal{K})$, then $\left(\nabla \cdot \boldsymbol{\delta}_{\boldsymbol{q}}, w_{h} f_{n}\right)_{\mathcal{K}}=0$. Also, since $\nabla w_{h} \in \widetilde{\boldsymbol{V}}_{h}(\mathcal{K})$, by eq. (5.22a), $\left(\boldsymbol{\delta}_{\boldsymbol{q}}, f_{n} \nabla w_{h}\right)_{\mathcal{K}}=0$. Thus,

$$
\begin{equation*}
\left\langle\boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n}, w_{h} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}=\quad \forall w_{h} \in W_{h}^{\perp}(\mathcal{K}) \tag{5.25}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
\left\langle\boldsymbol{I}_{\boldsymbol{q}} \cdot \boldsymbol{n}, w_{h} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}=\left(\nabla \cdot \boldsymbol{I}_{\boldsymbol{q}}, w_{h} f_{n}\right)_{\mathcal{K}}+\left(\boldsymbol{I}_{\boldsymbol{q}}, f_{n} \nabla w_{h}\right)_{\mathcal{K}} \quad \forall w_{h} \in W_{h}^{\perp}(\mathcal{K}) \tag{5.26}
\end{equation*}
$$

By definition of $\boldsymbol{I}_{\boldsymbol{q}}$, the second term on the right hand side is zero. Furthermore, note that

$$
\begin{equation*}
\left(\nabla \cdot \boldsymbol{I}_{\boldsymbol{q}}, w_{h} f_{n}\right)_{\mathcal{K}}=\left(\nabla \cdot \boldsymbol{q}, w_{h} f_{n}\right)_{\mathcal{K}}-\left(\nabla \cdot \boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}, w_{h} f_{n}\right)_{\mathcal{K}} \tag{5.27}
\end{equation*}
$$

and $\left(\nabla \cdot \boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}, w_{h} f_{n}\right)_{\mathcal{K}}=0$ since $\nabla \cdot \boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q} \in \widetilde{W}_{h}(\mathcal{K})$ and $w_{h} \in W_{h}^{\perp}(\mathcal{K})$. Therefore,

$$
\begin{equation*}
\left(\nabla \cdot \boldsymbol{I}_{\boldsymbol{q}}, w_{h} f_{n}\right)_{\mathcal{K}}=\left(\nabla \cdot \boldsymbol{q}, w_{h} f_{n}\right)_{\mathcal{K}} \tag{5.28}
\end{equation*}
$$

Also, since $P_{\widetilde{W}}^{f_{n}}(\nabla \cdot \boldsymbol{q}) \in \widetilde{W}_{h}(\mathcal{K})$ and $\left(\widetilde{w}_{h}, w_{h} f_{n}\right)_{\mathcal{K}}=0$ for all $\widetilde{w}_{h} \in \widetilde{W}_{h}(\mathcal{K})$, we can write

$$
\begin{equation*}
\left(\nabla \cdot \boldsymbol{I}_{\boldsymbol{q}}, w_{h} f_{n}\right)_{\mathcal{K}}=\left(\left(I d-P_{\widetilde{W}}^{f_{n}}\right) \nabla \cdot \boldsymbol{q}, w_{h} f_{n}\right)_{\mathcal{K}} \tag{5.29}
\end{equation*}
$$

where $I d$ denotes the identity operator. Thus,

$$
\begin{equation*}
\left\langle\boldsymbol{I}_{\boldsymbol{q}} \cdot \boldsymbol{n}, w_{h} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}=\left(\left(I d-P_{\widehat{W}}^{f_{n}}\right) \nabla \cdot \boldsymbol{q}, w_{h} f_{n}\right)_{\mathcal{K}} \quad \forall w_{h} \in W_{h}^{\perp}(\mathcal{K}) \tag{5.30}
\end{equation*}
$$

Using eq. (5.25) and eq. (5.30) in eq. (5.23) and rearranging terms, we obtain (5.31)

$$
\left\langle\tau \delta_{v}, w_{h} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}=\left\langle\tau I_{v}, w_{h} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}+\left(\left(P_{\widetilde{W}}^{f_{n}}-I d\right) \nabla \cdot \boldsymbol{q}, w_{h} f_{n}\right)_{\mathcal{K}} \quad \forall w_{h} \in W_{h}^{\perp}(\mathcal{K})
$$

By eq. (5.22b), $\delta_{v} \in W_{h}^{\perp}(\mathcal{K})$. Taking $w_{h}=\delta_{v}$ in eq. (5.31) we obtain

$$
\begin{equation*}
\left\langle\tau \delta_{v}, \delta_{v} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}=\left\langle\tau I_{v}, \delta_{v} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}+\left(\left(P_{\widetilde{W}}^{f_{n}}-I d\right) \nabla \cdot \boldsymbol{q}, \delta_{v} f_{n}\right)_{\mathcal{K}} \tag{5.32}
\end{equation*}
$$

Apply the Cauchy-Schwarz inequality to the right hand side of eq. (5.32):

$$
\begin{equation*}
\left\langle\tau \delta_{v}, \delta_{v} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}} \leq\left\|\tau I_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}\left\|\delta_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}+\left\|\left(I d-P_{\widetilde{W}}^{f_{n}}\right) \nabla \cdot \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}}\left\|\delta_{v}\right\|_{f_{n}, \mathcal{K}} \tag{5.33}
\end{equation*}
$$

Using eq. (5.5a) on the right hand side,

$$
\begin{equation*}
\left\langle\tau \delta_{v}, \delta_{v} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}} \leq\left\|\tau I_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}\left\|\delta_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}+C h_{K}^{1 / 2}\left\|\left(I d-P_{\widetilde{W}}^{f_{n}}\right) \nabla \cdot \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}}\left\|\delta_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}} \tag{5.34}
\end{equation*}
$$

Since $\tau$ is uniformly bounded above and below by constants $C_{\tau}^{\max }$ and $C_{\tau}^{\min }$, respectively,
$C_{\tau}^{\min }\left\|\delta_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}^{2} \leq C_{\tau}^{\max }\left\|I_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}\left\|\delta_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}+C h_{K}^{1 / 2}\left\|\left(I d-P{ }_{\widetilde{W}}^{f_{n}}\right) \nabla \cdot \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}}\left\|\delta_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}$.
Canceling terms and using eq. (5.5a) on the left hand side, we obtain the following bound

$$
\begin{equation*}
\frac{h_{K}^{-1 / 2}}{C} C_{\tau}^{\min }\left\|\delta_{v}\right\|_{f_{n}, \mathcal{K}} \leq C_{\tau}^{\max }\left\|I_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}+C h_{K}^{1 / 2}\left\|\left(I d-P \frac{f_{n}}{f_{W}}\right) \nabla \cdot \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}} \tag{5.36}
\end{equation*}
$$

Rearranging,

$$
\begin{equation*}
\left\|\delta_{v}\right\|_{f_{n}, \mathcal{K}} \leq C C_{\tau} h_{K}^{1 / 2}\left\|I_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}+\frac{C}{C_{\tau}^{\min }} h_{K}\left\|\left(I d-P \widetilde{W}_{\widetilde{W}}^{f_{n}}\right) \nabla \cdot \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}} \tag{5.37}
\end{equation*}
$$

The estimate eq. (5.21a) follows by applying the triangle inequality.
We next prove eq. (5.21b). We first find an estimate for $\boldsymbol{\delta}_{\boldsymbol{q}}$. Note that by eq. (5.22a), $\boldsymbol{\delta}_{\boldsymbol{q}}$ belongs to $\boldsymbol{V}_{h}^{\perp}(\mathcal{K})$. Therefore, by eq. (5.5b), we have

$$
\begin{equation*}
\left\|\boldsymbol{\delta}_{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{K}} \leq C h_{K}^{1 / 2}\left\|\boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}} \tag{5.38}
\end{equation*}
$$

Taking $\sigma_{h}=\boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n}$ in eq. (5.22c), we obtain

$$
\begin{equation*}
\left\|\boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}^{2}=\left\langle\boldsymbol{I}_{\boldsymbol{q}} \cdot \boldsymbol{n}, \boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}-\tau\left\langle I_{v}, \boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}+\tau\left\langle\delta_{v}, \boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}} \tag{5.39}
\end{equation*}
$$

Since $\delta_{v} \in W_{h}^{\perp}(\mathcal{K})$, by eq. (5.25), $\left\langle\delta_{v}, \boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}=0$. Thus,

$$
\begin{equation*}
\left\|\boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}^{2}=\left\langle\boldsymbol{I}_{\boldsymbol{q}} \cdot \boldsymbol{n}, \boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}}-\tau\left\langle I_{v}, \boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{Q}_{\mathcal{K}}} \tag{5.40}
\end{equation*}
$$

Applying the Cauchy-Schwarz inequality and substituting into eq. (5.38), we obtain

$$
\begin{equation*}
\left\|\boldsymbol{\delta}_{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{K}} \leq C h_{K}^{1 / 2}\left(\left\|\boldsymbol{I}_{\boldsymbol{q}} \cdot \boldsymbol{n}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}+C_{\tau}^{\max }\left\|I_{v}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}}\right) \tag{5.41}
\end{equation*}
$$

The estimate eq. (5.21b) follows by applying the triangle inequality.
Finally, we show eq. (5.21c). Note that, by the triangle inequality,

$$
\begin{equation*}
\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \partial \mathcal{K}_{0}} \leq\left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}\right\|_{f_{n}, \partial \mathcal{K}_{0}}+\left\|\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \partial \mathcal{K}_{0}} \tag{5.42}
\end{equation*}
$$

Next, we apply the inverse trace inequality in eq. (4.21b) to the second term on the right hand side to obtain:

$$
\begin{equation*}
\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \partial \mathcal{K}_{0}} \leq\left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}\right\|_{f_{n}, \partial \mathcal{K}_{0}}+C \Delta t^{-1 / 2}\left\|\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}} . \tag{5.43}
\end{equation*}
$$

The result follows after adding and subtracting $\boldsymbol{q}$ to the second term on the right hand side and applying the triangle inequality.

We next prove the equivalence between the standard and the weighted $L^{2}$ projections.

Lemma 5.4. Let $\mathcal{K} \in \mathcal{T}^{n}$ and $\mathcal{F} \in \mathcal{F}_{\mathcal{Q}}^{n}$. Let $P_{W}, \boldsymbol{P}_{\boldsymbol{V}}$ and $P_{M}$ denote the $L^{2}$ orthogonal projections onto $W_{h}, \boldsymbol{V}_{h}$ and $M_{h}$, respectively. If $f_{n}$ is uniformly bounded, then the following relations are satisfied

$$
\begin{align*}
& \left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \mathcal{K}} \leq C\left\|v-P_{W} v\right\|_{\mathcal{K}}  \tag{5.44a}\\
& \left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}} \leq C\left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{\mathcal{K}}  \tag{5.44b}\\
& \left\|v-P_{M}^{f_{n}} v\right\|_{f_{n}, \mathcal{F}} \leq C\left\|v-P_{M} v\right\|_{\mathcal{F}} . \tag{5.44c}
\end{align*}
$$

Proof. We will only show eq. (5.44a). The proofs for eq. (5.44b) and eq. (5.44c) are analogous. Note that by definition of $P_{W}^{f_{n}}$, for all $w_{h} \in W_{h}$

$$
\begin{equation*}
\left(P_{W}^{f_{n}} v-P_{W} v, f_{n} w_{h}\right)_{\mathcal{K}}=\left(v-P_{W} v, f_{n} w_{h}\right)_{\mathcal{K}} \tag{5.45}
\end{equation*}
$$

Let $w_{h}=P_{W}^{f_{n}} v-P_{W} v$, then,

$$
\begin{align*}
\left\|P_{W}^{f_{n}} v-P_{W} v\right\|_{f_{n}, \mathcal{K}}^{2} & =\left(v-P_{W} v, f_{n}\left(P_{W}^{f_{n}} v-P_{W} v\right)\right)_{\mathcal{K}}  \tag{5.46}\\
& \leq\left\|v-P_{W} v\right\|_{f_{n}, \mathcal{K}}\left\|P_{W}^{f_{n}} v-P_{W} v\right\|_{f_{n}, \mathcal{K}}
\end{align*}
$$

Thus,

$$
\begin{equation*}
\left\|P_{W}^{f_{n}} v-P_{W} v\right\|_{f_{n}, \mathcal{K}} \leq\left\|v-P_{W} v\right\|_{f_{n}, \mathcal{K}} \leq C\left\|v-P_{W} v\right\|_{\mathcal{K}} \tag{5.47}
\end{equation*}
$$

since $f_{n}$ is uniformly bounded. The result follows by the triangle inequality.
In order to obtain an equivalence between the standard and the weighted $L^{2}$ projections on the boundary $\mathcal{Q}_{\mathcal{K}}$ of a space-time element $\mathcal{K}$, we require the following continuous trace inequality:
Lemma 5.5 (Continuous trace inequality). Let $\mathcal{K}=K \times I_{n}$. For $\phi \in H^{(0,1)}(\mathcal{K})$, the following holds:

$$
\begin{equation*}
\|\phi\|_{\mathcal{Q}_{\mathcal{K}}}^{2} \leq C\left(\|\nabla \phi\|_{\mathcal{K}}+h_{K}^{-1}\|\phi\|_{\mathcal{K}}\right)\|\phi\|_{\mathcal{K}} . \tag{5.48}
\end{equation*}
$$

Proof. The proof is analogous to the proof of [20, Lemma 1.49].
We next find $L^{2}$ projection esimates of the different projection operators.
Theorem 5.6 ( $L^{2}$ orthogonal projection estimates). Let $\mathcal{K}=K \times I_{n}$ and $\mathcal{F}$ be a face on the free-surface boundary, i.e., $\mathcal{F} \in \mathcal{F}_{\mathcal{S}}^{n}$. Let $\partial \mathcal{F}_{0}$ denote the two edges of the face $\mathcal{F}$ that are on the time levels. Assume that the spatial shape-regularity condition eq. (4.5) holds and that the triangulation $\mathcal{T}^{n}$ does not have any hanging
nodes. Suppose that $(\boldsymbol{q}, v)$ are such that $\left.\boldsymbol{q}\right|_{\mathcal{K}} \in\left[H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})\right]^{2},\left.v\right|_{\mathcal{K}} \in H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})$, where $1 / 2<s_{t} \leq p+1$ and $1 \leq s_{s} \leq p+1$. Then we have the following estimates:

$$
\begin{gather*}
\left\|v-P_{W} v\right\|_{\mathcal{K}} \leq C\left(h_{K}^{s_{s}}+\Delta t^{s_{t}}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.49a}\\
\left\|v-P_{W} v\right\|_{\mathcal{Q}_{\mathcal{K}}} \leq C\left(h_{K}^{s_{s}-1 / 2}+h_{K}^{-1 / 2} \Delta t^{s_{t}}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.49b}\\
\left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{\mathcal{K}} \leq C\left(h_{K}^{s_{s}}+\Delta t^{s_{t}}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.49c}\\
\left\|\left(\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}\right) \cdot \boldsymbol{n}\right\|_{\mathcal{Q}_{\mathcal{K}}} \leq C\left(h_{K}^{s_{s}-1 / 2}+h_{K}^{-1 / 2} \Delta t^{s_{t}}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.49d}\\
\left\|v-P_{W} v\right\|_{\partial \mathcal{K}_{0}} \leq C\left(\Delta t^{-1 / 2} h_{K}^{s_{s}}+\Delta t^{s_{t}-1 / 2}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.49e}\\
\left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{\partial \mathcal{K}_{0}} \leq C\left(\Delta t^{-1 / 2} h_{K}^{s_{s}}+\Delta t^{s_{t}-1 / 2}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.49f}\\
\left\|v-P_{M} v\right\|_{\mathcal{F}} \leq C\left(h_{K}^{s_{s}}+\Delta t^{s_{t}}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{F})}  \tag{5.49~g}\\
\left\|v-P_{M} v\right\|_{\partial \mathcal{F}_{0}} \leq C\left(\Delta t^{-1 / 2} h_{K}^{s_{s}}+\Delta t^{s_{t}-1 / 2}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)(\mathcal{F})}} \tag{5.49h}
\end{gather*}
$$

Proof. First note that eq. (5.49c) and eq. (5.49f) result from applying eq. (5.49a) and eq. (5.49e), respectively, on each component of $\boldsymbol{q}$, so the proof for those estimates is not shown. Since the face $\mathcal{F}$ is a quadrilateral, the proof for eq. ( 5.49 g ) and eq. $(5.49 \mathrm{~h})$ can be found in [26, Lemma B.14].

Let $\pi^{t}$ and $\pi^{s}$ denote the orthogonal $L^{2}$-projections onto $L^{2}(K) \otimes P_{p}\left(I_{n}\right)$ and onto $P_{p}(K) \otimes L^{2}\left(I_{n}\right)$, respectively. We can define $P_{W}$ as $P_{W}:=\pi^{t} \circ \pi^{s}$.

We first show eq. (5.49a). Notice that

$$
\begin{align*}
\left\|v-P_{W} v\right\|_{\mathcal{K}}^{2}=\left\|v-\pi^{t} \circ \pi^{s} v\right\|_{\mathcal{K}}^{2}= & \left\|v-\pi^{t} v+\pi^{t}\left(v-\pi^{s} v\right)\right\|_{\mathcal{K}}^{2}  \tag{5.50}\\
& \leq C\left(\left\|v-\pi^{t} v\right\|_{\mathcal{K}}^{2}+\left\|\pi^{t}\left(v-\pi^{s} v\right)\right\|_{\mathcal{K}}^{2}\right)
\end{align*}
$$

Since $\pi^{t}$ is bounded and $\left\|\pi^{t}\right\|=1$, then

$$
\begin{equation*}
\left\|v-P_{W} v\right\|_{\mathcal{K}}^{2} \leq C\left(\left\|v-\pi^{t} v\right\|_{\mathcal{K}}^{2}+\left\|v-\pi^{s} v\right\|_{\mathcal{K}}^{2}\right) \tag{5.51}
\end{equation*}
$$

Let us treat each term separately. For the second term on the right hand side of eq. (5.51), by [20, Lemma 1.58], since we assume eq. (4.5) and no hanging nodes,

$$
\begin{align*}
\left\|v-\pi^{s} v\right\|_{\mathcal{K}}^{2}=\int_{t_{n}}^{t_{n+1}} \int_{K}\left(v-\pi^{s} v\right)^{2} \mathrm{~d} \boldsymbol{x} \mathrm{~d} x_{0} \leq C h_{K}^{2 s_{s}} & \int_{t_{n}}^{t_{n+1}}|v|_{H^{s_{s}(K)}}^{2} \mathrm{~d} x_{0}  \tag{5.52}\\
& \leq C h_{K}^{2 s_{s}}\|v\|_{H^{\left(0, s_{s}\right)}(\mathcal{K})}^{2}
\end{align*}
$$

where $0 \leq s_{s} \leq p+1$. Similarly, for the temporal projection we have

$$
\begin{equation*}
\left\|v-\pi^{t} v\right\|_{\mathcal{K}}^{2}=\int_{K} \int_{t_{n}}^{t_{n+1}}\left(v-\pi^{t} v\right)^{2} \mathrm{~d} x_{0} \mathrm{~d} \boldsymbol{x} \leq C \Delta t^{2 s_{t}}\|v\|_{H^{\left(s_{t}, 0\right)}(\mathcal{K})}^{2} \tag{5.53}
\end{equation*}
$$

where $0 \leq s_{t} \leq p+1$. Equation (5.49a) follows by combining eq. (5.52) and eq. (5.53).

Next, we show eq. (5.49b). Similarly as above, we have

$$
\begin{equation*}
\left\|v-P_{W} v\right\|_{\mathcal{Q}_{\mathcal{K}}}^{2} \leq C\left(\left\|v-\pi^{t} v\right\|_{\mathcal{Q}_{\mathcal{K}}}^{2}+\left\|v-\pi^{s} v\right\|_{\mathcal{Q}_{\mathcal{K}}}^{2}\right) \tag{5.54}
\end{equation*}
$$

For the spatial projection, using [20, Lemma 1.59], we have

$$
\begin{align*}
\left\|v-\pi^{s} v\right\|_{\mathcal{Q}_{\mathcal{K}}}^{2}= & \int_{t_{n}}^{t_{n+1}} \int_{\partial K}\left(v-\pi^{s} v\right)^{2} \mathrm{~d} \boldsymbol{x} \mathrm{~d} x_{0}  \tag{5.55}\\
& \leq C h_{K}^{2 s_{s}-1} \int_{t_{n}}^{t_{n+1}}|v|_{H^{s_{s}(K)}}^{2} \mathrm{~d} x_{0} \leq C h_{K}^{2 s_{s}-1}\|v\|_{H^{\left(0, s_{s}\right)}(\mathcal{K})}^{2}
\end{align*}
$$

For the temporal projection, similarly as above,

$$
\begin{equation*}
\left\|v-\pi^{t} v\right\|_{\mathcal{Q}_{\mathcal{K}}}^{2}=\int_{\partial K} \int_{t_{n}}^{t_{n+1}}\left(v-\pi^{t} v\right)^{2} \mathrm{~d} x_{0} \mathrm{~d} \boldsymbol{x} \leq C \Delta t^{2 s_{t}}\|v\|_{H^{\left(s_{t}, 0\right)}\left(\mathcal{Q}_{\mathcal{K}}\right)}^{2} \tag{5.56}
\end{equation*}
$$

Combining eq. (5.55) and eq. (5.56), we obtain

$$
\begin{equation*}
\left\|v-P_{W} v\right\|_{\mathcal{Q}_{\mathcal{K}}}^{2} \leq C h_{K}^{2 s_{s}-1}\|v\|_{H^{\left(0, s_{s}\right)}(\mathcal{K})}^{2}+C \Delta t^{2 s_{t}}\|v\|_{H^{\left(s_{t}, 0\right)}\left(\mathcal{Q}_{\mathcal{K}}\right)}^{2} \tag{5.57}
\end{equation*}
$$

Note that, by Lemma 5.5 , since $v \in H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})$ with $s_{s} \geq 1$, we have

$$
\begin{equation*}
\left\|\partial_{x_{0}}^{\alpha_{t}} v\right\|_{\mathcal{Q}_{\mathcal{K}}}^{2} \leq C\left\|\nabla \partial_{x_{0}}^{\alpha_{t}} v\right\|_{\mathcal{K}}\left\|\partial_{x_{0}}^{\alpha_{t}} v\right\|_{\mathcal{K}}+C h_{K}^{-1}\left\|\partial_{x_{0}}^{\alpha_{t}} v\right\|_{\mathcal{K}}^{2}, \tag{5.58}
\end{equation*}
$$

for all $0 \leq \alpha_{t} \leq s_{t}$. Thus,

$$
\begin{equation*}
\|v\|_{H^{\left(s_{t}, 0\right)}\left(\mathcal{Q}_{\mathcal{K}}\right)}^{2} \leq C\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}^{2}+C h_{K}^{-1}\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}^{2} \tag{5.59}
\end{equation*}
$$

and the result follows.
To show eq. (5.49d), we note the following

$$
\begin{equation*}
\left\|\left(\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}\right) \cdot \boldsymbol{n}\right\|_{\mathcal{Q}_{\mathcal{K}}} \leq\left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{\mathcal{Q}_{\mathcal{K}}} . \tag{5.60}
\end{equation*}
$$

The result follows by applying eq. (5.49b) component wise.
Finally, we show eq. (5.49e). For the spatial component of the projection, notice that

$$
\begin{align*}
\left\|v-\pi^{s} v\right\|_{\partial \mathcal{K}_{0}}^{2} & =\left\|v-\pi^{s} v\right\|_{K_{j}^{n+1}}^{2}+\left\|v-\pi^{s} v\right\|_{K_{j}^{n}}^{2} \\
& \leq C h_{K}^{2 s_{s}}\left(|v|_{H^{s_{s}}\left(K_{j}^{n+1}\right)}^{2}+|v|_{H^{s_{s}}\left(K_{j}^{n}\right)}^{2}\right)  \tag{5.61}\\
& \leq C h_{K}^{2 s_{s}}\|v\|_{H^{s_{s}}\left(\partial \mathcal{K}_{0}\right)}^{2}
\end{align*}
$$

By the Sobolev embedding theorem, the definition of fractional Sobolev norms [7, Chapter 14], and a standard scaling argument, we have for $s_{t}>1 / 2$,

$$
\begin{equation*}
\|v\|_{H^{s_{s}}\left(\partial \mathcal{K}_{0}\right)} \leq C \Delta t^{-1 / 2}\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})} \tag{5.62}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\left\|v-\pi^{s} v\right\|_{\partial \mathcal{K}_{0}}^{2} \leq C \Delta t^{-1} h_{K}^{2 s_{s}}\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}^{2} \tag{5.63}
\end{equation*}
$$

For the temporal projection, we have

$$
\begin{array}{rlrl}
\left\|v-\pi^{t} v\right\|_{\partial \mathcal{K}_{0}}^{2} & =\left\|v-\pi^{t} v\right\|_{K_{j}^{n+1}}^{2}+\left\|v-\pi^{t} v\right\|_{K_{j}^{n}}^{2} \\
& =\int_{K}\left(\left(v-\pi^{t} v\right)\left(t_{n+1}, \boldsymbol{x}\right)\right)^{2} \mathrm{~d} \boldsymbol{x}+\int_{K}\left(\left(v-\pi^{t} v\right)\left(t_{n}, \boldsymbol{x}\right)\right)^{2} \mathrm{~d} \boldsymbol{x} & & \text { (by def.) } \\
& \leq h_{K}^{2} \int_{\widehat{K}}\left(\left(v-\pi^{t} v\right)\left(t_{n+1}, \widehat{\boldsymbol{x}}\right)\right)^{2} \mathrm{~d} \widehat{\boldsymbol{x}}+h_{K}^{2} \int_{\widehat{K}}\left(\left(v-\pi^{t} v\right)\left(t_{n}, \widehat{\boldsymbol{x}}\right)\right)^{2} \mathrm{~d} \widehat{\boldsymbol{x}} & & \text { (by eq. (4.17a)) } \\
& =h_{K}^{2}\left\|\check{v}-\pi^{t} \check{v}\right\|_{\partial \check{\mathcal{K}}_{0}}^{2} & & \text { (by def.) } \\
& =h_{K}^{2}\left\|\widehat{v}-\widehat{\pi}^{t} \widehat{v}\right\|_{\partial \widehat{\mathcal{K}}_{0}}^{2} & & \text { (by eq. (4.7b)) } \\
& =h_{K}^{2} \int_{\widehat{K}}\left(\left(\widehat{v}-\widehat{\pi}^{t} \widehat{v}\right)(1, \widehat{\boldsymbol{x}})\right)^{2} \mathrm{~d} \widehat{\boldsymbol{x}}+h_{K}^{2} \int_{\widehat{K}}\left(\left(\widehat{v}-\widehat{\pi}^{t} \widehat{v}\right)(-1, \widehat{\boldsymbol{x}})\right)^{2} \mathrm{~d} \widehat{\boldsymbol{x}} & & \text { (by def.) } \\
& \leq C h_{K}^{2}\left\|\partial_{\widehat{x}_{0}}^{s_{t}} \widehat{v}\right\|_{\widehat{\mathcal{K}}}^{2} \\
& =C h_{K}^{2}\left(\frac{\Delta t}{2}\right)^{2 s_{t}-1}\left\|\partial_{\check{x}_{0}}^{s_{t}} \check{v}\right\|_{\check{\mathcal{K}}}^{2} & & \text { (by [15]) } \\
& \leq C h_{K}^{2}\left(\frac{\Delta t}{2}\right)^{2 s_{t}-1} h_{K}^{-2}\left\|\partial_{x_{0}}^{s_{t}} v\right\|_{\mathcal{K}}^{2} \\
& \leq C \Delta t^{2 s_{t}-1}\|v\|_{H^{\left(s_{t}, 0\right)}(\mathcal{K}) .}^{2} . & \text { (by eq. (4.7a)) } \tag{4.17a}
\end{array}
$$

Note that here we have used that $\left|\widehat{v}-\widehat{\pi}^{t} \widehat{v}( \pm 1)\right| \leq C\left\|\partial_{\widehat{x}_{0}}^{s_{t}} \widehat{v}\right\|_{\widehat{I}}$ which was shown in [15, Lemma 3.5]. This concludes the proof.

Corollary 5.7. Suppose that $f_{n}$ is uniformly bounded. Then, under the same assumptions as in Theorem 5.6, the following estimates are satisfied:

$$
\begin{gather*}
\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \mathcal{K}} \leq C\left(h_{K}^{s_{s}}+\Delta t^{s_{t}}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.65a}\\
\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}} \leq C\left(h_{K}^{s_{s}-1 / 2}+h_{K}^{-1 / 2} \Delta t^{s_{t}}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.65b}\\
\left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}} \leq C\left(h_{K}^{s_{s}}+\Delta t^{s_{t}}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.65c}\\
\left\|\left(\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}\right) \cdot \boldsymbol{n}\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}} \leq C\left(h_{K}^{s_{s}-1 / 2}+h_{K}^{-1 / 2} \Delta t^{s_{t}}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.65d}\\
\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \partial \mathcal{K}_{0}} \leq C\left(\Delta t^{-1 / 2} h_{K}^{s_{s}}+\Delta t^{s_{t}-1 / 2}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.65e}\\
\left\|\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}}^{f_{n}} \boldsymbol{q}\right\|_{f_{n}, \partial \mathcal{K}_{0}} \leq C\left(\Delta t^{-1 / 2} h_{K}^{s_{s}}+\Delta t^{s_{t}-1 / 2}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.65f}\\
\left\|v-P_{M}^{f_{n}} v\right\|_{f_{n}, \mathcal{F}} \leq C\left(h_{K}^{s_{s}}+\Delta t^{s_{t}}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{F})}  \tag{5.65~g}\\
\left\|v-P_{M}^{f_{n}} v\right\|_{f_{n}, \partial \mathcal{F}_{0}} \leq C\left(\Delta t^{-1 / 2} h_{K}^{s_{s}}+\Delta t^{s_{t}-1 / 2}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{F})} \tag{5.65h}
\end{gather*}
$$

Proof. Equation (5.65a), eq. (5.65c) and eq. (5.65g) follow directly from Lemma 5.4 and Theorem 5.6. Moreover, eq. (5.65d) and eq. (5.65f) follow from eq. (5.65b) and eq. ( 5.65 e ), respectively. Therefore, we only show eq. ( 5.65 b ), eq. ( 5.65 e ) and eq. (5.65h).

In order to prove eq. (5.65b), notice that since $f_{n}$ is uniformly bounded and using the triangle inequality, we obtain

$$
\begin{equation*}
\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}} \leq C\left\|v-P_{W} v\right\|_{\mathcal{Q}_{\mathcal{K}}}+C\left\|P_{W} v-P_{W}^{f_{n}} v\right\|_{\mathcal{Q}_{\mathcal{K}}} \tag{5.66}
\end{equation*}
$$

By eq. (4.21c), we obtain

$$
\begin{equation*}
\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}} \leq C\left\|v-P_{W} v\right\|_{\mathcal{Q}_{\mathcal{K}}}+C h_{K}^{-1 / 2}\left\|P_{W} v-P_{W}^{f_{n}} v\right\|_{\mathcal{K}} \tag{5.67}
\end{equation*}
$$

Using equivalence of norms in finite dimensional spaces, and by the triangle inequality, we have

$$
\begin{equation*}
\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \mathcal{Q}_{\mathcal{K}}} \leq C\left\|v-P_{W} v\right\|_{\mathcal{Q}_{\mathcal{K}}}+C h_{K}^{-1 / 2}\left\|P_{W} v-v\right\|_{f_{n}, \mathcal{K}}+C h_{K}^{-1 / 2}\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \mathcal{K}} \tag{5.68}
\end{equation*}
$$

Equation (5.65b) follows by recalling that $f_{n}$ is uniformly bounded and using the estimates eq. (5.49b), eq. (5.49a) and eq. (5.65a).

In order to show eq. (5.65e), since $f_{n}$ is uniformly bounded, we obtain by the triangle inequality

$$
\begin{equation*}
\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \partial \mathcal{K}_{0}} \leq C\left\|v-P_{W} v\right\|_{\partial \mathcal{K}_{0}}+C\left\|P_{W} v-P_{W}^{f_{n}} v\right\|_{\partial \mathcal{K}_{0}} \tag{5.69}
\end{equation*}
$$

Using eq. (4.21b) on the second term of the right hand side of eq. (5.69),

$$
\begin{equation*}
\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \partial \mathcal{K}_{0}} \leq C\left\|v-P_{W} v\right\|_{\partial \mathcal{K}_{0}}+C \Delta t^{-1 / 2}\left\|P_{W} v-P_{W}^{f_{n}} v\right\|_{\mathcal{K}} \tag{5.70}
\end{equation*}
$$

By the triangle inequality,

$$
\begin{equation*}
\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \partial \mathcal{K}_{0}} \leq C\left\|v-P_{W} v\right\|_{\partial \mathcal{K}_{0}}+C \Delta t^{-1 / 2}\left\|P_{W} v-v\right\|_{f_{n}, \mathcal{K}}+C \Delta t^{-1 / 2}\left\|v-P_{W}^{f_{n}} v\right\|_{f_{n}, \mathcal{K}} \tag{5.71}
\end{equation*}
$$

Then, eq. (5.65e) is obtained by eq. (5.49e), eq. (5.65a) and eq. (5.49a).
Finally, we show eq. (5.65h). Using that $f_{n}$ is uniformly bounded and by the triangle inequality,

$$
\begin{equation*}
\left\|v-P_{M}^{f_{n}} v\right\|_{f_{n}, \partial \mathcal{F}_{0}} \leq C\left\|v-P_{M} v\right\|_{\partial \mathcal{F}_{0}}+C\left\|P_{M} v-P_{M}^{f_{n}} v\right\|_{\partial \mathcal{F}_{0}} . \tag{5.72}
\end{equation*}
$$

By eq. (4.21a),

$$
\begin{equation*}
\left\|v-P_{M}^{f_{n}} v\right\|_{f_{n}, \partial \mathcal{F}_{0}} \leq C\left\|v-P_{M} v\right\|_{\partial \mathcal{F}_{0}}+C \Delta t^{-1 / 2}\left\|P_{M} v-P_{M}^{f_{n}} v\right\|_{\mathcal{F}} \tag{5.73}
\end{equation*}
$$

By the triangle inequality, we then obtain:
$\left\|v-P_{M}^{f_{n}} v\right\|_{f_{n}, \partial \mathcal{F}_{0}} \leq C\left\|v-P_{M} v\right\|_{\partial \mathcal{F}_{0}}+C \Delta t^{-1 / 2}\left\|P_{M} v-v\right\|_{f_{n}, \mathcal{F}}+C \Delta t^{-1 / 2}\left\|v-P_{M}^{f_{n}} v\right\|_{f_{n}, \mathcal{F}}$.
Equation (5.65h) follows by the uniform boundedness of $f_{n}$ and the estimates eq. ( 5.49 h ), eq. $(5.49 \mathrm{~g})$, and eq. $(5.65 \mathrm{~g})$.

To conclude this subsection, we show the error estimates of our projection $\Pi_{h}$.
Lemma 5.8. Assume that $\tau$ and $f_{n}$ are uniformly bounded. Under the same conditions as in Theorem 5.6, the following estimates hold:

$$
\begin{align*}
\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{K}} & \leq C\left(h_{K}^{s_{s}}+\Delta t^{s_{t}}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.75a}\\
& +C\left(h_{K}^{s_{s}}+\Delta t^{s_{t}}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})} \\
\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \partial \mathcal{K}_{0}} & \leq C\left(\Delta t^{-1 / 2} h_{K}^{s_{s}}+\Delta t^{s_{t}-1 / 2}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}  \tag{5.75b}\\
& +C\left(h_{K}^{s_{s}} \Delta t^{-1 / 2}+\Delta t^{s_{t}-1 / 2}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{K})}
\end{align*}
$$

Proof. These estimates follow from substituting eq. (5.65b), eq. (5.65c), eq. (5.65d), and eq. (5.65f) in Theorem 5.3.
5.2. The a priori error estimates. In this section we present the main result of this paper, namely a priori error estimates for the space-time HDG method eq. (3.6).

In order to obtain a priori error estimates, we first require to obtain the error equations and a bound for the projection errors. Define these projection errors as

$$
\begin{array}{ll}
\varepsilon_{h}^{\boldsymbol{q}}=\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}_{h}, \quad \varepsilon_{h}^{\boldsymbol{q}}=\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}_{h}^{-}, \quad \varepsilon_{h}^{v}=\Pi_{W} v-v_{h} \\
\varepsilon_{h}^{\lambda}=P_{M}^{f_{n}} v-\lambda_{h}, \quad \varepsilon_{h}^{\lambda-}=P_{M^{-}}^{f_{n-1}} v-\lambda_{h}^{-} \tag{5.76}
\end{array}
$$

where $\boldsymbol{\Pi}_{\boldsymbol{V}^{-}}$and $P_{M^{-}}^{f_{n-1}}$ denote the projection onto the spaces $\boldsymbol{V}_{h}$ and $M_{h}$, respectively, defined on the time slab $n-1$ for $n>0$. We furthermore remark that $\boldsymbol{\varepsilon}_{\boldsymbol{h}}^{\boldsymbol{q -}}$ $=0$ and $\varepsilon_{h}^{\lambda-}=0$ when $n=0$.

The following lemma describes the error equations.
Lemma 5.9. The error equations are given by

$$
\begin{align*}
& \quad\left(\varepsilon_{h}^{v}, f_{n} \nabla \cdot \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left(\varepsilon_{h}^{\boldsymbol{q}}, \boldsymbol{r}_{h} f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left\langle\varepsilon_{h}^{\boldsymbol{q}}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)} \\
& \quad-\left(\varepsilon_{h}^{\boldsymbol{q}}, f_{n} \partial_{t} \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left\langle\varepsilon_{h}^{\lambda}, \boldsymbol{r}_{h} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}  \tag{5.77a}\\
& =\left\langle\varepsilon_{h}^{\boldsymbol{q}-}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}-\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, f_{n} \partial_{t} \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}^{\prime}\right)_{\mathcal{T}^{n}} \\
& +\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}-\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)} \\
& \quad\left(\varepsilon_{h}^{\boldsymbol{q}}, f_{n} \nabla w_{h}\right)_{\mathcal{T}^{n}}-\left\langle\widehat{\varepsilon}_{h} \cdot \boldsymbol{n}, w_{h} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}=0,  \tag{5.77b}\\
& \left\langle\widehat{\varepsilon}_{h} \cdot \boldsymbol{n}, \mu_{h} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}-\left\langle\varepsilon_{h}^{\lambda}, f_{n} \partial_{t} \mu_{h}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}-\left\langle\varepsilon_{h}^{\lambda}, \mu_{h} f_{n}^{\prime}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}+\left\langle\left\langle\varepsilon_{h}^{\lambda}, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)} \\
& =  \tag{5.77c}\\
& =\left\langle\left\langle\varepsilon_{h}^{\lambda-}, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}+\left\langle\left\langle P_{M}^{f_{n}} v-v, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)} \\
& \quad-\left\langle\left\langle P_{M_{n-1}^{-1}}^{f_{n}} v-v, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}
\end{align*}
$$

where $\widehat{\varepsilon}_{h}=\boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}}-\tau\left(\varepsilon_{h}^{v}-\varepsilon_{h}^{\lambda}\right) \boldsymbol{n}$.
Proof. Substituting the exact solution ( $\boldsymbol{q}, v$ ) to eq. (3.1) into the space-time HDG method eq. (3.6), we find:

$$
\begin{align*}
&-\left(\boldsymbol{q}, f_{n} \partial_{t} \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left(\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left\langle\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}  \tag{5.78a}\\
&+\left(v, f_{n} \nabla \cdot \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left\langle v, \boldsymbol{r}_{h} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}=\left\langle\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)} \\
& \quad-\left(w_{h}, f_{n} \nabla \cdot \boldsymbol{q}\right)_{\mathcal{T}^{n}}=0,  \tag{5.78b}\\
&\left\langle\boldsymbol{q} \cdot \boldsymbol{n}, \mu_{h} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}-\left\langle v, f_{n} \partial_{t} \mu_{h}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}-\left\langle v, \mu_{h} f_{n}^{\prime}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}+\left\langle\left\langle v, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}  \tag{5.78c}\\
&=\left\langle\left\langle v, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}
\end{align*}
$$

Subtracting now (3.6) from (5.78), we obtain

$$
\begin{gather*}
-\left(\boldsymbol{q}-\boldsymbol{q}_{h}, f_{n} \partial_{t} \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left(\boldsymbol{q}-\boldsymbol{q}_{h}, \boldsymbol{r}_{h} f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left\langle\boldsymbol{q}-\boldsymbol{q}_{h}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}  \tag{5.79a}\\
+\left(v-v_{h}, f_{n} \nabla \cdot \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left\langle v-\lambda_{h}, \boldsymbol{r}_{h} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}=\left\langle\boldsymbol{q}-\boldsymbol{q}_{h}^{-}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)} \\
\quad-\left(w_{h} f_{n}, \nabla \cdot\left(\boldsymbol{q}-\boldsymbol{q}_{h}\right)\right)_{\mathcal{T}^{n}}-\left\langle\tau\left(v_{h}-\lambda_{h}\right), w_{h} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}=0 \tag{5.79b}
\end{gather*}
$$

$$
\begin{align*}
& \left\langle\left(\boldsymbol{q}-\boldsymbol{q}_{h}\right) \cdot \boldsymbol{n}+\tau\left(v_{h}-\lambda_{h}\right), \mu_{h} f_{n}\right\rangle_{\mathcal{F}_{\underline{2}}^{n}}-\left\langle v-\lambda_{h}, f_{n} \partial_{t} \mu_{h}\right\rangle_{\mathcal{F}_{s}^{n}}  \tag{5.79c}\\
& -\left\langle v-\lambda_{h}, \mu_{h} f_{n}^{\prime}\right\rangle_{\mathcal{F}_{s}^{n}}+\left\langle\left\langle v-\lambda_{h}, \mu_{h} f_{n}\right\rangle_{\partial \mathcal{E}_{s}^{n}\left(t_{n+1}\right)}=\left\langle\left\langle v-\lambda_{h}^{-}, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{s}^{n}\left(t_{n}\right)} .\right.
\end{align*}
$$

We next split the numerical errors as $\boldsymbol{q}-\boldsymbol{q}_{h}=\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}+\boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}}, \boldsymbol{q}-\boldsymbol{q}_{h}^{-}=\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}}-\boldsymbol{q}+$ $\varepsilon_{h}^{q-}, v-v_{h}=v-\Pi_{W} v+\varepsilon_{h}^{v}, v-\lambda_{h}=v-P_{M}^{f_{n}} v+\varepsilon_{h}^{\lambda}$ and $v-\lambda_{h}^{-}=v-P_{M^{-}}^{f_{n-1}} v+\varepsilon_{h}^{\lambda-}$. Note also that

$$
\begin{align*}
\boldsymbol{q}-\widehat{\boldsymbol{q}}_{h} & =\boldsymbol{q}-\boldsymbol{q}_{h}-\tau\left(\lambda_{h}-v_{h}\right) \boldsymbol{n}=\boldsymbol{q}-\boldsymbol{q}_{h}-\tau\left(v-v_{h}-v+\lambda_{h}\right) \boldsymbol{n} \\
& =\boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}}-\tau\left(\varepsilon_{h}^{v}-\varepsilon_{h}^{\lambda}\right) \boldsymbol{n}+\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\tau\left(v-\Pi_{W} v-\left(v-P_{M}^{f_{n}} v\right)\right) \boldsymbol{n}  \tag{5.80}\\
& =\widehat{\boldsymbol{\varepsilon}}_{h}+\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\tau\left(v-\Pi_{W} v-\left(v-P_{M}^{f_{n}} v\right)\right) \boldsymbol{n} .
\end{align*}
$$

We will write eq. (5.79) in terms of the projection and approximation errors.
Consider first eq. (5.79a):

$$
\begin{array}{r}
81)-\left(\varepsilon_{h}^{\boldsymbol{q}}, f_{n} \partial_{t} \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left(\varepsilon_{h}^{\boldsymbol{q}}, \boldsymbol{r}_{h} f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left\langle\varepsilon_{h}^{\boldsymbol{q}}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}+\left(\varepsilon_{h}^{v}, f_{n} \nabla \cdot \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}  \tag{5.81}\\
-\left\langle\varepsilon_{h}^{\lambda}, \boldsymbol{r}_{h} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{F}_{n}^{n}} \\
=-\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, f_{n} \partial_{t} \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)} \\
+\left(\Pi_{W} v-v, f_{n} \nabla \cdot \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left\langle P_{M}^{f_{n}} v-v, \boldsymbol{r}_{h} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}} \\
+\left\langle\varepsilon_{h}^{\boldsymbol{q}-}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}-\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)},
\end{array}
$$

which simplifies to

$$
\begin{align*}
& -\left(\varepsilon_{h}^{\boldsymbol{q}}, f_{n} \partial_{t} \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left(\varepsilon_{h}^{\boldsymbol{q}}, \boldsymbol{r}_{h} f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left\langle\varepsilon_{h}^{\boldsymbol{q}}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}  \tag{5.82}\\
& +\left(\varepsilon_{h}^{v}, f_{n} \nabla \cdot \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}}-\left\langle\varepsilon_{h}^{\lambda}, \boldsymbol{r}_{h} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}}=-\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, f_{n} \partial_{t} \boldsymbol{r}_{h}\right)_{\mathcal{T}^{n}} \\
& \quad-\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)} \\
& \quad+\left\langle\varepsilon_{h}^{\boldsymbol{q -}}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}-\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{r}_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)},
\end{align*}
$$

using the properties of the projections $\Pi_{W}$ and $P_{M}^{f_{n}}$. This proves eq. (5.77a).
We consider next eq. (5.79b). Integrating eq. (5.79b) by parts in space,

$$
\begin{equation*}
\left(\boldsymbol{q}-\boldsymbol{q}_{h}, f_{n} \nabla w_{h}\right)_{\mathcal{T}^{n}}-\left\langle\left(\boldsymbol{q}-\boldsymbol{q}_{h}\right) \cdot \boldsymbol{n}-\tau\left(\lambda_{h}-v_{h}\right), w_{h} f_{n}\right\rangle_{\mathcal{F}_{\underline{Q}}^{n}}=0 . \tag{5.83}
\end{equation*}
$$

We next write this equation in terms of the projection and approximation errors:

$$
\begin{align*}
\left(\varepsilon_{h}^{\boldsymbol{q}}, f_{n} \nabla w_{h}\right)_{\mathcal{T}^{n}}- & \left\langle\widehat{\varepsilon}_{h} \cdot \boldsymbol{n}, w_{h} f_{n}\right\rangle_{\mathcal{F}_{⿷}^{n}} \\
= & \left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, f_{n} \nabla w_{h}\right)_{\mathcal{T}^{n}}-\left\langle\tau\left(P_{M}^{f_{n}} v-v\right), w_{h} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}}  \tag{5.84}\\
& -\left\langle\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right) \cdot \boldsymbol{n}-\tau\left(\Pi_{W} v-v\right), w_{h} f_{n}\right\rangle_{\mathcal{F}_{\mathfrak{Q}}^{n}} .
\end{align*}
$$

Using the properties of the projections $\Pi_{h}$ and $P_{M}^{f_{n}}$, we obtain

$$
\begin{equation*}
\left(\varepsilon_{h}^{q}, f_{n} \nabla w_{h}\right)_{\mathcal{T}^{n}}-\left\langle\widehat{\varepsilon}_{h} \cdot \boldsymbol{n}, w_{h} f_{n}\right\rangle_{\mathcal{F}_{\underline{Q}}^{n}}=0, \tag{5.85}
\end{equation*}
$$

proving eq. (5.77b).

Finally, we write eq. (5.79c) in terms of the projection and approximation errors:

$$
\begin{align*}
\left\langle\widehat{\varepsilon}_{h} \cdot \boldsymbol{n},\right. & \left.\mu_{h} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}-\left\langle\varepsilon_{h}^{\lambda}, f_{n} \partial_{t} \mu_{h}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}-\left\langle\varepsilon_{h}^{\lambda}, \mu_{h} f_{n}^{\prime}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}+\left\langle\left\langle\varepsilon_{h}^{\lambda}, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}  \tag{5.86}\\
= & \left\langle\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right) \cdot \boldsymbol{n}-\tau\left(\Pi_{W} v-v\right), \mu_{h} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}+\left\langle\tau\left(P_{M}^{f_{n}} v-v\right), \mu_{h} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}} \\
& -\left\langle P_{M}^{f_{n}} v-v, f_{n} \partial_{t} \mu_{h}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}-\left\langle P_{M}^{f_{n}} v-v, \mu_{h} f_{n}^{\prime}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}+\left\langle\left\langle P_{M}^{f_{n}} v-v, \mu_{h} f_{n}\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}\right. \\
& +\left\langle\left\langle\varepsilon_{h}^{\lambda-}, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}-\left\langle\left\langle P_{M^{-}}^{f_{n-1}} v-v, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}
\end{align*}
$$

Using the properties of the projections $\Pi_{h}$ and $P_{M}^{f_{n}}$, we obtain

$$
\begin{align*}
&\left\langle\widehat{\varepsilon}_{h} \cdot \boldsymbol{n}, \mu_{h} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}-\left\langle\varepsilon_{h}^{\lambda}, f_{n} \partial_{t} \mu_{h}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}-\left\langle\varepsilon_{h}^{\lambda}, \mu_{h} f_{n}^{\prime}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}+\left\langle\left\langle\varepsilon_{h}^{\lambda}, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)} \\
&=\left\langle\left\langle\varepsilon_{h}^{\lambda-}, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}+\left\langle\left\langle P_{M}^{f_{n}} v-v, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}  \tag{5.87}\\
&-\left\langle\left\langle P_{M^{-}}^{f_{n-1}} v-v, \mu_{h} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}
\end{align*}
$$

proving eq. ( 5.77 c ).

Next, we prove a bound for the projection errors.
Lemma 5.10. The following bound holds for the projection errors:

$$
\begin{align*}
\frac{\alpha}{2}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2} & +\frac{\alpha}{2}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}^{2}+e^{-\alpha \Delta t}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}^{2}+e^{-\alpha \Delta t}\left\|\varepsilon_{h}^{\lambda}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}^{2}  \tag{5.88}\\
\leq & \left\|\varepsilon_{h}^{\boldsymbol{q}-}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}^{2}+\left\|\varepsilon_{h}^{\lambda-}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}^{2}+C \Delta t^{-2}\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{T}^{n}}^{2} \\
& +C\left\|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+C \Delta t^{-1}\left\|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}^{2} \\
& +C \Delta t^{-1}\left\|\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}^{2}+C \Delta t^{-1}\left\|P_{M}^{f_{n}} v-v\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}^{2} \\
& +C \Delta t^{-1}\left\|P_{M_{n}^{-1}}^{f_{n-1}} v-v\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}^{2}
\end{align*}
$$

Proof. Take $\boldsymbol{r}_{h}=\varepsilon_{h}^{\boldsymbol{q}}$ in eq. (5.77a), $w_{h}=\varepsilon_{h}^{v}$ in eq. (5.77b) and $\mu_{h}=\varepsilon_{h}^{\lambda}$ in eq. ( 5.77 c ). Adding the resulting equations we obtain

$$
\begin{align*}
& -\left(\varepsilon_{h}^{\boldsymbol{q}}, f_{n} \partial_{t} \varepsilon_{h}^{\boldsymbol{q}}\right)_{\mathcal{T}^{n}}-\left(f_{n}^{\prime} \varepsilon_{h}^{\boldsymbol{q}}, \varepsilon_{h}^{\boldsymbol{q}}\right)_{\mathcal{T}^{n}}+\left\langle f_{n} \varepsilon_{h}^{\boldsymbol{q}}, \varepsilon_{h}^{\boldsymbol{q}}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}  \tag{5.89}\\
& +\left(\varepsilon_{h}^{v}, f_{n} \nabla \cdot \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}}\right)_{\mathcal{T}^{n}}-\left\langle\varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\boldsymbol{q}} \cdot \boldsymbol{n} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}+\left(\varepsilon_{h}^{\boldsymbol{q}}, f_{n} \nabla \varepsilon_{h}^{v}\right)_{\mathcal{T}^{n}}-\left\langle\widehat{\varepsilon}_{h} \cdot \boldsymbol{n}, \varepsilon_{h}^{v} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}} \\
& +\left\langle\widehat{\varepsilon}_{h} \cdot \boldsymbol{n}, \varepsilon_{h}^{\lambda} f_{n}\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}-\left\langle\varepsilon_{h}^{\lambda}, f_{n} \partial_{t} \varepsilon_{h}^{\lambda}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}-\left\langle f_{n}^{\prime} \varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\lambda}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}+\left\langle\left\langle f_{n} \varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\lambda}\right\rangle\right\rangle_{\partial \varepsilon_{\mathcal{S}}^{n}\left(t_{n+1}\right)} \\
= & \left\langle\varepsilon_{h}^{\boldsymbol{q}-}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}+\left\langle\left\langle\varepsilon_{h}^{\lambda-}, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)} \\
& -\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, f_{n} \partial_{t} \varepsilon_{h}^{\boldsymbol{q}}\right)_{\mathcal{T}^{n}}-\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \varepsilon_{h}^{\boldsymbol{q}} f_{n}^{\prime}\right)_{\mathcal{T}^{n}} \\
& +\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}-\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}, \varepsilon_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)} \\
& +\left\langle\left\langle P_{M}^{f_{n}} v-v, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}-\left\langle\left\langle P_{M_{n-1}}^{f_{n}} v-v, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)} .
\end{align*}
$$

Applying integration by parts with respect to time on the first and ninth terms, integration by parts with respect to space on the sixth term, and expanding out
some terms:

$$
\begin{align*}
& -\frac{1}{2}\left(f_{n}^{\prime} \varepsilon_{h}^{\boldsymbol{q}}, \varepsilon_{h}^{\boldsymbol{q}}\right)_{\mathcal{T}^{n}}+\frac{1}{2}\left\langle f_{n} \varepsilon_{h}^{\boldsymbol{q}}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}+\frac{1}{2}\left\langle f_{n} \varepsilon_{h}^{\boldsymbol{q}}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}  \tag{5.90}\\
& +\left\langle\tau\left(\varepsilon_{h}^{v}-\varepsilon_{h}^{\lambda}\right), f_{n}\left(\varepsilon_{h}^{v}-\varepsilon_{h}^{\lambda}\right)\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}-\frac{1}{2}\left\langle f_{n}^{\prime} \varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\lambda}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}+\frac{1}{2}\left\langle\left\langle f_{n} \varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\lambda}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)} \\
& +\frac{1}{2}\left\langle\left\langle f_{n} \varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\lambda}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)} \\
= & \left\langle\varepsilon_{h}^{\boldsymbol{q}-}, \varepsilon_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}+\left\langle\left\langle\varepsilon_{h}^{\lambda-}, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}-\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, f_{n} \partial_{t} \varepsilon_{h}^{\boldsymbol{q}}\right)_{\mathcal{T}^{n}} \\
& -\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}} f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}-\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)} \\
& +\left\langle\left\langle P_{M}^{f_{n}} v-v, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}-\left\langle\left\langle P_{M^{-}}^{f_{n-1}} v-v, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)} .
\end{align*}
$$

Moving the first two terms on the right hand side to the left hand side, we obtain

$$
\begin{align*}
& -\frac{1}{2}\left(f_{n}^{\prime} \varepsilon_{h}^{\boldsymbol{q}}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}}\right)_{\mathcal{T}^{n}}+\frac{1}{2}\left\langle f_{n} \varepsilon_{h}^{\boldsymbol{q}}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}+\frac{1}{2}\left\langle f_{n} \varepsilon_{h}^{\boldsymbol{q}}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}-\left\langle\varepsilon_{h}^{\boldsymbol{q}-}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}  \tag{5.91}\\
& +\left\langle\tau\left(\varepsilon_{h}^{v}-\varepsilon_{h}^{\lambda}\right), f_{n}\left(\varepsilon_{h}^{v}-\varepsilon_{h}^{\lambda}\right)\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}-\frac{1}{2}\left\langle f_{n}^{\prime} \varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\lambda}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}+\frac{1}{2}\left\langle\left\langle f_{n} \varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\lambda}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)} \\
& +\frac{1}{2}\left\langle\left\langle f_{n} \varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\lambda}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}-\left\langle\left\langle\varepsilon_{h}^{\lambda-}, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)} \\
= & -\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, f_{n} \partial_{t} \varepsilon_{h}^{\boldsymbol{q}}\right)_{\mathcal{T}^{n}} \\
& -\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}} f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}-\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}, \varepsilon_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)} \\
& +\left\langle\left\langle P_{M}^{f_{n}} v-v, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}-\left\langle\left\langle P_{M-}^{f_{n-1}} v-v, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}
\end{align*}
$$

Notice that

$$
\begin{align*}
& \frac{1}{2}\left\langle f_{n} \varepsilon_{h}^{\boldsymbol{q}}, \varepsilon_{h}^{\boldsymbol{q}}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}-\left\langle\varepsilon_{h}^{\boldsymbol{q}-}, \varepsilon_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}=\frac{1}{2}\left\langle\varepsilon_{h}^{\boldsymbol{q}}-\varepsilon_{h}^{\boldsymbol{q}-}, \varepsilon_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}-\frac{1}{2}\left\langle\varepsilon_{h}^{\boldsymbol{q -}}, \varepsilon_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}  \tag{5.92}\\
& =\frac{1}{2}\left\langle\varepsilon_{h}^{\boldsymbol{q}}-\varepsilon_{h}^{\boldsymbol{q -}},\left(\varepsilon_{h}^{\boldsymbol{q}}-\varepsilon_{h}^{\boldsymbol{q -}}\right) f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}-\frac{1}{2}\left\langle\varepsilon_{h}^{\boldsymbol{q}-}, \varepsilon_{h}^{\boldsymbol{q}-} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)} .
\end{align*}
$$

Similarly,

$$
\begin{align*}
& \frac{1}{2}\left\langle\left\langle f_{n} \varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\lambda}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}-\left\langle\left\langle\varepsilon_{h}^{\lambda-}, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}  \tag{5.93}\\
& =\frac{1}{2}\left\langle\left\langle\varepsilon_{h}^{\lambda}-\varepsilon_{h}^{\lambda-},\left(\varepsilon_{h}^{\lambda}-\varepsilon_{h}^{\lambda-}\right) f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}-\frac{1}{2}\left\langle\left\langle\varepsilon_{h}^{\lambda-}, \varepsilon_{h}^{\lambda-} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}
\end{align*}
$$

Substituting these expressions in eq. (5.91) and rearranging terms, we obtain

$$
\begin{align*}
& -\frac{1}{2}\left(f_{n}^{\prime} \varepsilon_{h}^{\boldsymbol{q}}, \varepsilon_{h}^{\boldsymbol{q}}\right)_{\mathcal{T}^{n}}+\frac{1}{2}\left\langle f_{n} \varepsilon_{h}^{\boldsymbol{q}}, \varepsilon_{h}^{\boldsymbol{q}}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}+\frac{1}{2}\left\langle\varepsilon_{h}^{\boldsymbol{q}}-\boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}-},\left(\varepsilon_{h}^{\boldsymbol{q}}-\varepsilon_{h}^{\boldsymbol{q}-}\right) f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}  \tag{5.94}\\
& +\left\langle\tau\left(\varepsilon_{h}^{v}-\varepsilon_{h}^{\lambda}\right), f_{n}\left(\varepsilon_{h}^{v}-\varepsilon_{h}^{\lambda}\right)\right\rangle_{\mathcal{F}_{\mathcal{Q}}^{n}}-\frac{1}{2}\left\langle f_{n}^{\prime} \varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\lambda}\right\rangle_{\mathcal{F}_{\mathcal{S}}^{n}}+\frac{1}{2}\left\langle\left\langle f_{n} \varepsilon_{h}^{\lambda}, \varepsilon_{h}^{\lambda}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)} \\
& +\frac{1}{2}\left\langle\left\langle\varepsilon_{h}^{\lambda}-\varepsilon_{h}^{\lambda-},\left(\varepsilon_{h}^{\lambda}-\varepsilon_{h}^{\lambda-}\right) f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)} \\
= & \frac{1}{2}\left\langle\varepsilon_{h}^{\boldsymbol{q}-}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}-} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}+\frac{1}{2}\left\langle\left\langle\varepsilon_{h}^{\lambda-}, \varepsilon_{h}^{\lambda-} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)} \\
& -\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, f_{n} \partial_{t} \varepsilon_{h}^{\boldsymbol{q}}\right)_{\mathcal{T}^{n}} \\
& -\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}} f_{n}^{\prime}\right)_{\mathcal{T}^{n}}+\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}, \varepsilon_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}-\left\langle\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}, \varepsilon_{h}^{\boldsymbol{q}} f_{n}\right\rangle_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)} \\
& +\left\langle\left\langle P_{M}^{f_{n}} v-v, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}-\left\langle\left\langle P_{M^{-1}}^{f_{n-1}} v-v, \varepsilon_{h}^{\lambda} f_{n}\right\rangle\right\rangle_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}
\end{align*}
$$

Recall that $f_{n}=e^{-\alpha\left(t-t_{n}\right)}$, where $\alpha>0$ is constant, and so $f_{n}^{\prime}=-\alpha f_{n}$. Moreover, $f_{n}\left(t_{n}\right)=1$ and $f_{n}\left(t_{n+1}\right)=e^{-\alpha \Delta t}$. With these definitions and by the CauchySchwarz inequality applied to the right hand side of eq. (5.94), we obtain

$$
\begin{align*}
& \quad \frac{\alpha}{2}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+\frac{\alpha}{2}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}^{2}+\frac{e^{-\alpha \Delta t}}{2}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}^{2}+\frac{e^{-\alpha \Delta t}}{2}\left\|\varepsilon_{h}^{\lambda}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}^{2}  \tag{5.95}\\
& \leq \frac{1}{2}\left\|\varepsilon_{h}^{\boldsymbol{q}-}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}^{2}+\frac{1}{2}\left\|\varepsilon_{h}^{\lambda-}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}^{2} \\
& \quad+\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{T}^{n}}\left\|\partial_{t} \varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}+\alpha\left\|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{T}^{n}}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}} \\
& \quad+\left\|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}+\left\|\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n}\right)} \\
& \quad+\left\|P_{M}^{f_{n}} v-v\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}+\left\|P_{M_{n-}}^{f_{n-1}} v-v\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)} .
\end{align*}
$$

Note that, by eq. (4.21b) and eq. (4.21a),

$$
\begin{align*}
\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n}\right)} & \leq C \Delta t^{-1 / 2}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}} \\
\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)} & \leq C \Delta t^{-1 / 2}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}} \tag{5.96}
\end{align*}
$$

Furthermore, combining a standard inverse inequality and using equivalence of the norms $\|\cdot\|_{\mathcal{K}}$ and $\|\cdot\|_{f_{n}, \mathcal{K}}$ we also have

$$
\begin{equation*}
\left\|\partial_{t} \varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}} \leq C \Delta t^{-1}\left\|\varepsilon_{h}^{q}\right\|_{f_{n}, \mathcal{T}^{n}} \tag{5.97}
\end{equation*}
$$

Using eq. (5.96) and eq. (5.97) for the right hand side of eq. (5.95) and multiplying by 2 ,

$$
\begin{align*}
& \alpha\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+\alpha\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}^{2}+e^{-\alpha \Delta t}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}^{2}+e^{-\alpha \Delta t}\left\|\varepsilon_{h}^{\lambda}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}^{2} \\
\leq & \left\|\varepsilon_{h}^{\boldsymbol{q}-}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}^{2}+\left\|\varepsilon_{h}^{\lambda-}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}^{2} \\
& +C \Delta t^{-1}\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{T}^{n}}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}+2 \alpha\left\|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{T}^{n}}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}} \\
& +C \Delta t^{-1 / 2}\left\|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}  \tag{5.98}\\
& +C \Delta t^{-1 / 2}\left\|\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}} \\
& +C \Delta t^{-1 / 2}\left\|P_{M}^{f_{n}} v-v\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}} \\
& +C \Delta t^{-1 / 2}\left\|P_{M^{-}}^{f_{n-1}} v-v\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}
\end{align*}
$$

Applying Young's inequality to the right hand side of eq. (5.98), we obtain

$$
\begin{align*}
\alpha & \left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+\alpha\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}^{2}+e^{-\alpha \Delta t}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}^{2}+e^{-\alpha \Delta t}\left\|\varepsilon_{h}^{\lambda}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}^{2}  \tag{5.99}\\
\leq & \left\|\varepsilon_{h}^{\boldsymbol{q}-}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}^{2}+\left\|\varepsilon_{h}^{\lambda-}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}^{2}+\frac{C}{4 \delta_{1}} \Delta t^{-2}\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+\delta_{1}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2} \\
& +\frac{C}{4 \delta_{1}}\left\|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+\delta_{1}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+\frac{C}{4 \delta_{1}} \Delta t^{-1}\left\|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}^{2}+\delta_{1}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2} \\
& +\frac{C}{4 \delta_{1}} \Delta t^{-1}\left\|\boldsymbol{\Pi}_{\boldsymbol{V}}-\boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}^{2}+\delta_{1}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2} \\
& +\frac{C}{4 \delta_{2}} \Delta t^{-1}\left\|P_{M}^{f_{n}} v-v\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}^{2}+\delta_{2}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}^{2} \\
& +\frac{C}{4 \delta_{2}} \Delta t^{-1}\left\|P_{M_{n-}}^{f_{n-1}} v-v\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}^{2}+\delta_{2}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}^{2},
\end{align*}
$$

where $\delta_{1}, \delta_{2}>0$ are free to choose constants. Collecting terms,

$$
\begin{align*}
& \left(\alpha-4 \delta_{1}\right)\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+\left(\alpha-2 \delta_{2}\right)\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}^{2}+e^{-\alpha \Delta t}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}^{2}+e^{-\alpha \Delta t}\left\|\varepsilon_{h}^{\lambda}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}^{2}  \tag{5.100}\\
\leq & \left\|\varepsilon_{h}^{\boldsymbol{q}-}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}^{2}+\left\|\varepsilon_{h}^{\lambda-}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}^{2}+C \Delta t^{-2}\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{T}^{n}}^{2} \\
& +C\left\|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+C \Delta t^{-1}\left\|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}^{2} \\
& +C \Delta t^{-1}\left\|\boldsymbol{\Pi}_{\boldsymbol{V}^{-}} \boldsymbol{q}-\boldsymbol{q}\right\|_{f_{n}, \mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}^{2}+C \Delta t^{-1}\left\|P_{M}^{f_{n}} v-v\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}\left(t_{n+1}\right)}^{2} \\
& +C \Delta t^{-1}\left\|P_{M_{n-1}}^{f_{n-1}} v-v\right\|_{f_{n}, \partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}^{2}
\end{align*}
$$

The result follows by choosing $\delta_{1}=\alpha / 8$ and $\delta_{2}=\alpha / 4$.
Remark 5.11. If $(\boldsymbol{q}, v) \in\left[H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{n}\right)\right]^{2} \times H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{n}\right)$, with $1<s_{t} \leq p+1$ and $1 \leq s_{s} \leq p+1$, then combining Lemma 5.10, Corollary 5.7 and Lemma 5.8 gives the following leading order terms

$$
\begin{align*}
\frac{\alpha}{2}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2} & +\frac{\alpha}{2}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}^{2}+e^{-\alpha \Delta t}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}^{2}+e^{-\alpha \Delta t}\left\|\varepsilon_{h}^{\lambda}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}^{2} \\
\leq & \left\|\varepsilon_{h}^{\boldsymbol{q}-}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}^{2}+\left\|\varepsilon_{h}^{\lambda-}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}^{2} \\
& +C\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{n}\right)}^{2}  \tag{5.101}\\
& +C\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{n}\right)}^{2} \\
& +C\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}\left(\partial \mathcal{E}_{\mathcal{S}}^{n}\right)}^{2}
\end{align*}
$$

where $h=\max _{\mathcal{K}} h_{K}$.
To prove the a priori error estimates we will use the following lemma:
Lemma 5.12. Let $A_{n}, B_{n}, D_{n} \geq 0$ for all $n=0,1, \cdots N-1$, and $\alpha>0$. Moreover, assume that there exists a constant $C \geq 0$ such that $C \Delta t B_{n} \leq A_{n}$ for all $n$. If $B_{0}=0,(N-1) \Delta t=T$ and

$$
\begin{equation*}
A_{n}+e^{-\alpha \Delta t} B_{n} \leq B_{n-1}+D_{n} \tag{5.102}
\end{equation*}
$$

then,

$$
\begin{equation*}
A_{n} \leq C \sum_{i=1}^{n} D_{i} \tag{5.103}
\end{equation*}
$$

where $C>0$ depends on $\alpha$ and $T$.
Proof. Since $C \Delta t B_{n} \leq A_{n}$, by eq. (5.102), we have

$$
\begin{equation*}
\left(C \Delta t+e^{-\alpha \Delta t}\right) B_{n} \leq B_{n-1}+D_{n} \tag{5.104}
\end{equation*}
$$

Let $\gamma=C \Delta t+e^{-\alpha \Delta t}$. By induction, we have the following

$$
\begin{align*}
B_{n} & \leq \gamma^{-1} B_{n-1}+\gamma^{-1} D_{n} \\
& \leq \gamma^{-2} B_{n-2}+\gamma^{-2} D_{n-1}+\gamma^{-1} D_{n} \\
& \leq \gamma^{-3} B_{n-3}+\gamma^{-3} D_{n-2}+\gamma^{-2} D_{n-1}+\gamma^{-1} D_{n} \\
& \vdots  \tag{5.105}\\
& \leq \sum_{i=1}^{n} \gamma^{i-n-1} D_{i}
\end{align*}
$$

where we used that $B_{0}=0$. Note that $-n \leq i-n-1 \leq-1, i \in\{1, \ldots, n\}$. This implies that if $\gamma \geq 1$ then $\gamma^{i-n-1} \leq \gamma^{-1} \leq 1$ while if $\gamma<1$ then $\gamma^{i-n-1} \leq \gamma^{-n}$. We next bound $\gamma^{-n}$. Note that

$$
\begin{equation*}
e^{-\alpha \Delta t} \leq C_{1} \Delta t+e^{-\alpha \Delta t}=\gamma \tag{5.106}
\end{equation*}
$$

Since $n \leq N-1$, we have that $\gamma^{-n} \leq e^{\alpha n \Delta t} \leq e^{\alpha T} \leq C_{2}$. Therefore $\gamma^{i-n-1} \leq$ $\max \left\{1, C_{2}\right\}=C$.

We may therefore bound $B_{n}$ in eq. (5.105) as:

$$
\begin{equation*}
B_{n} \leq C \sum_{i=1}^{n} D_{i} \tag{5.107}
\end{equation*}
$$

In order to obtain the final result, recall that, by eq. (5.102), we have the following:

$$
\begin{equation*}
A_{n} \leq B_{n-1}+D_{n} \tag{5.108}
\end{equation*}
$$

Using eq. (5.107) we obtain

$$
\begin{equation*}
A_{n} \leq C \sum_{i=1}^{n-1} D_{i}+D_{n} \leq C \sum_{i=1}^{n} D_{i} \tag{5.109}
\end{equation*}
$$

which completes the proof.
The following theorem gives the final error bounds.
Theorem 5.13. Let $h=\max _{\mathcal{K} \in \mathcal{T}^{n}} h_{K}$. Assume that the spatial shape-regularity condition eq. (4.5) holds and that the triangulation $\mathcal{T}^{n}$ does not have any hanging nodes. Suppose that $(\boldsymbol{q}, v)$ solves eq. (3.1), with $\boldsymbol{q} \in\left[H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{n}\right)\right]^{2}$ and $v \in$ $H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{n}\right)$, with $1 / 2<s_{t} \leq p+1$ and $1 \leq s_{s} \leq p+1$. Then we have the following
estimates:

$$
\begin{align*}
& \sum_{n=0}^{N-1}\left\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{f_{n}, \mathcal{T}^{n}}+\sum_{n=0}^{N-1}\left\|v-\lambda_{h}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}} \\
& \quad \leq C\left(\Delta t^{-1} h^{s_{s}}+\Delta t^{s_{t}-1}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{n}\right)}  \tag{5.110}\\
& \quad+C\left(\Delta t^{-1} h^{s_{s}}+\Delta t^{s_{t}-1}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{n}\right)} \\
& \quad+C\left(\Delta t^{-1} h^{s_{s}}+\Delta t^{s_{t}-1}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}\left(\partial \mathcal{E}_{\mathcal{S}}^{n}\right)}
\end{align*}
$$

where $C>0$ depends on $\alpha$ and the final time $T$.
Proof. Let

$$
\begin{align*}
& A_{n}=\frac{\alpha}{2}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+\frac{\alpha}{2}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}^{2},  \tag{5.111}\\
& B_{n}=\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n+1}\right)}^{2}+\left\|\varepsilon_{h}^{\lambda}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n+1}\right)}^{2} .
\end{align*}
$$

Note that

$$
\begin{equation*}
B_{n-1}=\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{\mathcal{F}_{\Omega}^{n-1}\left(t_{n}\right)}^{2}+\left\|\varepsilon_{h}^{\lambda}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n-1}\left(t_{n}\right)}^{2}=\left\|\varepsilon_{h}^{\boldsymbol{q}-}\right\|_{\mathcal{F}_{\Omega}^{n}\left(t_{n}\right)}^{2}+\left\|\varepsilon_{h}^{\lambda-}\right\|_{\partial \mathcal{E}_{\mathcal{S}}^{n}\left(t_{n}\right)}^{2} \tag{5.112}
\end{equation*}
$$

By eq. (4.21b) and the equivalence of the norms $\|\cdot\|_{\mathcal{K}}$ and $\|\cdot\|_{f_{n}, \mathcal{K}}$, we have the following:

$$
\begin{equation*}
C \Delta t B_{n} \leq\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{\mathcal{T}^{n}}^{2}+\left\|\varepsilon_{h}^{\lambda}\right\|_{\mathcal{F}_{\mathcal{S}}^{n}}^{2} \leq C A_{n} \tag{5.113}
\end{equation*}
$$

Moreover, let $D_{n}$ be defined as follows:

$$
\begin{align*}
D_{n}= & C\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{n}\right)}^{2} \\
& +C\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{n}\right)}^{2}  \tag{5.114}\\
& +C\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}\left(\partial \mathcal{E}_{\mathcal{S}}^{n}\right)}^{2} .
\end{align*}
$$

With these definitions, Lemma 5.12 gives the following bound for the projection errors:

$$
\begin{align*}
& \frac{\alpha}{2}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+\frac{\alpha}{2}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}^{2} \\
& \quad \leq C\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right) \sum_{k=0}^{n}\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{k}\right)}^{2} \\
& \quad+C\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right) \sum_{k=0}^{n}\|v\|_{H^{\left(s_{t}, s_{s}\right)}\left(\mathcal{E}^{k}\right)}^{2}  \tag{5.115}\\
& \quad+C\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right) \sum_{k=0}^{n}\|v\|_{H^{\left(s_{t}, s_{s}\right)}\left(\partial \mathcal{E}_{\mathcal{S}}^{k}\right)}^{2} .
\end{align*}
$$

Summing over all time slabs, we obtain:

$$
\begin{align*}
& \frac{\alpha}{2} \sum_{n=0}^{N-1}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}^{2}+\frac{\alpha}{2} \sum_{n=0}^{N-1}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{s}^{n}}^{2} \\
& \quad \leq C(N-1)\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right)\|\boldsymbol{q}\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{E})}^{2}  \tag{5.116}\\
& \quad+C(N-1)\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}(\mathcal{E})}^{2} \\
& \quad+C(N-1)\left(\Delta t^{-2} h^{2 s_{s}}+\Delta t^{2 s_{t}-2}\right)\|v\|_{H^{\left(s_{t}, s_{s}\right)}\left(\partial \mathcal{E}_{s}\right)}^{2} .
\end{align*}
$$

By the triangle inequality note that

$$
\begin{align*}
& \sum_{n=0}^{N-1}\left\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{f_{n}, \mathcal{T}^{n}} \leq \sum_{n=0}^{N-1}\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{f_{n}, \mathcal{T}^{n}}+\sum_{n=0}^{N-1}\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|_{f_{n}, \mathcal{T}^{n}}  \tag{5.117}\\
& \sum_{n=0}^{N-1}\left\|v-\lambda_{h}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}} \leq \sum_{n=0}^{N-1}\left\|v-P_{M}^{f_{n}} v\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}+\sum_{n=0}^{N-1}\left\|\varepsilon_{h}^{\lambda}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}}
\end{align*}
$$

The result follows by Lemma 5.8 and eq. (5.116).
Remark 5.14. Assuming $\boldsymbol{q} \in\left[H^{(p+1, p+1)}(\mathcal{E})\right]^{2}$ and $v \in H^{(p+1, p+1)}(\mathcal{E})$, the error estimates in Theorem 5.13 give the following leading order terms:

$$
\begin{equation*}
\sum_{n=0}^{N-1}\left\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{f_{n}, \mathcal{T}^{n}}+\sum_{n=0}^{N-1}\left\|v-\lambda_{h}\right\|_{f_{n}, \mathcal{F}_{\mathcal{S}}^{n}} \leq C\left(\Delta t^{-1} h^{p+1}+\Delta t^{p}\right) \tag{5.118}
\end{equation*}
$$

## 6. Numerical results

In this section we verify the theoretical results of the previous sections. The space-time HDG method for the linear free-surface problem eq. (3.1) is implemented using the modular finite-element method (MFEM) library [1]. The linear systems of algebraic equations are solved by the direct solver MUMPS [3, 4] through PETSc $[5,6]$.

To obtain the space-time mesh, we first triangulate the spatial domain $\Omega$. We then extrude the spatial triangles in the time direction to obtain space-time prisms.

In all our simulations we take $\tau=5$ and $\alpha=0.1$.
6.1. Linear waves in an unbounded domain. We consider the time harmonic linear free-surface waves example in an unbounded domain from [31, Section 8]. We consider the domain $\Omega=[-1,1] \times[-1,0]$ and apply periodic boundary conditions at $x_{1}=-1$ and $x_{1}=1$. The analytical solution to this problem is given by

$$
\begin{align*}
\phi(\boldsymbol{x}, t) & =\phi_{0} \cosh \left(k\left(x_{2}+1\right)\right) \cos \left(\omega t-k x_{1}\right)  \tag{6.1a}\\
\zeta\left(x_{1}, t\right) & =-\partial_{t} \phi\left(x_{1}, 0, t\right)=\phi_{0} \omega \cosh (k) \sin \left(\omega t-k x_{1}\right) \tag{6.1b}
\end{align*}
$$

where $\phi_{0}$ denotes the amplitude of the velocity potential, $k$ is the wave number which is related to the wavelength $\lambda_{w}$ by $k=2 \pi / \lambda_{w}$, and $\omega$ is the frequency of the oscillations which satisfies the dispersion relation $\omega^{2}=k \tanh (k)$.

We take $\lambda_{w}=1$ and $\phi_{0}$ such that the maximum amplitude of the wave height is 0.05 . In table 1 , table 2 , and table 3 we show the approximation errors and convergence rates for the velocity $\boldsymbol{q}_{h}$ on the entire space-time domain $\mathcal{E}$ and for the free-surface height $\lambda_{h}$ on the entire free-surface boundary $\partial \mathcal{E}_{\mathcal{S}}$. We test convergence in space, in time, and in space-time separately.

We first test convergence in space. To ensure the spatial error dominates over the temporal error we take a small time step $\Delta t=10^{-5}$ when $p=1$ and $\Delta t=10^{-4}$ when $p=2$. We compute the error after 200 (when $p=1$ ) or 20 (when $p=2$ ) time steps. As observed in table 1, the error is of order $\mathcal{O}\left(h^{p+1}\right)$.

We next consider convergence in time. For this we compute up to a final time $T=1$. To ensure that the temporal error dominates over the spatial error we use a mesh consisting of 73728 elements when $p=1$ and 36864 elements when $p=2$. We observe in table 2 that the error is of order $\mathcal{O}\left(\Delta t^{p+1}\right)$.

Table 1. Spatial rates of convergence for linear waves in an unbounded domain, see section 6.1.

|  | $\boldsymbol{q}_{h}$ |  |  | $\lambda_{h}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | DOFs | $L^{2}(\mathcal{E})$-error | Order | $L^{2}\left(\partial \mathcal{E}_{\mathcal{S}}\right)$-error | Order |
|  | 228 | $1.1 \mathrm{e}-3$ | - | $2.5 \mathrm{e}-2$ | - |
| $p=1$ | 888 | $3.2 \mathrm{e}-4$ | 1.7 | $1.4 \mathrm{e}-2$ | 0.9 |
|  | 3504 | $8.5 \mathrm{e}-5$ | 1.9 | $3.4 \mathrm{e}-3$ | 2.0 |
|  | 13920 | $2.2 \mathrm{e}-5$ | 2.0 | $8.2 \mathrm{e}-4$ | 2.1 |
|  | 55488 | $5.4 \mathrm{e}-6$ | 2.0 | $1.9 \mathrm{e}-4$ | 2.1 |
|  |  |  |  |  |  |
| $p=2$ | 486 | $4.0 \mathrm{e}-4$ | - | $1.5 \mathrm{e}-3$ | - |
|  | 1890 | $6.0 \mathrm{e}-5$ | 2.7 | $2.3 \mathrm{e}-4$ | 2.8 |
|  | 7452 | $7.9 \mathrm{e}-6$ | 2.9 | $3.5 \mathrm{e}-5$ | 2.7 |
|  | 29592 | $1.0 \mathrm{e}-6$ | 3.0 | $4.8 \mathrm{e}-6$ | 2.9 |

Table 2. Time rates of convergence for linear waves in an unbounded domain, see section 6.1.

|  |  | $\boldsymbol{q}_{h}$ |  | $\lambda_{h}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\Delta t$ | $L^{2}(\mathcal{E})$-error | Order | $L^{2}\left(\partial \mathcal{E}_{\mathcal{S}}\right)$-error | Order |
|  | 1 | $1.7 \mathrm{e}-2$ | - | $1.7 \mathrm{e}-2$ | - |
| $p=1$ | $1 / 2$ | $5.1 \mathrm{e}-3$ | 1.8 | $5.1 \mathrm{e}-3$ | 1.8 |
|  | $1 / 4$ | $1.2 \mathrm{e}-3$ | 2.1 | $1.2 \mathrm{e}-3$ | 2.1 |
|  | $1 / 8$ | $3.0 \mathrm{e}-4$ | 2.0 | $3.0 \mathrm{e}-4$ | 2.0 |
|  | $1 / 16$ | $8.2 \mathrm{e}-5$ | 1.9 | $7.9 \mathrm{e}-5$ | 1.9 |
|  |  |  |  |  |  |
| $p=2$ | $1 / 2$ | $3.8 \mathrm{e}-3$ | - | $3.8 \mathrm{e}-3$ | - |
|  | $1 / 4$ | $5.9 \mathrm{e}-4$ | 3.0 | $4.8 \mathrm{e}-4$ | 3.0 |
|  | $1 / 8$ | $7.5 \mathrm{e}-6$ | 3.0 | $5.9 \mathrm{e}-5$ | 3.0 |
|  | $1 / 16$ | $1.6 \mathrm{e}-6$ | 3.0 | $7.5 \mathrm{e}-6$ | 3.0 |
|  |  |  |  |  | $1.3 \mathrm{e}-6$ |

We note that the rates of convergence in space and time separately are better than predicted from remark 5.14. We now consider convergence in space-time in which we refine the spatial mesh and time step simultaneously. We compute the solution up to a final time of $T=1$. The initial time step is $\Delta t=0.25$ and the initial mesh has 18 elements. We observe in table 3 that the error is of order $\mathcal{O}\left(\Delta t^{p}+h^{p}\right)$, as expected from our analysis, see remark 5.14.

Finally, we consider a case where $h$ is fixed so that the spatial mesh consists of 1152 triangles and the number of global degrees-of-freedom is 13920 . We take $p=1$ and solve the problem up to a final time $T=1$. From remark 5.14 , we see that if $h$ is fixed, eventually the dominant term in the error will be of order $\mathcal{O}\left(\Delta t^{-1} h^{p+1}\right)$, which results in divergence of the solution. This effect can be observed in table 4 where the errors start to increase after three levels of refinement in time. Unlike standard time stepping methods, for space-time methods $\Delta t$ has to be chosen carefully depending on its relation to the spatial mesh size $h$.

TABLE 3. Space-time rates of convergence for linear waves in an unbounded domain, see section 6.1.

|  | DOFs | $\underset{L^{2}(\mathcal{E}) \text {-error }}{\boldsymbol{q}_{h}}$ | Order | $\begin{gathered} \lambda_{h} \\ L^{2}\left(\partial \mathcal{E}_{\mathcal{S}}\right) \text {-error } \end{gathered}$ | Order |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $p=1$ | 228 | 3.5e-2 | - | $3.4 \mathrm{e}-2$ | - |
|  | 888 | $1.7 \mathrm{e}-2$ | 1.1 | $1.5 \mathrm{e}-2$ | 1.2 |
|  | 3504 | 7.2e-3 | 1.2 | $5.9 \mathrm{e}-3$ | 1.3 |
|  | 13920 | $3.2 \mathrm{e}-3$ | 1.2 | $2.7 \mathrm{e}-3$ | 1.2 |
|  | 55488 | $1.5 \mathrm{e}-3$ | 1.1 | 1.3e-3 | 1.1 |
| $p=2$ | 486 | 1.6e-2 | - | 1.3e-2 | - |
|  | 1890 | $3.4 \mathrm{e}-3$ | 2.2 | $2.3 \mathrm{e}-3$ | 2.5 |
|  | 7452 | $6.7 \mathrm{e}-4$ | 2.4 | $4.4 \mathrm{e}-4$ | 2.4 |
|  | 29592 | 1.4e-4 | 2.3 | 9.8e-5 | 2.2 |
|  | 117936 | $3.1 \mathrm{e}-5$ | 2.2 | $2.4 \mathrm{e}-5$ | 2.1 |

TABLE 4. Time rates of convergence for a coarse mesh for linear waves in an unbounded domain, see section 6.1.

| $\boldsymbol{q}_{h}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | 2 <br> $\Delta$ | $L^{2}(\mathcal{E})$-error | Order | $L^{2}\left(\partial \mathcal{E}_{\mathcal{S}}\right)$-error | Order | $\lambda_{h}$ |  |  |  |
| :---: | :---: | :---: | :---: |
| 1 | $1.8 \mathrm{e}-2$ | - | $1.8 \mathrm{e}-2$ |
| $1 / 2$ | $5.3 \mathrm{e}-3$ | 1.7 | $5.2 \mathrm{e}-3$ |
| $1 / 4$ | $1.8 \mathrm{e}-3$ | 1.6 | $1.5 \mathrm{e}-3$ |
| $1 / 8$ | $1.4 \mathrm{e}-3$ | 0.3 | $9.9 \mathrm{e}-4$ |
| $1 / 16$ | $2.0 \mathrm{e}-3$ | -0.5 | $1.5 \mathrm{e}-3$ |
| $1 / 32$ | $3.2 \mathrm{e}-3$ | -0.7 | $2.6 \mathrm{e}-3$ |
| $1 / 64$ | $5.6 \mathrm{e}-3$ | -0.8 | $5.0 \mathrm{e}-3$ |
| $1 / 128$ | $1.0 \mathrm{e}-2$ | -0.9 | $9.5 \mathrm{e}-3$ |
| $1 / 256$ | $1.8 \mathrm{e}-2$ | -0.8 | $1.8 \mathrm{e}-2$ |

6.2. Simulation of water waves in a water tank. In this example we consider waves generated by a piston-type wave maker. This test case is proposed in [35]. In this case we consider the spatial domain $\Omega=[0,10] \times[-1,0]$. We apply homogeneous Neumann boundary conditions on $x_{1}=10$ and $x_{2}=-1$. The wave maker is located on the left side of the domain, i.e., at $x_{1}=0$, where the boundary condition is given by

$$
\begin{equation*}
\boldsymbol{q} \cdot \boldsymbol{n}=T(t) \tag{6.2}
\end{equation*}
$$

where $T(t)=i a \exp (-a f t)$ with $a=0.05$ the amplitude of the wave and $f=1.8138$ the frequency of the wave. Only the real part of $T(t)$ produces physical solutions. We compute the solution for $t \in[0,53.4]$ and take $\Delta t=0.2$. The mesh consists of 512 prismatic elements which are constructed by extruding spatial triangles in the time direction. Figure 4 shows the free-surface elevation or wave height at different


Figure 4. Simulation of water waves in a water tank, see section 6.2. The free-surface elevation at different time levels for polynomial degree $p=1$ (solid line), $p=2$ (dashed line), and $p=3$ (dotted line).
time levels for polynomial orders $p=1, p=2$ and $p=3$. At time $t=4$, the first wave leaves the wave maker which is located at $x_{1}=0$. At $t=25.8$ the waves reach the right wall of the water tank. At time $t=53.4$, waves have hit the right wall and have started traveling in the opposite direction. We furthermore note that the discretization is less diffusive as the polynomial degree increases.

## 7. Conclusions

In this paper we presented a space-time hybridizible discontinuous Galerkin method for the mixed form of the linear free-surface problem on prismatic spacetime meshes. The use of weighted inner products allows us to show well-posedness of the discrete problem. An a priori error analysis was performed by using a projection operator tailored to the discretization. Additionally, this analysis explicitly specifies the dependency on the spatial mesh size and time step. Numerical tests verify our theoretical results.

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