

THE POINTWISE STABILITIES OF PIECEWISE LINEAR FINITE ELEMENT METHOD ON NON-OBTUSE TETRAHEDRAL MESHES OF NONCONVEX POLYHEDRA

HUADONG GAO AND WEIFENG QIU

ABSTRACT. Let Ω be a Lipschitz polyhedral (can be nonconvex) domain in \mathbb{R}^3 , and V_h denotes the finite element space of continuous piecewise linear polynomials. On non-obtuse quasi-uniform tetrahedral meshes, we prove that the finite element projection $R_h u$ of $u \in H^1(\Omega) \cap C(\bar{\Omega})$ (with $R_h u$ interpolating u at the boundary nodes) satisfies

$$\|R_h u\|_{L^\infty(\Omega)} \leq C |\log h| \|u\|_{L^\infty(\Omega)}.$$

If we further assume $u \in W^{1,\infty}(\Omega)$, then

$$\|R_h u\|_{W^{1,\infty}(\Omega)} \leq C |\log h| \|u\|_{W^{1,\infty}(\Omega)}.$$

1. INTRODUCTION

In this paper we consider the the Ritz projection $R_h u \in V_{r,h}$ of $u \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfying

$$(\nabla R_h u, \nabla v_h)_\Omega = (\nabla u, \nabla v_h)_\Omega, \quad \forall v_h \in V_{r,h}^0, \quad (1.1)$$

where $V_{r,h}$ is the finite element subspace of $H^1(\Omega)$ composed of piecewise polynomials of degree r ($r \geq 1$), $V_{r,h}^0 = H_0^1(\Omega) \cap V_{r,h}$, and $R_h u$ interpolates u at the boundary nodes on $\partial\Omega$. In fact, $R_h u$ is the finite element projection of u onto $V_{r,h}$ for the model problem

$$\Delta u = f \text{ in } \Omega, \quad (1.2)$$

with Dirichlet boundary condition on $\partial\Omega$.

Our motivation is to establish the stability in $L^\infty(\Omega)$

$$\|R_h u\|_{L^\infty(\Omega)} \leq C \|u\|_{L^\infty(\Omega)}, \quad (1.3a)$$

$$\text{or } \|R_h u\|_{L^\infty(\Omega)} \leq C |\log h| \|u\|_{L^\infty(\Omega)}; \quad (1.3b)$$

and the stability in $W^{1,\infty}(\Omega)$ (if $u \in W^{1,\infty}(\Omega)$)

$$\|R_h u\|_{W^{1,\infty}(\Omega)} \leq C \|u\|_{W^{1,\infty}(\Omega)}, \quad (1.4a)$$

$$\text{or } \|R_h u\|_{W^{1,\infty}(\Omega)} \leq C |\log h| \|u\|_{W^{1,\infty}(\Omega)}. \quad (1.4b)$$

There are a lot of important works for estimates (1.3) and (1.4). [9] and [16] are the first contributions for general quasi-uniform meshes. On convex polygonal domains, [9] considered piecewise linear ($r = 1$) approximation while [16] treated the finite element approximation (for any $r \geq 1$) to Neumann problem of (1.2). [13] proved (1.3) and (1.4) on polygonal (can be nonconvex) domains. When $r = 1$, the estimates provided in [13] are (1.3b) and (1.4b). Thus, estimates (1.3) and (1.4) are valid for most practical domains in \mathbb{R}^2 . On the contrast, in three dimensional space, all existing works [1, 3, 4, 7, 8, 11, 12, 14, 15] for estimates

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The second author is the corresponding author.

(1.3) and (1.4) are available on either domains with smooth boundary or convex polyhedral domains (Instead of explicit assumptions on domains, [1] needs $\|w\|_{W^{2,p}(\Omega)} \leq C\|\Delta w\|_{L^p(\Omega)}$ for some $p > 3$ in three dimensional space, for any function w with zero trace on $\partial\Omega$).

In this paper, we prove that if the meshes are non-obtuse (all internal dihedral angles of all tetrahedral elements are less than or equal to $\frac{\pi}{2}$), then estimates (1.3b) and (1.4b) hold for the finite element projection (1.1) with piecewise linear finite element space $V_h = V_{1,h}$ ($r = 1$).

In Section 2, we provide the main results and all assumptions. In Section 3, we show the proofs of our main results.

2. MAIN RESULTS

Let Ω be a Lipschitz polyhedra (can be nonconvex) in \mathbb{R}^3 . We denote by \mathcal{T}_h quasi-uniform conforming tetrahedral meshes of Ω . We define $V_h = H^1(\Omega) \cap P_1(\mathcal{T}_h)$ and $V_h^0 = H_0^1(\Omega) \cap V_h$. For any $u \in H^1(\Omega)$, we introduce the Ritz projection $R_h u \in V_h$ to satisfy

$$(\nabla R_h u, \nabla v_h)_\Omega = (\nabla u, \nabla v_h)_\Omega, \quad \forall v_h \in V_h^0, \quad (2.1)$$

where $R_h u$ interpolates u at the boundary nodes on $\partial\Omega$. In fact, $R_h u$ is the finite element projection of u onto V_h , and (2.1) is exactly the finite element projection (1.1) with $r = 1$.

Assumption 2.1. For any $T \in \mathcal{T}_h$, all internal dihedral angles of the tetrahedral element T are less than or equal to $\frac{\pi}{2}$. \mathcal{T}_h is called non-obtuse tetrahedral meshes of Ω .

Assumption 2.2. The mesh \mathcal{T}_h of Ω can be extended to a larger convex domain $\tilde{\Omega}$ quasi-uniformly with $\Omega \Subset \tilde{\Omega}$. We denote by $\tilde{\mathcal{T}}_h$ the extension of \mathcal{T}_h on $\tilde{\Omega}$.

Remark 2.1. We don't require $\tilde{\mathcal{T}}_h$ introduced in Assumption 2.2 to be non-obtuse for all tetrahedral elements. Only elements $T \in \mathcal{T}_h$ need to be non-obtuse.

Theorem 2.2. *If Assumption (2.1) and Assumption (2.2) hold, then there is a positive constant C such that for any $u \in H^1(\Omega) \cap C(\bar{\Omega})$,*

$$\|R_h u\|_{L^\infty(\Omega)} \leq C |\log h| \|u\|_{L^\infty(\Omega)}.$$

Theorem 2.3. *If Assumption (2.1) and Assumption (2.2) hold, then there is a positive constant C such that for any $u \in W^{1,\infty}(\Omega)$,*

$$\|R_h u\|_{W^{1,\infty}(\Omega)} \leq C |\log h| \|u\|_{W^{1,\infty}(\Omega)}.$$

3. ANALYSIS

Proof. (Proof of Theorem 2.2) Since $u \in C(\bar{\Omega})$, we denote by \tilde{u} the extension of u to $\tilde{\Omega}$, such that $u \in C_0(\tilde{\Omega})$ and $\|\tilde{u}\|_{L^\infty(\tilde{\Omega})} = \|u\|_{L^\infty(\Omega)}$. The existence of \tilde{u} satisfying the above two properties follows from the facts that $u \in C(\bar{\Omega})$ and the Whitney type extension operator \mathcal{E}_0 in Section 2.2 of Chapter 6 in [17] (see (8) and the proposition in Section 2.2 of Chapter 6 in [17]). We would like to emphasize that we don't need $\tilde{u} \in H^1(\tilde{\Omega})$.

We define $\tilde{V}_h^0 = H_0^1(\tilde{\Omega}) \cap P_1(\tilde{\mathcal{T}}_h)$. Let $\tilde{u}_h \in \tilde{V}_h^0$ satisfy

$$(\nabla \tilde{u}_h, \nabla \tilde{v}_h)_{\tilde{\Omega}} = \sum_{T \in \tilde{\mathcal{T}}_h} (-(\tilde{u}, \Delta \tilde{v}_h)_T + \langle \tilde{u}, \nabla \tilde{v}_h \cdot \vec{n} \rangle_{\partial T}), \quad \forall \tilde{v}_h \in \tilde{V}_h^0. \quad (3.1)$$

Here \vec{n} is the outward unit normal vector along ∂T for any $T \in \mathcal{T}_h$. For any $v_h \in V_h^0 = H_0^1(\Omega) \cap P_1(\mathcal{T}_h)$, we denote by $\tilde{v}_h \in \tilde{V}_h^0$ the zero extension of v_h to $\tilde{\Omega}$. By (3.1) and the definition of \tilde{v}_h , it is easy to see that

$$\begin{aligned} (\nabla \tilde{u}_h, \nabla v_h)_\Omega &= (\nabla \tilde{u}_h, \nabla \tilde{v}_h)_{\tilde{\Omega}} \\ &= \sum_{T \in \tilde{\mathcal{T}}_h} (-\langle \tilde{u}, \Delta \tilde{v}_h \rangle_T + \langle \tilde{u}, \nabla \tilde{v}_h \cdot \vec{n} \rangle_{\partial T}) \\ &= \sum_{T \in \mathcal{T}_h} (-\langle \tilde{u}, \Delta \tilde{v}_h \rangle_T + \langle \tilde{u}, \nabla \tilde{v}_h \cdot \vec{n} \rangle_{\partial T}) \\ &= \sum_{T \in \mathcal{T}_h} (-\langle u, \Delta v_h \rangle_T + \langle u, \nabla v_h \cdot \vec{n} \rangle_{\partial T}) = (\nabla u, \nabla v_h)_\Omega. \end{aligned} \quad (3.2)$$

The last equality holds since $u \in H^1(\Omega)$. On the other hand, since $\tilde{\Omega}$ is convex and $\tilde{u} \in C_0(\tilde{\Omega})$, (3.1) and [8, Theorem 12] imply that

$$\|\tilde{u}_h\|_{L^\infty(\tilde{\Omega})} \leq C |\log h| \|\tilde{u}\|_{L^\infty(\tilde{\Omega})} = C |\log h| \|u\|_{L^\infty(\Omega)}. \quad (3.3)$$

We notice that $R_h u \in V_h = H^1(\Omega) \cap P_1(\mathcal{T}_h)$ satisfies

$$(\nabla R_h u, \nabla v_h)_\Omega = (\nabla u, \nabla v_h)_\Omega, \quad \forall v_h \in V_h^0 = H_0^1(\Omega) \cap V_h.$$

Thus, by the above equation and (3.2), we have that $(R_h u - \tilde{u}_h)|_\Omega \in V_h$ and

$$(\nabla (R_h u - \tilde{u}_h), \nabla v_h)_\Omega = 0, \quad \forall v_h \in V_h^0 = H_0^1(\Omega) \cap V_h.$$

By Assumption (2.1) and [18, Theorem 3.2 and Lemma 5.1(iii)] (or by [2, 5, 6]), the above equation implies that

$$\|R_h u - \tilde{u}_h\|_{L^\infty(\Omega)} \leq \|R_h u - \tilde{u}_h\|_{L^\infty(\partial\Omega)} \leq \|u\|_{L^\infty(\partial\Omega)} + \|\tilde{u}_h\|_{L^\infty(\partial\Omega)}. \quad (3.4)$$

Thus, by (3.3) and (3.4), it is easy to see that

$$\begin{aligned} \|R_h u\|_{L^\infty(\Omega)} &\leq \|R_h u - \tilde{u}_h\|_{L^\infty(\Omega)} + \|\tilde{u}_h\|_{L^\infty(\Omega)} \\ &\leq \|u\|_{L^\infty(\partial\Omega)} + \|\tilde{u}_h\|_{L^\infty(\partial\Omega)} + \|\tilde{u}_h\|_{L^\infty(\Omega)} \\ &\leq \|u\|_{L^\infty(\Omega)} + 2\|\tilde{u}_h\|_{L^\infty(\Omega)} \leq C |\log h| \|u\|_{L^\infty(\Omega)}. \end{aligned}$$

The proof is complete. \square

Proof. (Proof of Theorem 2.3) We denote by $I_h u$ the standard interpolation of u on $V_h = H^1(\Omega) \cap P_1(\mathcal{T}_h)$.

By applying Theorem 2.2 to $u - I_h u$, we have

$$\|R_h u - I_h u\|_{L^\infty(\Omega)} \leq C |\log h| \|u - I_h u\|_{L^\infty(\Omega)}.$$

By inverse inequality and approximation properties of I_h ,

$$\begin{aligned} \|R_h u\|_{W^{1,\infty}(\Omega)} &\leq \|R_h u - I_h u\|_{W^{1,\infty}(\Omega)} + \|I_h u\|_{W^{1,\infty}(\Omega)} \\ &\leq C h^{-1} \|R_h u - I_h u\|_{L^\infty(\Omega)} + C \|u\|_{W^{1,\infty}(\Omega)} \leq C \|u\|_{W^{1,\infty}(\Omega)}. \end{aligned}$$

The proof is complete. \square

4. DECLARATIONS

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REFERENCES

- [1] S. Brenner and L. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, New York, 2002.
- [2] P.G. Ciarlet and P.A. Raviart, *Maximum principle and uniform convergence for the finite element method*, Comput. Methods Appl. Mech. Engrg., 2:17–31, 1973.
- [3] A. Demlow, D. Leykekhman, A.H. Schatz and L.B. Wahlbin, *Best approximation property in the W_∞^1 norm on graded meshes*, Math. Comp. 81:743–764, 2012.
- [4] J. Guzmán, D. Leykekhman, J. Rossmann and A.H. Schatz, *Hölder estimates for Green's functions on convex polyhedral domains and their applications to finite element methods*, Numer. Math., 112:221–243, 2009.
- [5] S. Korotov, M. Křížek and P. Neittaanmäki, *Weakened acute type condition for tetrahedral triangulations and the discrete maximum principle*, Math. Comp., 70(233):107–119, 2000.
- [6] M. Křížek and L. Qun, *On diagonal dominance of stiffness matrices in 3D*, East-West J. Numer. Math., 3(1):59–69, 1995.
- [7] D. Leykekhman and B. Li, *Weak discrete maximum principle of finite element methods in convex polyhedra*, Math. Comp., 90:1–18, 2021.
- [8] D. Leykekhman and B. Vexler, *Finite element pointwise results on convex polyhedral domains*, SIAM J Numer. Anal., 54(2):561–587, 2016.
- [9] F. Natterer, *Über die punktweise Konvergenz finiter Elemente*, Numer. Math. , 25:67–77, 1975.
- [10] J.A. Nitsche, *L_∞ Convergence of Finite Element Approximations*, Mathematical Aspects of Finite Element Methods. Lecture Notes in Math., vol. 606, pp. 261–274. Springer, Berlin (1977).
- [11] R. Rannacher, *Zur L^∞ -Konvergenz linearer finiter Elemente beim Dirichlet-Problem*, Math.Zeitschrift, 149:69–77, 1976.
- [12] R. Rannacher and R. Scott, *Some optimal error estimates for piecewise linear finite element approximations*, Math. Comp., 148:437–445, 1982.
- [13] A.H. Schatz, *A weak discrete maximum principle and stability of the finite element method in L^∞ on the plane polygonal domains. I*, Math. Comp., 34:77–91, 1980.
- [14] A.H. Schatz, *Pointwise error estimates and asymptotic error expansion inequalities for the finite element method on irregular grids: Part 1*, Math. Comp., 67:877–899, 1998.
- [15] A.H. Schatz and L.B. Wahlbin, *On the quasi-optimality in L_∞ of the \hat{H}^1 -projection into finite element spaces*, Math. Comp., 157:1–22, 1982.
- [16] R. Scott, *Optimal L_∞ estimates for the finite element method*, Math. Comp., 30:681–697, 1976.

- [17] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.
- [18] J. Wang and R. Zhang, *Maximum principles for P_1 -conforming finite element approximation of quasi-linear second order elliptic equations*, SIAM J Numer. Anal., 50(2):626–642, 2012.

SCHOOL OF MATHEMATICS AND STATISTICS, HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY,
WUHAN 430074, CHINA

Email address: huadong@hust.edu.cn

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF HONG KONG, KOWLOON, HONG KONG, CHINA

Email address: weifeqiu@cityu.edu.hk