Temporally semidiscrete approximation of a Dirichlet boundary control for a fractional/normal evolution equation with a final observation *

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Abstract

Optimal Dirichlet boundary control for a fractional/normal evolution with a final observation is considered. The unique existence of the solution and the first-order optimality condition of the optimal control problem are derived. The convergence of a temporally semidiscrete approximation is rigorously established, where the control is not explicitly discretized and the state equation is discretized by a discontinuous Galerkin method in time. Numerical results are provided to verify the theoretical results.

Keywords: Dirichlet boundary control; fractional evolution equation; discontinuous Galerkin method; convergence.

1 Introduction

There is an extensive literature on the numerical optimization with PDE constraints. So far, most of the literature focuses on the distributed control problems, and the works on the Dirichlet boundary control problems are rather limited. Compared with the distributed control problems, the Dirichlet boundary control problems are more challenging in the following senses. Firstly, the solution of the state equation of a Dirichlet boundary control problem is of significantly lower regularity than that of a distributed control problem, and this increases the difficulty in both theoretical and numerical analysis. Secondly, in the weak form of the solution of the state equation of a Dirichlet boundary control problem, the test function space is more regular than the trial function space, and hence the weak form is not appropriate for the discretization. Thirdly, since the normal derivative of the adjoint state occurs in the first-order optimality condition, the discrete first-order optimality condition will essentially involve the discrete normal derivative of the discrete adjoint state, and this increases the implementation difficulty.

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We summarize the works on the parabolic Dirichlet boundary control problems briefly as follows. Using an integral representation formula derived by the semigroup theory (cf. [3, Section 4.12] and [30]), Lasiecka [29, 31] analyzed spatial Galerkin approximations of an optimal Dirichlet boundary control problem and a time optimal Dirichlet boundary control problem for the parabolic equations. Kunisch and Vexler [28] analyzed constrained Dirichlet boundary control problems for a class of parabolic equations and derived the convergence of the PDAS strategy for two Dirichlet boundary control problems. Applying the Robin penalization method to a Dirichlet boundary control problem for a parabolic equation with a final observation, Belgacem et al. [4] obtained a penalized Robin boundary control problem. Gong et al. [16] analyzed a finite element approximation of a Dirichlet boundary control for a parabolic equation, where the variational discretization approach [19] was used and the state equation was discretized by the usual H^1 -conforming P1-element in space and discretized by the dG(0) scheme in time. Recently, Gong and Li [17] improved the spatial accuracy derived in [16], using the maximal L^p -regularity theory. We note that, for the state equation with rough Dirichlet boundary data, [29, 31] used the semigroup theory to define the solution whereas [16, 17, 28] used the transposition technique to define the solution (called the very weak solution).

For the numerical analysis of parabolic Neumann/Robin boundary control problems, we refer the reader to [1, 7, 27, 44]. For the numerical analysis of other optimal control problems for parabolic equations, we refer the reader to [8, 15, 34, 35, 47, 48, 49, 50] and the references therein. Although the spatial discretization is not considered in this paper, we would like to refer the reader to [5, 13, 14, 32] for the numerical analysis of elliptic and parabolic equations with rough Dirichlet boundary data.

To our best knowledge, no convergence result is available for the Galerkintype approximations of the Dirichlet boundary control problems governed by the parabolic equations with final observations. The fractional evolution equation is an extension of the normal evolution equation, widely used to describe the physical phenomena with memory effect [55]. Recently, Harbir et al. [2] studied an optimal distributed control problem for a space-time fractional diffusion equation. For the numerical analysis of the optimal distributed problems governed by the time fractional diffusion equations, we refer the reader to [18, 26, 38, 61]. To our knowledge, no numerical analysis is available for the Dirichlet boundary control problems governed by the fractional evolution equations. Hence, this paper tries to analyze the Dirichlet boundary control problems for the fractional and normal evolution equations in a unified way.

In this paper, we establish the convergence of a temporally semidiscrete approximation of an abstract optimal control problem governed by a fractional/normal evolution equation with a final observation. This approximation uses the variational discretization concept [19] and uses a discontinuous Galerkin method to discretize the state equation in time. The discontinuous Galerkin method is the famous dG(0) scheme for the normal evolution equation, and is equivalent to the well-known L1 scheme [39] with uniform temporal grids for the fractional evolution equations. The derived numerical analysis is applied to a Dirichlet boundary control problem. We note that there are many works (see [20, 21, 22, 23, 25, 36, 41, 43] and the references therein) devoted to the numerical analysis of the fractional diffusion equations with rough initial value and source term, but, to our knowledge, no numerical analysis is available for the fractional diffusion equation with rough Dirichlet boundary value. This paper also fills in this gap.

The rest of this paper is organized as follows. Section 2 establishes the convergence of a temporally semidiscrete approximation of an abstract optimal control problem. Section 3 applies the theory developed in the previous section to a Dirichlet boundary control problem. Section 4 performs three numerical experiments to confirm the theoretical results.

2 An abstract optimal control problem

2.1 Preliminaries

We will use the following conventions: for each linear vector space, the field of the scalars is \mathbb{C} ; for a Hilbert space \mathcal{X} , we use $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ to denote its inner product; for a Banach space \mathcal{B} , we use $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ to denote a duality paring between \mathcal{B}^* (the dual space of \mathcal{B}) and \mathcal{B} ; for a linear operator A, $\rho(A)$ denotes the resolvent set of A and R(z, A) denotes the inverse of z - A for each $z \in \rho(A)$; for two Banach spaces \mathcal{B}_1 and \mathcal{B}_2 , $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is the set of all bounded linear operators from \mathcal{B}_1 to \mathcal{B}_2 , and $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_1)$ is abbreviated to $\mathcal{L}(\mathcal{B}_1)$; I denotes the identity map; for a Lebesgue measurable subset $\mathcal{D} \subset \mathbb{R}^l$, $1 \leq l \leq 4$, $\langle p, q \rangle_{\mathcal{D}}$ means the integral $\int_{\mathcal{D}} p\overline{q}$, where \overline{q} is the conjugate of q; for a function v defined on (0, T), by v(t-), $0 < t \leq T$, we mean the limit $\lim_{s \to t^-} v(s)$; the notation C_{\times} means a positive constant, depending only on its subscript(s), and its value may differ at each occurrence; for any $0 < \theta < \pi$, define

$$\Sigma_{\theta} := \{ r e^{i\gamma} : r > 0, -\theta < \gamma < \theta \}, \tag{1}$$

$$\Gamma_{\theta} := \{ re^{-i\theta} : r \ge 0 \} \cup \{ re^{i\theta} : r > 0 \}$$

$$(2)$$

$$\Upsilon_{\theta} := \{ z \in \Gamma_{\theta} : -\pi \leqslant \Im z \leqslant \pi \}, \tag{3}$$

where *i* is the imaginary unit and Γ_{θ} and Υ_{θ} are so oriented that the negative real axis is to their left.

2.1.1 Time fractional Sobolev spaces

Assume that $-\infty < a < b < \infty$ and \mathcal{B} is a Banach space. Define

$${}_{0}H^{1}(a,b;\mathcal{B}) := \left\{ v \in L^{2}(a,b;\mathcal{B}) : v' \in L^{2}(a,b;\mathcal{B}), v(a) = 0 \right\},\$$

$${}^{0}H^{1}(a,b;\mathcal{B}) := \left\{ v \in L^{2}(a,b;\mathcal{B}) : v' \in L^{2}(a,b;\mathcal{B}), v(b) = 0 \right\},\$$

and endow them with the two norms

$$\begin{aligned} \|v\|_{0H^{1}(a,b;\mathcal{B})} &:= \|v'\|_{L^{2}(a,b;\mathcal{B})} \quad \forall v \in {}_{0}H^{1}(a,b;\mathcal{B}), \\ \|v\|_{0H^{1}(a,b;\mathcal{B})} &:= \|v'\|_{L^{2}(a,b;\mathcal{B})} \quad \forall v \in {}^{0}H^{1}(a,b;\mathcal{B}), \end{aligned}$$

respectively, where v^\prime is the first-order weak derivative of v.

For each
$$0 < \gamma < 1$$
, define

$${}_{0}H^{\gamma}(a,b;\mathcal{B}) := (L^{2}(a,b;\mathcal{B}), {}_{0}H^{1}(a,b;\mathcal{B}))_{\gamma,2},$$

$${}^{0}H^{\gamma}(a,b;\mathcal{B}) := (L^{2}(a,b;\mathcal{B}), {}^{0}H^{1}(a,b;\mathcal{B}))_{\gamma,2},$$

where $(\cdot, \cdot)_{\gamma,2}$ means the interpolation space defined by the *K*-method (cf. [42]). For convenience, the spaces $_{0}H^{\gamma}(a, b; \mathbb{C})$ and $^{0}H^{\gamma}(a, b; \mathbb{C})$ will be abbreviated to $_{0}H^{\gamma}(a, b)$ and $^{0}H^{\gamma}(a, b)$, respectively.

2.1.2 Riemann-Liouville fractional calculus operators

Assume that $-\infty < a < b < \infty$ and \mathcal{X} is a separable Hilbert space. For any $0 < \gamma < 1$, define

$$\left(\mathbf{D}_{a+}^{-\gamma} v \right)(t) := \frac{1}{\Gamma(\gamma)} \int_{a}^{t} (t-s)^{\gamma-1} v(s) \, \mathrm{d}s, \quad \text{a.e. } t \in (a,b),$$
$$\left(\mathbf{D}_{b-}^{-\gamma} v \right)(t) := \frac{1}{\Gamma(\gamma)} \int_{t}^{b} (s-t)^{\gamma-1} v(s) \, \mathrm{d}s, \quad \text{a.e. } t \in (a,b),$$

for all $v \in L^1(a, b; \mathcal{X})$, where $\Gamma(\cdot)$ is the gamma function. In addition, let D^0_{a+} and D_{b-}^0 be the identity operator on $L^1(a,b;\mathcal{X})$. For any $0 < \gamma \leq 1$, define

$$D_{a+}^{\gamma} v := D D_{a+}^{\gamma-1} v,$$
$$D_{b-}^{\gamma} v := -D D_{b-}^{\gamma-1} v.$$

for all $v \in L^1(a, b; \mathcal{X})$, where D is the first-order differential operator in the distribution sense.

Assume that $0 < \gamma < 1$. For any $v \in {}_{0}H^{\gamma}(a, b; \mathcal{X})$ and $w \in {}^{0}H^{\gamma}(a, b; \mathcal{X})$, we have

$$C_{1} \|v\|_{0H^{\gamma}(a,b;\mathcal{X})} \leq \|\mathbb{D}_{a+}^{\gamma}v\|_{L^{2}(a,b;\mathcal{X})} \leq C_{2} \|v\|_{0H^{\gamma}(a,b;\mathcal{X})},$$

$$C_{1} \|w\|_{0H^{\gamma}(a,b;\mathcal{X})} \leq \|\mathbb{D}_{b-}^{\gamma}w\|_{L^{2}(a,b;\mathcal{X})} \leq C_{2} \|w\|_{0H^{\gamma}(a,b;\mathcal{X})}.$$

where C_1 and C_2 are two positive constants depending only on γ . Let \mathcal{X}^* be the dual space of \mathcal{X} . For any $v \in {}_{0}H^{\gamma/2}(a,b;\mathcal{X}^{*})$ and $w \in {}^{0}H^{\gamma/2}(a,b;\mathcal{X})$, the equality

$$\int_0^T \langle \mathbf{D}_{a+}^{\gamma} v, w \rangle_{\mathcal{X}} \, \mathrm{d}t = \int_0^T \langle v, \mathbf{D}_{b-}^{\gamma} w \rangle_{\mathcal{X}} \, \mathrm{d}t \tag{4}$$

holds for the following two cases: $v \in {}_{0}H^{\gamma}(a, b; \mathcal{X}^{*})$ and $w \in {}^{0}H^{\gamma}(a, b; \mathcal{X});$ $\mathbf{D}_{a+}^{\gamma} v \in L^{2/(1+\gamma)}(a, b; \mathcal{X}^{*})$ and $\mathbf{D}_{b-}^{\gamma} v \in L^{2/(1+\gamma)}(a, b; \mathcal{X}).$ For the above theoretical results, we refer the reader to [11, 43].

2.1.3 Definitions of A and A^*

Assume that X and Y are two separable Hilbert spaces such that X is continuously embedded into Y and X is dense in Y. We will regard Y as a subspace of X^* , the dual space of X, in the sense that

$$\langle v, w \rangle_X := (v, w)_Y$$
 for all $v \in Y$ and $w \in X$.

Let \mathcal{A} and \mathcal{A}^* be two bounded linear operators from X to Y satisfying that

$$\rho(\mathcal{A}) \supset \Sigma_{\omega_0} \cup \{0\}, \quad \rho(\mathcal{A}^*) \supset \Sigma_{\omega_0} \cup \{0\}, \tag{5a}$$

$$\begin{cases} \|R(z,\mathcal{A})\|_{\mathcal{L}(Y)} \leqslant \frac{\mathcal{M}_{0}}{1+|z|} \quad \forall z \in \Sigma_{\omega_{0}}, \qquad (5b) \\ \|R(z,\mathcal{A}^{*})\|_{\mathcal{L}(Y)} \leqslant \frac{\mathcal{M}_{0}}{1+|z|} \quad \forall z \in \Sigma_{\omega_{0}}, \qquad (5c) \\ (\mathcal{A}v,w)_{Y} = (v,\mathcal{A}^{*}w)_{Y} \quad \forall v,w \in X, \qquad (5d) \\ c_{0}\|v\|_{X} \leqslant \|\mathcal{A}v\|_{Y} \leqslant c_{1}\|v\|_{X} \quad \forall v \in X, \qquad (5e) \\ c_{0}\|v\|_{X} \leqslant \|\mathcal{A}^{*}v\|_{Y} \leqslant c_{1}\|v\|_{X} \quad \forall v \in X, \qquad (5f) \\ (\mathcal{A}v,v)_{Y} \geqslant 0 \quad \forall v \in X, \qquad (5g) \end{cases}$$

$$\|R(z,\mathcal{A}^*)\|_{\mathcal{L}(Y)} \leqslant \frac{\mathcal{M}_0}{1+|z|} \quad \forall z \in \Sigma_{\omega_0},$$
(5c)

$$(\mathcal{A}v, w)_Y = (v, \mathcal{A}^* w)_Y \quad \forall v, w \in X,$$
(5d)

$$c_0 \|v\|_X \leqslant \|\mathcal{A}v\|_Y \leqslant c_1 \|v\|_X \quad \forall v \in X,$$
(5e)

(5f)

(5g)

where $\pi/2 < \omega_0 < \pi$, c_0 , c_1 and \mathcal{M}_0 are four positive constants. By the transposition technique, \mathcal{A} and \mathcal{A}^* can be extended as two bounded linear operators from Y to X^* by

$$\langle \mathcal{A}v, w \rangle_X := (v, \mathcal{A}^* w)_Y, \tag{6}$$

$$\langle \mathcal{A}^* v, w \rangle_X := (v, \mathcal{A}w)_Y, \tag{7}$$

for all $v \in Y$ and $w \in X$.

For each $0 \leq \theta \leq 1$, let $[X^*, Y]_{\theta}$ and $[X, Y]_{\theta}$ be the interpolation spaces defined by the famous complex interpolation method (cf. [42, Chapter 2]). We have that $[X^*, Y]_{\theta}$ is the dual space of $[X, Y]_{\theta}$ and vice versa (cf. [6]). By (5e), (5f) and [42, Theorem 2.6], a straightforward computation gives that

$$\|\mathcal{A}\|_{\mathcal{L}([X,Y]_{\theta},[X^*,Y]_{1-\theta})} \leqslant c_1 \quad \text{for all } 0 \leqslant \theta \leqslant 1.$$
(8)

Lemma 2.1. Assume that $0 \leq \theta \leq 1$ and $z \in \Sigma_{\omega_0}$. Then

$$\|R(z,\mathcal{A})\|_{\mathcal{L}(Y,[X,Y]_{\theta})} \leqslant \frac{C_{c_0,\mathcal{M}_0}}{1+|z|^{\theta}},\tag{9}$$

$$|R(z,\mathcal{A}^*)||_{\mathcal{L}(Y,[X,Y]_{\theta})} \leqslant \frac{C_{c_0,\mathcal{M}_0}}{1+|z|^{\theta}},\tag{10}$$

$$\|R(z,\mathcal{A})\|_{\mathcal{L}([X^*,Y]_{\theta},Y)} \leqslant \frac{C_{c_0,\mathcal{M}_0}}{1+|z|^{\theta}},\tag{11}$$

$$\|R(z,\mathcal{A}^*)\|_{\mathcal{L}([X^*,Y]_{\theta},Y)} \leqslant \frac{C_{c_0,\mathcal{M}_0}}{1+|z|^{\theta}}.$$
(12)

Moreover, for any $0 \leq \epsilon \leq 1$,

$$\|R(z,\mathcal{A})\|_{\mathcal{L}([X^*,Y]_{\theta},[X,Y]_{1-(1-\epsilon)\theta})} \leqslant \frac{C_{c_0,\mathcal{M}_0,\epsilon,\theta}}{1+|z|^{\epsilon\theta}}.$$
(13)

Proof. A straightforward computation gives

$$\|\mathcal{A}R(z,\mathcal{A})\|_{\mathcal{L}(Y)} = \|zR(z,\mathcal{A}) - I\|_{\mathcal{L}(Y)} \leq |z| \|R(z,\mathcal{A})\|_{\mathcal{L}(Y)} + 1 \leq 1 + \mathcal{M}_0,$$

by (5b), so that (5e) implies

$$\|R(z,\mathcal{A})\|_{\mathcal{L}(Y,X)} \leqslant \frac{1+\mathcal{M}_0}{c_0}.$$

By this estimate and (5b), using [42, Theorem 2.6] yields (9). Estimate (10) can be proved analogously.

Then let us prove (11). We first consider the following problem: seek $v \in Y$ such that

$$(v, (\overline{z} - \mathcal{A}^*)\varphi)_Y = \langle g, \varphi \rangle_X \quad \forall \varphi \in X,$$
(14)

where $g \in X^*$ is arbitrary but fixed. By (5f) and the fact that X is continuously embedded into Y, we conclude that $(\cdot, (\overline{z} - \mathcal{A}^*) \cdot)_Y$ is a continuous bilinear form on $Y \times X$. Inserting $\theta = 0$ into (10) implies that, for any $v \in Y \setminus \{0\}$,

$$\sup_{\varphi \in X \setminus \{0\}} \frac{|(v, (\overline{z} - \mathcal{A}^*)\varphi)_Y|}{\|\varphi\|_X} \ge \frac{\|v\|_Y^2}{\|R(\overline{z}, \mathcal{A}^*)v\|_X} \ge C_{c_0, \mathcal{M}_0} \|v\|_Y.$$

Since $z \in \rho(\mathcal{A}^*)$, it is evident that, for any $\varphi \in X \setminus \{0\}$,

$$\sup_{v \in Y} |(v, (\overline{z} - \mathcal{A}^*)\varphi)_Y| > 0$$

Consequently, the Babuska-Lax-Milgram theorem yields that problem (14) admits a unique solution $v \in Y$ and $||v||_Y \leq C_{c_0,\mathcal{M}_0}||g||_{X^*}$. Since (6) and (14) imply $v = R(z, \mathcal{A})g$, this indicates that

$$||R(z,A)g||_Y \leqslant C_{c_0,\mathcal{M}_0} ||g||_{X^*} \quad \forall g \in X^*,$$

and hence

$$||R(z,A)||_{\mathcal{L}(X^*,Y)} \leqslant C_{c_0,\mathcal{M}_0}.$$
(15)

By (5b) and (15), using [42, Theorem 2.6] yields (11). Estimate (12) is derived similarly.

Finally, let us prove (13). Inserting $\theta = \epsilon$ and $\theta = 0$ into (9) and (11) respectively yields

$$\|R(z,\mathcal{A})\|_{\mathcal{L}(Y,[X,Y]_{\epsilon})} \leqslant \frac{C_{c_0,\mathcal{M}_0}}{1+|z|^{\epsilon}}, \quad \|R(z,\mathcal{A})\|_{\mathcal{L}(X^*,Y)} \leqslant C_{c_0,\mathcal{M}_0}.$$

Using [42, Theorem 2.6] then gives

$$\|R(z,\mathcal{A})\|_{\mathcal{L}([X^*,Y]_{\theta},[Y,[X,Y]_{\epsilon}]_{\theta})} \leqslant \frac{C_{c_0,\mathcal{M}_0}}{1+|z|^{\epsilon\theta}}.$$

Hence, by the fact that (cf. [6])

 $[Y, [X, Y]_{\epsilon}]_{\theta} = [X, Y]_{1-(1-\epsilon)\theta}$ with equivalent norms,

we readily obtain (13). This completes the proof.

Remark 2.1. For any $z \in \Sigma_{\omega_0}$,

$$\begin{split} \|R(z,\mathcal{A})\|_{\mathcal{L}(X^*)} &= \|z^{-1}(z-\mathcal{A}+\mathcal{A})R(z,\mathcal{A})\|_{\mathcal{L}(X^*)} \\ &= \|I+\mathcal{A}R(z,\mathcal{A})\|_{\mathcal{L}(X^*)}/|z| \\ &\leqslant \frac{1+\|\mathcal{A}R(z,\mathcal{A})\|_{\mathcal{L}(X^*)}}{|z|} \\ &\leqslant \frac{1+c_1\|R(z,\mathcal{A})\|_{\mathcal{L}(X^*,Y)}}{|z|} \quad (by \ inserting \ \theta = 1 \ into \ (8)) \\ &\leqslant \frac{C_{c_0,c_1,\mathcal{M}_0}}{|z|} \quad (by \ inserting \ \theta = 0 \ into \ (11)). \end{split}$$

Also, we have

$$\begin{aligned} \|R(z,\mathcal{A})\|_{\mathcal{L}(X^*)} &= \|\mathcal{A}\mathcal{A}^{-1}R(z,\mathcal{A})\|_{\mathcal{L}(X^*)} \\ &\leqslant c_1 \|\mathcal{A}^{-1}R(z,\mathcal{A})\|_{\mathcal{L}(X^*,Y)} \quad (by \ inserting \ \theta = 1 \ into \ (8)) \\ &\leqslant C_{c_1,\mathcal{M}_0} \|R(z,\mathcal{A})\|_{\mathcal{L}(X^*,Y)} \quad (by \ (5b)) \\ &\leqslant C_{c_0,c_1,\mathcal{M}_0} \quad (by \ inserting \ \theta = 0 \ into \ (11)). \end{aligned}$$

Consequently,

$$\|R(z,\mathcal{A})\|_{\mathcal{L}(X^*)} \leqslant \frac{C_{c_0,c_1,\mathcal{M}_0}}{1+|z|} \quad \forall z \in \Sigma_{\omega_0}.$$
 (16)

2.1.4 Definitions of E_{α} and E_{α}^*

For any $0 < \alpha \leq 1$ and t > 0, define

$$E_{\alpha}(t) := \frac{1}{2\pi i} \int_{\Gamma_{\omega_0}} e^{tz} R(z^{\alpha}, \mathcal{A}) \,\mathrm{d}z, \qquad (17)$$

$$E^*_{\alpha}(t) := \frac{1}{2\pi i} \int_{\Gamma_{\omega_0}} e^{t\overline{z}} R(\overline{z^{\alpha}}, \mathcal{A}^*) \, \overline{\mathrm{d}z}.$$
 (18)

For any $v \in X^*$ and $w \in Y$, by the definitions of \mathcal{A} and \mathcal{A}^* and Lemma 2.1 we have that

$$(R(z,\mathcal{A})v,w)_Y = \langle v, R(\overline{z},\mathcal{A}^*)w \rangle_X \quad \forall z \in \Sigma_{\omega_0},$$

and hence from (17) and (18) we obtain

$$(E_{\alpha}(t)v, w)_{Y} = \langle v, E_{\alpha}^{*}(t)w \rangle_{X} \quad \forall t > 0.$$
(19)

Furthermore, by Lemma 2.1, a routine calculation (cf. [59, 23]) yields the following lemma.

Lemma 2.2. Assume that $0 < \alpha \leq 1$, $0 \leq \theta \leq 1$, t > 0, and $G = E_{\alpha}$ or E_{α}^* . Then

$$\|G(t)\|_{\mathcal{L}(Y,[X,Y]_{\theta})} \leqslant C_{c_0,\omega_0,\mathcal{M}_0} t^{\theta\alpha-1},$$
(20)

$$\|G(t)\|_{\mathcal{L}([X^*,Y]_{\theta},Y)} \leqslant C_{c_0,\omega_0,\mathcal{M}_0} t^{\theta\alpha-1},$$
(21)

$$\|G'(t)\|_{\mathcal{L}(Y,[X,Y]_{\theta})} \leqslant C_{c_0,\omega_0,\mathcal{M}_0} t^{\theta\alpha-2},$$
(22)

$$\|G'(t)\|_{\mathcal{L}([X^*,Y]_{\theta},Y)} \leqslant C_{c_0,\omega_0,\mathcal{M}_0} t^{\theta\alpha-2}.$$
(23)

Moreover, for any $0 \leq \epsilon \leq 1$,

$$E_{\alpha} \in C((0,\infty); \mathcal{L}([X^*,Y]_{\theta}, [X,Y]_{1-(1-\epsilon)\theta})$$
(24)

and

$$\|E_{\alpha}(t)\|_{\mathcal{L}([X^*,Y]_{\theta},[X,Y]_{1-(1-\epsilon)\theta})} \leqslant C_{c_0,\omega_0,\mathcal{M}_0,\epsilon,\theta} t^{\epsilon\theta\alpha-1}.$$
(25)

Remark 2.2. By (16) we have

$$||E_{\alpha}(t)||_{\mathcal{L}(X^*)} \leqslant C_{c_0,c_1,\omega_0,\mathcal{M}_0} t^{\alpha-1}, \quad \forall t > 0.$$

2.1.5 Solutions of the fractional evolution equations

Following the mild solution theory of fractional/normal evolution equations in [54, 41, 46, 23], we introduce the following mild solutions. Assume that $0 < \alpha \leq 1$ and $0 < T < \infty$. For any

$$g \in L^1(0,T; [X^*,Y]_{\theta}) \text{ with } 0 \leqslant \theta \leqslant 1,$$

we call

$$(\mathcal{S}_{\alpha}g)(t) := \int_0^t E_{\alpha}(t-s)g(s) \,\mathrm{d}s, \quad \text{a.e. } 0 < t \leqslant T,$$
(26)

the mild solution to the following fractional evolution equation:

$$(D_{0+}^{\alpha} - \mathcal{A})w = g, \quad w(0) = 0.$$
 (27)

For any $v \in [X^*, Y]_{\theta}$, $0 \leq \theta \leq 1$, we call

$$(\mathcal{S}_{\alpha}(v\delta_0))(t) := E_{\alpha}(t)v, \quad 0 < t \leq T,$$
(28)

the mild solution to (27) with $g = v\delta_0$, where δ_0 is the Dirac measure in time concentrated at t = 0. Symmetrically, for any

$$g \in L^1(0,T; [X^*,Y]_{\theta}) \text{ with } 0 \leq \theta \leq 1,$$

we call

$$(\mathcal{S}^*_{\alpha}g)(t) := \int_t^T E^*_{\alpha}(s-t)g(s) \,\mathrm{d}s, \quad \text{a.e. } 0 < t < T,$$
(29)

the mild solution to the following backward fractional evolution equation:

$$(D_{T-}^{\alpha} - \mathcal{A}^*)w = g, \quad w(T) = 0.$$
 (30)

For any $v \in [X^*, Y]_{\theta}$, $0 \leq \theta \leq 1$, we call

$$(\mathcal{S}^*_{\alpha}(v\delta_T))(t) := E^*_{\alpha}(T-t)v, \quad 0 < t \leq T,$$
(31)

the mild solution to equation (30) with $g = v\delta_T$, where δ_T is the Dirac measure in time concentrated at t = T.

Lemma 2.3. Assume that $0 < \alpha, \theta \leq 1$ and $q > 1/(\theta \alpha)$. Then

$$\mathcal{S}_{\alpha} \in \mathcal{L}(L^2(0,T;[X^*,Y]_{\theta}), L^2(0,T;Y)), \tag{32}$$

$$\mathcal{S}_{\alpha} \in \mathcal{L}(L^q(0,T; [X^*,Y]_{\theta}), C([0,T];Y)).$$
(33)

Moreover, for any $g \in L^q(0,T; [X^*,Y]_{\theta})$ and $v \in Y$,

$$((\mathcal{S}_{\alpha}g)(T), v)_Y = \int_0^T \langle g(t), (\mathcal{S}^*_{\alpha}(v\delta_T))(t) \rangle_X \, \mathrm{d}t.$$
(34)

Proof. By (21) and (26), a routine argument (cf. [9, Theorem 2.6]) yields (32) and (33). Note that (26) and (33) imply

$$(\mathcal{S}_{\alpha}g)(T) = \int_0^T E_{\alpha}(T-t)g(t) \,\mathrm{d}t,$$

and hence

$$((\mathcal{S}_{\alpha}g)(T), v)_{Y} = \left(\int_{0}^{T} E_{\alpha}(T-t)g(t) \,\mathrm{d}t, v\right)_{Y} = \int_{0}^{T} (E_{\alpha}(T-t)g(t), v)_{Y} \,\mathrm{d}t$$
$$= \int_{0}^{T} \langle g(t), E_{\alpha}^{*}(T-t)v \rangle_{X} \,\mathrm{d}t \quad (\mathrm{by} \ (19))$$
$$= \int_{0}^{T} \langle g(t), (\mathcal{S}_{\alpha}^{*}(v\delta_{T}))(t) \rangle_{X} \,\mathrm{d}t \quad (\mathrm{by} \ (31)).$$

This proves (34) and hence this lemma.

Lemma 2.4. For any $0 < \theta \leq 1$,

$$\lim_{\alpha \to 1-} \|\mathcal{S}_{\alpha} - \mathcal{S}_{1}\|_{\mathcal{L}(L^{1}(0,T;[X^{*},Y]_{\theta}), L^{1}(0,T;Y))} = 0,$$
(35)

$$\lim_{\alpha \to 1^{-}} \|\mathcal{S}_{\alpha} - \mathcal{S}_{1}\|_{\mathcal{L}(L^{\infty}(0,T;[X^{*},Y]_{\theta}), C([0,T];Y))} = 0.$$
(36)

Proof. Since

$$R(z,\mathcal{A}) - R(z^{\alpha},\mathcal{A}) = (z^{\alpha} - z)R(z,\mathcal{A})R(z^{\alpha},\mathcal{A}) \quad \text{for all } z \in \Sigma_{\omega_0},$$

a straightforward calculation gives, by (17) and Lemma 2.1, that

$$\|(E_{\alpha} - E_{1})(t)\|_{\mathcal{L}([X^{*}, Y]_{\theta}, Y)} \leqslant C_{c_{0}, \mathcal{M}_{0}} \int_{0}^{\infty} e^{t \cos \omega_{0} r} \frac{|re^{i\omega_{0}} - (re^{i\omega_{0}})^{\alpha}|}{(1+r)(1+r^{\theta\alpha})} \,\mathrm{d}r$$

for all t > 0. It follows that

$$\|E_{\alpha} - E_1\|_{L^1(0,T;\mathcal{L}([X^*,Y]_{\theta},Y))} \leqslant C_{c_0,\omega_0,\mathcal{M}_0} \int_0^{\infty} \frac{|re^{i\omega_0} - (re^{i\omega_0})^{\alpha}|}{r(1+r)(1+r^{\theta\alpha})} \,\mathrm{d}r.$$

Then Lebesgue's dominated convergence theorem yields

$$\lim_{\alpha \to 1^{-}} \|E_{\alpha} - E_{1}\|_{L^{1}(0,T;\mathcal{L}([X^{*},Y]_{\theta},Y))} = 0.$$
(37)

Since Young's inequality implies

$$\|\mathcal{S}_{\alpha} - \mathcal{S}_{1}\|_{\mathcal{L}(L^{1}(0,T;[X^{*},Y]_{\theta}),L^{1}(0,T;Y)} \leq \|\mathcal{E}_{\alpha} - E_{1}\|_{L^{1}(0,T;\mathcal{L}([X^{*},Y]_{\theta}))},$$

by (37) we readily obtain (35). Moreover, by (26) and (33) we have

$$\|\mathcal{S}_{\alpha} - \mathcal{S}_{1}\|_{\mathcal{L}(L^{\infty}(0,T;[X^{*},Y]_{\theta},Y),C([0,T];Y))} \leq \|E_{\alpha} - E_{1}\|_{L^{1}(0,T;\mathcal{L}([X^{*},Y]_{\theta},Y))},$$

so that (37) proves (36). This completes the proof.

Lemma 2.5. Assume that $0 < \alpha, \theta \leq 1$. Then for any $g \in C([0,T]; [X^*,Y]_{\theta})$ we have

$$(\mathcal{D}^{\alpha}_{0+} - \mathcal{A})\mathcal{S}_{\alpha}g = g \tag{38}$$

and

$$\| \mathcal{D}_{0+}^{\alpha} \mathcal{S}_{\alpha} g \|_{C([0,T];[X^*,Y]_{(1-\epsilon)\theta})} + \| \mathcal{A} \mathcal{S}_{\alpha} g \|_{C([0,T];[X^*,Y]_{(1-\epsilon)\theta})}$$

$$\leq c \| g \|_{C([0,T];[X^*,Y]_{\theta})},$$
(39)

where $0 < \epsilon < 1$ and c is a positive constant independent of g.

Proof. Since a complete rigorous proof of this lemma is tedious but standard (cf. [54]), we only present briefly the key ingredients of the proof.

Step 1. Define

$$\eta(t) := \frac{1}{2\pi i} \int_{\Gamma_{\omega_0}} e^{tz} z^{\alpha - 1} R(z^\alpha, \mathcal{A}) \,\mathrm{d}z, \quad t > 0.$$

$$\tag{40}$$

A straightforward computation gives that, for any $0 < t \leq T$,

$$\begin{split} \eta(t) &= \frac{1}{2\pi i} \int_{\Gamma_{\omega_0}} e^{tz} z^{-1} (z^{\alpha} - \mathcal{A} + \mathcal{A}) R(z^{\alpha}, \mathcal{A}) \, \mathrm{d}z \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\omega_0}} e^{tz} z^{-1} I \, \mathrm{d}z + \frac{1}{2\pi i} \int_{\Gamma_{\omega_0}} e^{tz} z^{-1} \mathcal{A} R(z^{\alpha}, \mathcal{A}) \, \mathrm{d}z \\ &= I + \frac{1}{2\pi i} \mathcal{A} \int_{\Gamma_{\omega_0}} e^{tz} z^{-1} R(z^{\alpha}, \mathcal{A}) \, \mathrm{d}z, \end{split}$$

where Γ_{ω_0} is deformed so that the origin is to its left. Hence, we conclude from (8) and Lemma 2.1 the following properties:

$$\begin{split} \eta &\in C([0,T]; \mathcal{L}([X^*,Y]_{\theta},X^*)) \cap C^1((0,T]; \mathcal{L}([X^*,Y]_{\theta},X^*)); \\ \eta(0) &= I; \\ \eta'(t) &= \frac{1}{2\pi i} \mathcal{A} \int_{\Gamma_{\omega_0}} e^{tz} R(z^{\alpha},\mathcal{A}) \, \mathrm{d}z, \quad t > 0; \\ \|\eta'(t)\|_{\mathcal{L}([X^*,Y]_{\theta},X^*)} &\leq C_{c_0,c_1,\omega_0,\mathcal{M}_0} t^{\theta\alpha-1}, \quad t > 0. \end{split}$$

Step 2. By the theory of Laplace transform, from (26) we obtain that

$$(\mathcal{D}_{0+}^{\alpha-1} \mathcal{S}_{\alpha} g)(t) = \int_0^t \eta(t-s)g(s) \,\mathrm{d}s, \quad 0 < t \leqslant T.$$

$$\tag{41}$$

Hence, by the properties of η presented in Step 1,

$$(\mathcal{D}_{0+}^{\alpha} \mathcal{S}_{\alpha} g)(t) = \frac{\mathrm{d}}{\mathrm{d}t} (\mathcal{D}_{0+}^{\alpha-1} \mathcal{S}_{\alpha} g)(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{0}^{t} \eta(t-s)g(s) \,\mathrm{d}s$$
$$= g(t) + \mathcal{A} \int_{0}^{t} \frac{1}{2\pi i} \int_{\Gamma_{\omega_{0}}} e^{(t-s)z} R(z^{\alpha}, \mathcal{A}) \,\mathrm{d}z \, g(s) \,\mathrm{d}s$$
$$= g(t) + \mathcal{A} \int_{0}^{t} E_{\alpha}(t-s)g(s) \,\mathrm{d}s \quad (\mathrm{by} \ (17))$$
$$= g(t) + \mathcal{A}(\mathcal{S}_{\alpha}g)(t) \quad (\mathrm{by} \ (26))$$

for each $0 \leq t \leq T$. This implies equality (38).

Step 3. For convenience, we will use c to denote a positive constant, whose value is independent of g but may differ at each occurrence. A routine calculation gives, by (24), (25) and (26), that

$$\|\mathcal{S}_{\alpha}g\|_{C([0,T];[X,Y]_{1-(1-\epsilon)\theta})} \leq c\|g\|_{C([0,T];[X^*,Y]_{\theta})},$$

so that (8) implies

$$\|\mathcal{AS}_{\alpha}g\|_{C([0,T];[X^*,Y]_{(1-\epsilon)\theta})} \leqslant c \|g\|_{C([0,T];[X^*,Y]_{\theta})}.$$
(42)

Since $[X^*, Y]_{\theta}$ is continuously embedded into $[X^*, Y]_{(1-\epsilon)\theta}$, we have

$$\|g\|_{C([0,T];[X^*,Y]_{(1-\epsilon)\theta})} \leq c \|g\|_{C([0,T];[X^*,Y]_{\theta})}.$$
(43)

Combining (38), (42) and (43) proves (39) and thus concludes the proof.

2.2 Continuous problem

Let Z be a Hilbert space and let $U_{ad} \subset L^{\infty}(0,T;Z)$ be a convex, bounded and closed subset of $L^{2}(0,T;Z)$. We consider the following abstract optimal control problem:

$$\min_{u \in U_{\rm ad}} J_{\alpha}(u) := \frac{1}{2} \| (\mathcal{S}_{\alpha} \mathcal{R}_{\theta_0} u)(T) - y_d \|_Y^2 + \frac{\nu}{2} \| u \|_{L^2(0,T;Z)}^2, \tag{44}$$

where $0 < \alpha \leq 1$, $y_d \in Y$, $\nu > 0$ is a regularization parameter, and $\mathcal{R}_{\theta_0} : Z \to [X^*, Y]_{\theta_0}$ is a bounded linear operator for some $0 < \theta_0 \leq 1$.

Define $\mathcal{R}^*_{\theta_0} : [X, Y]_{\theta_0} \to Z$ by

$$(\mathcal{R}^*_{\theta_0}v, w)_Z := \overline{\langle \mathcal{R}_{\theta_0}w, v \rangle_{[X,Y]_{\theta_0}}}$$

for all $v \in [X, Y]_{\theta_0}$ and $w \in Z$. Assume that $q > \max\{1/(\theta_0 \alpha), 2\}$. By (33), $(\mathcal{S}_{\alpha}\mathcal{R}_{\theta_0})(T)$ is a bounded linear operator from $L^q(0, T; Z)$ to Y. Clearly, J_{α} in (44) is a strictly convex functional on $L^q(0, T; Z)$, and U_{ad} is a convex, bounded and closed subset of $L^q(0, T; Z)$. By (34), a routine argument (cf. [60, Theorems 2.14 and 2.21]) yields the following theorem.

Theorem 2.1. Problem (44) admits a unique solution $u \in U_{ad}$, and the following first-order optimality condition holds:

$$\int y = \mathcal{S}_{\alpha} \mathcal{R}_{\theta_0} u, \tag{45a}$$

$$\begin{cases} p = \mathcal{S}^*_{\alpha} \big((y(T) - y_d) \delta_T \big), \tag{45b} \end{cases}$$

$$\int_{0}^{T} \left(\mathcal{R}_{\theta_{0}}^{*} p(t) + \nu u(t), v(t) - u(t) \right)_{Z} \mathrm{d}t \ge 0 \text{ for all } v \in U_{ad}.$$
(45c)

2.3 Temporally discrete problem

Let J > 1 be an integer and define $t_j := j\tau$ for each $j = 0, 1, 2, \ldots, J$, where $\tau := T/J$. For each Banach space \mathcal{X} , define

$$W_{\tau}(\mathcal{X}) := \{ V \in L^{\infty}(0,T;\mathcal{X}) : V \text{ is constant on } (t_{j-1},t_j) \quad \forall 1 \leq j \leq J \}.$$

For any $0 < \alpha < 1$ and $g \in L^1(0,T;X^*)$, define $\mathcal{S}_{\alpha,\tau}g \in W_{\tau}(Y)$ and $\mathcal{S}^*_{\alpha,\tau}g \in W_{\tau}(Y)$, respectively, by that

$$\int_{0}^{T} \langle (\mathbf{D}_{0+}^{\alpha} - \mathcal{A}) \mathcal{S}_{\alpha,\tau} g, V \rangle_{X} \, \mathrm{d}t = \int_{0}^{T} \langle g, V \rangle_{X} \, \mathrm{d}t, \tag{46}$$

$$\int_{0}^{T} \langle (\mathbf{D}_{T-}^{\alpha} - \mathcal{A}^{*}) \mathcal{S}_{\alpha,\tau}^{*} g, V \rangle_{X} \, \mathrm{d}t = \int_{0}^{T} \langle g, V \rangle_{X} \, \mathrm{d}t, \tag{47}$$

for all $V \in W_{\tau}(X)$. For any $g \in L^1(0,T;X^*)$, define $\mathcal{S}_{1,\tau}g \in W_{\tau}(Y)$ and $\mathcal{S}_{1,\tau}^*g \in W_{\tau}(Y)$, respectively, by that

$$\left((\mathcal{S}_{1,\tau}g)(0+), V(0+) \right)_{Y} + \sum_{j=1}^{J-1} \left((\mathcal{S}_{1,\tau}g)(t_{j}+) - (\mathcal{S}_{1,\tau}g)(t_{j}-), V(t_{j}+) \right)_{Y} - \int_{0}^{T} \langle \mathcal{AS}_{1,\tau}g, V \rangle_{X} \, \mathrm{d}t = \int_{0}^{T} \langle g, V \rangle_{X} \, \mathrm{d}t,$$

$$(48)$$

$$\left((\mathcal{S}_{1,\tau}^*g)(T-), V(T-) \right)_Y + \sum_{j=1}^{J-1} \left((\mathcal{S}_{1,\tau}^*g)(t_j-) - (\mathcal{S}_{1,\tau}^*g)(t_j+), V(t_j-) \right)_Y - \int_0^T \langle \mathcal{A}^* \mathcal{S}_{1,\tau}^*g, V \rangle_X \, \mathrm{d}t = \int_0^T \langle g, V \rangle_X \, \mathrm{d}t,$$
(49)

for all $V \in W_{\tau}(X)$. We will present some properties of $\mathcal{S}_{\alpha,\tau}$, $0 < \alpha \leq 1$, in Section 2.4.

Remark 2.3. Scheme (48) is a famous discontinuous Galerkin method for parabolic equations (cf. [10]), and this scheme is a variant of the backward Euler difference scheme.

Remark 2.4. We note that the idea of using the Galerkin methods to discretize the time fractional calculus operators was firstly developed by McLean and Mustapha [45, 52, 53, 51]. The L1 scheme [39, 57] is widely used for the discretizations of the fractional diffusion equations. Jin et al. [24, Remark 3] discovered that the L1 scheme is equivalent to discretization (46) with uniform temporal grids. For the numerical analysis of discretization (46) with nonuniform temporal grids, we refer the reader to [36, 37].

Using the variational discretization concept proposed in [19], we consider the following temporally discrete problem:

$$\min_{U \in U_{\rm ad}} J_{\alpha,\tau}(U) := \frac{1}{2} \| (\mathcal{S}_{\alpha,\tau} \mathcal{R}_{\theta_0} U)(T-) - y_d \|_Y^2 + \frac{\nu}{2} \| U \|_{L^2(0,T;Z)}^2.$$
(50)

Note that (56) implies that $(S_{\alpha,\tau}\mathcal{R}_{\theta_0})(T-)$ is a bounded linear operator from $L^2(0,T;Z)$ to Y. In addition, U_{ad} is a convex, bounded and closed subset of $L^2(0,T;Z)$. Hence, applying [60, Theorems 2.14 and 2.21] to problem (50) yields the following theorem, by Lemma 2.6.

Theorem 2.2. Problem (50) admits a unique solution $U \in U_{ad}$, and the following first-order optimality condition holds:

$$Y = \mathcal{S}_{\alpha,\tau} \mathcal{R}_{\theta_0} U, \tag{51a}$$

$$P = \mathcal{S}^*_{\alpha,\tau} \left((Y(T-) - y_d) \hat{\delta}_T \right), \tag{51b}$$

$$\int_{0}^{T} \left(\mathcal{R}_{\theta_{0}}^{*} P(t) + \nu U(t), V(t) - U(t) \right)_{Z} \mathrm{d}t \ge 0 \quad \text{for all } V \in U_{ad},$$
(51c)

where

$$\widehat{\delta}_T := \begin{cases} 0 & \text{if } 0 < t < T - \tau, \\ \tau^{-1} & \text{if } T - \tau < t < T. \end{cases}$$
(52)

A simple modification of the proof of [38, Theorem 4.3] yields the following error estimate, by Lemma 2.7.

Theorem 2.3. Assume that $0 < \alpha \leq 1$. Let u and y be defined in Theorem 2.1, and let U and Y be defined in Theorem 2.2. Then

$$\|(y-Y)(T-)\|_{Y} + \sqrt{\nu} \|u-U\|_{L^{2}(0,T;Z)}$$

$$\leq C_{c_{0},\omega_{0},\mathcal{M}_{0},T} \left(\|y_{d}\|_{Y} + \|\mathcal{R}_{\theta_{0}}\|_{\mathcal{L}(Z,[X^{*},Y]_{\theta_{0}})}\|u\|_{L^{\infty}(0,T;Z)}\right) \times \left(1/(\theta_{0}\alpha) + \sqrt{\varepsilon(\alpha,\theta_{0},J)} + \varepsilon(\alpha,\theta_{0},J)\tau^{\theta_{0}\alpha/2}\right)\tau^{\theta_{0}\alpha/2}.$$
(53)

where

$$\varepsilon(\alpha, \theta, J) := \begin{cases} \frac{1}{\theta\alpha} + \frac{1 - J^{\theta\alpha - 1}}{1 - \theta\alpha} & \text{if } \theta\alpha \neq 1, \\ \ln J & \text{if } \theta\alpha = 1. \end{cases}$$
(54)

2.4 Properties of $S_{\alpha,\tau}$

Assume that $0 < \alpha \leq 1$ and $g \in L^1(0,T;X^*)$. Define $\{W_j\}_{j=1}^J \subset Y$ as follows: for any $1 \leq k \leq J$,

$$b_1 W_k + \sum_{j=1}^{k-1} (b_{k-j+1} - 2b_{k-j} + b_{k-j-1}) W_j - \tau^{\alpha} \mathcal{A} W_k = \tau^{\alpha-1} \int_{t_{k-1}}^{t_k} g(t) \, \mathrm{d}t \quad (55)$$

in X^* , where $b_j := j^{1-\alpha}/\Gamma(2-\alpha)$ for each $0 \leq j \leq J$. A straightforward computation yields that (cf. [24, Remark 3])

$$(\mathcal{S}_{\alpha,\tau}g)(t_j-) = W_j \quad \forall 1 \leqslant j \leqslant J.$$

Hence, we conclude from (55) and Lemma 2.1 that, for any $0 \leq \beta \leq 1$,

$$\mathcal{S}_{\alpha,\tau} \in \mathcal{L}\big(L^1(0,T;[X^*,Y]_{1-\beta}), L^\infty(0,T;[X,Y]_\beta)\big)$$
(56)

and

$$\lim_{\alpha \to 1^{-}} \| \mathcal{S}_{\alpha,\tau} - \mathcal{S}_{1,\tau} \|_{\mathcal{L}(L^{1}(0,T;[X^{*},Y]_{1-\beta}), L^{\infty}(0,T;[X,Y]_{\beta}))} = 0.$$
(57)

Symmetrically, for any $0 \leq \beta \leq 1$ we have that

$$S_{\alpha,\tau}^* \in \mathcal{L}(L^1(0,T; [X^*,Y]_{1-\beta}), L^{\infty}(0,T; [X,Y]_{\beta}))$$
(58)

and

$$\lim_{\alpha \to 1^{-}} \|\mathcal{S}_{\alpha,\tau}^{*} - \mathcal{S}_{1,\tau}^{*}\|_{\mathcal{L}(L^{1}(0,T;[X^{*},Y]_{1-\beta}), L^{\infty}(0,T;[X,Y]_{\beta}))} = 0.$$
(59)

Lemma 2.6. Assume that $0 < \alpha \leq 1$. For any $f \in L^1(0,T;X^*)$ and $g \in L^1(0,T;Y)$,

$$\int_0^T (\mathcal{S}_{\alpha,\tau}f, g)_Y \,\mathrm{d}t = \int_0^T \langle f, \, \mathcal{S}_{\alpha,\tau}^*g \rangle_X \,\mathrm{d}t.$$
(60)

Proof. Assume that $0 < \alpha < 1$. By (58) we have $\mathcal{S}^*_{\alpha,\tau}g \in W_{\tau}(X)$, and then (47) and the density of X in Y yield that

$$\int_0^T \left((\mathcal{D}_{T-}^\alpha - \mathcal{A}^*) \mathcal{S}_{\alpha,\tau}^* g, V \right)_Y \mathrm{d}t = \int_0^T (g, V)_Y \mathrm{d}t \tag{61}$$

for all $V \in W_{\tau}(Y)$. Hence,

$$\int_{0}^{T} (\mathcal{S}_{\alpha,\tau}f, g)_{Y} dt = \int_{0}^{T} \left(\mathcal{S}_{\alpha,\tau}f, (D_{T-}^{\alpha} - \mathcal{A}^{*})\mathcal{S}_{\alpha,\tau}^{*}g \right)_{Y} dt$$
$$= \int_{0}^{T} \langle (D_{0+}^{\alpha} - \mathcal{A})\mathcal{S}_{\alpha,\tau}f, \mathcal{S}_{\alpha,\tau}^{*}g \rangle_{X} dt \quad (by (4) \text{ and } (6))$$
$$= \int_{0}^{T} \langle f, \mathcal{S}_{\alpha,\tau}^{*}g \rangle_{X} dt \quad (by (46)).$$

This proves (60) for $0 < \alpha < 1$. For the proof of (60) with $\alpha = 1$, we refer the reader to [59, Chapter 12].

Lemma 2.7. Assume that $0 < \alpha, \theta \leq 1$ and $p \in \{1, \infty\}$. For any $g \in L^p(0,T; [X^*,Y]_{\theta})$ we have

$$\|(\mathcal{S}_{\alpha} - \mathcal{S}_{\alpha,\tau})g\|_{L^{p}(0,T;Y)} \leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}}\varepsilon(\alpha,\theta,J)\tau^{\theta\alpha}\|g\|_{L^{p}(0,T;[X^{*},Y]_{\theta})},$$
(62)

and for any $v \in Y$ we have

$$\|\mathcal{S}_{\alpha}(v\delta_{0}) - \mathcal{S}_{\alpha,\tau}(v\widehat{\delta}_{0})\|_{L^{1}(0,T;[X,Y]_{\theta})} \leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}}\varepsilon(\alpha,\theta,J)\tau^{\theta\alpha}\|v\|_{Y},$$
(63)

where $\varepsilon(\cdot, \cdot, \cdot)$ is defined by (54) and

$$\widehat{\delta}_0(t) := \begin{cases} t_1^{-1} & \text{if } 0 < t < t_1, \\ 0 & \text{if } t_1 < t < T. \end{cases}$$
(64)

The main task of the rest of this subsection is to prove Lemma 2.7. Firstly, we summarize some auxiliary results in [38]. Assume that $0 < \alpha < 1$. For any $z \in \mathbb{C} \setminus \{0\}$ with $-\pi \leq \Im z \leq \pi$, define

$$\psi_{\alpha}(z) := (e^{z} - 1)^{2} \sum_{k = -\infty}^{\infty} (z + 2k\pi i)^{\alpha - 2}.$$
 (65)

There exists $\pi/2 < \omega^* \leqslant \omega_0$, depending only on ω_0 , such that

$$e^{-z}\psi_{\alpha}(z) \in \Sigma_{\omega_0} \text{ for all } z \in \Sigma_{\omega^*} \text{ with } -\pi \leqslant \Im z \leqslant \pi$$
 (66)

and that, for any $z \in \Upsilon_{\omega^*} \setminus \{0\}$,

$$|e^{-z}\psi_{\alpha}(z)| \geqslant C_{\omega_0}|z|^{\alpha},\tag{67}$$

$$|\psi_{\alpha}(z) - z^{\alpha}| \leqslant C_{\omega_0} |z|^{\alpha+1}.$$
(68)

Define

$$\mathcal{E}_{\alpha}(t) := \tau^{-1} \mathcal{E}_{\alpha, \lfloor t/\tau \rfloor}, \quad t > 0,$$
(69)

where $|\cdot|$ is the floor function and

$$\mathcal{E}_{\alpha,j} := \frac{1}{2\pi i} \int_{\Upsilon_{\omega^*}} e^{jz} R(\tau^{-\alpha} e^{-z} \psi_{\alpha}(z), \mathcal{A}) \,\mathrm{d}z, \quad j \in \mathbb{N}.$$
(70)

Following the proof of [38, Lemma 3.5], we obtain that, for any $g \in L^1(0,T;Y)$,

$$(\mathcal{S}_{\alpha,\tau}g)(t_j-) = \int_0^{t_j} \mathcal{E}_{\alpha}(t_j-t)g(t) \,\mathrm{d}t \quad \forall 1 \leqslant j \leqslant J.$$
(71)

Since (11), (66), (67), (69) and (70) imply $\|\mathcal{E}_{\alpha}\|_{L^{\infty}(0,T;\mathcal{L}(X^*,Y))} < \infty$, from (56) and the fact that $L^1(0,T;Y)$ is dense in $L^1(0,T;X^*)$ we conclude that (71) holds for all $g \in L^1(0,T;X^*)$.

Secondly, we present some auxiliary estimates in the following three lemmas.

Lemma 2.8. For any $0 < \alpha < 1$, $0 \leq \theta \leq 1$ and $z \in \Upsilon_{\omega^*} \setminus \{0\}$,

$$\|e^{z}R(\tau^{-\alpha}z^{\alpha},\mathcal{A}) - R(\tau^{-\alpha}e^{-z}\psi_{\alpha}(z),\mathcal{A})\|_{\mathcal{L}(Y,[X,Y]_{\theta})} \leq \frac{C_{c_{0},\omega_{0},\mathcal{M}_{0}}|z|}{1 + (\tau^{-\alpha}|z|^{\alpha})^{\theta}},$$
(72)

$$\|e^{z}R(\tau^{-\alpha}z^{\alpha},\mathcal{A}) - R(\tau^{-\alpha}e^{-z}\psi_{\alpha}(z),\mathcal{A})\|_{\mathcal{L}([X^{*},Y]_{\theta},Y)} \leqslant \frac{C_{c_{0},\omega_{0},\mathcal{M}_{0}}|z|}{1 + (\tau^{-\alpha}|z|^{\alpha})^{\theta}}.$$
 (73)

Proof. A straightforward computation gives

$$e^{z}R(\tau^{-\alpha}z^{\alpha},\mathcal{A}) - R(\tau^{-\alpha}e^{-z}\psi_{\alpha}(z),\mathcal{A})$$

= $(\tau^{-\alpha}(\psi_{\alpha}(z) - z^{\alpha}) + (1 - e^{z})\mathcal{A})R(\tau^{-\alpha}z^{\alpha},\mathcal{A})R(\tau^{-\alpha}e^{-z}\psi_{\alpha}(z),\mathcal{A})$
= $\mathbb{I}_{1} + \mathbb{I}_{2},$

where

$$\begin{split} \mathbb{I}_1 &:= \tau^{-\alpha}(\psi_\alpha(z) - z^\alpha) R(\tau^{-\alpha} z^\alpha, \mathcal{A}) R(\tau^{-\alpha} e^{-z} \psi_\alpha(z), \mathcal{A}), \\ \mathbb{I}_2 &:= (1 - e^z) \mathcal{A} R(\tau^{-\alpha} z^\alpha, \mathcal{A}) R(\tau^{-\alpha} e^{-z} \psi_\alpha(z), \mathcal{A}). \end{split}$$

We conclude from (9), (66) and (67) that, for any $0 \leq \beta \leq 1$,

$$\|R(\tau^{-\alpha}z^{\alpha},\mathcal{A})\|_{\mathcal{L}(Y,[X,Y]_{\beta})} \leqslant C_{\mathcal{M}_{0}}(1+(\tau^{-\alpha}|z|^{\alpha})^{\beta})^{-1},\tag{74}$$

$$\|R(\tau^{-\alpha}e^{-z}\psi_{\alpha}(z),\mathcal{A})\|_{\mathcal{L}(Y,[X,Y]_{\beta})} \leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}}(1+(\tau^{-\alpha}|z|^{\alpha})^{\beta})^{-1}.$$
 (75)

For \mathbb{I}_1 we have, by (68), (74) and (75),

$$\begin{aligned} &\|\mathbb{I}_1\|_{\mathcal{L}(Y,[X,Y]_{\theta})} \\ &\leqslant C_{\omega_0}\tau^{-\alpha}|z|^{\alpha+1}\|R(\tau^{-\alpha}z^{\alpha},\mathcal{A})\|_{\mathcal{L}(Y,[X,Y]_{\theta})}\|R(\tau^{-\alpha}e^{-z}\psi_{\alpha}(z),\mathcal{A})\|_{\mathcal{L}(Y)} \\ &\leqslant \frac{C_{c_0,\omega_0,\mathcal{M}_0}\tau^{-\alpha}|z|^{\alpha+1}}{\left(1+\tau^{-\alpha}|z|^{\alpha}\right)^{\theta}} \leqslant \frac{C_{c_0,\omega_0,\mathcal{M}_0}|z|}{1+(\tau^{-\alpha}|z|^{\alpha})^{\theta}}. \end{aligned}$$

Since

$$\begin{split} &\|\mathcal{A}R(\tau^{-\alpha}z^{\alpha},\mathcal{A})R(\tau^{-\alpha}e^{-z}\psi_{\alpha}(z),\mathcal{A})\|_{\mathcal{L}(Y,[X,Y]_{\theta})} \\ &= \|(\tau^{-\alpha}z^{\alpha}R(\tau^{-\alpha}z^{\alpha},\mathcal{A})-I)R(\tau^{-\alpha}e^{-z}\psi_{\alpha}(z),\mathcal{A})\|_{\mathcal{L}(Y,[X,Y]_{\theta})} \\ &\leqslant \|\tau^{-\alpha}z^{\alpha}R(\tau^{-\alpha}z^{\alpha},\mathcal{A})\|_{\mathcal{L}(Y,[X,Y]_{\theta})}\|R(\tau^{-\alpha}e^{-z}\psi_{\alpha}(z),\mathcal{A})\|_{\mathcal{L}(Y)} \\ &\quad + \|R(\tau^{-\alpha}e^{-z}\psi_{\alpha}(z),\mathcal{A})\|_{(Y,[X,Y]_{\theta})} \\ &\leqslant \frac{C_{c_{0},\omega_{0},\mathcal{M}_{0}}}{1+(\tau^{-\alpha}|z|^{\alpha})^{\theta}} \quad (\text{by (74) and (75)}), \end{split}$$

we obtain

$$\|\mathbb{I}_2\|_{\mathcal{L}(Y,[X,Y]_{\theta})} \leqslant \frac{C_{c_0,\omega_0,\mathcal{M}_0}|z|}{1+(\tau^{-\alpha}|z|^{\alpha})^{\theta}}.$$

Combining the above estimates of \mathbb{I}_1 and \mathbb{I}_2 proves (72). Since (73) can be derived analogously, this completes the proof.

Lemma 2.9. Assume that $0 < \alpha < 1$ and $0 \leq \theta \leq 1$. Then

$$\max_{1 \leq j \leq J} j^{2-\theta\alpha} \| E_{\alpha}(t_j) - \mathcal{E}_{\alpha}(t_j) - \|_{\mathcal{L}(Y, [X,Y]_{\theta})} \leq C_{c_0, \omega_0, \mathcal{M}_0} \tau^{\theta\alpha - 1},$$
(76)

$$\max_{1 \leq j \leq J} j^{2-\theta\alpha} \| E_{\alpha}(t_j) - \mathcal{E}_{\alpha}(t_j-) \|_{\mathcal{L}([X^*,Y]_{\theta},Y)} \leq C_{c_0,\omega_0,\mathcal{M}_0} \tau^{\theta\alpha-1}.$$
(77)

Proof. For each $1 \leq j \leq J$, inserting $t = t_j$ into (17) yields

$$E_{\alpha}(t_j) = \frac{1}{2\pi i} \int_{\Gamma_{\omega^*}} e^{t_j z} R(z^{\alpha}, \mathcal{A}) \, \mathrm{d}z = \frac{\tau^{-1}}{2\pi i} \int_{\Gamma_{\omega^*}} e^{j z} R(\tau^{-\alpha} z^{\alpha}, \mathcal{A}) \, \mathrm{d}z,$$

and so we conclude from (69) and (70) that

$$E_{\alpha}(t_j) - \mathcal{E}_{\alpha}(t_j -) = \mathbb{I}_1 + \mathbb{I}_2,$$

where

$$\mathbb{I}_{1} := \frac{\tau^{-1}}{2\pi i} \int_{\Gamma_{\omega^{*}} \setminus \Upsilon_{\omega^{*}}} e^{jz} R(\tau^{-\alpha} z^{\alpha}, \mathcal{A}) \,\mathrm{d}z, \\
\mathbb{I}_{2} := \frac{\tau^{-1}}{2\pi i} \int_{\Upsilon_{\omega^{*}}} e^{(j-1)z} \left(e^{z} R(\tau^{-\alpha} z^{\alpha}, \mathcal{A}) - R(\tau^{-\alpha} e^{-z} \psi_{\alpha}(z), \mathcal{A}) \right) \,\mathrm{d}z.$$

A straightforward computation gives

$$\begin{split} \|\mathbb{I}_{1}\|_{\mathcal{L}(Y,[X,Y]_{\theta})} \\ \leqslant C_{c_{0},\mathcal{M}_{0}}\tau^{-1}\int_{\pi/\sin\omega^{*}}^{\infty}e^{j\cos\omega^{*}r}\left(1+(\tau^{-\alpha}r^{\alpha})^{\theta}\right)^{-1}\mathrm{d}r \quad (\mathrm{by}\ (9)) \\ < C_{c_{0},\mathcal{M}_{0}}\tau^{\theta\alpha-1}\int_{\pi/\sin\omega^{*}}^{\infty}e^{j\cos\omega^{*}r}r^{-\theta\alpha}\,\mathrm{d}r \\ < C_{c_{0},\mathcal{M}_{0}}\tau^{\theta\alpha-1}\int_{\pi/\sin\omega^{*}}^{\infty}e^{j\cos\omega^{*}r}\,\mathrm{d}r \\ < C_{c_{0},\mathcal{M}_{0}}\tau^{\theta\alpha-1}j^{-1}e^{j\pi\cot\omega^{*}} \end{split}$$

and

$$\begin{split} \|\mathbb{I}_{2}\|_{\mathcal{L}(Y,[X,Y]_{\theta})} \\ < C_{c_{0},\omega_{0},\mathcal{M}_{0}}\tau^{-1}\int_{0}^{\pi/\sin\omega^{*}}e^{(j-1)\cos\omega^{*}r}r(1+(\tau^{-\alpha}r^{\alpha})^{\theta})^{-1}\,\mathrm{d}r \quad (\mathrm{by}\ (72)) \\ < C_{c_{0},\omega_{0},\mathcal{M}_{0}}\tau^{\theta\alpha-1}\int_{0}^{\pi/\sin\omega^{*}}e^{(j-1)\cos\omega^{*}r}r^{1-\theta\alpha}\,\mathrm{d}r \\ < C_{c_{0},\omega_{0},\mathcal{M}_{0}}\tau^{\theta\alpha-1}j^{\theta\alpha-2}. \end{split}$$

Together the above estimates of \mathbb{I}_1 and \mathbb{I}_2 yields (76). Since (77) can be proved analogously by (11) and (73), this completes the proof.

Lemma 2.10. For any $0 < \alpha < 1$ and $0 < \theta \leq 1$,

$$\|E_{\alpha} - \mathcal{E}_{\alpha}\|_{L^{1}(0,T;\mathcal{L}(Y,[X,Y]_{\theta}))} \leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}} \Big(\frac{1}{\theta\alpha} + \frac{1 - J^{\theta\alpha - 1}}{1 - \theta\alpha}\Big)\tau^{\theta\alpha}, \qquad (78)$$

$$\|E_{\alpha} - \mathcal{E}_{\alpha}\|_{L^{1}(0,T;\mathcal{L}([X^{*},Y]_{\theta},Y))} \leq C_{c_{0},\omega_{0},\mathcal{M}_{0}} \left(\frac{1}{\theta\alpha} + \frac{1 - J^{\theta\alpha-1}}{1 - \theta\alpha}\right) \tau^{\theta\alpha}.$$
 (79)

Proof. By (20) we have

$$\|E_{\alpha} - E_{\alpha}(t_1)\|_{L^1(0,t_1;\mathcal{L}(Y,[X,Y]_{\theta}))} < C_{c_0,\omega_0,\mathcal{M}_0}\tau^{\theta\alpha}/(\theta\alpha),$$
(80)

and a straightforward calculation gives

$$\sum_{j=2}^{J} \|E_{\alpha} - E_{\alpha}(t_{j})\|_{L^{1}(t_{j-1}, t_{j}; \mathcal{L}(Y, [X, Y]_{\theta}))}$$

$$\leq \tau \|E_{\alpha}'\|_{L^{1}(t_{1}, T; \mathcal{L}(Y, [X, Y]_{\theta}))} \leq C_{c_{0}, \omega_{0}, \mathcal{M}_{0}} \tau \int_{t_{1}}^{T} t^{\theta \alpha - 2} dt \quad (by \ (22))$$

$$= C_{c_{0}, \omega_{0}, \mathcal{M}_{0}} \tau^{\theta \alpha} (1 - J^{\theta \alpha - 1}) / (1 - \theta \alpha). \tag{81}$$

It follows that

$$\sum_{j=1}^{J} \|E_{\alpha} - E_{\alpha}(t_{j})\|_{L^{1}(t_{j-1}, t_{j}; \mathcal{L}(Y, [X, Y]_{\theta}))}$$
$$\leqslant C_{c_{0}, \omega_{0}, \mathcal{M}_{0}} \Big(\frac{1}{\theta \alpha} + \frac{1 - J^{\theta \alpha - 1}}{1 - \theta \alpha}\Big) \tau^{\theta \alpha}.$$

In addition, by (76),

$$\sum_{j=1}^{J} \tau \| E_{\alpha}(t_{j}) - \mathcal{E}_{\alpha}(t_{j}) - \mathcal{E}_{\alpha}(t_{j}) \|_{\mathcal{L}(Y, [X, Y]_{\theta})} \leq C_{c_{0}, \omega_{0}, \mathcal{M}_{0}} \tau^{\theta \alpha} \sum_{j=1}^{J} j^{\theta \alpha - 2} \leq C_{c_{0}, \omega_{0}, \mathcal{M}_{0}} \tau^{\theta \alpha} (1 - J^{\theta \alpha - 1}) / (1 - \theta \alpha).$$

Consequently,

$$\begin{split} \|E_{\alpha} - \mathcal{E}_{\alpha}\|_{L^{1}(0,T;\mathcal{L}(Y,[X,Y]_{\theta}))} \\ \leqslant \sum_{j=1}^{J} \left(\|E_{\alpha} - E_{\alpha}(t_{j})\|_{L^{1}(t_{j-1},t_{j};\mathcal{L}(Y,[X,Y]_{\theta}))} + \tau \|E_{\alpha}(t_{j}) - \mathcal{E}_{\alpha}(t_{j}-)\|_{L^{1}(t_{j-1},t_{j};\mathcal{L}(Y,[X,Y]_{\theta}))} \right) \\ \leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}} \left(\frac{1}{\theta\alpha} + \frac{1 - J^{\theta\alpha-1}}{1 - \theta\alpha} \right) \tau^{\theta\alpha}, \end{split}$$

which proves (78). Since (79) can be proved analogously by (21), (23) and (77), this completes the proof.

Thirdly, we prove that (62) holds for $0 < \alpha < 1$ and p = 1.

Lemma 2.11. Assume that $0 < \alpha < 1$ and $0 < \theta \leq 1$. For any $g \in L^1(0,T; [X^*,Y]_{\theta})$, we have

$$\|(\mathcal{S}_{\alpha} - \mathcal{S}_{\alpha,\tau})g\|_{L^{1}(0,T;Y)} \leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}} \Big(\frac{1}{\theta\alpha} + \frac{1 - J^{\theta\alpha - 1}}{1 - \theta\alpha}\Big)\tau^{\theta\alpha}\|g\|_{L^{1}(0,T;[X^{*},Y]_{\theta})}.$$
(82)

Proof. Step 1. Let us prove

$$\int_{0}^{T} \left\| \int_{0}^{t} (E_{\alpha}(t-s) - E_{\alpha}(t-s+\tau))g(s) \,\mathrm{d}s \right\|_{Y} \,\mathrm{d}t$$

$$\leq C_{c_{0},\omega_{0},\mathcal{M}_{0}} \left(\frac{1}{\theta\alpha} + \frac{1-J^{\theta\alpha-1}}{1-\theta\alpha} \right) \tau^{\theta\alpha} \|g\|_{L^{1}(0,T;[X^{*},Y]_{\theta})}.$$
(83)

A straightforward computation gives

$$\int_0^\tau \|E_\alpha(t) - E_\alpha(t+\tau)\|_{\mathcal{L}([X^*,Y]_\theta,Y)} \,\mathrm{d}t \leqslant C_{c_0,\omega_0,\mathcal{M}_0} \tau^{\theta\alpha}/(\theta\alpha) \quad (by \ (21))$$

and

$$\int_{\tau}^{T} \|E_{\alpha}(t) - E_{\alpha}(t+\tau)\|_{\mathcal{L}([X^*,Y]_{\theta},Y)} dt$$
$$\leq C_{c_{0},\omega_{0},\mathcal{M}_{0}}\tau \int_{\tau}^{T} t^{\theta\alpha-2} dt \quad (\text{by (23)})$$
$$\leq C_{c_{0},\omega_{0},\mathcal{M}_{0}}\tau^{\theta\alpha}(1-J^{\theta\alpha-1})/(1-\theta\alpha).$$

It follows that

$$\int_0^T \|E_\alpha(t) - E_\alpha(t+\tau)\|_{\mathcal{L}([X^*,Y]_\theta,Y)} \leq C_{c_0,\omega_0,\mathcal{M}_0} \left(\frac{1}{\theta\alpha} + \frac{1-J^{\theta\alpha-1}}{1-\theta\alpha}\right) \tau^{\theta\alpha}.$$

Hence, (83) follows from the estimate

$$\begin{split} &\int_{0}^{T} \left\| \int_{0}^{t} \left(E_{\alpha}(t-s) - E_{\alpha}(t-s+\tau) \right) g(s) \, \mathrm{d}s \right\|_{Y} \mathrm{d}t \\ &\leqslant \int_{0}^{T} \int_{0}^{t} \| E_{\alpha}(t-s) - E_{\alpha}(t-s+\tau) \|_{\mathcal{L}([X^{*},Y]_{\theta},Y)} \| g(s) \|_{[X^{*},Y]_{\theta}} \, \mathrm{d}s \, \mathrm{d}t \\ &= \int_{0}^{T} \int_{s}^{T} \| E_{\alpha}(t-s) - E_{\alpha}(t-s+\tau) \|_{\mathcal{L}([X^{*},Y]_{\theta},Y)} \| g(s) \|_{[X^{*},Y]_{\theta}} \, \mathrm{d}t \, \mathrm{d}s \\ &= \int_{0}^{T} \int_{0}^{T-s} \| E_{\alpha}(r) - E_{\alpha}(r+\tau) \|_{\mathcal{L}([X^{*},Y]_{\theta},Y)} \, \mathrm{d}r \, \| g(s) \|_{[X^{*},Y]_{\theta}} \, \mathrm{d}s \\ &\leqslant \| g \|_{L^{1}(0,T;[X^{*},Y]_{\theta})} \int_{0}^{T} \| E_{\alpha}(r) - E_{\alpha}(r+\tau) \|_{\mathcal{L}([X^{*},Y]_{\theta},Y)} \, \mathrm{d}s. \end{split}$$

Step 2. Let us prove that

$$\int_{0}^{T} \left\| \int_{0}^{t} E_{\alpha}(t-s+\tau)g(s) - G(t) \right\|_{Y} \mathrm{d}t \leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}} \frac{1-J^{\theta\alpha-1}}{1-\theta\alpha} \tau^{\theta\alpha} \|g\|_{L^{1}(0,T;[X^{*},Y]_{\theta})},$$
(84)

where $G \in W_{\tau}(Y)$ is defined by

$$G(t_j -) := \sum_{k=1}^{j} E_{\alpha}(t_j - t_k + \tau) \int_{t_{k-1}}^{t_k} g(s) \,\mathrm{d}s, \quad 1 \le j \le J.$$
(85)

For any $t_{j-1} < t < t_j$ with $1 \leq j \leq J$, by (21) we have

$$\left\| \int_{t_{j-1}}^{t} E_{\alpha}(t-s+\tau)g(s) \,\mathrm{d}s - E_{\alpha}(\tau) \int_{t_{j-1}}^{t_{j}} g(s) \,\mathrm{d}s \right\|_{Y}$$

$$\leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}} \tau^{\theta\alpha-1} \|g\|_{L^{1}(t_{j-1},t_{j};[X^{*},Y]_{\theta})},$$

and by (23) we have

$$\begin{split} & \left\|\sum_{k=1}^{j-1} \int_{t_{k-1}}^{t_k} \left(E_{\alpha}(t-s+\tau) - E_{\alpha}(t_j-t_k+\tau) \right) g(s) \, \mathrm{d}s \right\|_Y \\ & \leq C_{c_0,\omega_0,\mathcal{M}_0} \sum_{k=1}^{j-1} \tau(t_j-t_k+\tau)^{\theta\alpha-2} \|g\|_{L^1(t_{k-1},t_k;[X^*,Y]_{\theta})} \\ & = C_{c_0,\omega_0,\mathcal{M}_0} \tau^{\theta\alpha-1} \sum_{k=1}^{j-1} (j-k+1)^{\theta\alpha-2} \|g\|_{L^1(t_{k-1},t_k;[X^*,Y]_{\theta})}. \end{split}$$

Hence, for each $t_{j-1} < t < t_j$ with $1 \leq j \leq J$,

$$\begin{split} \left\| \int_{0}^{t} E_{\alpha}(t-s+\tau)g(s) \,\mathrm{d}s - G(t_{j}-) \right\|_{Y} \\ &\leqslant \left\| \int_{t_{j-1}}^{t} E_{\alpha}(t-s+\tau)g(s) \,\mathrm{d}s - E_{\alpha}(\tau) \int_{t_{j-1}}^{t_{j}} g(s) \,\mathrm{d}s \right\|_{Y} + \\ &+ \left\| \sum_{k=1}^{j-1} \int_{t_{k-1}}^{t_{k}} \left(E_{\alpha}(t-s+\tau) - E_{\alpha}(t_{j}-t_{k}+\tau) \right) g(s) \,\mathrm{d}s \right\|_{Y} \quad (by \ (85)) \\ &\leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}} \tau^{\theta\alpha-1} \sum_{k=1}^{j} (j-k+1)^{\theta\alpha-2} \|g\|_{L^{1}(t_{k-1},t_{k};[X^{*},Y]_{\theta})}. \end{split}$$

It follows that, for each $1 \leqslant j \leqslant J$,

$$\int_{t_{j-1}}^{t_j} \left\| \int_0^t E_\alpha(t+\tau-s)g(s) \,\mathrm{d}s - G(t) \right\|_Y \mathrm{d}t$$

$$\leqslant C_{c_0,\omega_0,\mathcal{M}_0} \tau^{\theta\alpha} \sum_{k=1}^j (j-k+1)^{\theta\alpha-2} \|g\|_{L^1(t_{k-1},t_k;[X^*,Y]_\theta)}.$$

Therefore,

$$\begin{split} &\int_{0}^{T} \left\| \int_{0}^{t} E_{\alpha}(t+\tau-s)g(s) \,\mathrm{d}s - G(t) \right\|_{Y} \mathrm{d}t \\ &\leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}} \tau^{\theta\alpha} \sum_{j=1}^{J} \sum_{k=1}^{j} (j-k+1)^{\theta\alpha-2} \|g\|_{L^{1}(t_{k-1},t_{k};[X^{*},Y]_{\theta})} \\ &= C_{c_{0},\omega_{0},\mathcal{M}_{0}} \tau^{\theta\alpha} \sum_{k=1}^{J} \sum_{j=k}^{J} (j-k+1)^{\theta\alpha-2} \|g\|_{L^{1}(t_{k-1},t_{k};[X^{*},Y]_{\theta})} \\ &\leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}} \tau^{\theta\alpha} \|g\|_{L^{1}(0,T;[X^{*},Y]_{\theta})} \sum_{m=1}^{J} m^{\theta\alpha-2}. \end{split}$$

The desired estimate (84) then follows from the simple inequality

$$\sum_{m=1}^{J} m^{\theta \alpha - 2} < 1 + \frac{1 - J^{\theta \alpha - 1}}{1 - \theta \alpha}.$$
 (86)

Step 3. Let us prove that

$$\int_{0}^{T} \| (\mathcal{S}_{\alpha,\tau}g)(t) - G(t) \|_{Y} \, \mathrm{d}t \leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}} \frac{1 - J^{\theta\alpha - 1}}{1 - \theta\alpha} \tau^{\theta\alpha} \| g \|_{L^{1}(0,T;[X^{*},Y]_{\theta})}.$$
(87)

Noting that $S_{\alpha,\tau}g$ and G are piecewise constant, we have

$$\int_{0}^{T} \|(\mathcal{S}_{\alpha,\tau}g)(t) - G(t)\|_{Y} dt
= \sum_{j=1}^{J} \tau \|(\mathcal{S}_{\alpha,\tau}g)(t_{j}-) - G(t_{j}-)\|_{Y}
\leqslant \sum_{j=1}^{J} \tau \Big\| \sum_{k=1}^{j} \left(\mathcal{E}_{\alpha}((t_{j}-t_{k}+\tau)-) - E_{\alpha}(t_{j}-t_{k}+\tau) \right) \int_{t_{k-1}}^{t_{k}} g(s) ds \Big\|_{Y} (by (71) and (85))
\leqslant \sum_{j=1}^{J} \tau \sum_{k=1}^{j} \|\mathcal{E}_{\alpha}((t_{j}-t_{k}+\tau)-) - E_{\alpha}(t_{j}-t_{k}+\tau)\|_{\mathcal{L}([X^{*},Y]_{\theta},Y)} \int_{t_{k-1}}^{t_{k}} \|g(s)\|_{[X^{*},Y]_{\theta}} ds
= \tau \sum_{k=1}^{J} \sum_{j=k}^{J} \|\mathcal{E}_{\alpha}((t_{j}-t_{k}+\tau)-) - E_{\alpha}(t_{j}-t_{k}+\tau)\|_{\mathcal{L}([X^{*},Y]_{\theta},Y)} \int_{t_{k-1}}^{t_{k}} \|g(s)\|_{[X^{*},Y]_{\theta}} ds
\leqslant \tau \|g\|_{L^{1}(0,T;[X^{*},Y]_{\theta})} \sum_{m=1}^{J} \|\mathcal{E}_{\alpha}(t_{m}-) - E(t_{m})\|_{\mathcal{L}([X^{*},Y]_{\theta},Y)}
\leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}} \tau^{\theta_{\alpha}} \|g\|_{L^{1}(0,T;[X^{*},Y]_{\theta})} \sum_{m=1}^{J} m^{\theta_{\alpha}-2} (by (77)).$$

Hence, (87) follows from (86).

Step 4. By (26) we have

$$\begin{aligned} \|(\mathcal{S}_{\alpha} - \mathcal{S}_{\alpha,\tau})g\|_{L^{1}(0,T;Y)} \\ &\leqslant \int_{0}^{T} \left\| \int_{0}^{t} (E_{\alpha}(t-s) - E_{\alpha}(t-s+\tau))g(s) \,\mathrm{d}s \right\|_{Y} \mathrm{d}t \\ &+ \int_{0}^{T} \left\| \int_{0}^{t} E_{\alpha}(t-s+\tau)g(s) \,\mathrm{d}s - G(t) \right\|_{Y} \mathrm{d}t \\ &+ \int_{0}^{T} \|(\mathcal{S}_{\alpha,\tau}g)(t) - G(t)\|_{Y} \,\mathrm{d}t. \end{aligned}$$

Therefore, combining (83), (84) and (87) proves (82) and thus concludes the proof. $\hfill\blacksquare$

Fourthly, let us prove that (62) holds for $0 < \alpha < 1$ and $p = \infty$.

Lemma 2.12. Assume that $0 < \alpha < 1$ and $0 < \theta \leq 1$. For any $g \in L^{\infty}(0,T;[X^*,Y]_{\theta})$ we have

$$\|(\mathcal{S}_{\alpha} - \mathcal{S}_{\alpha,\tau})g\|_{L^{\infty}(0,T;Y)}$$

$$\leq C_{c_{0},\omega_{0},\mathcal{M}_{0}}\left(\frac{1}{\theta\alpha} + \frac{1 - J^{\theta\alpha - 1}}{1 - \theta\alpha}\right)\tau^{\theta\alpha}\|g\|_{L^{\infty}(0,T;[X^{*},Y]_{\theta})}.$$
(88)

Proof. By virtue of (26) and (71) we have that

$$\max_{1 \leq j \leq J} \| (\mathcal{S}_{\alpha}g)(t_j) - (\mathcal{S}_{\alpha,\tau}g)(t_j) - \|_Y$$
$$\leq \| E_{\alpha} - \mathcal{E}_{\alpha} \|_{L^1(0,T;\mathcal{L}([X^*,Y]_{\theta},Y))} \| g \|_{L^{\infty}(0,T;[X^*,Y]_{\theta})},$$

so that (79) implies that

$$\max_{1 \leq j \leq J} \| (\mathcal{S}_{\alpha}g)(t_{j}) - (\mathcal{S}_{\alpha,\tau}g)(t_{j}) - \|_{Y}$$
$$\leq C_{c_{0},\omega_{0},\mathcal{M}_{0}} \left(\frac{1}{\theta\alpha} + \frac{1 - J^{\theta\alpha - 1}}{1 - \theta\alpha} \right) \| g \|_{L^{\infty}(0,T;[X^{*},Y]_{\theta})}.$$

It remains therefore to prove that

$$\max_{1 \leqslant j \leqslant J} \|\mathcal{S}_{\alpha}g - (\mathcal{S}_{\alpha}g)(t_{j})\|_{L^{\infty}(t_{j-1},t_{j};Y)}$$
$$\leqslant C_{c_{0},\omega_{0},\mathcal{M}_{0}}\left(\frac{1}{\theta\alpha} + \frac{1 - J^{\theta\alpha - 1}}{1 - \theta\alpha}\right)\tau^{\theta\alpha}\|g\|_{L^{\infty}(0,T;[X^{*},Y]_{\theta})}.$$

But this is easily derived by (21), (23) and (26); see the proof of [9, Theorem 2.6] for the relevant techniques. This completes the proof.

Finally, we are in a position to conclude the proof of Lemma 2.7 as follows.

Proof of Lemma 2.7. Let us first prove (62). By Lemmas 2.11 and 2.12 we have that (62) holds for all $0 < \alpha < 1$ and $p \in \{1, \infty\}$, so that passing to the limit $\alpha \to 1$ yields, by Lemma 2.4 and (57), that (62) holds for all $0 < \alpha \leq 1$ and $p \in \{1, \infty\}$.

Then let us prove (63). Assume that $0 < \alpha < 1$. Combining (64) and (71) gives that

$$(\mathcal{S}_{\alpha,\tau}(v\delta_0))(t_j-) = \mathcal{E}_{\alpha}(t_j-)v, \quad 1 \leq j \leq J,$$

and so (28) implies

$$\begin{aligned} \|\mathcal{S}_{\alpha}(v\delta_{0}) - \mathcal{S}_{\alpha,\tau}(v\delta_{0})\|_{L^{1}(0,T;[X,Y]_{\theta})} \\ \leqslant \|E_{\alpha} - \mathcal{E}_{\alpha}\|_{L^{1}(0,T;\mathcal{L}(Y,[X,Y]_{\theta}))} \|v\|_{Y}. \end{aligned}$$

Therefore, (78) proves that (63) holds for each $0 < \alpha < 1$. A simple modification of the proof of (37) gives

$$\lim_{\alpha \to 1^{-}} \|E_{\alpha} - E_1\|_{L^1(0,T;\mathcal{L}(Y,[X,Y]_{\theta}))} = 0,$$

so that (28) implies

$$\lim_{\alpha \to 1^{-}} \| \mathcal{S}_{\alpha}(v\delta_0) - \mathcal{S}_1(v\delta_0) \|_{L^1(0,T;[X,Y]_{\theta})} = 0.$$

Moreover, (57) yields

$$\lim_{\alpha \to 1^{-}} \|\mathcal{S}_{\alpha,\tau}(v\widehat{\delta}_0) - \mathcal{S}_{1,\tau}(v\widehat{\delta}_0)\|_{L^1(0,T;[X,Y]_{\theta})} = 0.$$

Therefore, passing to the limit $\alpha \to 1-$ in (63) yields that (63) holds with $\alpha = 1$. This completes the proof of Lemma 2.7.

3 A Dirichlet boundary control problem

Assume that $0 < \alpha \leq 1$, $0 < T < \infty$, and $\Omega \subset \mathbb{R}^d$ (d = 2, 3) is a bounded convex polygonal domain with boundary $\partial \Omega$. Define

$$U_{\mathrm{ad}} := \left\{ v \in L^2(0,T;L^2(\partial\Omega)): \ u_* \leqslant v(x,t) \leqslant u^* \text{ for a.e. } (x,t) \in \partial\Omega \times (0,T) \right\},$$

where u_* and u^* are two given constants. For any $y \in C((0,T]; L^2(\Omega))$ and $u \in L^2(0,T; L^2(\partial\Omega))$, define

$$J_{\alpha}(y,u) := \frac{1}{2} \|y(T) - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|u\|_{L^2(0,T;L^2(\partial\Omega))}^2,$$
(89)

where $y_d \in L^2(\Omega)$ and $\nu > 0$ is a regularization parameter. We are concerned with the following optimal Dirichlet boundary control problem:

Minimize
$$J_{\alpha}(y, u)$$
 subject to $u \in U_{ad}$ and

$$\begin{cases}
(\partial_{0+}^{\alpha} - \Delta)y = 0 & \text{in } \Omega \times (0, T) \\
y = u & \text{on } \partial\Omega \times (0, T) \\
y(\cdot, 0) = 0 & \text{in } \Omega.
\end{cases}$$
(90)

Here, ∂_{0+}^{α} , a fractional partial differential operator, is the scalar-valued version of D_{0+}^{α} with respect to the time variable t.

To apply the theory in the previous section to problem (90), we will use the following settings:

$$\mathcal{A} := \Delta; \quad \mathcal{A}^* := \Delta; \quad X := H_0^1(\Omega) \cap H^2(\Omega); \quad Y := L^2(\Omega); \quad Z := L^2(\partial \Omega);$$

the operator $\mathcal{R}_{\theta_0}: Z \to [X^*, Y]_{\theta_0}, 0 < \theta_0 < 1/4$, is defined by that

$$\langle \mathcal{R}_{\theta_0} w, v \rangle_{[X,Y]_{\theta_0}} := -\langle w, \partial_{\boldsymbol{n}} v \rangle_{\partial \Omega} \tag{91}$$

for all $w \in Z$ and $v \in [X, Y]_{\theta_0}$, where $\partial_n v$ is the outward normal derivative of v on $\partial\Omega$. By the well-known trace inequality that

$$\|v\|_{Z} \leqslant \frac{C_{\Omega}}{\sqrt{\epsilon}} \|v\|_{[X,Y]_{3/4-\epsilon}} \quad \text{for all } v \in [X,Y]_{3/4-\epsilon} \text{ with } 0 < \epsilon \leqslant 3/4, \qquad (92)$$

we readily conclude that, for any $0 < \theta_0 < 1/4$,

$$\|\mathcal{R}_{\theta_0}\|_{\mathcal{L}(Z,[X^*,Y]_{\theta_0})} \leqslant \frac{C_\Omega}{\sqrt{1-4\theta_0}}.$$
(93)

Remark 3.1. For the techniques to prove (92), we refer the reader to [56, 3, Ch. VI], [58, Lemmas 16.1 and 23.1] and [42, Corollary 4.37].

Let u and y be defined in Theorem 2.1, and let U and Y be defined in Theorem 2.2. A straightforward calculation gives, by (53) and (93), that

$$\|(y-Y)(T-)\|_{Y} + \sqrt{\nu} \|u-U\|_{L^{2}(0,T;Z)}$$

$$\leq C_{u_{*},u^{*},T,\Omega} \left(\|y_{d}\|_{Y} + (1-4\theta_{0})^{-1/2} \right) (\theta_{0}\alpha)^{-1} \tau^{\theta_{0}\alpha/2}$$

for all $0 < \theta_0 < 1/4$. Assuming $\tau < \exp(-4)$ and inserting $\theta_0 = 1/4 - 1/\ln(1/\tau)$ into the above inequality, we then obtain

$$\|(y-Y)(T-)\|_{Y} + \sqrt{\nu}\|u-U\|_{L^{2}(0,T;Z)}$$

$$\leq C_{u_{*},u^{*},T,\Omega} \alpha^{-1} \left(\|y_{d}\|_{L^{2}(\Omega)} + \sqrt{\ln(1/\tau)}\right) \tau^{\alpha/8}.$$
(94)

Remark 3.2. For the case $\alpha = 1$, [29, 30, 33, 31, 32] used

$$-\mathcal{A}\int_0^t E_1(t-s)(-\mathcal{A})^{-1}\mathcal{R}_{\theta_0}u(s)\,\mathrm{d} s, \quad 0\leqslant t\leqslant T,$$

as the solution of the state equation of problem (90) with $u \in L^2(0,T;Z)$. It is evident that the above solution is exactly $S_1 \mathcal{R}_{\theta_0} u$.

It remains to prove that $S_{\alpha}\mathcal{R}_{\theta_0}u$ is a sensible solution to the state equation of problem (90) for each $u \in L^2(0,T;Z)$. To this end, we first introduce the very weak solution concept of the state equation, following the idea in [40]. For any $0 < \alpha < 1$ and $g \in L^2(0,T;Y)$, there exists a unique $w \in {}^0H^{\alpha}(0,T;Y) \cap$ $L^2(0,T;X)$ such that (cf. [36, 43])

$$(\mathcal{D}_{T-}^{\alpha} - \mathcal{A})w = g$$

and

$$||w||_{{}^{0}H^{\alpha}(0,T;Y)} + ||w||_{L^{2}(0,T;X)} \leq C_{\alpha} ||g||_{L^{2}(0,T;L^{2}(\Omega))}$$

For $\alpha = 1$ the above results are standard (cf. [12]). Hence, by the method of transposition (cf. [40]), we define the very weak solution $y \in L^2(0,T;Y)$ to the state equation of problem (90) with $u \in L^2(0,T;Z)$ by that

$$\int_0^T \left(y, \, (\mathcal{D}_{T-}^{\alpha} - \mathcal{A})\varphi \right)_Y \mathrm{d}t = -\langle u, \partial_{\boldsymbol{n}}\varphi \rangle_{\partial\Omega \times (0,T)}$$

for all $\varphi \in {}^{0}H^{\alpha}(0,T;Y) \cap L^{2}(0,T;X).$

Then we will prove that, for any $0 < \theta_0 < 1/4$ and $u \in L^2(0,T;Z)$, $S_{\alpha}\mathcal{R}_{\theta_0}u$ is identical to the very weak solution to the state equation of problem (90), and hence the application of the theory in the previous section to problem (90) is reasonable.

Lemma 3.1. Assume that $0 < \alpha \leq 1$ and $0 < \theta_0 < 1/4$. Then $S_{\alpha}\mathcal{R}_{\theta_0}u$ is the very weak solution to the state equation of problem (90) for each $u \in L^2(0,T;Z)$.

Proof. We only prove the case $0 < \alpha < 1$, the proof of the case $\alpha = 1$ being easier. Assume first that $u \in C([0,T];Z)$. By Lemma 2.5 we have

$$(\mathcal{D}_{0+}^{\alpha} - \mathcal{A})\mathcal{S}_{\alpha}\mathcal{R}_{\theta_0}u = \mathcal{R}_{\theta_0}u$$

and

$$D_{0+}^{\alpha} \mathcal{S}_{\alpha} \mathcal{R}_{\theta_0} u, \, \mathcal{A} \mathcal{S}_{\alpha} \mathcal{R}_{\theta_0} u \in C([0,T]; X^*).$$
(95)

Hence, for any $\varphi \in {}^0H^{\alpha}(0,T;Y) \cap L^2(0,T;X)$ we have

$$\int_0^T \langle (\mathbf{D}_{0+}^{\alpha} - \mathcal{A}) \mathcal{S}_{\alpha} \mathcal{R}_{\theta_0} u, \varphi \rangle_X \, \mathrm{d}t = \int_0^T \langle \mathcal{R}_{\theta_0} u, \varphi \rangle_X \, \mathrm{d}t.$$
(96)

Because (95) implies $D_{0+}^{\alpha} S_{\alpha} \mathcal{R}_{\theta_0} u \in L^2(0,T;X^*)$, by [43, Lemma 3.4] we have

$$\mathcal{S}_{\alpha}\mathcal{R}_{\theta_0}u \in {}_0H^{\alpha}(0,T;X^*).$$

Also, (95) and (15) imply

$$\mathcal{S}_{\alpha}\mathcal{R}_{\theta_0}u \in L^2(0,T;Y).$$

Consequently, by (4) we have

$$\int_0^T \langle \mathcal{D}_{0+}^{\alpha} \, \mathcal{S}_{\alpha} \mathcal{R}_{\theta_0} u, \varphi \rangle_X \, \mathrm{d}t = \int_0^T (\mathcal{S}_{\alpha} \mathcal{R}_{\theta_0} u, \, \mathcal{D}_{T-}^{\alpha} \, \varphi)_Y \, \mathrm{d}t,$$

and it is evident by (6) that

$$\int_0^T \langle -\mathcal{A}\mathcal{S}_\alpha \mathcal{R}_{\theta_0} u, \varphi \rangle_X \, \mathrm{d}t = \int_0^T (\mathcal{S}_\alpha \mathcal{R}_{\theta_0} u, -\mathcal{A}^* \varphi)_Y \, \mathrm{d}t.$$

Combining (96) and the above two equations gives

$$\int_0^T \left(\mathcal{S}_\alpha \mathcal{R}_{\theta_0} u, \, (\mathcal{D}_{T-}^\alpha - \mathcal{A}^*) \varphi \right)_Y \mathrm{d}t = \int_0^T \langle \mathcal{R}_{\theta_0} u, \varphi \rangle_X \, \mathrm{d}t.$$

The arbitrariness of $\varphi \in {}^{0}H^{\alpha}(0,T;Y) \cap L^{2}(0,T;X)$ proves that $\mathcal{S}_{\alpha}\mathcal{R}_{\theta_{0}}u$ is indeed the very weak solution. The general case $u \in L^{2}(0,T;Z)$ then follows from a standard density argument by

$$\mathcal{S}_{\alpha}\mathcal{R}_{\theta_0} \in \mathcal{L}(L^2(0,T;Z), L^2(0,T;Y)),$$

which is a direct consequence of (32) and the fact $\mathcal{R}_{\theta_0} \in \mathcal{L}(Z, [X^*, Y]_{\theta_0})$. This completes the proof.

4 Numerical results

This section performs three numerical experiments in two-dimensional space to verify the theoretical results. We will use the following settings: $\Omega := (0,1) \times (0,1)$; T = 0.1; $X, Y, Z, \mathcal{R}_{\theta_0}$ and U_{ad} are defined as in Section 3.

Experiment 1. Define

$$g(t) := \begin{cases} 1 & \text{if } 0 < t < 2/3T, \\ 3 & \text{if } 2/3T < t < T, \end{cases}$$
$$v(x,y) := \begin{cases} y^{-1/2} & \text{if } (x,y) \in \{(0,y) : 0 < y < 1\}, \\ 0 & \text{if } (x,y) \in \partial\Omega \setminus \{(0,y) : 0 < y < 1\} \end{cases}$$

To approximate $S_{\alpha}\mathcal{R}_{\theta_0}(gv)$, we use discretization (46)($0 < \alpha < 1$) or (48)($\alpha = 1$) in time and use the usual $H^1(\Omega)$ -conforming P1-element method in space. Let U^M be the corresponding numerical approximation with time step $\tau = T/2^M$ and spatial mesh size $h = 2^{-9}$. Estimates (62) and (93) predict that $\|U^M - U^{13}\|_{L^{\infty}(0,T;L^2(\Omega))}$ is close to $O(\tau^{0.125})$ for $\alpha = 0.5$ and close to $O(\tau^{0.25})$ for $\alpha = 1$, and this is confirmed by the numerical results in Table 1.

	$\alpha = 0.5$		$\alpha = 1$	
M	$ U^M - U^{13} _{L^{\infty}(L^2)}$	Order	$ U^M - U^{13} _{L^{\infty}(L^2)}$	Order
4	6.93e-1	_	4.34e-1	_
6	6.26e-1	0.07	3.23e-1	0.21
8	5.51e-1	0.09	2.24e-1	0.26
10	4.96e-1	0.08	1.62e-1	0.24

Table 1: $\|\cdot\|_{L^{\infty}(L^2)}$ means the norm $\|\cdot\|_{L^{\infty}(0,T;L^2(\Omega))}$

Experiment 2. Define

$$v(x,y) := x^{-1/2}$$
 for all $(x,y) \in \Omega$.

To approximate $S_{\alpha}(v\delta_0)$, we use discretization (46)($0 < \alpha < 1$) or (48)($\alpha = 1$) in time and use the usual $H^1(\Omega)$ -conforming P1-element method in space. Let U^M be the corresponding numerical approximation with time step $\tau = T/2^M$ and spatial mesh size $h = 2^{-9}$. Table 2 illustrates that $\|U^M - U^{13}\|_{L^1(0,T;H^1_0(\Omega))}$ is close to $O(\tau^{0.25})$ for $\alpha = 0.5$ and close to $O(\tau^{0.5})$ for $\alpha = 1$, which agrees well with estimate (63).

	$\alpha = 0.5$		$\alpha = 1$	
M	$ U^M - U^{13} _{L^1(H^1_0)}$	Order	$ U^M - U^{13} _{L^1(H^1_0)}$	Order
4	1.71e-0	_	4.24e-1	_
5	1.46e-0	0.23	3.01e-1	0.49
6	1.23e-0	0.25	2.10e-1	0.52
7	1.02e-0	0.27	1.45e-1	0.54

Table 2: $\|\cdot\|_{L^{1}(H_{0}^{1})}$ means the norm $\|\cdot\|_{L^{1}(0,T;H_{0}^{1}(\Omega))}$

Experiment 3. Let $\nu := 10, u_* := 0, u^* := 20$ and

$$y_d(x,y) := 1$$
 for all $(x,y) \in \Omega$.

To approximate problem (90), we will use the temporal discretization in Section 2.3 and the $H^1(\Omega)$ -conforming P1-element method to discretize the state equation in time and space, respectively; see [16] for the implementation details. Let U^M be the corresponding numerical solution with time step $\tau = T/2^M$ and spatial mesh size $h = 2^{-8}$. The numerical results in Table 3 show that $\|U^M - U^{13}\|_{L^2(0,T;L^2(\partial\Omega))}$ is close to $O(\tau^{0.125})$, which agrees with error estimate (94).

M	$ U^M - U^{12} _{L^2(0,T;L^2(\partial\Omega))}$	Order
4	2.44e-1	_
5	2.08e-1	0.23
6	1.88e-1	0.15
7	1.66e-1	0.18

Table 3: Numerical results for Experiment 3 with $\alpha = 1$

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