Stability and conservation properties of Hermite-based approximations of the Vlasov-Poisson system

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Abstract

Spectral approximation based on Hermite-Fourier expansion of the Vlasov-Poisson model for a collisionless plasma in the electrostatic limit is provided, by including high-order artificial collision operators of Lenard-Bernstein type. These differential operators are suitably designed in order to preserve the physically-meaningful invariants (number of particles, momentum, energy). In view of time-discretization, stability results in appropriate norms are presented. In this study, necessary conditions link the magnitude of the artificial collision term, the number of spectral modes of the discretization, as well as the time-step. The analysis, carried out in full for the Hermite discretization of a simple linear problem in one-dimension, is then partly extended to cover the complete nonlinear Vlasov-Poisson model.

Key words: Vlasov equation, spectral methods, conservation laws, Hermite polyomials

1991 MSC: 65N35, 35Q83

1. Introduction

The numerical approximation of physical systems described by kinetic equations is a formidable challenge [32]. These equations are, indeed, highly dimensional, strongly non-linear, and describe phenomena that are extremely multi-scale, as the behavior of the physical system at macroscopic scales is influenced by the microscopic particle dynamics. In plasma physics, scale separation occurs at the kinetic level because of the difference in mass between electrons and ions [18]. Other important applications that may be worth mentioning can be found in fluid dynamics, particularly, atmospheric and climate research [1], and multidimensional radiative transfer problems [24]. In all these fields, performing macroscale simulations that accurately include effects from the underlying microscale particle dynamics is still an open challenge.

In this work, we focus on the numerical approximation of the kinetics equation describing the behavior of electrically charged particle in a noncollisional plasmas, also known as *the Vlasov equation*. Such equation governs the time evolution of the distribution function of the plasma particles, through the action of an electromagnetic field generated by the charge and current densities of the same moving particles. The resulting coupling through Maxwell's equations (or the Poisson's equation in the electrostatic limit) is highly nonlinear since the electromagnetic sources in such equations, i.e., charge and current densities, depend on the same distribution functions [15].

In his historical and pioneering paper, cf. [16], Grad proposed to expand the velocity distribution function of a noncollisional plasma at equilibrium using Hermite functions. Hermite functions are Hermite polynomials multiplied by the Gaussian exponential function, $w(v) = \exp(-v^2)$, where v is the velocity of the plasma particles. Such a weight w is indeed the velocity distribution of a plasma at equilibrium and is a steady state solution of the Vlasov equation. Since a plasma at equilibrium is described by the first mode of the Hermite expansion, we expect that only a few modes may be needed to describe a plasma in a perturbed state but still close to the equilibrium. Moreover, when the solution of the Vlasov equation is expanded on the Hermite basis functions, the equations for the first three coefficients correspond to the conservation laws for the number of particles, momentum and energy, and determine the macroscopic (i.e., fluid) behavior of a plasma. The following terms of the Hermite expansion introduces kinetics effects in the model in a very straighforward manner, thus providing a strategy to realize the coupling between microand macro-physics. Thus, the micro/macro coupling is an intrinsic and specific feature of the Hermite approach, which cannot be replicated if we choose a different set of basis functions. For these reasons, Hermite functions are a sort of "ideal" basis for solving numerically Vlasov-based models of noncollisional plasmas.

Since late sixties throughout the last five decades, Grad's idea has extensively been applied to the development of plasma simulators; see, for example, [2, 14, 20, 19, 35, 6, 33], where the Hermite basis for velocity is coupled with the Fourier basis in space. A renewed interest has been manifested in very recent years towards these approximation methods [4, 5, 9, 10], as the excellent properties mentioned above make them the natural numerical framework of high resolution and computationally efficient solvers [39, 34]. Moreover, the accuracy of Hermite's approximations can be improved by order of magnitudes by introducing a translation factor, u, and a scaling factor, α , in the so-called generalized weight, $w(v) = exp\left(-((v-u)/\alpha)^2\right)$, cf. [37]. Empirical evidence that a convenient choice of the scaling factor α can improve the accuracy in Hermite discretizations of the Vlasov equation was shown in [35]. Generalized basis function of Hermite type has been investigated for solving time-dependent parabolic problems in [25] and, more recently, in [11] for the approximation of the Vlasov phase space. An adaptive strategy is currently under investigation, see [31], where both u and u may change through momentum and energy following how the plasma evolves in time during a numerical simulation. Such adaptive strategy is sought to improve the computational efficiency by using only a few spectral modes where a macroscopic description of the system is appropriate and adding more modes where the microscopic physics is important [38]. This aspect offers the possibility of selecting the most meaningful number of spectral modes for a given resolution in phase space.

The strong point in favour of spectral schemes is that such schemes can be extremely accurate because of their exceptional convergence rate, see, for example, the books referenced in [8, 7, 3, 13, 12, 36]. Their stability for Vlasovbased systems can be ensured in different ways. If we assume that the velocity domain remains bounded during a plasma simulation, we can use the different basis provided by Legendre polynomials and stability can be enforced somehow through a penalty technique acting on boundary terms, see for example [28, 29]. Relaxing this assumption yields an unbounded velocity domain and this approach is no longer feasible. In the more general case, the Vlasov equation describe a collisionless transport phenomenon in a six-dimensional phase space, and a straightforward way to enforce numerical stability to the discretization of an advection equation is by adding a suitable artificial dissipation to its, otherwise zero, right-hand side. However, in the case of the Vlasov equation, using an artificial dissipation term introduces a major issue because such modification must not destroy the conservation properties of the original method. Discrete analogs of the total number of particles (also proportional to mass and charge of the plasma particles), the total momentum and the total energy may indeed exist in spectral-based discretizations using, for the space term, the Fourier expansion [19, 35, 5, 9], or the discontinuous Galerkin method [27, 26, 30, 21, 22] Conservation properties are fundamental in long-time integration runs since they provide physically meaningful constraints on the numerical approximation of the plasma behavior. Such constraints are strongly related to significant properties like the wellposedness and robustness of the method, and the reliability of the numerical simulation. This fact justifies the great effort that has been devoted in design spectral methods with such discrete conservation properties.

In the spectral discretizations of the Vlasov equation using Hermite basis functions, the conservation of number of particles, momentum and energy is strictly related to the lowest-order modes and can be destroyed by the numerical dissipation term. A possible way to maintain a perfect preservation of low modes, is to design such dissipation terms through Lenard-Bernstein-like operators (see [23]) of order 2k, with integer $k \ge 1$. In this case, the 1D-1D Vlasov-Poisson system of equations takes the form

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial v} = -(-1)^k \nu \widetilde{L}^{(k)} L^{(k)} f \qquad \text{in } \Omega \times [0, T], \tag{1}$$

$$\frac{\partial E}{\partial x} = 1 - \int_{\Omega_v} f \, dv \qquad \text{in } \Omega \times [0, T], \tag{2}$$

where f is the distribution function, E the electric field, $\widetilde{L}^{(k)}$ and $L^{(k)}$ are the Lenard-Bernstein-like operators only acting onto the velocity variable v. The positive parameter ν is a sort of *artificial viscosity* used to tune the action of the differential operator $\widetilde{L}^{(k)}L^{(k)}$ on f. The combination $\widetilde{L}^{(k)}L^{(k)}$ is the Lenard-Bernstein-like operator of order 2k and introduces a sort of *artificial collisional term*, i.e., a numerical dissipation, in the equation. This kind of dissipation terms were proposed in previous works to control the filamentation process based on an empirical argument, cf. [4, 5, 9, 10, 28, 29].

Commonly, there are two different choices of Hermite functions, which are Hermite polynomials multiplied by a suitable weight function. The classical polynomial orthogonality weighted by $w(v) = e^{-v^2}$ leads to the so called asymmetrically weighted (AW) case, whereas the orthogonality of Hermite functions, each one weighted by $w(v) = e^{-v^2/2}$, leads to the symmetrically weighted (SW) case. This terminology will be better clarified in the coming sections. Accordingly, we have two different definitions of the Lenard-Bernstein differential operators $\widetilde{L}^{(k)}$ and $L^{(k)}$. In both cases, the basis elements are eigenfunctions of the combined operator, and the corresponding eigenvalues are zero regarding the first k-1 modes. This actually says that the action of these operators does not modify such modes, or, in other words, $\widetilde{L}^{(k)}L^{(k)}$ induces dissipation only for the modes starting from k. Despite these common properties, the two discrete formulations resulting from using AW and SW Hermite functions are substantially different. In fact, it turns out that, concerning time-discretization, the SW formulation can easily be proven to be algebraically stable with or without the diffusive term (see [19, 35]), while for the AW formulation the issue is far more delicate. More precisely, the stability result in the $L^2(\Omega)$ norm that we are interested to investigate reads as

$$\frac{d}{dt} \left\| f(\cdot, \cdot, t) \right\|_{L^2(\Omega)}^2 \le 0.$$

This inequality trivially implies the boundedness in time of f. The main criticism to the SW formulation is that, although stable, it does not effectively preserve the lowest modes during time evolution. On the contrary, the AW formulation perfectly conserves all the basic invariants, but its stability needs a deeper analysis. What we are able to prove in our work is an $L^2(\Omega)$ stability result when ν is sufficiently large thanks to a suitable extension of the Poincarè inequality in weighted norms defined on the real line. The property of stability then follows by classical estimates for bilinear forms in Sobolev spaces. When instead ν is small, the result is certainly not true in the continuous case, but still holds in the framework of numerical discretizations, by suitably linking ν to the time discretization parameter Δt , the final time T, and the maximum integer N used for the Hermite truncation in the variable ν . We show how to get these relations for a simple linear advection-diffusion model problem, and successively we partly extend our arguments to equation (1).

A stability result for the Hermite approximation of 1D-1V Vlasov-Poisson model was provided in [14], where L^2 boundedness is proven with respect to the parameter N. However, that paper fails in proving absolute stability with respect to t, since the estimate there provided contains an exponential growth in time on the right-hand side of the estimate inequality. The major result of our work is in achieving a stability estimate where boundedness in time is guaranteed for all t.

The outline of the paper is as follows. In Section 2, we introduce the Hermite-based discretization framework and discuss some useful relations. In Sections 3 and 4, we introduce the Lenard-Bernstein-like operators for the spectral method using the asymmetrically weighted (AW) Hermite functions, and study their actions on the conservation property of the Vlasov-Poisson system. In Sections 5 and 6, we do the same for the spectral method using the symmetrically weighted (SW) Hermite functions. In Section 7 we introduce the SW and AW Hermite discretization of the advection problem

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} = -(-1)^k \nu \widetilde{L}^{(k)} L^{(k)} f,\tag{3}$$

for the unknown scalar field f(v,t), with the initial condition $f(v,0)=f_0(v)$, and in Section 8, we study how the stabilization operator $\widetilde{L}^{(k)}L^{(k)}$ impact on its spectral discretization. In Section 9, we apply the implicit time

discretization to the system of coefficient resulting from the Hermite discretization and investigate its stability using a suitable weighted norm. In Section 10, we extend our approach to the full spectral discretization of the Vlasov-Poisson system of equations, and derive sufficient condition to guarantee the stability of the method. In Section 11 we offer our final remarks and conclusions.

2. Preliminary properties of the Hermite polynomials

We start by pointing out some well-known relations concerning Hermite polynomials, that, as usual, are denoted by $H_n(v)$ and we consider as functions of the independent variable $v \in \mathbb{R}$, the integer number n being the degree of the polynomial. First of all, we have the three-point recursion formula that links H_{n+1} to H_n and H_{n-1} :

$$H_0 = 1, \quad H_1 = 2v,$$
 (4)

$$H_{n+1} = 2vH_n - 2nH_{n-1}, \qquad n \ge 1 \tag{5}$$

and the differential equation for H_n

$$H_n'' - 2vH_n' + 2nH_n = 0, (6)$$

which holds for $n \in \mathbb{N}$ and where \prime and $\prime\prime$ denote the first and second derivatives with respect to v. Moreover, the next formulas link Hermite polynomials of different degrees n:

$$H_n' = 2vH_n - H_{n+1}, (7)$$

$$H'_0 = 0$$
 and $H'_n = 2nH_{n-1}, \quad \forall n \ge 1.$ (8)

The relation between the Hermite polynomials and their first derivative in (8) can recursively be generalized as follows:

$$H_n^{(m)} = \begin{cases} 0 & n < m, \\ 2^m \frac{n!}{(n-m)!} H_{n-m} & n \ge m. \end{cases}$$
 (9)

Hermite polynomials are orthogonal with respect to the weight function e^{-v^2} and are normalized in such a way that:

$$\int_{\mathbb{R}} H_n^2 e^{-v^2} \, dv = \sqrt{\pi} \, 2^n \, n!. \tag{10}$$

By examining relation (8), it turns out that the derivatives of the Hermite polynomials are also orthogonal with respect to the weight e^{-v^2} . Using (8) and (10) for $n \ge 1$, we can find that:

$$\int_{\mathbb{R}} (H'_n)^2 e^{-v^2} dv = 4n^2 \int_{\mathbb{R}} (H_{n-1})^2 e^{-v^2} dv = 4n^2 \sqrt{\pi} \, 2^{n-1} \, (n-1)!$$

$$= 2n \sqrt{\pi} \, 2^n \, n! = 2n \int_{\mathbb{R}} H_n^2 e^{-v^2} dv. \tag{11}$$

The above relation is trivially satisfied also for n = 0. For n > m, we recursively find that

$$\int_{\mathbb{R}} \left(H_n^{(m)} \right)^2 e^{-v^2} \, dv = 2^m \frac{n!}{(n-m)!} \int_{\mathbb{R}} H_n^2 e^{-v^2} \, dv. \tag{12}$$

Consider the generic function φ that can be expanded as a series of Hermite polynomials $\varphi = \sum_{n=0}^{\infty} C_n H_n$ and its first derivative $\varphi' = \sum_{n=1}^{\infty} C_n H'_n$. The Fourier coefficients C_n of φ are obtained as usual:

$$C_n = \frac{1}{\sqrt{\pi} \, 2^n \, n!} \int_{\mathbb{R}} \varphi H_n e^{-v^2} \, dv. \tag{13}$$

Of course, φ has to be such that all the above integrals are finite. From the orthogonality of Hermite polynomials and their derivatives, it follows that:

$$\int_{\mathbb{R}} \varphi^2 e^{-v^2} dv = \sum_{n=0}^{\infty} C_n^2 \int_{\mathbb{R}} H_n^2 e^{-v^2} dv,$$
$$\int_{\mathbb{R}} (\varphi')^2 e^{-v^2} dv = \sum_{n=0}^{\infty} C_n^2 \int_{\mathbb{R}} (H_n')^2 e^{-v^2} dv.$$

The last summation can also start from n = 1 since $H'_0 = 0$.

We show a few inequalities that will be used later in this paper. By isolating the effect of the first Fourier coefficients, we can prove Poincaré-type inequalities for a linear combination of Hermite polynomials and their first derivatives with respect to the norm induced by the weighted L^2 inner product where the weight is equal to e^{-v^2} . Indeed, the orthogonality of the first derivatives of the Hermite polynomials, equation (11), and the fact that $2n \geq 2$ for $n \geq 1$, imply that:

$$\int_{\mathbb{R}} (\varphi')^{2} e^{-v^{2}} dv = \int_{\mathbb{R}} \left(\sum_{n=1}^{\infty} C_{n} H'_{n} \right)^{2} e^{-v^{2}} dv = \sum_{n=1}^{\infty} C_{n}^{2} \int_{\mathbb{R}} \left(H'_{n} \right)^{2} e^{-v^{2}} dv$$

$$= \sum_{n=1}^{\infty} C_{n}^{2} 2n \int_{\mathbb{R}} H_{n}^{2} e^{-v^{2}} dv \ge 2 \sum_{n=1}^{\infty} C_{n}^{2} \int_{\mathbb{R}} H_{n}^{2} e^{-v^{2}} dv, \tag{14}$$

where all summations start from n = 1 since $H_0 = 1$ and $H'_0 = 0$. Then, we add and subtract the weighted integral of the zeroth-order mode, i.e, $C_0^2 H_0^2$, to the last member of inequality (14) and use the expansion of φ , so to have

$$\int_{\mathbb{R}} (\varphi')^{2} e^{-v^{2}} dv \ge 2 \sum_{n=0}^{\infty} C_{n}^{2} \int_{\mathbb{R}} H_{n}^{2} e^{-v^{2}} dv - 2C_{0}^{2} \int_{\mathbb{R}} H_{0}^{2} e^{-v^{2}} dv$$

$$= 2 \int_{\mathbb{R}} \varphi^{2} e^{-v^{2}} dv - 2\sqrt{\pi} C_{0}^{2}.$$
(15)

By reversing this inequality we find that

$$\int_{\mathbb{R}} \varphi^2 e^{-v^2} \, dv \le \frac{1}{2} \int_{\mathbb{R}} \left(\varphi' \right)^2 e^{-v^2} \, dv + \sqrt{\pi} C_0^2. \tag{16}$$

This inequality can be generalized to derivatives of order m > 1. Since $H_n^{(m)} = 0$ for n < m, using formulas (9) and (12), we find that

$$\int_{\mathbb{R}} (\varphi^{(m)})^2 e^{-v^2} dv = \int_{\mathbb{R}} \left(\sum_{n=m}^{\infty} C_n H_n^{(m)} \right)^2 e^{-v^2} dv = \sum_{n=m}^{\infty} C_n^2 \int_{\mathbb{R}} (H_n^{(m)})^2 e^{-v^2} dv$$

$$= \sum_{n=m}^{\infty} C_n^2 2^m \frac{n!}{(n-m)!} \int_{\mathbb{R}} H_n^2 e^{-v^2} dv \ge 2^m m! \sum_{n=m}^{\infty} C_n^2 \int_{\mathbb{R}} H_n^2 e^{-v^2} dv, \tag{17}$$

as $n!/(n-m)! \ge m!$ when $n \ge m$. Now, we add and subtract the weighted integral of the first m modes, i.e., $\left(C_{\ell}H_{\ell}\right)^2$, $\ell=0,\ldots,m-1$, to the last member of (17), and use the normalization of the Hermite polynomials to find that

$$\int_{\mathbb{R}} (\varphi^{(m)})^2 e^{-v^2} dv \ge 2^m m! \left(\sum_{n=0}^{\infty} C_n^2 \int_{\mathbb{R}} H_n^2 e^{-v^2} dv - \sum_{\ell=0}^{m-1} C_\ell^2 \int_{\mathbb{R}} H_\ell^2 e^{-v^2} dv \right)
= 2^m m! \left(\int_{\mathbb{R}} \varphi^2 e^{-v^2} dv - \sqrt{\pi} \sum_{\ell=0}^{m-1} 2^\ell \ell! C_\ell^2 \right).$$
(18)

By reversing this inequality we find that

$$\int_{\mathbb{R}} \varphi^2 e^{-v^2} \, dv \le \frac{1}{2^m \, m!} \int_{\mathbb{R}} \left(\varphi^{(m)} \right)^2 e^{-v^2} \, dv + \sqrt{\pi} \sum_{\ell=0}^{m-1} \, 2^\ell \, \ell! \, C_\ell^2. \tag{19}$$

The most general Poincaré-type inequality is the one involving derivatives of order m and p. Assuming that m > p and noting that $H^m = H^{(p+(m-p))} = (H^{(p)})^{(m-p)}$, a straightforward calculation exploiting the orthogonality of the derivatives of the Hermite polynomials yields

$$\int_{\mathbb{R}} (\varphi^{(m)})^{2} e^{-v^{2}} dv = \int_{\mathbb{R}} \left(\sum_{n=m}^{\infty} C_{n} H_{n}^{(m)} \right)^{2} e^{-v^{2}} dv = \int_{\mathbb{R}} \left(\sum_{n=m}^{\infty} C_{n} H_{n}^{(p+(m-p))} \right)^{2} e^{-v^{2}} dv$$

$$= \sum_{n=m}^{\infty} C_{n}^{2} 2^{m-p} \frac{n!}{(n-(m-p))!} \int_{\mathbb{R}} (H_{n}^{(p)})^{2} e^{-v^{2}} dv$$

$$\geq 2^{m-p} (m-p)! \sum_{n=m}^{\infty} C_{n}^{2} \int_{\mathbb{R}} (H_{n}^{(p)})^{2} e^{-v^{2}} dv, \tag{20}$$

where we also used the fact that n!/(n-(m-p))! > (m-p)! for n>1. Then, we add and subtract the weighted integrals of $C_\ell^2 \big(H_\ell^{(p)}\big)^2$ for $\ell=p,\ldots,p+(m-p)-1$, to the last member of (20) and we repeat the same argument as above to obtain

$$\int_{\mathbb{R}} (\varphi^{(p)})^{2} e^{-v^{2}} dv \leq \frac{1}{2^{m-p} (m-p)!} \int_{\mathbb{R}} (\varphi^{(m)})^{2} e^{-v^{2}} dv + \sum_{\ell=p}^{m-1} C_{\ell}^{2} \int_{\mathbb{R}} (H_{\ell}^{(p)})^{2} e^{-v^{2}} dv
= \frac{1}{2^{m-p} (m-p)!} \int_{\mathbb{R}} (\varphi^{(m)})^{2} e^{-v^{2}} dv + \sum_{\ell=p}^{m-1} C_{\ell}^{2} 2^{p} \frac{\ell!}{(\ell-p)!} \int_{\mathbb{R}} H_{\ell}^{2} e^{-v^{2}} dv
= \frac{1}{2^{m-p} (m-p)!} \int_{\mathbb{R}} (\varphi^{(m)})^{2} e^{-v^{2}} dv + 2^{p} \sqrt{\pi} \sum_{\ell=p}^{m-1} 2^{\ell} \frac{(\ell!)^{2}}{(\ell-p)!} C_{\ell}^{2}.$$
(21)

In particular, if φ belongs to the space of polynomials of degree at most N, we have $2n \leq 2N$, so that the relations in (14) can be adjusted to obtain the so called *inverse inequality*

$$\int_{\mathbb{R}} (\varphi')^2 e^{-v^2} dv \le 2N \int_{\mathbb{R}} \varphi^2 e^{-v^2} dv. \tag{22}$$

Another useful inequality can be derived as follows. First of all, from (7) and (8), we know that:

$$2vH_n = H'_n + H_{n+1} = 2nH_{n-1} + H_{n+1} \quad \forall n \ge 1.$$
 (23)

Afterwords, we start by showing that:

$$\int_{\mathbb{R}} v^{2} H_{n}^{2} e^{-v^{2}} dv = \int_{\mathbb{R}} n^{2} H_{n-1}^{2} e^{-v^{2}} dv + \frac{1}{4} \int_{\mathbb{R}} H_{n+1}^{2} e^{-v^{2}} dv$$

$$= \sqrt{\pi} \left[n^{2} 2^{n-1} (n-1)! + \frac{1}{4} 2^{n+1} (n+1)! \right] = \sqrt{\pi} \left[2^{n-1} n n! + 2^{n-1} (n+1) n! \right]$$

$$= \sqrt{\pi} 2^{n-1} (2n+1) n! \le \sqrt{\pi} \frac{3}{4} 2^{n+1} n n! = \frac{3}{4} \int_{\mathbb{R}} \left(H_{n}' \right)^{2} e^{-v^{2}} dv, \quad \forall n \ge 1, \quad (24)$$

where we noted that $2n+1 \le 2n+n=3n$, since $n \ge 1$. The last equality follows from (11). In short, we can write:

$$\int_{\mathbb{R}} v^2 H_n^2 e^{-v^2} \, dv \le \frac{3}{4} \int_{\mathbb{R}} \left(H_n' \right)^2 e^{-v^2} \, dv, \qquad \forall n \ge 1.$$
 (25)

In general, let us suppose that φ is a polynomial of degree N with $C_0=0$. Thus, φ has an expansion of the type $\varphi=\sum_{n=1}^N C_n H_n$. For a given set of values α_n , the following relation is a consequence of the Schwartz inequality:

$$\left(\sum_{n=1}^{N} \alpha_n\right)^2 = \left(\sum_{n=1}^{N} 1 \cdot \alpha_n\right)^2 \le \sum_{n=1}^{N} 1^2 \sum_{n=1}^{N} \alpha_n^2 = N \sum_{n=1}^{N} \alpha_n^2.$$
 (26)

With the help of the above inequality, the orthogonality of the Hermite polynomials implies that:

$$\int_{\mathbb{R}} v^{2} \varphi^{2} e^{-v^{2}} dv = \int_{\mathbb{R}} v^{2} \left(\sum_{n=1}^{N} C_{n} H_{n} \right)^{2} e^{-v^{2}} dv \le N \sum_{n=1}^{N} C_{n}^{2} \int_{\mathbb{R}} v^{2} H_{n}^{2} e^{-v^{2}} dv$$

$$\le \frac{3}{4} N \sum_{n=1}^{N} C_{n}^{2} \int_{\mathbb{R}} \left(H_{n}' \right)^{2} e^{-v^{2}} dv = \frac{3}{4} N \int_{\mathbb{R}} \left(\varphi' \right)^{2} e^{-v^{2}} dv, \tag{27}$$

which holds for every polynomial φ with degree less or equal to N and $C_0=0$.

We end this preliminary section by introducing a few definitions concerning the *Hermite functions*, i.e., those functions that can be written as a linear combination (finite or infinite) of the elements of the *Hermite basis functions* $\{\psi_n\}$. Following the current literature, we will adopt a suitable notation in order to distinguish the so-called *symmetrically-weighted* (SW) case, from the *asymmetrically-weighted* (AW) one. The reason of this setting will be made clear as we proceed with the exposition. We then consider the following definition:

$$\psi_n(v) = \begin{cases} \gamma_n^{SW} H_n(v) e^{-v^2/2} & \text{symmetrically-weighted case,} \\ \gamma_n^{AW} H_n(v) e^{-v^2} & \text{asymmetrically-weighted case,} \end{cases}$$
 (28)

for some suitable choice of the real scalar coefficients γ_n^{SW} and γ_n^{AW} (see below). Besides, we introduce the dual basis functions defined by:

$$\psi^{n}(v) = \begin{cases} \widetilde{\gamma}_{n}^{SW} H_{n}(v) e^{-v^{2}/2} & \text{symmetrically-weighted case,} \\ \widetilde{\gamma}_{n}^{AW} H_{n}(v) & \text{asymmetrically-weighted case.} \end{cases}$$
 (29)

The coefficients $\widetilde{\gamma}_n^{SW}$ and $\widetilde{\gamma}_n^{AW}$ are obtained from the orthogonality relation:

$$\langle \psi_n, \psi^m \rangle = \delta_{n,m}. \tag{30}$$

We have:

$$\gamma_n^{SW} = \tilde{\gamma}_n^{SW} = (\sqrt{\pi} 2^n \, n!)^{-\frac{1}{2}},\tag{31}$$

and

$$\gamma_n^{AW} = (\pi 2^n \, n!)^{-\frac{1}{2}}, \qquad \widetilde{\gamma}_n^{AW} = (2^n \, n!)^{-\frac{1}{2}}.$$
 (32)

3. Diffusive operators in the AW case

Throughout the paper we will use indifferently the notation $\partial f/\partial v$ and f' to denote the partial derivative of functions like f(v) or f(t,v), regardless of their possible dependence on time.

We begin with the study of the second-order (k = 1) differential operator that appears in the Vlasov equation (1) and the simplified model equation (3). In the asymmetric case, this operator can be decomposed as the functional product of the two first-order operators:

$$L = \frac{1}{2} \frac{\partial}{\partial v} + v \mathcal{I}, \qquad \qquad \widetilde{L} = \frac{\partial}{\partial v}, \tag{33}$$

with \mathcal{I} the identity operator. The second operator, i.e., \widetilde{L} , is just the derivative with respect to the variable v.

We investigate the action of $\widetilde{L}L$ on Hermite functions that we write in the form:

$$f(v) = h(v)e^{-v^2},$$
 (34)

where h is a generic polynomial. For the operator L, we have:

$$Lf = \left(\frac{1}{2}\frac{\partial}{\partial v} + v\mathcal{I}\right)f = \frac{1}{2}h'e^{-v^2} - vhe^{-v^2} + vhe^{-v^2} = \frac{1}{2}h'e^{-v^2}.$$
 (35)

Clearly, Lf is identically zero if h is a constant. Therefore, by taking h=1 we find that $L(e^{-v^2})=0$.

Similarly, for k = 2 we have

$$L^{2}f = L(Lf) = L\left(\frac{1}{2}h'e^{-v^{2}}\right) = \frac{1}{4}h''e^{-v^{2}} - \frac{1}{2}vh'e^{-v^{2}} + \frac{1}{2}vh'e^{-v^{2}} = \frac{1}{4}h''e^{-v^{2}},\tag{36}$$

and, in general, for $k \geq 2$ we have

$$L^{k}f = L(L^{k-1}f) = \frac{1}{2^{k}}h^{(k)}e^{-v^{2}}.$$
(37)

Equation (37) can be proved recursively by using (35) for the first step, assuming that $L^{k-1} = (1/2^{k-1})h^{(k-1)}e^{-v^2}$ and applying the definition of L given in (33) to derive the relation at step k.

The combination of L and \widetilde{L} provides the so called second-order Lenard-Bernstein-like operator [23]:

$$\widetilde{L}Lf = \widetilde{L}\left(\frac{1}{2}\frac{\partial}{\partial v} + v\mathcal{I}\right)f = \widetilde{L}\left(\frac{1}{2}h'e^{-v^2}\right) = \frac{1}{2}h''e^{-v^2} - h've^{-v^2}.$$
(38)

Within the space of polynomials, $\widetilde{L}Lf$ is zero if and only if h is constant. The combined operator is diffusive. To prove this statement, we consider the time dependent problem for the unknown function $f(v,t) = h(v,t)e^{-v^2}$:

$$\frac{\partial f}{\partial t} - \widetilde{L}Lf = \frac{\partial f}{\partial t} - \frac{\partial Lf}{\partial v} = 0,$$
(39)

where again we assume that h is a polynomial with respect to v. We multiply (39) by h, integrate over \mathbb{R} , and, then, integrate by parts the second integrand. The boundary terms are zero since they can be expressed as a polynomial multiplied by e^{-v^2} , which tends to zero for $|v| \to \infty$. Considering the expression of Lf given in (35), we obtain:

$$0 = \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \widetilde{L}Lf \right) h \, dv = \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial Lf}{\partial v} \right) h \, dv = \int_{\mathbb{R}} \frac{\partial f}{\partial t} h + \int_{\mathbb{R}} \left(Lf \right) h' \, dv - \left[\left(Lf \right) h \right]_{-\infty}^{+\infty}$$
$$= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} \, dv + \frac{1}{2} \int_{\mathbb{R}} \left(h' \right)^2 e^{-v^2} \, dv. \tag{40}$$

From the equation above it follows that:

$$\frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} \, dv = -\int_{\mathbb{R}} \left(h' \right)^2 e^{-v^2} \, dv \le 0, \tag{41}$$

so that $\widetilde{L}Lf$ can be considered a dissipative operator for the weighted $L^2(\mathbb{R})$ norm.

Next, we repeat the same analysis for the fourth-order operator (k=2). Consider again $f(v,t)=h(v,t)e^{-v^2}$ with h polynomial, and the time dependent problem:

$$\frac{\partial f}{\partial t} + \widetilde{L}^2 L^2 f = \frac{\partial f}{\partial t} + \frac{\partial^2 L^2 f}{\partial v^2} = 0 \tag{42}$$

(note the change of sign with respect to Eq. (39)). As before, we multiply (42) by h and integrate over \mathbb{R} . Using the integration by parts (twice), we note that all the boundary terms are zero since they always consist of a polynomial function in v multiplied by the Gaussian function e^{-v^2} , which tends to zero for $|v| \to \infty$. Omitting the boundary terms and using (36) in the next calculation, we obtain:

$$0 = \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} + \frac{\partial^2 L^2 f}{\partial v^2} \right) h \, dv = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} \, dv + \int_{\mathbb{R}} \left(L^2 f \right) h'' \, dv$$
$$= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} \, dv + \frac{1}{4} \int_{\mathbb{R}} \left(h'' \right)^2 e^{-v^2} \, dv. \tag{43}$$

The equations above imply that $-\widetilde{L}^2L^2f$ plays the role of a diffusive term, since:

$$\frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} dv = -\frac{1}{2} \int_{\mathbb{R}} (h'')^2 e^{-v^2} dv \le 0.$$
 (44)

The general case can be handled in a very similar way. We write the time-dependent problem with the 2k-th order operator as follows:

$$\frac{\partial f}{\partial t} + (-1)^k \widetilde{L}^k L^k f = \frac{\partial f}{\partial t} + (-1)^k \frac{\partial^k L^k f}{\partial v^k} = 0 \qquad (k \ge 1).$$
 (45)

Repeating the same arguments it follows that $-(-1)^k \widetilde{L}^k L^k f$ is a diffusive operator. Indeed, applying the integration by parts (k times) and recalling (37), yields:

$$-(-1)^k \int_{\mathbb{R}} \left(\widetilde{L}^k L^k f \right) h \, dv = -(-1)^k \int_{\mathbb{R}} \frac{\partial^k L^k f}{\partial v^k} h \, dv$$
$$= -(-1)^k (-1)^k \int_{\mathbb{R}} \left(L^k f \right) h^{(k)} \, dv = -\frac{1}{2^k} \int_{\mathbb{R}} \left(h^{(k)} \right)^2 e^{-v^2} \, dv \le 0, \tag{46}$$

where $h^{(k)}$ is the k-th derivative of h with respect to v. The operators of order 2k for $k \ge 1$ so far examined are not strictly negative definite, since their kernel is not empty.

4. Action of the diffusive operators in the AW Hermite case

Consider (38) in terms of the Hermite functions' basis. A direct calculation yields:

$$\widetilde{L}L\psi_n = \widetilde{L}L(\gamma_n^{AW}H_ne^{-v^2}) = \frac{\gamma_n^{AW}}{2}(H_n'' - 2vH_n')e^{-v^2} = \frac{\gamma_n^{AW}}{2}(-2nH_ne^{-v^2})$$

$$= -n\gamma_n^{AW}H_ne^{-v^2} = -n\psi_n, \tag{47}$$

where we used the differential equation (6). In other words, the function ψ_n is the eigenfunction of the differential operator $\widetilde{L}L$ with eigenvalue -n. As the corresponding eigenvalue is zero for n=0, it follows that $\widetilde{L}L$ acts on Hermite functions without altering the equation for the first Hermite coefficient C_0 . This is a further confirmation of the diffusive nature of the operator regarding the Hermite modes that are higher than 1.

A similar relation holds also for \widetilde{L}^2L^2 and for the more general operator \widetilde{L}^kL^k . First, we consider the case k=2. Using (36) with $h=H_n$, a straightforward calculation yields:

$$\widetilde{L}^{2}L^{2}\psi_{n} = \widetilde{L}^{2}L^{2}\left(\gamma_{n}^{AW}H_{n}e^{-v^{2}}\right) = \widetilde{L}^{2}\left(\gamma_{n}^{AW}\frac{1}{4}H_{n}^{"}e^{-v^{2}}\right) = \frac{\gamma_{n}^{AW}}{4}\left(H_{n}^{"}e^{-v^{2}}\right)^{"}.$$
(48)

To compute the last term in the equation above, we proceed in two steps, starting from the first derivative of $H''e^{-v^2}$. Using (6), we have that:

$$(H_n''e^{-v^2})' = ((2vH_n' - 2nH_n)e^{-v^2})'$$

$$= (2H_n' + 2vH_n'' - 2nH_n')e^{-v^2} - 2v(2vH_n' - 2nH_n)e^{-v^2}$$

$$= (2H_n' + 2vH_n'' - 2nH_n')e^{-v^2} - 2vH_n''e^{-v^2}$$

$$= 2(1-n)H_n'e^{-v^2}.$$
(49)

Using again (6), we have that:

$$\left(H_n''e^{-v^2}\right)' = H_n'''e^{-v^2} - 2vH_n''e^{-v^2} = \left(H_n'' - 2vH_n'\right)e^{-v^2} = -2nH_ne^{-v^2}.$$
 (50)

Hence, the second derivative of $H''e^{-v^2}$ with respect to v is readily given by collecting the results of (49) and (50), and reads as:

$$\left(H_n''e^{-v^2}\right)'' = \left(\left(H_n''e^{-v^2}\right)'\right)' = \left(2(1-n)H_n'e^{-v^2}\right)' = 4n(n-1)H_ne^{-v^2}.$$
 (51)

Replacing (51) in (48), finally yields:

$$\widetilde{L}^{2}L^{2}\psi_{n} = \frac{\gamma_{n}^{AW}}{4} 4n(n-1)H_{n}e^{-v^{2}} = n(n-1)\gamma_{n}^{AW}H_{n}e^{-v^{2}} = n(n-1)\psi_{n},$$
(52)

which shows that ψ_n is an eigenfunction of \widetilde{L}^2L^2 corresponding to the eigenvalue n(n-1). Note that such eigenvalue is zero for n=0 and n=1, which means that the fourth-order operator \widetilde{L}^2L^2 does not modify the equations for the first two modes of the AW Hermite expansion of f.

Repeating the same argument for a general integer $k \ge 1$, we find out that:

$$\widetilde{L}^k L^k \psi_n = (-1)^k n(n-1) \dots (n-(k-1))\psi_n = (-1)^k \frac{n!}{(n-k)!} \psi_n.$$
(53)

Therefore, we conclude that every element of the AW Hermite function's basis is an eigenfunction of the 2k-th operator $\widetilde{L}^k L^k$ with eigenvalue $(-1)^k n!/(n-k)!$, which takes the value of zero for $0 \le n \le k-1$.

We conclude this section by investigating the action of the Lenard-Bernstein-like operators on Hermite functions expressed as linear combinations of the AW Hermite functions' basis and the implications on the conservation properties of the discretization. Similar topics were considered in the more specific context of Vlasov-based models, cf. [9, 4].

To this end, we consider again the expansion $f(v) = h(v)e^{-v^2}$, where the polynomial function is given by (see (13)):

$$h = \sum_{n=0}^{\infty} C_n H_n. \tag{54}$$

By multiplying and dividing by the normalization factor γ_n^{AW} , and, then, using the definition of the AW basis (see (28)-(29)) we find that:

$$f = he^{-v^2} = \left(\sum_{n=0}^{\infty} C_n H_n\right) e^{-v^2} = \sum_{n=0}^{\infty} \frac{C_n}{\gamma_n^{AW}} \left(\gamma_n^{AW} H_n e^{-v^2}\right) = \sum_{n=0}^{\infty} C_n^{\star} \psi_n,\tag{55}$$

where $C_n^{\star} = C_n/\gamma_n^{AW}$. Since ψ_n is an eigenfunction of the generalized Lenard-Bernstein operators, we obtain the following relations:

$$\widetilde{L}Lf = \sum_{n=0}^{\infty} C_n^* \widetilde{L}L\psi_n = \sum_{n=0}^{\infty} (-n)C_n^* \psi_n,$$
(56)

$$\widetilde{L}^{2}L^{2}f^{2} = \sum_{n=0}^{\infty} C_{n}^{\star} \widetilde{L}^{2}L^{2}\psi_{n} = \sum_{n=0}^{\infty} n(n-1)C_{n}^{\star}\psi_{n},$$
(57)

. . .

$$\widetilde{L}^{k}L^{k}f = \sum_{n=0}^{\infty} C_{n}^{\star}\widetilde{L}^{k}L^{k}\psi_{n} = \sum_{n=0}^{\infty} (-1)^{k} \frac{n!}{(n-k)!} C_{n}^{\star}\psi_{n}.$$
(58)

From the identities above, it follows immediately that:

$$\widetilde{L}Lf = \sum_{n=0}^{\infty} D_n^{(1)} \psi_n \quad \text{with } D_n^{(1)} = -nC_n^{\star},$$
 (59)

$$\widetilde{L}^{2}L^{2}f = \sum_{n=0}^{\infty} D_{n}^{(2)}\psi_{n} \qquad \text{with } D_{n}^{(2)} = n(n-1)C_{n}^{\star}, \tag{60}$$

. .

$$\widetilde{L}^k L^k f = \sum_{n=0}^{\infty} D_n^{(k)} \psi_n \qquad \text{with } D_n^{(k)} = (-1)^k \frac{n!}{(n-k)!} C_n^{\star}. \tag{61}$$

By definition, it holds that $D_0^{(k)} = D_1^{(k)} = \ldots = D_{k-1}^{(k)} = 0$ for a generic $k \ge 1$. The case k = 3 corresponds to the operator used in Refs. [5, 9].

Using the properties that we have established so far, we are able to prove some conservation properties for problems of parabolic type like those considered in (39) (using $\widetilde{L}Lf$), (42) (using $-\widetilde{L}^2L^2f$), (45) (using $-(-1)^k\widetilde{L}^kL^kf$). The mass conservation for a distribution function f(t,v) is expressed by:

$$\frac{d}{dt} \int_{\mathbb{R}} f \, dv = 0. \tag{62}$$

In the first case, we integrate (62) on \mathbb{R} , use (39), apply the fundamental theorem of calculus and substitute the expression of Lf in (35) to obtain:

$$\frac{d}{dt} \int_{\mathbb{R}} f \, dv = \int_{\mathbb{R}} \widetilde{L} L f \, dv = \int_{\mathbb{R}} \frac{\partial L f}{\partial v} \, dv = \left[L f \right]_{-\infty}^{\infty} = \frac{1}{2} \left[h' e^{-v^2} \right]_{-\infty}^{\infty} = 0, \tag{63}$$

since e^{-v^2} times a polynomial of any degree tends to zero for $v \to \pm \infty$.

In the second case, we integrate (62) on \mathbb{R} , use (42), and apply the fundamental theorem of calculus to obtain:

$$\frac{d}{dt} \int_{\mathbb{R}} f \, dv = -\int_{\mathbb{R}} \widetilde{L}^2 L^2 f \, dv = -\int_{\mathbb{R}} \frac{\partial}{\partial v} (\widetilde{L} L^2 f) \, dv = -\left[\widetilde{L} L^2 f\right]_{-\infty}^{\infty}. \tag{64}$$

Furthermore, by using (36), we find that:

$$\widetilde{L}L^2f = \frac{\partial L^2f}{\partial v} = \frac{1}{4}\frac{\partial}{\partial v}\left(h''e^{-v^2}\right) = \frac{1}{4}\left(h''' - 2vh''\right)e^{-v^2}.$$
(65)

Therefore, the last term above provides zero in (64), since the Gaussian function e^{-v^2} multiplied by any polynomial tends to zero for $v \to \pm \infty$.

Finally, to obtain the general result for $\widetilde{L}^k L^k f$, we integrate (62) on \mathbb{R} , use (45), and apply the fundamental theorem of calculus. We obtain:

$$\frac{d}{dt} \int_{\mathbb{R}} f \, dv = -(-1)^k \int_{\mathbb{R}} \widetilde{L}^k L^k f \, dv = -(-1)^k \int_{\mathbb{R}} \frac{\partial}{\partial v} (\widetilde{L}^{k-1} L^k f) \, dv$$

$$= -(-1)^k \left[\widetilde{L}^{k-1} L^k f \right]_{-\infty}^{\infty} = 0, \tag{66}$$

since we can prove recursively that $\widetilde{L}^{k-1}L^kf$ is equal to a polynomial times e^{-v^2} , which tends to zero for $v\to\pm\infty$. Another important issue is the *momentum conservation*, which is expressed by:

$$\frac{d}{dt} \int_{\mathbb{R}} v f \, dv = 0. \tag{67}$$

We start by noting that there is no momentum conservation for the operator $\widetilde{L}L$. We then consider the two other cases in which f is the solution of (42) (using $-\widetilde{L}^2L^2f$), and (45) (using $-(-1)^k\widetilde{L}^kL^kf$).

In the first case, momentum conservation is achieved because, in view of (42), we know that:

$$\frac{d}{dt} \int_{\mathbb{R}} v f \, dv = -\int_{\mathbb{R}} v \frac{\partial^2 L^2 f}{\partial v^2} \, dv. \tag{68}$$

Then, we integrate by parts the right-hand side, apply the fundamental theorem of calculus and arrive at:

$$\frac{d}{dt} \int_{\mathbb{R}} v f \, dv = \int_{\mathbb{R}} \frac{\partial L^2 f}{\partial v} \, dv - \left[v \frac{\partial L^2 f}{\partial v} \right]_{-\infty}^{\infty} = \left[L^2 f \right]_{-\infty}^{\infty} - \left[v \frac{\partial L^2 f}{\partial v} \right]_{-\infty}^{\infty} = 0, \tag{69}$$

As in the previous situations, the arguments in the square brackets are of the form of a polynomial multiplied by the Gaussian function e^{-v^2} .

Through very similar steps, we can easily arrive at a general statement regarding the conservation of the m-th moment, m > 1. Indeed, we have:

$$\frac{d}{dt} \int_{\mathbb{R}} v^m f \, dv = 0 \tag{70}$$

in presence of the operator $\widetilde{L}^k L^k f$, and provided that the condition k > m is satisfied. The conservation of the velocity moments of the distribution function f implies the conservation of physical quantities such as momentum and energy in Vlasov models. We will discuss this topic at the beginning of Section 10.

5. Diffusive operators in the SW case

Differently from the AW case, the generalized Lenard-Bernstein operators that we consider in the SW case read as follows:

$$L = \frac{\partial}{\partial v} + v\mathcal{I}, \qquad \widetilde{L} = \frac{\partial}{\partial v} - v\mathcal{I}. \tag{71}$$

We investigate the action of the $\widetilde{L}L$ operator on Hermite functions of the form $f = he^{-v^2/2}$, where h is once again a polynomial in v. The weighted L^2 inner product for such functions is:

$$(f,g) = \int_{\mathbb{R}} fg \, dv = \int_{\mathbb{R}} h_f h_g e^{-v^2} \, dv, \tag{72}$$

where $f = h_f e^{-v^2/2}$ and $g = h_g e^{-v^2/2}$, and h_f and h_g are polynomials. This somehow justifies the adoption of the term "symmetric".

The results will be analogous to those presented in the previous sections. We briefly review the main points. From straightforward calculations it follows that:

$$Lf = f' + vf = h'e^{-v^2/2} - vhe^{-v^2/2} + vhe^{-v^2/2} = h'e^{-v^2/2},$$
(73)

$$\widetilde{L}Lf = \widetilde{L}(h'e^{-v^2/2}) = (h'' - 2vh')e^{-v^2/2}.$$
 (74)

These relations imply that the operator $\widetilde{L}L$ is diffusive. In fact, consider again the time dependent problem:

$$\frac{\partial f}{\partial t} - \widetilde{L}Lf = 0, (75)$$

where, now, we choose $f(v,t) = h(v,t)e^{-v^2/2}$. We multiply equation (75) by f and integrate over \mathbb{R} . Thus, we end up with the equality:

$$\int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \widetilde{L}Lf \right) f \, dv = 0, \tag{76}$$

and using the definition of \widetilde{L} given in (71), we have that

$$\frac{1}{2} \int_{\mathbb{R}} \frac{\partial}{\partial t} \left(h^2 e^{-v^2} \right) dv - \int_{\mathbb{R}} \left((Lf)' - vLf \right) f \, dv = 0, \tag{77}$$

where again we denoted the derivative with respect to v of Lf by (Lf)'. Then, we integrate by parts the second integral of (77) and note again that the boundary terms for $v \to \pm \infty$ are zero. This leads us to:

$$0 = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} dv + \int_{\mathbb{R}} (Lf) f' dv - [(Lf)f]_{-\infty}^{\infty} + \int_{\mathbb{R}} v(Lf) f dv$$

$$= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} dv + \int_{\mathbb{R}} (Lf) (f' + vf) dv = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} dv + \int_{\mathbb{R}} (Lf)^2 dv.$$
 (78)

The last relation shows that the operator $\widetilde{L}L$ introduces a dissipation.

The same result holds for the fourth-order operator and the related time dependent problem:

$$\frac{\partial f}{\partial t} + \tilde{L}^2 L^2 f = 0. ag{79}$$

Here, the proof is a bit more involved, but still elementary. We first note that $\widetilde{L}^2g=\widetilde{L}(\widetilde{L}g)=\widetilde{L}(g'-vg)$, from which it follows that:

$$\widetilde{L}^2 g = (g' - vg)' - v(g' - vg) = g'' - 2vg' + (v^2 - 1)g, \tag{80}$$

and

$$L^{2}f = f'' + 2vf' + (v^{2} + 1)f = f'' + 2(vf)' + (v^{2} - 1)f.$$
(81)

By multiplying equation (79) by f and integrating over \mathbb{R} , we find that:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}h^2e^{-v^2}dv + \int_{\mathbb{R}}\left(\widetilde{L}^2L^2f\right)f\,dv = 0.$$
(82)

From straightforward calculations using integration by parts, (80) (with $g=L^2f$) and (81), we get the following relation:

$$\int_{\mathbb{R}} (\widetilde{L}^{2}L^{2}f) f \, dv = \int_{\mathbb{R}} \left((L^{2}f)'' - 2v(L^{2}f)' + (v^{2} - 1)L^{2}f \right) f \, dv$$

$$= \int_{\mathbb{R}} (L^{2}f) f'' \, dv + 2 \int_{\mathbb{R}} (L^{2}f) (vf)' \, dv + \int_{\mathbb{R}} (v^{2} - 1)(L^{2}f) f \, dv$$

$$= \int_{\mathbb{R}} (L^{2}f) \left(f'' + 2(vf)' + (v^{2} - 1)f \right) \, dv = \int_{\mathbb{R}} (L^{2}f)^{2} \, dv. \tag{83}$$

Therefore, also this time-dependent equation is dissipative, from the viewepoint of the $L^2(\mathbb{R})$ -weighted norm.

In general, we may consider the time dependent problem:

$$\frac{\partial f}{\partial t} + (-1)^k \widetilde{L}^k L^k f = 0. \tag{84}$$

With the same considerations as above, we find the relation:

$$\frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} dv = -\int_{\mathbb{R}} \left(L^k f \right)^2 dv \le 0, \tag{85}$$

which shows the dissipative nature of the second term of (84).

Regarding the expansion in the Hermite basis functions, after the application of the diffusive operators, we also obtain straightforward results. First, we write the function $f(v) = h(v)e^{-v^2/2}$ in the SW Hermite basis, by using the expansion:

$$h = \sum_{n=0}^{\infty} C_n H_n, \quad \text{with} \quad C_n = \gamma_n^{SW} \tilde{\gamma}_n^{SW} \int_{\mathbb{R}} h H_n e^{-v^2} dv.$$
 (86)

The corresponding coefficients $D_m^{(1)}$ are such that:

$$\widetilde{L}Lf = \widetilde{L}\left(\sum_{n=0}^{\infty} C_n H_n' e^{-v^2/2}\right) = \sum_{m=1}^{\infty} D_m^{(1)} H_m e^{-v^2/2},\tag{87}$$

which allows us to express $\widetilde{L}Lf$ in terms of Hermite functions. In practice:

$$\widetilde{L}L\left(H_{n}e^{-v^{2}/2}\right) = \widetilde{L}\left(H'_{n}e^{-v^{2}/2}\right) = H''_{n}e^{-v^{2}/2} - vH'_{n}e^{-v^{2}/2} - vH'_{n}e^{-v^{2}/2}$$

$$= \left(H''_{n} - 2vH'_{n}\right)e^{-v^{2}/2} = -2nH_{n}e^{-v^{2}/2},$$
(88)

where we used again the differential equation for Hermite polynomials (see (6)). Therefore, we obtain:

$$D_m^{(1)} = -2mC_m \qquad \text{for } m \ge 1, \tag{89}$$

Going to the general case, we want to compute the coefficients $\mathcal{D}_m^{(k)}$ such that:

$$\widetilde{L}^k L^k f = \sum_{m=0}^{\infty} D_m^{(k)} H_m e^{-v^2/2},\tag{90}$$

One finally obtains:

$$D_m^{(k)} = (-1)^k 2^k m(m-1) \cdots (m-k) C_m = (-1)^k 2^k \frac{m!}{(m-k-1)!} C_m \quad \text{for } m \ge 0.$$
 (91)

As for the AW case, the first k + 1 coefficients are automatically zero.

6. Action of the diffusive operators in the SW Hermite case

We recall that $L = \partial/\partial v + v\mathcal{I}$ and $\widetilde{L} = \partial/\partial v - v\mathcal{I}$. Consider the SW Hermite basis functions: $\psi_n = \gamma_n^{SW} H_n e^{-v^2/2}$. A straightforward calculation yields:

$$L\psi_{n} = \gamma_{n}^{SW} \left(\frac{\partial}{\partial v} + v\mathcal{I}\right) H_{n} e^{-v^{2}/2}$$

$$= \gamma_{n}^{SW} \left(H'_{n} e^{-v^{2}/2} - vH_{n} e^{-v^{2}/2} + vH_{n} e^{-v^{2}/2}\right) = \gamma_{n}^{SW} H'_{n} e^{-v^{2}/2}.$$
(92)

Using the result above we obtain:

$$L^{2}\psi_{n} = L(L\psi_{n}) = L\left(\gamma_{n}^{SW}H'_{n}e^{-v^{2}/2}\right) = \gamma_{n}^{SW}\left(\frac{\partial}{\partial v} + v\mathcal{I}\right)H'_{n}e^{-v^{2}/2}$$
$$= \gamma_{n}^{SW}\left(H''_{n}e^{-v^{2}/2} - vH'_{n}e^{-v^{2}/2} + vH'_{n}e^{-v^{2}/2}\right) = \gamma_{n}^{SW}H''_{n}e^{-v^{2}/2}.$$
 (93)

A simple recursive argument allows us to prove the formula for a generic k:

$$L^{k}\psi_{n} = \gamma_{n}^{SW} H_{n}^{(k)} e^{-v^{2}/2}, \tag{94}$$

where we recall that $H_n^{(k)}=d^kH_n/dv^k$. Indeed, we have already proved that the formula is true for k=1 and k=2. Since $L^{k-1}\psi_n=\gamma_n^{SW}H_n^{(k-1)}e^{-v^2/2}$, a straightforward calculation yields:

$$L^{k}\psi_{n} = L(L^{(k-1)}\psi_{n}) = L\left(\gamma_{n}^{SW}H_{n}^{(k-1)}e^{-v^{2}/2}\right) = \gamma_{n}^{SW}\left(\frac{\partial}{\partial v} + v\mathcal{I}\right)(H_{n}^{(k-1)}e^{-v^{2}/2})$$

$$= \gamma_{n}^{SW}\left(H_{n}^{(k)}e^{-v^{2}/2} - vH_{n}^{(k-1)}e^{-v^{2}/2} + vH_{n}^{(k-1)}e^{-v^{2}/2}\right) = \gamma_{n}^{SW}H_{n}^{(k)}e^{-v^{2}/2}.$$
(95)

Now, we compute the action of \widetilde{L} , \widetilde{L}^2 , and \widetilde{L}^k on $L\psi_n$, $L^2\psi_n$, and $\widetilde{L}^k\psi_n$, respectively. In the first case, we recover the relation:

$$\widetilde{L}L\psi_{n} = \gamma_{n}^{SW} \left(\frac{\partial}{\partial v} - v\mathcal{I} \right) (H'_{n}e^{-v^{2}/2}) = \gamma_{n}^{SW} \left(H''_{n}e^{-v^{2}/2} - vH'_{n}e^{-v^{2}/2} - vH'_{n}e^{-v^{2}/2} \right)
= \gamma_{n}^{SW} (H''_{n} - 2vH'_{n}) e^{-v^{2}/2} = \gamma_{n}^{SW} (-2n)H_{n}e^{-v^{2}/2} = -2n\psi_{n}, \qquad n \ge 1.$$
(96)

In the second case, first we obtain:

$$\widetilde{L}L^{2}\psi_{n} = \gamma_{n}^{SW} \left(\frac{\partial}{\partial v} - v\mathcal{I} \right) (H_{n}^{"}e^{-v^{2}/2}) = \gamma_{n}^{SW} \left((H_{n}^{"})^{'}e^{-v^{2}/2} - vH_{n}^{"}e^{-v^{2}/2} - vH_{n}^{"}e^{-v^{2}/2} \right)
= \gamma_{n}^{SW} \left((H_{n}^{"})^{'} - 2vH_{n}^{"} \right) e^{-v^{2}/2} = \gamma_{n}^{SW} \left((2vH_{n}^{'} - 2nH_{n})^{'} - 2vH_{n}^{"} \right) e^{-v^{2}/2}
= \gamma_{n}^{SW} \left(2H_{n}^{'} + 2vH_{n}^{"} - 2nH_{n}^{'} - 2vH_{n}^{"} \right) e^{-v^{2}/2} = \gamma_{n}^{SW} 2(1-n)H_{n}^{'}e^{-v^{2}/2}, \quad n \ge 2,$$
(97)

and then:

$$\widetilde{L}^{2}L^{2}\psi_{n} = \widetilde{L}(\widetilde{L}L^{2}\psi_{n}) = \widetilde{L}(\gamma_{n}^{SW} 2(1-n)H'_{n}e^{-v^{2}/2})$$

$$= \gamma_{n}^{SW} 2(1-n)\left(\frac{\partial}{\partial v} - v\mathcal{I}\right)(H'_{n}e^{-v^{2}/2}) = \gamma_{n}^{SW} 2(1-n)(H''_{n} - vH'_{n} - vH'_{n})e^{-v^{2}/2}$$

$$= \gamma_{n}^{SW} 2(1-n)(H''_{n} - 2vH'_{n})e^{-v^{2}/2} = \gamma_{n}^{SW} 2(1-n)(-2n)H_{n}e^{-v^{2}/2} = 4n(n-1)\psi_{n}. \tag{98}$$

The final case, for a generic k, follows by a recursive argument, allowing us to prove that:

$$\widetilde{L}^k L^k \psi_n = (-1)^k 2^k \frac{n!}{(n-k)!} \psi_n, \qquad n \ge k.$$
 (99)

Except for the factor 2^k , this expression is the same as that in (53). Therefore, we conclude that every element of the SW Hermite functions' basis is an eigenfunction of the 2k-th operator $\widetilde{L}^k L^k$ with eigenvalue $(-1)^k 2^k n!/(n-k)!$, for $n \ge k$. The eigenvalue is zero for $0 \le n \le k-1$. We can similarly conclude that the 2k-th operator does not modify the equations for the first k modes of the expansion of f.

We end this section by investigating the action of the generalized Lenard-Bernstein operators on Hermite functions expressed as linear combinations of SW Hermite basis functions. To this purpose, we consider the expansion:

$$f = he^{-v^2/2} = \left[\sum_{n=0}^{\infty} C_n H_n\right] e^{-v^2/2} = \sum_{n=0}^{\infty} \frac{C_n}{\gamma_n^{SW}} \left[\gamma_n^{SW} H_n e^{-v^2/2}\right] = \sum_{n=0}^{\infty} C_n^{\star} \psi_n, \tag{100}$$

where $C_n^{\star} = C_n/\gamma_n^{SW}$. Since ψ_n is an eigenfunction of the generalized Lenard-Bernstein operators, we readily find the following relations:

$$\widetilde{L}Lf = \sum_{n=0}^{\infty} C_n^{\star} \widetilde{L}L\psi_n = \sum_{n=0}^{\infty} (-2n)C_n^{\star} \psi_n, \tag{101}$$

$$\widetilde{L}^{2}L^{2}f = \sum_{n=0}^{\infty} C_{n}^{\star} \widetilde{L}^{2}L^{2}\psi_{n} = \sum_{n=0}^{\infty} 4n(n-1)C_{n}^{\star}\psi_{n}, \tag{102}$$

. . .

$$\widetilde{L}^k L^k f = \sum_{n=0}^{\infty} C_n^* \widetilde{L}^k L^k \psi_n = \sum_{n=0}^{\infty} (-1)^k 2^k \frac{n!}{(n-k)!} C_n^* \psi_n, \tag{103}$$

from which we deduce that:

$$\widetilde{L}Lf = \sum_{n=0}^{\infty} D_n^{(1)} \psi_n \quad \text{with } D_n^{(1)} = -2nC_n^{\star},$$
 (104)

$$\widetilde{L}^2 L^2 f = \sum_{n=0}^{\infty} D_n^{(2)} \psi_n \qquad \text{with } D_n^{(2)} = 4n(n-1)C_n^{\star}, \tag{105}$$

. . .

$$\widetilde{L}^k L^k f = \sum_{n=0}^{\infty} D_n^{(k)} \psi_n \qquad \text{with } D_n^{(k)} = (-1)^k 2^k \frac{n!}{(n-k)!} C_n^{\star}. \tag{106}$$

By definition, it holds that $D_0^{(k)} = D_1^{(k)} = \ldots = D_{k-1}^{(k)} = 0$ for a generic $k \ge 1$.

As far as mass and momentum conservations are concerned, we do not have the same results of the AW Hermite discretization. Indeed, we can check that equations (62) and (70) do not hold anymore in the symmetric case. Instead, we can prove the conservation of the weighted integrals:

$$\int_{\mathbb{R}} f(v,t)e^{-v^2/2} dv \quad \text{and} \quad \int_{\mathbb{R}} v f(v,t)e^{-v^2/2} dv,$$

which however are not associated with physical, conserved quantities of interest in the continuous setting.

7. Hermite approximations of the advection equation

We take into account the following time-dependent problem for the unknown scalar field f(v,t):

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} = 0,\tag{107}$$

supplemented with the initial condition:

$$f(v,0) = f_0(v). (108)$$

We start with the study of the stability of the SW Hermite variational formulation of equation (107). To this end, we set $f(v,t) = h(v,t)e^{-v^2/2}$ (where h is a polynomial in v). Take f as the test function, and integrate over \mathbb{R} . We obtain:

$$0 = \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} \right) f \, dv = \int_{\mathbb{R}} \left(\frac{\partial}{\partial t} \left(\frac{f^2}{2} \right) - \frac{\partial}{\partial v} \left(\frac{f^2}{2} \right) \right) \, dv = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} \, dv, \tag{109}$$

since the integral of $\partial f^2/\partial v$ is zero because $f(v,t)\to 0$ for $v\to \pm \infty$. The relation above shows that the weighted norm of the function f, solving equation (107) in weak form, is conserved (i.e. it does not change in time).

The same is not going to be true for the AW case. In fact, we may try to study the stability with the same approach followed before. This time we set $f(v,t) = h(v,t)e^{-v^2}$ (where h is a polynomial in v). We then take h as test function and integrate over \mathbb{R} . We obtain:

$$0 = \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} \right) h \, dv = \int_{\mathbb{R}} h \frac{\partial h}{\partial t} e^{-v^2} \, dv - \int_{\mathbb{R}} h \frac{\partial f}{\partial v} \, dv.$$
 (110)

Successively, we integrate by parts the last term, substitute $f = he^{-v^2}$ and integrate by parts again. All the boundary terms are zero since they involve a polynomial in v multiplied by a decaying exponential and are omitted. This procedure yields:

$$-\int_{\mathbb{R}} h \frac{\partial f}{\partial v} dv = \int_{\mathbb{R}} f \frac{\partial h}{\partial v} dv = \int_{\mathbb{R}} \frac{\partial h}{\partial v} h e^{-v^2} dv = \int_{\mathbb{R}} \frac{\partial}{\partial v} \left(\frac{h^2}{2}\right) e^{-v^2} dv = \int_{\mathbb{R}} h^2 v e^{-v^2} dv.$$
 (111)

Finally, we find that:

$$0 = \frac{d}{dt} \int_{\mathbb{R}} \frac{h^2}{2} e^{-v^2} dv + \int_{\mathbb{R}} v h^2 e^{-v^2} dv.$$
 (112)

Since $v \in \mathbb{R}$ can assume positive or negative values, the sign of the second integral is undetermined, and therefore, the AW Hermite variational formulation is not absolutely stable in the weighted $L^2(\mathbb{R})$ norm. Note, however, that the weighted norm in the AW case does not have a direct physical meaning as in the SW case. In both the continuous case and its SW Hermite discretization, the quantity $\int_{\mathbb{R}} f^2 dv$ is preserved. This quantity is not preserved in the AW discretization. In fact, we are in the situation in which neither the weighted L^2 -norm nor the unweighted one are preserved.

Now, we derive the recursive equation for the coefficients of the Hermite expansion in both AW and SW cases. In order to simplify the notation, in the expressions below, we set $\gamma_n = \gamma_n^{SW}$ when we deal with the SW case or $\gamma_n = \gamma_n^{AW}$ when we deal with the AW case (we recall that these coefficients are defined in (31) and (32)). Also, we

use the notation $C_n^* = C_n/\gamma_n$ to denote the coefficients of the expansion in the Hermite functions ψ_n . As usual, we have:

$$f(v,t) = \sum_{n=0}^{\infty} C_n(t) H_n(v) e^{-v^2} = \sum_{n=0}^{\infty} C_n^{\star}(t) \psi_n(v).$$
 (113)

Accordingly, the initial condition is set through the relation:

$$\sum_{n=0}^{\infty} C_{n,0} H_n(v) e^{-v^2} = \sum_{n=0}^{\infty} C_{n,0}^{\star} \psi_n(v) = f_0(v).$$
(114)

To derive the system of equations for the coefficients C_n^* related to the solution of (107), we multiply (107) by ψ^m and integrate in v over \mathbb{R} . All integrals can easily be computed using the orthogonality of the Hermite functions' basis. In view of expansion (113), we have that:

$$0 = \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} \right) \psi^m \, dv = \sum_{n=0}^{\infty} \dot{C}_n^{\star}(t) \int_{\mathbb{R}} \psi_n \psi^m \, dv - \sum_{n=0}^{\infty} C_n^{\star}(t) \int_{\mathbb{R}} \frac{d\psi_n}{dv} \psi^m \, dv$$
$$= \dot{C}_m^{\star}(t) - \sum_{n=0}^{\infty} C_n^{\star}(t) \int_{\mathbb{R}} \frac{d\psi_n}{dv} \psi^m \, dv, \tag{115}$$

where the upper dot indicates the derivative with respect to t. The equation for each coefficient $C_n^{\star}(t)$ can be recovered by reformulating $d\psi_n/dv$ in terms of the basis functions ψ_n and using the orthogonality against ψ^m . We discuss the two cases for the AW and SW Hermite approximation in the following subsections.

7.1. Symmetrically-weighted case

To ease the notation in the developments of this section, we continue using the symbol γ_n instead of γ_n^{SW} , which is defined in (31). For $n \ge 1$, using (7)-(8), we compute $d\psi_n/dv$ as follows:

$$\frac{d\psi_n}{dv} = \frac{d}{dv} \left(\gamma_n H_n e^{-v^2/2} \right) = \gamma_n \left(H'_n - v H_n \right) e^{-v^2/2} = \gamma_n \left(\frac{1}{2} H'_n + \frac{1}{2} H'_n - v H_n \right) e^{-v^2/2}
= \gamma_n \left(n H_{n-1} - \frac{1}{2} H_{n+1} \right) e^{-v^2/2} = \frac{n \gamma_n}{\gamma_{n-1}} \psi_{n-1} - \frac{1}{2} \frac{\gamma_n}{\gamma_{n+1}} \psi_{n+1}.$$
(116)

Thus, equation (115) implies that:

$$\dot{C}_{n}^{\star}(t) = \frac{(n+1)\gamma_{n+1}}{\gamma_{n}} C_{n+1}^{\star}(t) - \frac{1}{2} \frac{\gamma_{n-1}}{\gamma_{n}} C_{n-1}^{\star}(t), \tag{117}$$

that we supplement with the initial condition $C_n^{\star}(0) = C_{n,0}^{\star}$. Equivalently, one has for $n \geq 1$:

$$\dot{C}_n(t) = (n+1)C_{n+1}(t) - \frac{1}{2}C_{n-1}(t), \tag{118}$$

with the (obvious) initial condition $C_n(0) = C_{n,0}$. The case n = 0 can be treated separately, by observing that:

$$\frac{d\psi_0}{dv} = \frac{d}{dv} \left(\gamma_0 H_0 e^{-v^2/2} \right) = -\gamma_0 v e^{-v^2/2} = -\frac{\gamma_0}{2\gamma_1} \left(\gamma_1 2v e^{-v^2/2} \right) = -\frac{\gamma_0}{2\gamma_1} \left(\gamma_1 H_1 e^{-v^2/2} \right) \\
= -\frac{\gamma_0}{2\gamma_1} \psi_1 = -\frac{1}{\sqrt{2}} \psi_1 \quad \Rightarrow \quad \int_{\mathbb{R}} \frac{d\psi_0}{dv} \psi^0 dv = 0,$$

since $H_0(v) = 1$, $H_1(v) = 2v$, and $\gamma_0/\gamma_1 = \sqrt{2}$, so obtaining from (115) that

$$\dot{C}_0(t) = 0 \quad \Rightarrow \quad C_0(t) = C_{0,0} \quad \forall t.$$
 (119)

We proved above that the system associated with equations (117)-(119) is stable in the L^2 -weighted norm.

7.2. Asymmetrically-weighted case

As in the previous section we ease the notation by writing the symbol γ_n instead of γ_n^{AW} , which is defined in (32). In this case, using (7), multiplying and dividing by γ_{n+1} , and using the definition of ψ_{n+1} , we have:

$$\frac{d\psi_n}{dv} = \frac{d}{dv} \left[\gamma_n H_n e^{-v^2} \right] = \gamma_n \left(H'_n - 2v H_n \right) e^{-v^2} = -\gamma_n H_{n+1} e^{-v^2} = -\frac{\gamma_n}{\gamma_{n+1}} \psi_{n+1}, \tag{120}$$

which now provides the differential equation, for $n \ge 1$:

$$\dot{C}_{n}^{\star}(t) = -\frac{\gamma_{n-1}}{\gamma_{n}} C_{n-1}^{\star}(t), \tag{121}$$

supplemented with the initial condition $C_n^{\star}(0) = C_{n,0}^{\star}$. This is equivalent to:

$$\dot{C}_n(t) = -C_{n-1}(t) \tag{122}$$

For n=0 we have again (119). Moreover we have the initial conditions $C_n(0)=C_{n,0}$; hence, $C_0(t)=C_{0,0}$ for every $t\geq 0$.

We now provide a solution to such a system of equations. For instance, when n=1, we need to solve:

$$\dot{C}_1(t) = -C_0(t) \quad \Rightarrow \quad C_1(t) = C_{1,0} - C_{0,0}t. \tag{123}$$

Clearly, this coefficient grows in magnitude with t. By successive integrations, one can prove that the n-th coefficient behaves as t^n . In practice, it is possible to find numbers $\alpha_\ell^{(n)}$ in such a way that:

$$C_n(t) = \gamma_n C_n^{\star}(t) = \sum_{\ell=0}^n \alpha_{\ell}^{(n)} t^{\ell},$$
 (124)

which is clearly unbounded for t tending to infinity. We already proved that the Galerkin approximation of the advection problem in the AW case is not unconditionally stable in the $L^2(\mathbb{R})$ -weighted norm. For a polynomial of degree at most N, such a norm with respect to t is given by the sum $\left(\sum_{n=0}^N \left(C_n^\star(t)\right)^2\right)^{1/2}$. A way to stabilize the approximation scheme is to introduce some numerical dissipation. We note, however, that this may not be the only option. We will study this problem in the next section.

7.3. Some additional considerations on the SW and AW Hermite approximations

We consider two exact solutions of equation (107) that are well-suited for the treatment with Hermite functions (in the SW case and the AW case, respectively) and see how their expansion coefficients look like, in particular with respect to the time variable t. It has to be remarked, however, that the truncated series of an exact solution does not coincide, in general, with the discrete solution obtained by the Galerkin process. So, the purpose of the following computation is only to illustrate why the approximations based on the SW or the AW Hermite functions may behave rather differently.

First, we consider the exact solution of $\ (107)$ given by $f(v,t)=e^{-\frac{(v+t)^2}{2}}$ and denote its coefficients with respect to the SW Hermite functions by $C_n^{SW,ex}$, where the superscript "ex" stands for "exact". At t=0, only one coefficient is nonzero, i.e., $C_0^{SW,ex}(0)=\pi^{\frac{1}{4}}$. For a generic t>0, the expansion coefficients of f are, for $n\geq 0$:

$$C_n^{SW,ex}(t) = \int_{\mathbb{R}} e^{-\frac{(v+t)^2}{2}} \psi_n(v) \, dv = \gamma_n^{SW} e^{-\frac{t^2}{4}} \int_{\mathbb{R}} e^{-\left(v+\frac{t}{2}\right)^2} H_n(v) \, dv = \sqrt{\pi} 2^n \gamma_n^{SW} \left(-\frac{t}{2}\right)^n e^{-\frac{t^2}{4}}, \quad (125)$$

where, for the integration, we used the convolution formula [17]:

$$\int_{\mathbb{R}} e^{-(x-y)^2} H_n(x) \, dx = \sqrt{\pi} 2^n y^n. \tag{126}$$

Formula (125) shows that all the expansion coefficients $C_n^{SW,ex}(t)$ converge to zero for $t \to \infty$ including the one with n=0 (note that the coefficient provided by the Galerkin approximation, namely $C_0(0)$, is instead constant in time).

For the AW case we consider the exact solution of (107) given by $f(v,t)=e^{-(v+t)^2}$ and we similarly denote its expansion coefficients as $C_n^{AW,ex}(t)$. The expansion of f on the AW Hermite basis functions still contains only one coefficient at the initial time t=0, i.e., $C_0^{AW,ex}(0)=\sqrt{\pi}$. The new coefficients look as follows:

$$C_n^{AW,ex}(t) = \int_{\mathbb{R}} e^{-(v+t)^2} \psi_n(v) \, dv = \sqrt{\pi} 2^n \gamma_n^{AW} \left(-t\right)^n, \tag{127}$$

where, for the integration, we used again formula (126). Formula (127) shows that the expansion coefficients $C_n^{AW,ex}(t)$ diverge to $\pm \infty$ when $t \to \infty$, the sign depending on n being even or odd. Of course, in these circumstances a remedy can be easily found by introducing a shift in the Hermite basis as mentioned in the introduction. The fact that the expansion needs to be centered and rescaled properly has been known for a long time but complicates the analysis and so it will be considered in future work.

8. The advection equation with the stabilization term in the AW case

We start our analysis by adding the second-order (k=1) operator $\nu \widetilde{L}L$ to the right-hand side of the advection equation:

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} = \nu \widetilde{L} L f,\tag{128}$$

which we solve for f(v,t). We will prove that the new term acts like a stabilization term. To this end, we set $f = he^{-v^2}$, take h as the test function, (we assume that h is a polynomial in v at every time), and integrate (128) over \mathbb{R} . We substitute the stabilization term $\nu \widetilde{L} L f$ with the expression given in (40) (or (46) with k=1) to obtain:

$$\int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} \right) h \, dv = \nu \int_{\mathbb{R}} \left(\widetilde{L} L f \right) h \, dv = -\frac{\nu}{2} \int_{\mathbb{R}} \left(h' \right)^2 e^{-v^2} \, dv. \tag{129}$$

We integrate by parts the second integral term and apply the Young inequality (with constant σ) to obtain:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^{2} e^{-v^{2}} dv = -\int_{\mathbb{R}} h' h e^{-v^{2}} dv - \frac{\nu}{2} \int_{\mathbb{R}} (h')^{2} e^{-v^{2}} dv$$

$$\leq \left| \int_{\mathbb{R}} h' h e^{-v^{2}} dv \right| - \frac{\nu}{2} \int_{\mathbb{R}} (h')^{2} e^{-v^{2}} dv$$

$$\leq \frac{1}{2\sigma} \int_{\mathbb{R}} h^{2} e^{-v^{2}} dv + \frac{1}{2} (\sigma - \nu) \int_{\mathbb{R}} (h')^{2} e^{-v^{2}} dv, \tag{130}$$

where we used the fact that the boundary contributions from the integration by parts are zero. From the Poincarè inequality (16) (take $\varphi = h$) we have that

$$-\frac{1}{2} \int_{\mathbb{R}} (h')^2 e^{-v^2} dv \le -\int_{\mathbb{R}} h^2 e^{-v^2} dv + \sqrt{\pi} C_0^2.$$
 (131)

Using this inequality with $\nu > \sigma$, we find that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}h^{2}e^{-v^{2}}dv \leq \left(\frac{1}{2\sigma} - (\nu - \sigma)\right)\int_{\mathbb{R}}h^{2}e^{-v^{2}}dv + (\nu - \sigma)\sqrt{\pi}C_{0}^{2}.$$
(132)

The coefficient $(1/(2\sigma) - (\nu - \sigma))$ is negative if $\nu > \sigma + 1/(2\sigma)$. For example, by taking $\sigma = 1$ and $\nu > 3/2$, we find:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}h^{2}e^{-v^{2}}dv \le -\left(\nu - \frac{3}{2}\right)\int_{\mathbb{R}}h^{2}e^{-v^{2}}dv + (\nu - 1)\sqrt{\pi}C_{0}^{2}.$$
(133)

Now, we consider $C_0(t) = C_0(0) = C_{0,0}$ and introduce the quantities:

$$K = \frac{\nu - 1}{\nu - \frac{3}{2}} \sqrt{\pi} C_{0,0}^2 \quad \text{and} \quad Y(t) = \int_{\mathbb{R}} h^2 e^{-v^2} \, dv - K, \tag{134}$$

so we can rewrite (133) as

$$\frac{1}{2}\frac{d}{dt}Y(t) \le -\left(\nu - \frac{3}{2}\right)Y(t) \tag{135}$$

since K is constant. Note that for t = 0 we have

$$Y(0) = \int_{\mathbb{R}} h_0^2 \, dv - K,\tag{136}$$

where $h_0 = h(v, 0)$, which is provided by the expansion of the initial solution f_0 . Finally, an application of the Gronwall's inequality yields

$$Y(t) \le Y(0) \exp\left(-2\left(\nu - \frac{3}{2}\right)t\right) \le Y(0),\tag{137}$$

since the argument of the exponential is negative. Using the expression of Y(t) and Y(0), respectively given in (134) and (136), the condition $Y(t) \le Y(0)$ implies that

$$\int_{\mathbb{R}} h^2 e^{-v^2} \, dv \le \int_{\mathbb{R}} h_0^2 \, dv = \int_{\mathbb{R}} h(v,0)^2 \, dv, \tag{138}$$

which is the stability in the weighted L^2 norm. Note that $\nu > \frac{3}{2}$ is a sufficient but not necessary conditions for stability.

Concerning the case k > 1, a proof of stability for ν sufficiently large, can be given following the same steps of the case for k = 1. We just provide here a sketch of the main steps for the classical L^2 -weighted norm. Thanks to (46), formula (129) can be rewritten as

$$\int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} \right) h \, dv = -(-1)^k \nu \int_{\mathbb{R}} \left(\widetilde{L}^{(k)} L^{(k)} f \right) h \, dv = -\frac{\nu}{2^k} \int_{\mathbb{R}} \left(h^{(k)} \right)^2 e^{-v^2} \, dv. \tag{139}$$

As in (130) we use the Schwarz and Young inequality; then, we estimate the right-hand side of (139) by using (21) with p = 1 and m = k. By using (131), we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}h^{2}e^{-v^{2}}dv \le \Phi_{1}\int_{\mathbb{R}}h^{2}e^{-v^{2}}dv + \Phi_{2}$$
(140)

where

$$\Phi_1 = \frac{1}{2\sigma} - \nu(k-1)! + \sigma \quad \text{and} \quad \Phi_2 = \left(\nu(k-1)! - \sigma\right)\sqrt{\pi}C_0^2 + \nu(k-1)!\sqrt{\pi}\sum_{\ell=1}^{k-1} 2^{\ell} \frac{(\ell!)^2}{(\ell-1)!}C_{\ell}^2. \tag{141}$$

Now, we redefine

$$K = \frac{\left(\nu(k-1)! - \sigma\right)}{\nu(k-1)! - \sigma - \frac{1}{2\sigma}} \sqrt{\pi} C_0^2 \quad \text{and} \quad Y(t) = \int_{\mathbb{R}} h_0^2 \, dv - K, \tag{142}$$

so that

$$\frac{1}{2}\frac{d}{dt}Y(t) \le \Phi_1 Y(t) + \Psi_1(t), \quad \text{where} \quad \Psi_1(t) = \nu(k-1)! \sqrt{\pi} \sum_{\ell=1}^{k-1} 2^\ell \frac{(\ell!)^2}{(\ell-1)!} C_\ell^2, \quad (143)$$

since $C_0 = C_{0,0}$ is independent of t. An application of the Gronwall's Lemma leads to

$$Y(t) \le Y(0)e^{-2\Phi_1 t} + \int_0^t \Psi_1(\tau)d\tau$$

Choosing, for example, $\sigma=1$ and taking $\nu(k-1)!>3/2$, it is easy now to get the stability estimate that generalizes (137) to any $k\geq 1$. We also note that the diffusion parameter ν is now multiplied by (k-1)!. So, if we increase k, the numerical diffusion due to the Lenard-Bernstein operators acts only on higher terms in the expansion of f and we may probably take smaller values for ν .

We confirm the stability result for k=1 by deriving the explicit recursive formula for the Hermite expansion coefficients and providing their explicit form. To this end, we consider the second expansion of f given in (113) and

repeat the calculation of Section 7.2 by including now the stabilization term $\nu \widetilde{L}Lf$, which can be treated in the AW case with the help of (47):

$$0 = \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} - \nu \widetilde{L} L f \right) \psi^m \, dv$$

$$= \sum_{n=0}^{\infty} \dot{C}_n^{\star}(t) \int_{\mathbb{R}} \psi_n \psi^m \, dv - \sum_{n=0}^{\infty} C_n^{\star}(t) \int_{\mathbb{R}} \frac{d\psi_n}{dv} \psi^m \, dv + \nu \sum_{n=0}^{\infty} n C_n^{\star}(t) \int_{\mathbb{R}} \psi_n \psi^m \, dv$$

$$= \dot{C}_m^{\star}(t) - \sum_{n=0}^{\infty} C_n^{\star}(t) \int_{\mathbb{R}} \frac{d\psi_n}{dv} \psi^m \, dv + \nu m C_m^{\star}(t). \tag{144}$$

We compute the last integral using again (120) to obtain:

$$\dot{C}_{n}^{\star}(t) = -\frac{\gamma_{n-1}}{\gamma_{n}} C_{n-1}^{\star}(t) - \nu n C_{n}^{\star}(t), \tag{145}$$

which holds for $n \ge 1$, while for n = 0 we find that $C_0^{\star}(t) = C_{0,0}^{\star}$ is constant. We rewrite the above system of equations as follows (compare with (122)):

$$\dot{C}_n(t) = -C_{n-1}(t) - \nu n C_n(t). \tag{146}$$

For instance, for n = 1, we find the ordinary differential equation:

$$\dot{C}_1(t) = -C_0(t) - \nu C_1(t) = -C_{0,0} - \nu C_1(t), \tag{147}$$

the solution of which is:

$$C_1(t) = \left(C_{1,0} + \frac{1}{\nu}C_{0,0}\right)e^{-\nu t} - \frac{1}{\nu}C_{0,0}.$$
(148)

Since ν is positive, $C_1(t)$ is clearly bounded with respect to t.

It is not hard to show that, for a generic n, the expression of the n-th coefficient takes the form:

$$C_n(t) = \sum_{\ell=0}^{n} \alpha_{\ell}^{(n)} e^{-\ell \nu t},$$
(149)

where the constants $\alpha_{\ell}^{(n)}$ depend on n and the diffusion parameter ν . It is important to analyze such a dependence on the diffusion parameter. Indeed, using (149) in (146) for $n \geq 1$ and $0 \leq \ell \leq n-1$ yields the recursive relation

$$\alpha_{\ell}^{(n)} = -\frac{1}{\nu(n-\ell)}\alpha_{\ell}^{(n-1)},$$

from which a straightforward calculation yields:

$$\alpha_{\ell}^{(n)} = \frac{(-1)^{n-\ell}}{(n-\ell)! \, \nu^{n-\ell}} \alpha_{\ell}^{(\ell)}.$$

From the initial condition

$$C_{n,0} = C_n(0) = \sum_{\ell=0}^{n} \alpha_{\ell}^{(n)} = \alpha_n^{(n)} + \sum_{\ell=0}^{n-1} \alpha_{\ell}^{(n)},$$

we find the expression of $\alpha_n^{(n)}$, which is given by

$$\alpha_n^{(n)} = C_{n,0} - \sum_{\ell=0}^{n-1} \alpha_\ell^{(n)} = C_{n,0} + \sum_{\ell=0}^{n-1} \frac{(-1)^{n-\ell}}{(n-\ell)! \nu^{n-\ell}} \alpha_\ell^{(\ell)}.$$

For example, starting from $\alpha_0^0 = C_{0,0}$, for n=1, we find that $\alpha_1^{(1)} = C_{1,0} - \alpha_0^0/\nu = C_{1,0} - C_{0,0}/\nu$. Similarly, $\alpha_2^{(2)}$ is computed from $\alpha_0^{(0)}$ and $\alpha_1^{(1)}$, and the following coefficients are obtained from the ones that have already been computed. One can realize that ν appears at the denominator to the n-th power. It turns out that the coefficients $C_n(t)$

in (149) are of the form $C_n(0)$ plus a dissipative term. The stronger dissipation is obtained when $\ell=1$, which provides a contribution like $e^{-\nu t}/\nu$ (see (148)). If we do not want this dissipation to be too heavy so that the perturbation is of order ε when we integrate until the final time T, we can consider $e^{-\nu T} \approx \nu \varepsilon$ and take $T \approx |\ln(\nu \epsilon)|/\nu$.

9. Time discretization of the 1-D problem

We study the numerical approximation of the system of differential equations in (146). We use an implicit conservative method in time such as the trapezoidal rule. For a time-step $\Delta t > 0$, we write for $j \geq 1$:

$$\frac{C_n^j - C_n^{j-1}}{\Delta t} = -\frac{C_{n-1}^j + C_{n-1}^{j-1}}{2} - \nu n \frac{C_n^j + C_n^{j-1}}{2},\tag{150}$$

with the initial condition $C_n^0 = C_{n,0}$. For n = 0 we have instead $C_0^j = C_{0,0}, \forall j \geq 0$. For instance, we can make the formula explicit for n = 1:

$$C_1^j \left(1 + \frac{\nu}{2} \Delta t \right) = C_1^{j-1} \left(1 - \frac{\nu}{2} \Delta t \right) - \Delta t C_{0,0}. \tag{151}$$

After defining $\chi_n = (1 - \frac{1}{2}\nu n\Delta t)/(1 + \frac{1}{2}\nu n\Delta t), n \ge 1$, we get $|\chi_n| < 1$. By recursive arguments, one can show that the expression for C_1^j takes the form of a linear combination of powers of χ_1 , i.e.:

$$C_1^j = \sum_{\ell=0}^j (\chi_1)^\ell \alpha_\ell, \tag{152}$$

where the numbers α_{ℓ} depend on ν and Δt . This expression is inserted in (150) in order to compute the sequence $C_2^j, \forall j \geq 0$, and so on.

We may assume that the solution $h = fe^{v^2}$ of (128) belongs to the space of polynomials of degree less or equal to N. When n reaches the value N, the expression of the corresponding coefficients $C_N^j, \forall j \geq 0$ is a combination of all the powers $(\chi_n)^\ell$ with $1 \leq n \leq N$ and $0 \leq \ell \leq j$.

Since $|\chi_N|<1$, the discretization method is always unconditionally stable. However, a wise relation between the parameters N, ν and Δt should be set up in order to avoid unpleasant numerical effects due to the *stiffness* of the originating differential system (146) for N large. A rule of thumb is to require that the product $\nu N \Delta t$ is of the order of unity. Actually, if we analyze (149) when n=N, the most significant term is that given by the exponential $e^{-N\nu t}$, displaying a very steep tangent for t=0. Although there are in principle no restrictions on Δt for the trapezoidal scheme, such quick variations in time are well resolved only if the time-step is maintained suitably small.

The last arguments show that stability holds for any $\nu>0$, whereas in (138) the proof was only provided for $\nu>3/2$. Indeed, we conjecture that the stability in the L^2 -weighted norm is not verified for values of ν less than a certain constant. However, it is possible to construct milder weighted norms where a result of stability can still be achieved for any ν . We show how to do this by starting from the differential system (146). For any $n\geq 1$, we multiply (146) by the Hermite coefficient C_n and use the Young inequality on the right-hand side to obtain

$$\frac{1}{2}\frac{d}{dt}C_n^2 = -C_nC_{n-1} - \nu nC_n^2 \le \frac{1}{2\sigma_n}C_n^2 + \frac{\sigma_n}{2}C_{n-1}^2 - \nu nC_n^2.$$
(153)

The family of parameters $\sigma_n > 0$ will be decided later on. We multiply both sides of the inequality above by a weight $w_n > 0$ and sum over index n, so obtaining

$$\frac{1}{2} \frac{d}{dt} \sum_{n=1}^{\infty} w_n C_n^2 \le \sum_{n=1}^{\infty} \frac{1}{2\sigma_n} w_n C_n^2 + \sum_{n=1}^{\infty} \frac{\sigma_n}{2} w_n C_{n-1}^2 - \sum_{n=1}^{\infty} \nu n w_n C_n^2.$$
 (154)

By shifting the index in the sum containing C_{n-1} and collecting the corresponding terms under the same symbol of summation, we get

$$\frac{1}{2} \frac{d}{dt} \sum_{n=1}^{\infty} w_n C_n^2 \le \sum_{n=1}^{\infty} \left[\left(\frac{1}{2\sigma_n} - \nu n \right) w_n + \frac{\sigma_{n+1}}{2} w_{n+1} \right] C_n^2 + \frac{\sigma_1}{2} w_1 C_0^2.$$
 (155)

For example, we can consider to choose $\sigma_n = 1/(\nu n)$. Successively, we impose that the expression in the square brackets is equal to $-(\nu/4)w_n$, which implies that

$$w_{n+1} = \nu^2(n+1)\left(n - \frac{1}{2}\right)w_n,\tag{156}$$

and starting for example from $w_1 = 1$ provides the full weight sequence (altough a different starting value produce a different sequence of coefficients w_n , our argument is independent of such starting):

$$\frac{1}{2}\frac{d}{dt}\sum_{n=1}^{\infty}w_nC_n^2 \le -\frac{\nu}{4}\sum_{n=1}^{\infty}w_nC_n^2 + \frac{1}{2\nu}w_1C_0^2. \tag{157}$$

Finally, by setting

$$Y(t) = \sum_{n=1}^{\infty} w_n C_n^2 - (2/\nu^2) w_1 C_0^2,$$
(158)

we obtain

$$\frac{1}{2}Y'(t) \le -\frac{\nu}{4}Y(t). \tag{159}$$

Thus, by applying the Gronwall's lemma, we conclude with the estimate

$$Y(t) \le Y(0)e^{-(\nu/2)t} \le Y(0),\tag{160}$$

for all $t \geq 0$, from which we can find our stability result

$$\sum_{n=1}^{\infty} w_n C_n^2(t) \le \sum_{n=1}^{\infty} w_n C_n^2(0),$$

since the term $(2/\nu^2)w_1C_0^2$ in (158) is independent of time and can be removed.

The next step is to characterize the weights w_n which are required to satisfy the recursive relation (156). Assuming that $w_1 = 1$, from a straightforward calculation, we find

$$w_n = (2\nu^2)^{n-1} n! \frac{(2n-3)!}{2^{n-2}(n-2)!} = 2(\nu^2)^{n-1} n(n-1)(2n-3)!$$
(161)

By substituting into (158), we are finally able to give an expression to the stability norm. Note that it depends on ν . We can go ahead with our computations by noting that

$$w_n \ge 2(\nu^2)^{n-1} n(n-1) 2^{n-2} (n-2)! = \frac{1}{2} (\nu^2)^{n-1} 2^n n!$$
 (162)

Therefore, if $\nu \geq 1$, and, hence, $\nu^{2n} \geq 1$, we discover that

$$Y(t) = \sum_{n=1}^{\infty} w_n C_n^2 + \frac{2}{\nu^2} C_0^2 \ge \frac{1}{2} \sum_{n=1}^{\infty} (\nu^2)^{n-1} 2^n \, n! \, C_n^2 + \frac{2}{\nu^2} C_0^2 \ge \frac{1}{2\nu^2} \sum_{n=1}^{\infty} \nu^{2n} 2^n \, n! C_n^2 + \frac{1}{2\nu^2} C_0^2$$

$$\ge \frac{1}{2\sqrt{\pi}\nu^2} \left(\sqrt{\pi} \sum_{n=0}^{\infty} 2^n \, n! \, C_n^2 \right). \tag{163}$$

Thus, if we can bound Y, we automatically bound the last term in parenthesis, which corresponds to the square of the classical L^2 -weighted norm of the solution f expanded as in (113). This confirms that, if ν is sufficiently large, stability is ensured in the standard way. On the other hand, when $\nu < 1$, we can only rely on the stability result involving the weights w_n .

If we are in finite dimension $(n \le N)$, the norms are equivalent for any $\nu > 0$, but with constants heavily dependent on N. For example, for $\nu \le 1$, which implies that $(\nu^2)^{n-1} \ge (\nu^2)^{N-1}$, we can write

$$Y(t) = \sum_{n=1}^{N} w_n C_n^2 + \frac{2}{\nu^2} C_0^2 \ge \frac{\left(\nu^2\right)^{N-1}}{2} \sum_{n=1}^{N} 2^n \, n! \, C_n^2 + \frac{2}{\nu^2} C_0^2 \ge \frac{\left(\nu^2\right)^{N-1}}{2\sqrt{\pi}} \left(\sqrt{\pi} \sum_{n=0}^{N} 2^n \, n! \, C_n^2\right). \tag{164}$$

This shows that, when Y is bounded by a constant, the classical L^2 -weighted norm of the solution f is bounded by that constant multiplied by a factor behaving as the inverse of ν^{2N} . If we choose $\nu < 1$, such a constant grows to infinity as $\mathcal{O}(\nu^{2N})$, and the stability control on the L^2 -weighted norm of the solution f provided by inequality (164) is lost.

10. Full discretization of the Vlasov-Poisson equation

We consider the AW Hermite-based discretization of the Vlasov-Poisson problem (1)-(2) for the distribution function $f(x, v, t) = h(x, v, t)e^{-v^2}$ stabilized by the Lenard-Bernstein-like operator of order 2k with $k \ge 1$, which we rewrite here for convenience of exposition:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial v} = -(-1)^k \nu \widetilde{L}^{(k)} L^{(k)} f \qquad \text{in } \Omega \times [0, T], \tag{165}$$

$$\frac{\partial E}{\partial x} = 1 - \int_{\Omega} f \, dv \qquad \text{in } \Omega \times [0, T]. \tag{166}$$

System (165)-(166) is completed by assigning a sufficiently regular initial solution $f(\mathbf{x}, \mathbf{v}, 0) = f_0(x, v)$. We specialize the discussion to periodic boundary conditions in space, i.e., at the boundaries of Ω_x .

Some of the reasons for approaching the Vlasov problem by Hermite discretizations have been pointed out in the introduction. The AW context is the one that guarantees a large number of conservation properties, even with the addition of the diffusion term discussed so far. By the way, from the practical viewpoint the use of the viscous term should not just be interpreted as a way to improve the time-stability of the schemes, but has an important role in the reduction of the negative phenomenon known as *filamentation*, cf. [4], which shows up as a polluting effect on the computed solutions, due to the nonlinearity of the problem in conjunction with the truncation of the high modes.

To discretize the Vlasov-Poisson equations in time, we integrate equation (165) with respect to the independent unknown t between t^{j-1} and t^j by applying the trapezoidal rule and we evaluate equation (166) at t^j . To ease the exposition, we assume a constant time step $\Delta t = t^j - t^{j-1}$. At the timestep $j \ge 1$, system (165)-(166) yields

$$\frac{f^{j} - f^{j-1}}{\Delta t} + v \frac{\partial}{\partial x} \left(\frac{f^{j} + f^{j-1}}{2} \right) - \frac{E^{j} + E^{j-1}}{2} \frac{\partial}{\partial v} \left(\frac{f^{j} + f^{j-1}}{2} \right)$$

$$= -(-1)^{k} \nu \widetilde{L}^{(k)} L^{(k)} \left(\frac{f^{j} + f^{j-1}}{2} \right) \tag{167}$$

$$\frac{\partial E^j}{\partial x} = 1 - \int_{\Omega_n} f^j \, dv. \tag{168}$$

For j = 0 we impose the value of f at time t = 0 as initial datum.

Following the guidelines of the previous section, a proof of the absolute stability in time of this scheme can be provided for a sufficiently large parameter ν . The situation gets more technically involved if ν is relatively small. We remind you that in Section 8 we distinguished between $\nu \geq 1$ and $\nu < 1$. In the latter case, stability is achieved in a suitable norm and the generalization of this proof to the Vlasov-Poisson system becomes rather complicated.

Here, our goal is to derive stability conditions that relate the time step Δt , the collisional factor ν and the degree of the Hermite polynomial N. To this end, we write (167) in operator form by collecting all the terms involving the unknown variable f^j on the left-hand side and denoting all other terms that are computable from what is known from the previous time step in the right-hand side term g^{j-1} :

$$\left[\mathcal{I} + \frac{\Delta t}{2}v\frac{\partial}{\partial x} - \frac{\Delta t}{2}\left(\frac{E^j + E^{j-1}}{2}\right)\frac{\partial}{\partial v} + (-1)^k\frac{\Delta t}{2}\nu\widetilde{L}^{(k)}L^{(k)}\right]f^j = g^{j-1}.$$
 (169)

In this preliminary analysis, we will not take into consideration that the problem is actually nonlinear. Indeed, the value E^j has still to be computed, since it is strictly linked to f^j through the relation (168).

We first set $f^j = h^j e^{-v^2}$. To simplify the exposition, we remove the label j from h^j and introduce the notation:

$$\mathcal{A}(x) = \left(\int_{\Omega_v} h^2 e^{-v^2} dv\right)^{\frac{1}{2}}, \qquad \overline{\mathcal{A}} = \left(\int_{\Omega_x} \mathcal{A}^2 dx\right)^{\frac{1}{2}}, \tag{170}$$

$$\mathcal{B}(x) = \left(\int_{\Omega_v} \left| \frac{\partial h}{\partial v} \right|^2 e^{-v^2} dv \right)^{\frac{1}{2}}, \qquad \overline{\mathcal{B}} = \left(\int_{\Omega_x} \mathcal{B}^2 dx \right)^{\frac{1}{2}}. \tag{171}$$

Then, we rewrite problem (169) in weak form. To this end, we multiply (169) by the test function ϕ , integrate over domain $\Omega = \Omega_x \times \Omega_v$, and define the bilinear form:

$$B(h,\phi) = \int_{\Omega} h\phi e^{-v^2} dv dx + \frac{\Delta t}{2} \int_{\Omega} \phi v \left(\frac{\partial h}{\partial x}\right) e^{-v^2} dv dx$$
$$-\frac{\Delta t}{4} \int_{\Omega_x} \left(E^j + E^{j-1}\right) \left[\int_{\Omega_v} \frac{\partial \left(he^{-v^2}\right)}{\partial v} \phi dv \right] dx + \frac{\nu \Delta t}{2^{k+1}} \int_{\Omega} \frac{\partial^k h}{\partial v^k} \frac{\partial^k \phi}{\partial v^k} e^{-v^2} dv dx, \tag{172}$$

where the last term is obtained after successive integration by parts as done in (46) and using formula (37) for $L^{(k)}f$. Now, we consider the problem of finding $f = he^{-v^2}$ such that:

$$B(h,\phi) = \int_{\Omega} g^{j-1}\phi \, dv \, dx,\tag{173}$$

for every ϕ . Both h and ϕ will be represented as a suitable expansion (finite or infinite) of Hermite polynomials. We skip the details concerning the formulation in the proper functional spaces, since this aspect is not relevant for the analysis we are carrying out in this paper.

We want the bilinear form B to be positive definite. First, we discuss the case k = 1, and note that the last integral term in (172) can be transformed as follows

$$\frac{\nu \Delta t}{4} \int_{\Omega} \frac{\partial h}{\partial v} \frac{\partial h}{\partial v} e^{-v^2} dv dx = \frac{\nu \Delta t}{4} \overline{\mathcal{B}}^2.$$
 (174)

In this way, we get:

$$B(h,h) = \overline{\mathcal{A}}^{2} + \int_{\Omega_{v}} \frac{\Delta t}{2} v \underbrace{\left(\frac{1}{2} \int_{\Omega_{x}} \frac{\partial h^{2}}{\partial x} dx\right)}_{=0} e^{-v^{2}} dv$$
$$- \frac{\Delta t}{4} \int_{\Omega_{x}} \left(E^{j} + E^{j-1}\right) \left[\int_{\Omega_{v}} \frac{\partial}{\partial v} \left(he^{-v^{2}}\right) h dv\right] dx + \nu \frac{\Delta t}{4} \overline{\mathcal{B}}^{2}, \tag{175}$$

where we noted that the integral of $\partial h^2/\partial x$ over Ω_x is zero because we assumed periodicity in space. We successively integrate by parts the third term on the right:

$$-\frac{\Delta t}{4} \int_{\Omega_{\tau}} \left(E^{j} + E^{j-1} \right) \left[\int_{\Omega_{\tau}} \frac{\partial}{\partial v} \left(h e^{-v^{2}} \right) h \, dv \right] \, dx = \frac{\Delta t}{4} \int_{\Omega_{\tau}} \left(E^{j} + E^{j-1} \right) \left[\int_{\Omega_{\tau}} h \frac{\partial h}{\partial v} e^{-v^{2}} \, dv \right] \, dx. \tag{176}$$

Let us now define:

$$\mathcal{M} = \max_{x \in \Omega_x} \left| E^j + E^{j-1} \right|. \tag{177}$$

Since \mathcal{M} depends on E^j (and, consequently, on f^j through (168), we may assume that for Δt sufficiently small, $E^j \approx E^{j-1}$. Thus, $\mathcal{M} \approx 2 \max_{x \in \Omega_x} \left| E^{j-1} \right|$. This makes the following evaluation of Δt practically possible (see the estimate in (183) below).

We estimate (176) by applying the Schwartz and Young inequalities as follows:

$$-\frac{\Delta t}{4} \int_{\Omega_{x}} \left(E^{j} + E^{j-1} \right) \left[\int_{\Omega_{v}} h \frac{\partial h}{\partial v} e^{-v^{2}} dv \right] dx$$

$$\geq -\frac{\Delta t}{4} \int_{\Omega_{x}} \left| E^{j} + E^{j-1} \right| \left[\int_{\Omega_{v}} h^{2} e^{-v^{2}} dv \right]^{\frac{1}{2}} \left[\int_{\Omega_{v}} \left(\frac{\partial h}{\partial v} \right)^{2} e^{-v^{2}} dv \right]^{\frac{1}{2}} dx$$

$$\geq -\frac{\Delta t}{4} \mathcal{M} \int_{\Omega_{x}} \mathcal{A} \mathcal{B} dx \geq -\frac{\Delta t}{4} \mathcal{M} \int_{\Omega_{x}} \left(\frac{\sigma}{2} \mathcal{A}^{2} + \frac{1}{2\sigma} \mathcal{B}^{2} \right) dx$$

$$= -\frac{\Delta t}{4} \mathcal{M} \left(\frac{\sigma}{2} \overline{\mathcal{A}}^{2} + \frac{1}{2\sigma} \overline{\mathcal{B}}^{2} \right), \tag{178}$$

where $\sigma > 0$ is an arbitrary parameter. Using this estimate in (175), we find the inequality

$$B(h,h) \ge \overline{\mathcal{A}}^2 + \nu \frac{\Delta t}{4} \overline{\mathcal{B}}^2 - \frac{\Delta t}{4} \mathcal{M} \left(\frac{\sigma}{2} \overline{\mathcal{A}}^2 + \frac{1}{2\sigma} \overline{\mathcal{B}}^2 \right). \tag{179}$$

To derive sufficient conditions for the positivity of the bilinear form $B(\cdot,\cdot)$, i.e., $B(h,h) \ge 0$, we can proceed in different ways. First, for every strictly positive quantity σ , we can impose that

$$\frac{\Delta t}{4} \mathcal{M} \left(\frac{\sigma}{2} \overline{\mathcal{A}}^2 + \frac{1}{2\sigma} \overline{\mathcal{B}}^2 \right) \le \overline{\mathcal{A}}^2 + \nu \frac{\Delta t}{4} \overline{\mathcal{B}}^2. \tag{180}$$

To this end, we note that:

$$\frac{\sigma}{2}\overline{\mathcal{A}}^2 + \frac{1}{2\sigma}\overline{\mathcal{B}}^2 = \frac{\sigma}{2}\left(\overline{\mathcal{A}}^2 + \frac{1}{\sigma^2}\overline{\mathcal{B}}^2\right).$$

Comparing the expression above with the right-hand side of inequality (180), suggests us to set $1/\sigma^2 = \nu \Delta t/4$, or, equivalently that $\sigma = 2/\sqrt{\nu \Delta t}$. We set this value of σ back into inequality (180) to find that

$$\frac{\Delta t}{4} \mathcal{M} \frac{\sigma}{2} \overline{\mathcal{A}}^2 + \frac{1}{2\sigma} \overline{\mathcal{B}}^2 = \frac{\Delta t \mathcal{M}}{4\sqrt{\nu \Delta t}} \left(\overline{\mathcal{A}}^2 + \nu \frac{\Delta t}{4} \overline{\mathcal{B}}^2 \right) \le \left(\overline{\mathcal{A}}^2 + \nu \frac{\Delta t}{4} \overline{\mathcal{B}}^2 \right), \tag{181}$$

from which we immediately have the condition:

$$\frac{\Delta t \mathcal{M}}{4\sqrt{\nu \Delta t}} \le 1,\tag{182}$$

that we can rewrite as

$$\Delta t \le \frac{16\nu}{M^2},\tag{183}$$

after renormalizing the factor $\sqrt{\Delta t}$ in the denominator of (182) and squaring the resulting inequality. Such a constraint on Δt constitutes a sufficient condition to realize the invertibility of problem (169) for k=1. Unfortunately, we are unable to provide a similar result in the case when k>1. The problem is that inequality (180) becomes of the form:

$$\frac{\Delta t}{4} \mathcal{M} \left(\frac{\sigma}{2} \overline{\mathcal{A}}^2 + \frac{1}{2\sigma} \overline{\mathcal{B}}^2 \right) \le \overline{\mathcal{A}}^2 + \frac{\nu \Delta t}{2^{k+1}} \int_{\Omega} \left(\frac{\partial^k h}{\partial v^k} \right)^2 e^{-v^2} \, dv \, dx. \tag{184}$$

We can bound \overline{B} , that contains only first derivatives, by an expression containing higher order derivatives, only if a certain number of low modes of h is set to zero. This is certainly not consistent with the freedom we would like to leave to these coefficients.

To recover an alternative estimate of the time step Δt that does not involve the diffusion parameter ν , we suppose that h is a linear combination of a finite number of Hermite polynomials. In practice, h is going to be a polynomial of degree less than or equal to N. In this situation, we can rely on the inverse type inequality:

$$\overline{\mathcal{B}} \le \sqrt{2N}\,\overline{\mathcal{A}},\tag{185}$$

which is easily deducible from (22). Thus, to control the last term at the end of (179) we proceed by writing:

$$-\frac{\Delta t}{4}\mathcal{M}\left(\frac{\sigma}{2}\overline{\mathcal{A}}^2 + \frac{1}{2\sigma}\overline{\mathcal{B}}^2\right) \ge -\frac{\Delta t}{4}\mathcal{M}\left(\frac{\sigma}{2} + \frac{N}{\sigma}\right)\overline{\mathcal{A}}^2 = -\frac{\Delta t}{4}\mathcal{M}\sqrt{2N}\overline{\mathcal{A}}^2,\tag{186}$$

where we noticed that the absolute value of the term in the middle is minimized by the choice $\sigma = \sqrt{2N}$. In this way, the positivity of the bilinear form is realized by requiring that the last term in (186) is less than $\overline{\mathcal{A}}^2 + \frac{1}{4}\nu\Delta t\overline{\mathcal{B}}^2$. This is true by choosing:

$$\Delta t \le \frac{4}{\mathcal{M}\sqrt{2N}},\tag{187}$$

and, now, the bound on Δt depends on N but not on ν . Moreover, this calculation does not involve any explicit expression from the Lenard-Bernestein diffusion operators on the right-hand side of (172) since this term was just eliminated because of its positivity for $\phi = h$. This means that this time the relation between N, Δt , and \mathcal{M} holds for any value of $k \geq 1$.

We can make further considerations by putting together inequalities (183) and (187). If Δt is chosen in order to be consistent with both of them, we get:

$$\Delta t \approx \frac{16\nu}{\mathcal{M}^2}$$
 and $\Delta t \approx \frac{4}{\mathcal{M}\sqrt{2N}}$ imply that $\nu \approx \frac{\mathcal{M}}{4\sqrt{2N}}$. (188)

Similarly, setting $1/\mathcal{M} = \sqrt{2N}\Delta t/4$ in $\Delta t \approx 16\nu/\mathcal{M}^2$ above implies that

$$\Delta t \approx 16\nu \frac{1}{\mathcal{M}^2} = 16\nu \frac{2N\Delta t^2}{16} = 2N\nu \Delta t^2, \tag{189}$$

from which we derive the relation

$$\nu N \Delta t \approx \frac{1}{2}.\tag{190}$$

The last relation agrees with the suggestion, made in the previous section, that the product $N\nu\Delta t$ should be of order of the unity.

We can say something more if the electric field is treated explicitly, i.e.: $(E^j + E^{j-1})/2 \approx E^{j-1}$, with $\mathcal{M} = 2 \max_{x \in \Omega_x} |E^{j-1}|$. The maximum norm can be bounded through the first derivative. This is done in the following way:

$$\mathcal{M}^2 \le |\Omega_x| \int_{\Omega_x} \left(\frac{\partial E^{j-1}}{\partial x}\right)^2 dx = |\Omega_x| \int_{\Omega_x} \left(2 - \int_{\Omega_v} f^{j-1} dv\right)^2 dx,\tag{191}$$

where $|\Omega_x|$ denotes the measure of Ω_x . Next, we use a standard inequality and the Schwartz inequality to obtain:

$$\frac{\mathcal{M}^2}{|\Omega_x|} \le \int_{\Omega_x} \left(2 - \int_{\Omega_v} f^{j-1} dv\right)^2 dx \le 2 \int_{\Omega_x} \left[4 + \left(\int_{\Omega_v} f^{j-1} dv\right)^2\right] dx$$

$$\le 8|\Omega_x| + \int_{\Omega_x} \left[\int_{\Omega_v} (f^{j-1})^2 e^{v^2} dv \int_{\Omega_v} e^{-v^2} dv\right] dx$$

$$\le 8|\Omega_x| + \sqrt{\pi} \int_{\Omega_x} \int_{\Omega_v} (h^{j-1})^2 e^{-v^2} dv dx = 8|\Omega_x| + \sqrt{\pi} \mathcal{H}, \tag{192}$$

where we denoted the last integral by \mathcal{H} . Then, we consider again inequality (179) from which we remove the non-negative term $\nu \Delta t \overline{\mathcal{B}}^2/4$ to obtain a sufficient condition that is independent of ν . Using (192) in the right-hand side of (186), we end up with

$$\frac{\Delta t}{4} |\Omega_x|^{1/2} \left(8|\Omega_x| + \sqrt{\pi} \mathcal{H} \right)^{1/2} \sqrt{2N} \overline{\mathcal{A}}^2 \le \overline{\mathcal{A}}^2, \tag{193}$$

which implies

$$\Delta t |\Omega_x|^{1/2} \left(8|\Omega_x| + \sqrt{\pi} \mathcal{H} \right)^{1/2} \le \frac{4}{\sqrt{2N}},\tag{194}$$

This last condition is substantially similar to (187). However, this derivation implies that having a knowledge of either \mathcal{M} or \mathcal{H} at the step j-1, we have an idea on how to set up the new time-step for the successive iteration.

In the final part of our study, we put together what we have learned in the previous sections, and investigate the interplay between time stability and conservation properties. We consider, first, the conservation of the mass, which is the zero-th order moment of the Vlasov distribution function f. After discretization in time, we assume that f^j is expanded on the Hermite functions' basis:

$$f^{j}(x,v) = \sum_{n=0}^{\infty} C_{n}^{\star,j}(x)\psi_{n}(v).$$
 (195)

The variational formulation for the expansion coefficients $C_n^{\star,j}$ is obtained by substituting (195) in (167), multiplying by the test function ψ^m and integrating on $\Omega = \Omega_x \times \Omega_v$:

$$\sum_{n=0}^{\infty} \left[\int_{\Omega} \frac{C_n^{\star,j} - C_n^{\star,j-1}}{\Delta t} \psi_n \psi^m dx dv \right] + \sum_{n=0}^{\infty} \left[\int_{\Omega} \frac{\partial}{\partial x} \left(\frac{C_n^{\star,j} + C_n^{\star,j-1}}{2} \right) v \psi_n \psi^m dx dv \right]$$

$$- \sum_{n=0}^{\infty} \left[\int_{\Omega} \frac{E^j + E^{j-1}}{2} \frac{C_n^{\star,j} + C_n^{\star,j-1}}{2} \frac{\partial \psi_n}{\partial v} \psi^m dx dv \right]$$

$$+ (-1)^k \nu \sum_{n=0}^{\infty} \left[\int_{\Omega} \widetilde{L}^{(k)} L^{(k)} \left(\frac{C_n^{\star,j} + C_n^{\star,j-1}}{2} \right) \psi_n \psi^m dx dv \right] = 0.$$

$$(196)$$

We separate the integration with respect to x from that with respect to v, obtaining:

$$\int_{\Omega_{x}} \frac{C_{m}^{\star,j} - C_{m}^{\star,j-1}}{\Delta t} dx + \sum_{n=0}^{\infty} \left[\int_{\Omega_{x}} \frac{\partial}{\partial x} \left(\frac{C_{n}^{\star,j} + C_{n}^{\star,j-1}}{2} \right) dx \int_{\Omega_{v}} v \psi_{n} \psi^{m} dv \right]
+ \frac{\gamma_{m}}{\gamma_{m+1}} \int_{\Omega_{x}} \frac{E^{j} + E^{j-1}}{2} \frac{C_{m-1}^{\star,j} + C_{m-1}^{\star,j-1}}{2} dx
- m(m-1) \cdots (m-k+1) \nu \int_{\Omega_{x}} \frac{C_{m}^{\star,j} + C_{m}^{\star,j-1}}{2} dx = 0.$$
(197)

We further note that, due to the periodic boundary conditions, the integral in the variable x of the second term is zero. In terms of the coefficients in the Hermite polynomial basis, Eq. (197) becomes:

$$\int_{\Omega_x} \frac{C_n^j - C_n^{j-1}}{\Delta t} dx + \sqrt{\frac{n+1}{n}} \int_{\Omega_x} \frac{E^j + E^{j-1}}{2} \frac{C_{n-1}^j + C_{n-1}^{j-1}}{2} dx$$
$$-n(n-1)\cdots(n-k+1)\nu \int_{\Omega_x} \frac{C_n^j + C_n^{j-1}}{2} dx = 0.$$
(198)

This system of equations is coupled with (168). As a consequence of the orthogonality, we have:

$$\int_{\Omega_v} f^j \, dv = \sum_{n=0}^{\infty} \int_{\Omega_v} C_n^j H_n e^{-v^2} = \sqrt{\pi} \, C_0^j. \tag{199}$$

Thus, the discretized Poisson equation takes the form:

$$\frac{\partial E^j}{\partial x} = 1 - \sqrt{\pi} \, C_0^j. \tag{200}$$

By integrating this last relation with respect to x and using the boundary conditions for E^j , we discover that $\int_{\Omega_x} C_0^j dx$ is constant for all $j \geq 0$. This condition is maintained by the scheme (198), whatever is $k \geq 1$. More in general,

conservation of momenta $\int_{\Omega} v^m f^j dx dv$, $j \ge 0$, is guaranteed up to $m \le k-1$. This corresponds to the generalization for arbitrary k of the conservation properties that were proven in Ref. [5] for k=3.

11. Conclusion

We investigated the role of Lenard-Bernstein-like pseudo-collisional operators in conjunction with spectral approximations of the Vlasov equation for a collisionless plasma in the electrostatic limit. In particular, we analyzed the spectral approximation of some one-dimensional, simplified model problems based on different families of Hermite basis functions using the symmetric and the asymmetric formulations. In the asymmetric case, we were able to prove the absolute stability in time in an L^2 -weighted norm, a problem that has been unresolved for many years. The results have partially been extended to the case of the full Vlasov-Poisson model.

Acknowledgements

This work was supported by the LDRD program of Los Alamos National Laboratory under project number 20170207ER. Los Alamos National Laboratory is operated by Triad National Security, LLC, for the National Nuclear Security Administration of U.S. Department of Energy (Contract No. 89233218CNA000001). The authors are affiliated to the Italian Istituto Nazionale di Alta Matematica (INdAM). This manuscript has no associated data.

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