Superconvergence of Discontinuous Galerkin methods for Elliptic Boundary Value Problems

Limin Ma*

Abstract

In this paper, we present a unified analysis of the superconvergence property for a large class of mixed discontinuous Galerkin methods. This analysis applies to both the Poisson equation and linear elasticity problems with symmetric stress formulations. Based on this result, some locally postprocess schemes are employed to improve the accuracy of displacement by order $\min(k+1, 2)$ if polynomials of degree k are employed for displacement. Some numerical experiments are carried out to validate the theoretical results.

Keywords. superconvergence, postprocessing, discontinuous Galerkin, linear elasticity problem

1 Introduction and Notation

1.1 Introduction

In this work, we investigate the superconvergence property and postprocess schemes of mixed discontinuous Galerkin methods for two classes of problems. One is the second order model problem

$$\begin{cases} c\boldsymbol{p} - \nabla u = 0 & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{p} = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \boldsymbol{p} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ and $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$. Here c is a bounded and positive definite matrix from $\mathbb{R}^n$ to $\mathbb{R}^n$, u is a scalar function and $\boldsymbol{p}$ is a vector-valued function. The other one is the linear elasticity problem

$$\begin{cases} A\boldsymbol{\sigma} - \epsilon(\boldsymbol{u}) = 0 & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\sigma} = f & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \Gamma_D, \\ \boldsymbol{\sigma}\boldsymbol{n} = 0 & \text{on } \Gamma_N, \end{cases} \tag{1.2}$$

*Department of Mathematics, Pennsylvania State University, University Park, PA, 16802, USA. maliminpku@gmail.com

with $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) and $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$. Here the displacement is denoted by $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ and the stress tensor is denoted by $\boldsymbol{\sigma} : \Omega \to \mathcal{S}$, where $\mathcal{S}$ is the set of symmetric $n \times n$ tensors. The linearized strain tensor $\epsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$. The compliance tensor $A : \mathcal{S} \to \mathcal{S}$

$$A\boldsymbol{\sigma} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E} tr(\boldsymbol{\sigma}) I \tag{1.3}$$

is assumed to be bounded and symmetric positive definite, where E and ν are the Young's modulus and Poisson ratio of the elastic material under consideration, respectively.

Postprocessing type of superconvergence property was discussed in literature, see [2, 6, 18, 35] for instance. There are two main ingredients for this kind of superconvergence. One is the superclose property of the projection of the exact solution, and the other one is an appropriate postprocess scheme which is performed seperately on each element. For the scalar elliptic problem (1.1), the postprocessing type of superconvergence has been analyzed for the conforming elements, mixed element and nonconforming elements with superclose property, see for instance, [2, 5, 7–10, 17, 22, 24, 26, 35] and the references therein. For some nonconforming elements, the lack of this superclose property of the canonical interpolation leads to the difficulty in analyzing the superconvergence result. Recently, a superconvergence of two nonconforming elements in this case was analyzed in [28, 29] by employing the superclose property of a related mixed element. The superconvergence property was also analyzed for various discontinuous Galerkin methods [14, 15, 37]. For the linear elasticity problem (1.2), the strong symmetry of the stress tensor causes a substantial additional difficulty for developing stable mixed elements for elasticity problem [1, 4, 23, 30–33]. The mixed methods in [12, 20] and hybridizable discontinuous Galerkin methods in [16] imposed weak symmetry on the stress tensor, and achieved optimal convergence for stress and superconvergence for displacement by post processing. A postprocessing schemes was analyzed for a mixed element methods solving the linear elasticity problems (1.2) in [34]. A superconvergent hybridizable discontinuous Galerkin method with strong symmetry was analyzed in [11].

In this paper, a unified superconvergence analysis of a large class of mixed discontinuous Galerkin methods is presented for both the scalar elliptic problem (1.1) and linear elasticity problem (1.2) in [21, 25] . Mixed discontinuous Galerkin methods employ discontinuous polynomials with degree k and $k + 1$ for the displacement $\boldsymbol{u}$ and the stress $\boldsymbol{\sigma}$, respectively. Thanks to a conforming projection and the corresponding commuting diagram, the L^2 projections of u for (1.1) and $\boldsymbol{u}$ for (1.2) admit a superclose property. Note that this property can be advantageously exploited to design a high accuracy approximation to u and $\boldsymbol{u}$. Indeed, following the idea in [2, 14, 15, 17, 35, 37], we propose four postprocessing schemes for the mixed discontinuous Galerkin method in [25] and get new approximations to u with high accuracy for second order scalar elliptic problem (1.1). For some special choices of parameters, the mixed discontinuous Galerkin method in [25] is hybridizable and leads to a much smaller system. The variable $\hat{\boldsymbol{p}}_h$ in the hybridized formulation is an approximation to $\boldsymbol{p}$ on edges. This $\hat{\boldsymbol{p}}_h$, together with the aforementioned postprocessing scheme, gives rise to a superconvergent approximation to the solution u of (1.1). For the elasticity problem (1.2), a post processing scheme in [34] was analyzed for a mixed element method. In this paper, a similar scheme is proposed for the discontinuous Galerkin method with symmetric stress in [21]. The proposed postprocessing scheme is analyzed to admit a desirable superconvergence property when $k \geq n$, which improves the accuracy of displacement by order $\min(k + 1, 2)$ if polynomials of degree k is employed for displacement. The current result provides the first analysis for a number of new methods [13, 21, 24, 36]. The numerical tests for linear elasticity problems also indicate that there is no such conforming interpolation which admits the commuting diagram when $k < n$.

The rest of the paper is organized as follows. Section 2 and 3 analyze the postprocessing schemes and the superconvergence property for scalar elliptic problems and linear elasticity problems, respectively. Some numerical examples are tested in Section 4 to verify the theoretical results.

1.2 Notation

Given a nonnegative integer m and a bounded domain $D \subset \mathbb{R}^n$, let $H^m(D)$, $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$ be the usual Sobolev space, norm and semi-norm, respectively. The L^2-inner product on D and ∂D are denoted by $(\cdot,\cdot)_D$ and $\langle\cdot,\cdot\rangle_{\partial D}$, respectively. Let $\|\cdot\|_{0,D}$ and $\|\cdot\|_{0,\partial D}$ be the norms of $L^2(D)$ and $L^2(\partial D)$, respectively. The norms $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$ are abbreviated as $\|\cdot\|_m$ and $|\cdot|_m$, respectively, when D is chosen as Ω. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded polygonal domain covered exactly by a shape-regular partition $\mathcal{T}$ into polyhedrons. Let h_K be the diameter of element $K \in \mathcal{T}$ and $h = \max_{K\in\mathcal{T}} h_K$. Denote the set of all interior edges/faces of $\mathcal{T}$ by $\mathcal{E}_h^I$, and all edges/faces on boundary Γ_D and Γ_N by $\mathcal{E}_h^D$ and $\mathcal{E}_h^N$, respectively. Let $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^D \cup \mathcal{E}_h^N$ and h_e be the diameter of edge/face $e \in \mathcal{E}_h$. For any interior edge/face $e = K^+ \cap K^-$, let $\boldsymbol{n}^i = \boldsymbol{n}|_{\partial K^i}$ be the unit outward normal vector on ∂K^i with $i = +, -$. For $K \subset \mathbb{R}^n$ and any nonnegative integer r, let $P_r(K, \mathbb{R})$ be the space of all polynomials of degree not greater than r on K.

Throughout this paper, we shall use letter C, which is independent of mesh-size h, stabilization parameters η, τ, γ, to denote a generic positive constant which may stand for different values at different occurrences. Following [38], the notations $x \lesssim y$ and $x \gtrsim y$ mean $x \le Cy$ and $x \ge Cy$, respectively. Denote $x \lesssim y \lesssim x$ by $x \cong y$.

2 Scalar elliptic problems

This section analyzes the postprocessing schemes and the superconvergence result for the scalar elliptic problem (2.5).

2.1 Discontinuous Galerkin methods for scalar elliptic problems

Consider the second order elliptic model problem (1.1). For any scalar-valued function v_h and vector-valued function $\boldsymbol{q}_h$ that are piecewise smooth with respect to $\mathcal{T}$, let $v_h^\pm = v_h|_{\partial K^\pm}$, $\boldsymbol{q}_h^\pm = \boldsymbol{q}_h|_{\partial K^\pm}$. Define the average $\{\cdot\}$ and the jump $[\cdot]$ on interior edges/faces $e \in \mathcal{E}_h^I$ as follows:

$$
\begin{aligned}
\{\boldsymbol{q}_h\} &= \tfrac{1}{2}(\boldsymbol{q}_h^+ + \boldsymbol{q}_h^-), \quad [\boldsymbol{q}_h] = \boldsymbol{q}_h^+ \cdot \boldsymbol{n}^+ + \boldsymbol{q}_h^- \cdot \boldsymbol{n}^-,\\
\{v_h\} &= \tfrac{1}{2}(v_h^+ + v_h^-), \quad [v_h] = v_h^+ \boldsymbol{n}^+ + v_h^- \boldsymbol{n}^-.
\end{aligned}
\tag{2.1}
$$

For any boundary edge/face $e \subset \partial\Omega$, define

$$
\begin{aligned}
&\{\boldsymbol{q}_h\} = \boldsymbol{q}_h, \quad [\boldsymbol{q}_h] = 0, \qquad\quad \{v_h\} = v_h, \quad [v_h] = v_h \boldsymbol{n}, \quad \text{on } \Gamma_D,\\
&\{\boldsymbol{q}_h\} = \boldsymbol{q}_h, \quad [\boldsymbol{q}_h] = \boldsymbol{q}_h \cdot \boldsymbol{n}, \quad \{v_h\} = v_h, \quad [v_h] = 0, \qquad \text{on } \Gamma_N.
\end{aligned}
\tag{2.2}
$$

For any scalar-valued function v_h and vector-valued function $\boldsymbol{q}_h$, define the piecewise gradient ∇_h and piecewise divergence div_h by

$$\nabla_h v_h\big|_K = \nabla(v_h|_K), \quad \mathrm{div}_h \boldsymbol{q}_h\big|_K = \mathrm{div}(\boldsymbol{q}_h|_K) \quad \forall K \in \mathcal{T}.$$

Define some inner products as follows:

$$(\cdot,\cdot)_{\mathcal{T}} = \sum_{K\in\mathcal{T}} (\cdot,\cdot)_K, \quad \langle\cdot,\cdot\rangle = \sum_{e\in\mathcal{E}_h} \langle\cdot,\cdot\rangle_e, \quad \langle\cdot,\cdot\rangle_{\partial\mathcal{T}} = \sum_{K\in\mathcal{T}} \langle\cdot,\cdot\rangle_{\partial K}. \tag{2.3}$$

Whenever there is no ambiguity, we simplify $(\cdot,\cdot)_{\mathcal{T}}$ as $(\cdot,\cdot)$. With the aforementioned definitions, the following DG identity [3] holds:

$$(\boldsymbol{q}_h, \nabla_h v_h) = -(\mathrm{div}_h \boldsymbol{q}_h, v_h) + \langle [\boldsymbol{q}_h], \{v_h\}\rangle + \langle \{\boldsymbol{q}_h\}, [v_h]\rangle. \tag{2.4}$$

The four-field extended Galerkin formulation in [25] seeks $(\boldsymbol{p}_h, \check{\boldsymbol{p}}_h, u_h, \check{u}_h) \in \boldsymbol{Q}_h \times \check{\boldsymbol{Q}}_h \times V_h \times \check{V}_h$ such that

$$\begin{cases} (c\boldsymbol{p}_h, \boldsymbol{q}_h) + (u_h, \mathrm{div}_h \boldsymbol{q}_h) - \langle \{u_h\} + \check{u}_h - \gamma[u_h], [\boldsymbol{q}_h]\rangle & = 0, \qquad \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h, \\ -(\mathrm{div}_h \boldsymbol{p}_h, v_h) - \langle \gamma[\boldsymbol{p}_h] + \check{\boldsymbol{p}}_h, [v_h]\rangle + \langle [\boldsymbol{p}_h], \{v_h\}\rangle & = -(f, v_h) \quad \forall v_h \in V_h, \\ -\langle \tau^{-1}\check{\boldsymbol{p}}_h + [u_h], \check{\boldsymbol{q}}_h\rangle_e & = 0, \qquad \forall \check{\boldsymbol{q}}_h \in \check{\boldsymbol{Q}}_h, \\ \langle \eta^{-1}\check{u}_h + [\boldsymbol{p}_h], \check{v}_h\rangle_e & = 0, \qquad \forall \check{v}_h \in \check{V}_h, \end{cases} \tag{2.5}$$

where

$$\begin{aligned} \boldsymbol{Q}_h &: = \{\boldsymbol{q}_h \in L^2(\Omega, \mathbb{R}^n) : \boldsymbol{q}_h|_K \in \boldsymbol{Q}(K),\ \forall K \in \mathcal{T}_h\}, \\ \check{\boldsymbol{Q}}_h &: = \{\check{\boldsymbol{q}}_h \in L^2(\mathcal{E}_h, \mathbb{R}^n) : \boldsymbol{q}_h|_K \in \check{\boldsymbol{Q}}(K),\ \forall K \in \mathcal{T}_h\}, \\ V_h &: = \{v_h \in L^2(\Omega, \mathbb{R}) : \boldsymbol{q}_h|_K \in V(K),\ \forall K \in \mathcal{T}_h\}, \\ \check{V}_h &: = \{\check{v}_h \in L^2(\mathcal{E}_h, \mathbb{R}) : \boldsymbol{q}_h|_K \in \check{V}(K),\ \forall K \in \mathcal{T}_h\}. \end{aligned}$$

Here γ is constant, $\check{\boldsymbol{p}}_h$ and $\check{u}_h$ are the modifications to $\boldsymbol{p}_h$ and u_h on elementary boundaries, respectively. Define the discontinuous spaces $\boldsymbol{Q}_h$, $\check{\boldsymbol{Q}}_h$, V_h and $\check{V}_h$ with

$$\boldsymbol{Q}(K) = P_k(K, \mathbb{R}^n),\ \check{\boldsymbol{Q}}(K) = P_k(K, \mathbb{R}^n),\ V(K) = P_k(K, \mathbb{R}),\ \check{V}(K) = P_k(K, \mathbb{R})$$

by $\boldsymbol{Q}_h^k$, $\check{\boldsymbol{Q}}_h^k$, V_h^k and $\check{V}_h^k$, respectively. Define

$$\begin{aligned} \|\boldsymbol{q}_h\|_{\mathrm{div,h}}^2 &= (c\boldsymbol{q}_h, \boldsymbol{q}_h) + \|\mathrm{div}_h \boldsymbol{q}_h\|_0^2 + \|\eta^{1/2}[\boldsymbol{q}_h]\|_0^2, \quad \|\check{\boldsymbol{q}}_h\|_{0,h}^2 = \|\tau^{-1/2}\check{\boldsymbol{q}}_h\|_0^2, \\ \|v_h\|_{0,h}^2 &= \|v_h\|_0^2 + \|\tau^{1/2}[v_h]\|_0^2 + \|\eta^{-1/2}\{v_h\}\|_0^2, \qquad \|\check{v}_h\|_{0,h}^2 = \|\eta^{-1/2}\check{v}_h\|_0^2. \end{aligned} \tag{2.6}$$

For $H(\mathrm{div})$-based formulations (2.5), the well-posedness and the error estimate is analyzed in [25] under a set of assumptions as presented below. The error estimate of $\boldsymbol{p}_h$ in L^2-norm is similar to the one for the stress tensor in [21], thus the details of the proof is omitted here.

Lemma 2.1. *For $H(\mathrm{div})$-based four-field formulation* (2.5) *with $\eta = (\rho h_e)^{-1}$, $\tau \cong \eta^{-1} = \rho h_e$, if the spaces $\boldsymbol{Q}_h, V_h, \check{V}_h$ satisfy the conditions*

(C1) Let $\boldsymbol{R}_h := \boldsymbol{Q}_h \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ and $\boldsymbol{R}_h \times V_h$ is a stable pair for mixed method;

(C2) $\mathrm{div}_h \boldsymbol{Q}_h = V_h$

(C3) $\{V_h\} \subset \check{V}_h$

Then, the formulation (2.5) *is uniformly well-posed with respect to the norms* (2.6) *when* $\rho \in (0, \rho_0]$. *Namely, if* $(\boldsymbol{p}_h, \check{\boldsymbol{p}}_h, u_h, \check{u}_h) \in \boldsymbol{Q}_h \times \check{\boldsymbol{Q}}_h \times V_h \times \check{V}_h$ *is the solution of* (2.5)*, it holds that*

$$\|\boldsymbol{p}_h\|_{\mathrm{div,h}} + \|\check{\boldsymbol{p}}_h\|_{0,h} + \|u_h\|_{0,h} + \|\check{u}_h\|_{0,h} \lesssim \|f\|_{0,\Omega}.$$

If $\boldsymbol{p} \in \boldsymbol{H}^{k+2}(\Omega, \mathbb{R}^n)$, $u \in H^{k+1}(\Omega, \mathbb{R})(k \geq 0)$, *and* $\boldsymbol{Q} \times \check{\boldsymbol{Q}}_h \times V_h \times \check{V}_h = \boldsymbol{Q}_h^{k+1} \times \check{\boldsymbol{Q}}_h^k \times V_h^k \times \check{V}_h^{k+1}$,

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{div,h}} + \|\check{\boldsymbol{p}}_h\|_{0,h} + \|u - u_h\|_{0,h} + \|\check{u}_h\|_{0,h} \lesssim h^{k+1}(|\boldsymbol{p}|_{k+2} + |u|_{k+1}). \tag{2.7}$$

Furthermore, if $\boldsymbol{p} \in \boldsymbol{H}^{k+2}(\Omega, \mathbb{R}^n)$,

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_0 \lesssim h^{k+2}(|\boldsymbol{p}|_{k+2} + |u|_{k+1}). \tag{2.8}$$

We can establish the following superclose property for the extended Galerkin formulation (2.5).

Theorem 2.2. *Suppose* $\boldsymbol{p} \in H^{k+2}(\Omega, \mathbb{R}^n)$, $u \in H^{k+1}(\Omega, \mathbb{R})(k \geq 0)$, *and* $(\boldsymbol{p}_h, \check{\boldsymbol{p}}_h, u_h, \check{u}_h) \in \boldsymbol{Q}_h^{k+1} \times \check{\boldsymbol{Q}}_h^k \times V_h^k \times \check{V}_h^{k+1}$ *is the solution of the four-field formulation* (2.5) *with* $\eta = \left(\rho h_e\right)^{-1}$, $\tau \cong \eta^{-1} = \rho h_e$. *It holds that*

$$\|\mathrm{P}_h^k u - u_h\|_{0,\Omega} \lesssim h^{\min(2k+2,k+3)}(|\boldsymbol{p}|_{k+2} + |u|_{k+1}),$$

where P_h^k *is the* L^2*-projection onto* V_h^k.

We omit the proof here since it is similar to the analysis for linear elasticity problems (1.2) in the next section.

2.2 Postprocess techniques for scalar elliptic problems

Consider the $H(\mathrm{div})$-based four-field formulation (2.5) with $\boldsymbol{Q}_h = \boldsymbol{Q}_h^{k+1}$, $\check{\boldsymbol{Q}}_h = \check{\boldsymbol{Q}}_h^k$, $V_h = V_h^k$ and $\check{V}_h = \check{V}_h^{k+1}$. Define

$$\hat{\boldsymbol{p}}_h = \{\boldsymbol{p}_h\} + \gamma[\boldsymbol{p}_h] + \check{\boldsymbol{p}}_h. \tag{2.9}$$

Note that $\hat{\boldsymbol{p}}_h$ is an approximation to $\boldsymbol{p}$ on elementary boundaries.

We list three postprocessing techniques [2, 14, 15, 19, 34, 35] for the elliptic problem (1.1). Here there are two choices of the projection operator P_h, one is the L^2 projection to piecewise constant space, namely

$$\mathrm{P}_h u = \mathrm{P}_h^0 u,$$

where P_h^k is the L^2 projection to V_h^k, and the other one is the L^2 projection to the discrete space V_h, namely

$$\int_\Omega \mathrm{P}_h u v_h \, dx = \int_\Omega u v_h \, dx, \quad \forall v_h \in V_h.$$

For either choice of P_h, consider the following three postprocessing schemes:

1. Let $u_{1,h}^* \in V_h^{k+2}$ be the solution of

$$\begin{cases} \int_K \alpha \nabla u_{1,h}^* \cdot \nabla v_h dx = -\int_K f v_h dx + \int_{\partial K} \boldsymbol{p}_h \cdot \boldsymbol{n} v_h ds, \quad \forall v_h \in (I - \mathrm{P}_h) V_h^{k+2}|_K, \\ \mathrm{P}_h(u_{1,h}^* - u_h) = 0. \end{cases} \tag{2.10}$$

   with $\alpha = c^{-1}$.

2. Let $u_{2,h}^* \in V_h^{k+2}$ be the solution of

$$\begin{cases} \int_K \alpha \nabla u_{2,h}^* \cdot \nabla v_h dx = -\int_K f v_h dx + \int_{\partial K} \hat{\boldsymbol{p}}_h \cdot \boldsymbol{n} v_h ds, \quad \forall v_h \in (I - \mathrm{P}_h) V_h^{k+2}|_K, \\ \mathrm{P}_h(u_{2,h}^* - u_h) = 0. \end{cases} \tag{2.11}$$

   with $\alpha = c^{-1}$ and $\hat{\boldsymbol{p}}_h$ defined in (2.9).

3. Let $u_{3,h}^* \in V_h^{k+2}$ be the solution of

$$\begin{cases} \int_K \nabla u_{3,h}^* \cdot \nabla v_h dx = \int_K c \boldsymbol{p}_h \cdot \nabla v_h dx, \quad \forall v_h \in (I - \mathrm{P}_h^0) V_h^{k+2}|_K, \\ \mathrm{P}_h^0(u_{3,h}^* - u_h) = 0. \end{cases} \tag{2.12}$$

Note that the schemes (2.11) and (2.12) are identical in some special cases. If α is a constant matrix, the first equation in (2.12) is equivalent to

$$\int_K \alpha \nabla u_{3,h}^* \cdot \nabla v_h dx = \int_K \boldsymbol{p}_h \cdot \nabla v_h dx = -\int_K \mathrm{div}_h \boldsymbol{p}_h v_h dx + \int_{\partial K} \boldsymbol{p}_h \boldsymbol{n} v_h ds$$

for any $v_h \in (I - \mathrm{P}_h^0) V_h^{k+2}|_K$. By (2.5), the above equation reads

$$\int_K \alpha \nabla u_{3,h}^* \cdot \nabla v_h dx = -\int_K f v_h dx + \int_{\partial K} \hat{\boldsymbol{p}}_h \boldsymbol{n} v_h ds.$$

It implies that for this particular α, the postprocess algorithms (2.11) with $\mathrm{P}_h = \mathrm{P}_h^0$ and (2.12) are the same.

We analyze in the following theorem that the above postprocessing techniques can improve the accuracy for the mixed discontinuous Galerkin formulation (2.5).

Theorem 2.3. *Suppose $\boldsymbol{p} \in \boldsymbol{H}^{k+2}(\Omega)$, $u \in H^{k+3}(\Omega)(k \geq 0)$, and $(\boldsymbol{p}_h, \check{\boldsymbol{p}}_h, u_h, \check{u}_h) \in \boldsymbol{Q}_h^{k+1} \times \check{\boldsymbol{Q}}_h^k \times V_h^k \times \check{V}_h^{k+1}$ is the solution of the four-field formulation* (2.5) *with $\eta = (\rho h_e)^{-1}$, $\tau \cong \eta^{-1} = \rho h_e$. It holds that*

$$\|u - u_h^*\|_0 \lesssim h^{\min(2k+2, k+3)} |u|_{k+3},$$

where $u_h^* = u_{1,h}^*$ in (2.10)*, $u_{2,h}^*$ in* (2.11) *or $u_{3,h}^*$ in* (2.12)*.*

Proof. Let $v_h = (I - \mathrm{P}_h)(\mathrm{P}_h^{k+2} u - u_h^*)$. Since $\mathrm{P}_h^0 v_h = 0$,

$$\|v_h\|_{0,\Omega} = \|v_h - \mathrm{P}_h^0 v_h\|_{0,\Omega} \lesssim h |v_h|_{1,h}. \tag{2.13}$$

It follows from the trace inequality that

$$\|[v_h]\|_{\mathcal{E}_h} + \|\{v_h\}\|_{\mathcal{E}_h} \lesssim h^{-1/2}\|v_h\|_{0,\Omega} \lesssim h^{1/2}|v_h|_{1,h}. \tag{2.14}$$

By (2.10) and (2.11),

$$\begin{aligned}|v_h|_{1,h}^2 =&(\alpha\nabla_h(\mathrm{P}_h^{k+2}u - u_h^*), \nabla_h v_h) - (\alpha\nabla_h \mathrm{P}_h(\mathrm{P}_h^{k+2}u - u_h^*), \nabla_h v_h)\\ =&(\alpha\nabla_h \mathrm{P}_h^{k+2}u, \nabla_h v_h) + (f, v_h) - \langle[\tilde{\boldsymbol{p}}_h], \{v_h\}\rangle - \langle\{\tilde{\boldsymbol{p}}_h\}, [v_h]\rangle\\ &- (\alpha\nabla_h \mathrm{P}_h(\mathrm{P}_h^{k+2}u - u_h^*), \nabla_h v_h),\end{aligned}$$

where $\tilde{\boldsymbol{p}}_h = \boldsymbol{p}_h$ if $u_h^* = u_{1,h}^*$, and $\tilde{\boldsymbol{p}}_h = \hat{\boldsymbol{p}}_h$ if $u_h^* = u_{2,h}^*$. Since $f = \nabla\cdot(\alpha\nabla u)$ and $\boldsymbol{p} = \alpha\nabla u$,

$$\begin{aligned}|v_h|_{1,h}^2 =&(\alpha\nabla_h(\mathrm{P}_h^{k+2} - I)u, \nabla v_h) - \langle[\tilde{\boldsymbol{p}}_h], \{v_h\}\rangle + \langle\{\boldsymbol{p} - \tilde{\boldsymbol{p}}_h\}, [v_h]\rangle\\ &- (\alpha\nabla_h \mathrm{P}_h(\mathrm{P}_h^{k+2}u - u_h^*), \nabla_h v_h).\end{aligned} \tag{2.15}$$

If $\mathrm{P}_h = \mathrm{P}_h^0$, the last term on the right hand side of the above equation equals zero. If P_h is the L^2 projection to V_h, namely $\mathrm{P}_h = \mathrm{P}_h^k$, by the triangle inequality and the inverse estimate,

$$|(\alpha\nabla_h \mathrm{P}_h(\mathrm{P}_h^{k+2}u - u_h^*), \nabla_h v_h)| \lesssim h^{-1}\|\mathrm{P}_h(\mathrm{P}_h^{k+2}u - u_h^*)\|_0 |v_h|_{1,h}.$$

Since $\mathrm{P}_h u_h^* = \mathrm{P}_h u_h$,

$$|(\alpha\nabla_h \mathrm{P}_h(\mathrm{P}_h^{k+2}u - u_h^*), \nabla_h v_h)| \lesssim h^{-1}\|\mathrm{P}_h(\mathrm{P}_h^{k+2}u - u_h)\|_0 |v_h|_{1,h}.$$

If $\tilde{\boldsymbol{p}}_h = \boldsymbol{p}_h$, by the error estimates in (2.7) and (2.14),

$$\begin{aligned}|\langle[\tilde{\boldsymbol{p}}_h], \{v_h\}\rangle| &\le \eta^{-1/2}\|\eta^{1/2}[\boldsymbol{p}_h]\|_{\mathcal{E}_h}\|\{v_h\}\|_{\mathcal{E}_h} \le h^{k+2}|v_h|_{1,h}(|\boldsymbol{p}|_{k+2} + |u|_{k+1}),\\ |\langle\{\boldsymbol{p} - \tilde{\boldsymbol{p}}_h\}, [v_h]\rangle| &\le h^{-1/2}\|\boldsymbol{p} - \boldsymbol{p}_h\|_0\|\{v_h\}\|_{\mathcal{E}_h} \le h^{k+2}|v_h|_{1,h}(|\boldsymbol{p}|_{k+2} + |u|_{k+1}).\end{aligned}$$

If $\tilde{\boldsymbol{p}}_h = \hat{\boldsymbol{p}}_h$, $\tilde{\boldsymbol{p}}_h\cdot\boldsymbol{n}$ is continuous on interior edges. Thus,

$$\langle[\tilde{\boldsymbol{p}}_h], \{v_h\}\rangle = 0.$$

The error estimates in (2.7) and (2.14) imply that

$$|\langle\{\boldsymbol{p} - \tilde{\boldsymbol{p}}_h\}, [v_h]\rangle| \le |\langle\{\boldsymbol{p} - \boldsymbol{p}_h\}, [v_h]\rangle| + \left|\langle\gamma[\boldsymbol{p}_h] + \check{\boldsymbol{p}}_h, [v_h]\rangle\right| \lesssim h^{k+2}|v_h|_{1,h}(|\boldsymbol{p}|_{k+2} + |u|_{k+1}).$$

Substituting the above estimates, Theorem 2.2 and

$$|(\mathrm{P}_h^{k+2} - I)u|_{1,h} \lesssim h^{k+2}|u|_{k+3}$$

into (2.15),

$$|v_h|_{1,h} \lesssim \|\alpha\nabla_h(\mathrm{P}_h^{k+2} - I)u\|_{0,\Omega} + h^{k+2} + h^{-1}\|\mathrm{P}_h(\mathrm{P}_h^{k+2}u - u_h)\|_0 \lesssim h^{k+2}|u|_{k+3}. \tag{2.16}$$

By the definition of P_h^k, u_h^* and the superconvergence result in Theorem 2.2,

$$\|\mathrm{P}_h(\mathrm{P}_h^{k+2}u - u_h^*)\|_0 = \|\mathrm{P}_h(\mathrm{P}_h^k u - u_h)\|_0 \lesssim h^{\min(2k+2,k+3)}(|\boldsymbol{p}|_{k+2} + |u|_{k+1}).$$

It follows (2.13), (2.16) and the above estimate that

$$\|u - u_h^*\|_0 \le \|u - \mathrm{P}_h^{k+2}u\|_0 + \|\mathrm{P}_h(\mathrm{P}_h^{k+2}u - u_h^*)\|_0 + \|v_h\|_0 \lesssim h^{\min(2k+2,k+3)}|u|_{k+3},$$

which completes the proof for $u_h^* = u_{1,h}^*$ and $u_{2,h}^*$. The proof for $u_h^* = u_{3,h}^*$ is similar to the analysis in Theorem 3.5 for linear elasticity problem, which is omitted here. □

Remark 2.1. *Similar to the analysis in [21] which is also presented in Section 3.1 for the linear elasticity problem, the formulation* (2.5) *with*

$$\tau = O(h),\ \eta = \tau^{-1},\ \gamma = 0 \tag{2.17}$$

is hybridizable, and can be reduced to a formulation with $\hat{\boldsymbol{p}}_h$. By solving this reduced formulation with much less computational cost, we can construct an approximation $u_{2,h}^*$ to u with accuracy $O(h^{\min(2k+2,k+3)})$ if the solution u is smooth enough.

Remark 2.2. *For the first two postprocessing procedures, we let $u_{1,h}^*$ and $u_{2,h}^*$ in the discrete space V_h^{k+2} to guarantee the superconvergence in Theorem 2.3. For a general mixed discontinuous Galerkin formulation with the conditions (C1)-(C3), if u_h superconverges to the projection of the exact displacement, namely*

$$\|\mathrm{P}_h u - u_h\|_{0,\Omega} \lesssim h^r \inf_{\boldsymbol{q}_h\in \boldsymbol{Q}_h, v_h\in V_h} (\|\boldsymbol{p}-\boldsymbol{q}_h\|_{\mathrm{div,h}} + \|u - v_h\|_{0,h}),$$

and $\|\boldsymbol{p}-\boldsymbol{p}_h\|_0 \lesssim h^{\min(1,r-1)} \inf_{\boldsymbol{q}_h\in \boldsymbol{Q}_h, v_h\in V_h} (\|\boldsymbol{p}-\boldsymbol{q}_h\|_{\mathrm{div,h}} + \|u - v_h\|_{0,h})$. We can choose a similar postprocessing technique by replacing V_h^{k+2} in (2.10) *and* (2.11) *by a large enough discrete space $\tilde{V}_h$ with $V_h \subset \tilde{V}_h$ and*

$$\inf_{v_h\in\tilde{V}_h} \|u - v_h\|_0 + h|u - v_h|_{1,h} \lesssim h^{\min(2,r)} \inf_{\boldsymbol{q}_h\in \boldsymbol{Q}_h, v_h\in V_h} (\|\boldsymbol{p}-\boldsymbol{q}_h\|_{\mathrm{div,h}} + \|u - v_h\|_{0,h}).$$

Then, a similar analysis proves the superconvergence result

$$\|u - u_h^*\|_0 \lesssim h^{\min(2,r)} \inf_{\boldsymbol{q}_h\in \boldsymbol{Q}_h, v_h\in V_h} (\|\boldsymbol{p}-\boldsymbol{q}_h\|_{\mathrm{div,h}} + \|u - v_h\|_{0,h}).$$

Next we introduce a Taylor expansion type postprocessing scheme, which follows [6]. Recall that P_h^k is the L^2-projection onto V_h^k. Define the operator $\tilde{\mathrm{P}}_h^{k+2}$ onto V_h^{k+2} by

$$\begin{cases} \displaystyle\int_K \partial^\alpha (u - \tilde{\mathrm{P}}_h^{k+2} u)\, dx = 0, \quad \forall\, k+1 \le |\alpha| \le k+2, \\ \qquad \mathrm{P}_h^k (u - \tilde{\mathrm{P}}_h^{k+2} u) = 0. \end{cases}$$

Define ϕ_α by $\phi_\alpha|_K = \frac{1}{\alpha!}(x - M_K)^\alpha$, where M_K is the centroid of element K. There exists the Taylor expansion

$$(\tilde{\mathrm{P}}_h^{k+2} - \mathrm{P}_h^k) u = (I - \mathrm{P}_h^k)\tilde{\mathrm{P}}_h^{k+2} u = \sum_{|\alpha|=k+1}^{k+2} c_\alpha (I - \mathrm{P}_h^k)\phi_\alpha \tag{2.18}$$

with constants c_α to be determined. Since

$$\mathrm{P}_h^0 \partial^\beta \phi_\alpha = \delta_{\alpha\beta}, \quad \partial^\beta \tilde{\mathrm{P}}_h^{k+2} u = \sum_{|\alpha|=k+1}^{k+2} c_\alpha \partial^\beta \phi_\alpha$$

for any $k+1 \le |\alpha|, |\beta| \le k+2$, it holds that

$$c_\alpha = \mathrm{P}_h^0 \partial^\alpha \tilde{\mathrm{P}}_h^{k+2} u = \mathrm{P}_h^0 \partial^\alpha u, \tag{2.19}$$

which can be written as a function of $\boldsymbol{p} = c\nabla u$, namely, $c_\alpha = c_\alpha(\boldsymbol{p})$. Define the Taylor expansion type postprocessing $u_{4,h}^* \in V_h^{k+2}$ in [6] by

$$u_{4,h}^* = u_h + \sum_{|\alpha|=k+1}^{k+2} c_\alpha(\boldsymbol{p}_h)(I - \mathrm{P}_h^k)\phi_\alpha. \tag{2.20}$$

The proof for the following theorem indicates that the same superconvergence result can be obtained if $\boldsymbol{p}_h$ in (2.20) is replaced by any high accuracy approximation to $\boldsymbol{p}$.

Theorem 2.4. *Suppose $\boldsymbol{p} \in \boldsymbol{H}^{k+2}(\Omega)$, $u \in H^{k+3}(\Omega)(k \geq 0)$, and $(\boldsymbol{p}_h, \check{\boldsymbol{p}}_h, u_h, \check{u}_h) \in \boldsymbol{Q}_h^{k+1} \times \check{\boldsymbol{Q}}_h^k \times V_h^k \times \check{V}_h^{k+1}$ is the solution of the four-field formulation* (2.5) *with $\eta = \left(\rho h_e\right)^{-1}$, $\tau \cong \eta^{-1} = \rho h_e$. It holds that*

$$\|u - u_{4,h}^*\|_0 \lesssim h^{\min(2k+2,k+3)}|u|_{k+3}.$$

Proof. Note that

$$\begin{aligned} u - u_{4,h}^* &= (\mathrm{P}_h^k u - u_h) + (u - \tilde{\mathrm{P}}_h^{k+2}u) + (\tilde{\mathrm{P}}_h^{k+2}u - \mathrm{P}_h^k u - \sum_{|\alpha|=k+1}^{k+2} c_\alpha(\boldsymbol{p}_h)(I - \mathrm{P}_h^k)\phi_\alpha) \\ &= (\mathrm{P}_h^k u - u_h) + (u - \tilde{\mathrm{P}}_h^{k+2}u) + \sum_{|\alpha|=k+1}^{k+2} (c_\alpha(\boldsymbol{p}) - c_\alpha(\boldsymbol{p}_h))(I - \mathrm{P}_h^k)\phi_\alpha. \end{aligned}$$

By the definition of $c_\alpha(\cdot)$,

$$\|c_\alpha(\boldsymbol{p}) - c_\alpha(\boldsymbol{p}_h)\|_0 \lesssim h^{-|\alpha|+1}\|\boldsymbol{p} - \boldsymbol{p}_h\|_0. \tag{2.21}$$

It follows from the above equation and the fact $\|(I - \mathrm{P}_h^k)\phi_\alpha\|_0 \lesssim h^{|\alpha|}$ that

$$\begin{aligned} \|u - u_{4,h}^*\|_0 \leq& \|\mathrm{P}_h^k u - u_h\|_0 + \|u - \mathrm{P}_h^{k+2}u\|_0 + \sum_{|\alpha|=k+1}^{k+2} \|c_\alpha(\boldsymbol{p}) - c_\alpha(\boldsymbol{p}_h)\|_0 \|(I - \mathrm{P}_h^k)\phi_\alpha\|_0 \qquad (2.22) \\ \leq& \|\mathrm{P}_h^k u - u_h\|_0 + h^{k+3}|u|_{k+3} + h\|\boldsymbol{p} - \boldsymbol{p}_h\|_0. \qquad (2.23) \end{aligned}$$

A substitution of (2.8) and Theorem 2.2 into the above inequality leads to

$$\|u - u_{4,h}^*\|_0 \lesssim h^{\min(2k+2,k+3)}(|\boldsymbol{p}|_{k+2} + |u|_{k+1}), \tag{2.24}$$

which completes the proof. $\square$

3 Linear elasticity problems

This section analyzes the superconvergence result for the linear elasticity problem (3.5).

3.1 Discontinuous Galerkin method for linear elasticity problems

Consider the linear elasticity problem (1.2). Let

$$
\begin{aligned}
\boldsymbol{\Sigma}_h :&= \{\boldsymbol{q}_h \in L^2(\Omega, \mathcal{S}) : \boldsymbol{q}_h|_K \in \boldsymbol{Q}(K),\ \forall K \in \mathcal{T}_h\},\\
\boldsymbol{V}_h :&= \{v_h \in L^2(\Omega, \mathbb{R}^n) : \boldsymbol{q}_h|_K \in V(K),\ \forall K \in \mathcal{T}_h\},\\
\check{\boldsymbol{\Sigma}}_h :&= \{\check{\boldsymbol{q}}_h \in L^2(\mathcal{E}_h, \mathcal{S}) : \boldsymbol{q}_h|_K \in \check{\boldsymbol{Q}}(K),\ \forall K \in \mathcal{T}_h\},\\
\check{\boldsymbol{V}}_h :&= \{\check{v}_h \in L^2(\mathcal{E}_h, \mathbb{R}^n) : \boldsymbol{q}_h|_K \in \check{V}(K),\ \forall K \in \mathcal{T}_h\},
\end{aligned}
$$

where $\boldsymbol{\Sigma}_h, \check{\boldsymbol{\Sigma}}_h, \boldsymbol{V}_h, \check{\boldsymbol{V}}_h$ are subspaces of $L^2(\Omega, \mathcal{S})$, $L^2(\mathcal{E}_h, \mathcal{S})$, $L^2(\Omega, \mathbb{R}^n)$ and $L^2(\mathcal{E}_h, \mathbb{R}^n)$, respectively. For any vector-valued function $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and tensor-valued function $\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h$, let $\boldsymbol{v}_h^{\pm} = \boldsymbol{v}_h|_{\partial K^{\pm}}$, $\boldsymbol{\tau}_h^{\pm} = \boldsymbol{\tau}_h|_{\partial K^{\pm}}$. Define the average $\{\cdot\}$ and the jump $[\cdot]$ on interior edges/faces $e \in \mathcal{E}_h^I$ as follows:

$$
\begin{aligned}
&\{\boldsymbol{\tau}_h\} = \tfrac{1}{2}(\boldsymbol{\tau}_h^+ + \boldsymbol{\tau}_h^-), \quad [\boldsymbol{\tau}_h] = \boldsymbol{\tau}_h^+\boldsymbol{n}^+ + \boldsymbol{\tau}_h^-\boldsymbol{n}^-,\\
&\{\boldsymbol{v}_h\} = \tfrac{1}{2}(\boldsymbol{v}_h^+ + \boldsymbol{v}_h^-), \quad [\boldsymbol{v}_h] = \boldsymbol{v}_h^+ \odot \boldsymbol{n}^+ + \boldsymbol{v}_h^- \odot \boldsymbol{n}^- - (\boldsymbol{v}_h^+ \cdot \boldsymbol{n}^+ + \boldsymbol{v}_h^- \cdot \boldsymbol{n}^-)I,
\end{aligned}
\tag{3.1}
$$

where $\boldsymbol{v}_h \odot \boldsymbol{n} = \boldsymbol{v}_h\boldsymbol{n}^T + \boldsymbol{n}\boldsymbol{v}_h^T$. For any boundary edge/face $e \subset \partial\Omega$, define

$$
\begin{aligned}
&\{\boldsymbol{\tau}_h\} = \boldsymbol{\tau}_h, \quad [\boldsymbol{\tau}_h] = 0, \qquad \{\boldsymbol{v}_h\} = \boldsymbol{v}_h, \quad [\boldsymbol{v}_h] = \boldsymbol{v}_h \odot \boldsymbol{n} - (\boldsymbol{v}_h \cdot \boldsymbol{n})I, \quad \text{on } \Gamma_D,\\
&\{\boldsymbol{\tau}_h\} = \boldsymbol{\tau}_h, \quad [\boldsymbol{\tau}_h] = \boldsymbol{\tau}_h\boldsymbol{n}, \quad \{\boldsymbol{v}_h\} = \boldsymbol{v}_h, \quad [\boldsymbol{v}_h] = 0, \qquad \text{on } \Gamma_N.
\end{aligned}
\tag{3.2}
$$

With the aforementioned definitions, the following identities [3] holds:

$$
\langle \boldsymbol{\tau}_h\boldsymbol{n}, \boldsymbol{v}_h\rangle_{\partial\mathcal{T}_h} = \langle \{\boldsymbol{\tau}_h\}, [\boldsymbol{v}_h]\rangle + \langle [\boldsymbol{\tau}_h], \{\boldsymbol{v}_h\}\rangle, \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h,\ \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.3}
$$

For any vector-valued function $\boldsymbol{v}_h$ and tensor-valued function $\boldsymbol{\tau}_h$, define the piecewise symmetric strain tensor ϵ_h and piecewise divergence div_h by

$$
\epsilon_h(\boldsymbol{v}_h)\big|_K = \epsilon(\boldsymbol{v}_h|_K), \quad \mathrm{div}_h\boldsymbol{\tau}_h\big|_K = \mathrm{div}(\boldsymbol{\tau}_h|_K), \quad \forall K \in \mathcal{T}_h.
$$

There exists a similar DG identity to (2.4) as below

$$
(\boldsymbol{\tau}_h, \epsilon_h(\boldsymbol{v}_h)) = -(\mathrm{div}_h\boldsymbol{\tau}_h, \boldsymbol{v}_h) + \langle [\boldsymbol{\tau}_h], \{\boldsymbol{v}_h\}\rangle + \langle \{\boldsymbol{\tau}_h\}\boldsymbol{n}, [\boldsymbol{v}_h]\boldsymbol{n}\rangle, \quad \text{if } \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h. \tag{3.4}
$$

For the linear elasticity problem (1.2), consider the four-field extended Galerkin formulation in [21], which seeks $(\boldsymbol{\sigma}_h, \check{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h, \check{\boldsymbol{u}}_h) \in \boldsymbol{\Sigma}_h \times \check{\boldsymbol{\Sigma}}_h \times \boldsymbol{V}_h \times \check{\boldsymbol{V}}_h$ such that

$$
\left\{
\begin{aligned}
(A\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\boldsymbol{u}_h, \mathrm{div}_h\boldsymbol{\tau}_h) - \langle \{\boldsymbol{u}_h\} + \check{\boldsymbol{u}}_h - (\gamma \cdot \boldsymbol{n})[\boldsymbol{u}_h]\boldsymbol{n}, [\boldsymbol{\tau}_h]\rangle &= 0, && \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h,\\
(\mathrm{div}_h\boldsymbol{\sigma}_h, \boldsymbol{v}_h) - \langle [\boldsymbol{\sigma}_h], \{\boldsymbol{v}_h\}\rangle + \langle \check{\boldsymbol{\sigma}}_h + [\boldsymbol{\sigma}_h]\gamma^T, [\boldsymbol{v}_h]\rangle &= (f, \boldsymbol{v}_h) && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\\
\langle \tau^{-1}\check{\boldsymbol{\sigma}}_h + [\boldsymbol{u}_h], \check{\boldsymbol{\tau}}_h\rangle_e &= 0, && \forall \check{\boldsymbol{\tau}}_h \in \check{\boldsymbol{\Sigma}}_h,\\
\langle \eta^{-1}\check{\boldsymbol{u}}_h + [\boldsymbol{\sigma}_h], \check{\boldsymbol{v}}_h\rangle_e &= 0, && \forall \check{\boldsymbol{v}}_h \in \check{\boldsymbol{V}}_h.
\end{aligned}
\right.
\tag{3.5}
$$

For any $(\boldsymbol{\tau}_h, \check{\boldsymbol{\tau}}_h, \boldsymbol{v}_h, \check{\boldsymbol{v}}_h) \in \boldsymbol{\Sigma}_h \times \check{\boldsymbol{\Sigma}}_h \times \boldsymbol{V}_h \times \check{\boldsymbol{V}}_h$, define

$$
\begin{aligned}
\|\boldsymbol{\tau}_h\|_{\mathrm{div,h}}^2 &= (A\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) + \|\mathrm{div}_h\boldsymbol{\tau}_h\|_0^2 + \|\eta^{1/2}[\boldsymbol{\tau}_h]\|_{\mathcal{E}_h}^2, & \|\check{\boldsymbol{\tau}}_h\|_{0,h}^2 &= \|\tau^{-1/2}\check{\boldsymbol{\tau}}_h\|_{\mathcal{E}_h}^2,\\
\|\boldsymbol{v}_h\|_{0,h}^2 &= \|\boldsymbol{v}_h\|_0^2 + \|\tau^{1/2}[\boldsymbol{v}_h]\|_{\mathcal{E}_h}^2, & \|\check{\boldsymbol{v}}_h\|_{0,h}^2 &= \|\eta^{-1/2}\check{\boldsymbol{v}}_h\|_{\mathcal{E}_h}^2,
\end{aligned}
$$

and the L^2 norm of $\boldsymbol{\tau}_h$ by

$$\|\boldsymbol{\tau}_h\|_A^2 = (A\boldsymbol{\tau}_h, \boldsymbol{\tau}_h), \quad \forall \boldsymbol{\tau}_h \in L^2(\Omega, \mathcal{S}).$$

For $H(\text{div})$-based formulations (3.5), the well-posedness and the error estimate are analyzed in [21] under a set of assumptions as listed in the following lemma.

Lemma 3.1. *The four-field formulation* (3.5) *which satisfies the conditions*

(A1) $\boldsymbol{\Sigma}_h = \boldsymbol{\Sigma}_h^{k+1}$, $\text{div}_h \boldsymbol{\Sigma}_h = \boldsymbol{V}_h \subset \boldsymbol{V}_h^k$, $k \geq 0$;

(A2) $\check{\boldsymbol{V}}_h^{k+1} \subset \check{\boldsymbol{V}}_h$;

(A3) $\tau = \rho_1 h_e$, $\eta = \rho_2^{-1} h_e^{-1}$ *and there exist positive constants* C_1, C_2, C_3 *and* C_4 *such that*

$$0 \leq \rho_1 \leq C_1, \quad C_2 \leq \rho_2 \leq C_3, \quad 0 \leq \gamma \leq C_4,$$

is uniformly well-posed with respect to the norms when ρ_1 *and* ρ_2*. Namely, if* $\boldsymbol{\sigma} \in H^{k+2}(\Omega, \mathcal{S})$, $u \in H^{k+1}(\Omega, \mathbb{R}^n)(k \geq 0)$ *and let* $(\boldsymbol{\sigma}_h, \check{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h, \check{\boldsymbol{u}}_h) \in \boldsymbol{\Sigma}_h^{k+1} \times \check{\boldsymbol{\Sigma}}_h^k \times \boldsymbol{V}_h^k \times \check{\boldsymbol{V}}_h^{k+1}$ *be the solution of* (3.5)*, then we have the following error estimate:*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\text{div,h}} + \|\check{\boldsymbol{\sigma}}_h\|_{0,h} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,h} + \|\check{\boldsymbol{u}}_h\|_{0,h} \lesssim h^{k+1}(|\boldsymbol{\sigma}|_{k+2} + |\boldsymbol{u}|_{k+1}). \tag{3.6}$$

Furthermore, if $k \geq n$*, it holds that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_A \lesssim h^{k+2}(|\boldsymbol{\sigma}|_{k+2} + |\boldsymbol{u}|_{k+1}). \tag{3.7}$$

Here discrete spaces $\boldsymbol{\Sigma}_h^k$, $\check{\boldsymbol{\Sigma}}_h^k$, $\boldsymbol{V}_h^k$ and $\check{\boldsymbol{V}}_h^k$ are subspaces of $L^2(\Omega, \mathcal{S})$, $L^2(\mathcal{E}_h, \mathcal{S})$, $L^2(\Omega, \mathbb{R}^n)$ and $L^2(\mathcal{E}_h, \mathbb{R}^n)$, respectively, and contain all piecewise polynomials of degree not larger than k.

The analysis in [21] shows that a special case of (3.5) is hybridizable as presented below. Denote

$$\boldsymbol{Z}_h = \{\boldsymbol{u}_h \in \boldsymbol{V}_h : \epsilon_h(\boldsymbol{u}_h) = 0\},$$

$$\boldsymbol{V}_h^{\perp} = \{\boldsymbol{u}_h \in \boldsymbol{V}_h : (\boldsymbol{u}_h, \boldsymbol{v}_h) = 0, \ \forall \boldsymbol{v}_h \in \boldsymbol{Z}_h\}.$$

Theorem 3.2. *The formulation* (3.5) *with discrete spaces satisfying the assumptions in Lemma 3.1 and condition* (2.17) *can be decomposed into two sub-problems as:*

(I) Local problems. For each element K, given $\hat{\boldsymbol{\sigma}}_h \in \check{\boldsymbol{\Sigma}}_h$*, find* $(\boldsymbol{\sigma}_h^K, \boldsymbol{u}_h^K) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h^{\perp}$ *such that for any* $(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h^{\perp}$

$$\begin{cases} (A\boldsymbol{\sigma}_h^K, \boldsymbol{\tau}_h)_K - (\epsilon_h(\boldsymbol{u}_h^K), \boldsymbol{q}_h)_K + \langle \eta \boldsymbol{\sigma}_h^K \boldsymbol{n}, \boldsymbol{\tau}_h \boldsymbol{n} \rangle_{\partial K} &= \langle \eta \hat{\boldsymbol{\sigma}}_h \boldsymbol{n}, \boldsymbol{\tau}_h \boldsymbol{n} \rangle_{\partial K}, \\ -(\boldsymbol{\sigma}_h^K, \epsilon_h(\boldsymbol{v}_h))_K &= (f, \boldsymbol{v}_h)_K - \langle \hat{\boldsymbol{\sigma}}_h \boldsymbol{n}, \boldsymbol{v}_h \rangle_{\partial K}. \end{cases} \tag{3.8}$$

Denote $W_{\boldsymbol{\Sigma}} : \check{\boldsymbol{\Sigma}}_h \to \boldsymbol{\Sigma}_h$ *and* $W_{\boldsymbol{V}} : \check{\boldsymbol{\Sigma}}_h \to \boldsymbol{V}_h^{\perp}$ *by*

$$W_{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\sigma}}_h)|_K = \boldsymbol{\sigma}_h^K \quad \textit{and} \quad W_{\boldsymbol{V}}(\hat{\boldsymbol{\sigma}}_h)|_K = \boldsymbol{u}_h^K,$$

respectively.

(II) Global problem. Find $(\hat{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h^0) \in \check{\boldsymbol{\Sigma}}_h \times Z_h$ *such that for any* $\boldsymbol{v}_h^0 \in Z_h$ *and* $\check{\boldsymbol{\tau}}_h \in \check{\boldsymbol{\Sigma}}_h$,

$$\begin{cases} \langle \eta(\hat{\boldsymbol{\sigma}}_h - W_Q(\hat{\boldsymbol{\sigma}}_h))\boldsymbol{n}, (\check{\boldsymbol{\tau}}_h - W_Q(\check{\boldsymbol{\tau}}_h))\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{u}_h^0, W_V(\check{\boldsymbol{\tau}}_h)\rangle_{\partial\mathcal{T}_h} & = -(f, W_{\boldsymbol{V}}(\check{\boldsymbol{\tau}}_h)), \\ \langle \check{\boldsymbol{\sigma}}_h\boldsymbol{n}, \boldsymbol{v}_h^0\rangle_{\partial\mathcal{T}_h}, & = (f, \boldsymbol{v}_h^0). \end{cases} \tag{3.9}$$

Let $(\hat{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h^0)$ *be the solution of* (3.9), $(\boldsymbol{\sigma}_h^K, \boldsymbol{u}_h^K)$ *be the solution of* (3.8), *and* $(\boldsymbol{\sigma}_h, \check{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h, \check{\boldsymbol{u}}_h)$ *be the solution of* (3.5). *Then,*

$$\boldsymbol{\sigma}_h^K = \boldsymbol{\sigma}_h|_K, \ \boldsymbol{u}_h^K + \boldsymbol{u}_h^0 = \boldsymbol{u}_h|_K, \ \hat{\boldsymbol{\sigma}}_h = \check{\boldsymbol{\sigma}}_h + \{\boldsymbol{\sigma}_h\}.$$

Theorem 3.2 indicates that the discontinuous Galerkin formulation (3.5) with this special choice (2.17) of parameters can be written as a system of $\hat{\boldsymbol{\sigma}}_h$ and $\boldsymbol{u}_h^0$, which reduces the degree of freedom and the computational cost.

3.2 Superclose analysis for linear elasticity problems

This section considers the superclose result for linear elasticity problems (3.5). The analysis for the superclose property requires two main ingredients: a conforming interpolation onto $\boldsymbol{\Sigma}_h$, and the commuting property of this interpolation.

Let $\mathbf{P}_h$ be the standard L^2-projection onto $\boldsymbol{V}_h$, namely

$$(\mathbf{P}_h\boldsymbol{u}, \boldsymbol{v}_h) = (\boldsymbol{u}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

For $\boldsymbol{V}_h = \boldsymbol{V}_h^k$, denote the L^2-projection by $\mathbf{P}_h^k$. The analysis for the linear elasticity problem requires the following assumption

Assumption 3.1. *There exists a projection* $\boldsymbol{\Pi}_h$ *onto a conforming subspace* $\boldsymbol{\Sigma}_h^c$ *of* $\boldsymbol{\Sigma}_h$, *and the projection* $\boldsymbol{\Pi}_h$ *admits the commuting diagram*

$$\mathrm{div}\boldsymbol{\Pi}_h\boldsymbol{\tau} = \boldsymbol{P}_h\mathrm{div}\boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, \Omega). \tag{3.10}$$

Let $(\boldsymbol{\sigma}_h, \check{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h, \check{\boldsymbol{u}}_h) \in \boldsymbol{\Sigma}_h \times \check{\boldsymbol{\Sigma}}_h \times \boldsymbol{V}_h \times \check{\boldsymbol{V}}_h$ be the solution of the four-field formulation (3.5). Define

$$\boldsymbol{e}_{\boldsymbol{u}} = \mathbf{P}_h\boldsymbol{u} - \boldsymbol{u}_h, \quad \boldsymbol{d}_h = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h. \tag{3.11}$$

Lemma 3.3. *Suppose that the conditions (A1)-(A3) and the Assumption 3.1 hold. For any* $\boldsymbol{\psi} \in L^2(\Omega, \mathbb{R}^n)$, *let* $\boldsymbol{\phi}$ *be the solution of Problem* (1.2) *with* $f = \boldsymbol{\psi}$, *which implies that* $\mathrm{div}(A^{-1}\epsilon(\boldsymbol{\phi})) = \boldsymbol{\psi}$. *It holds that*

$$\begin{aligned}(\boldsymbol{e}_{\boldsymbol{u}}, \boldsymbol{\psi}) =& (\mathrm{div}_h\boldsymbol{d}_h, (I - \boldsymbol{P}_h)\boldsymbol{\phi}) + (A\boldsymbol{d}_h, (I - \boldsymbol{\Pi}_h)(A^{-1}\epsilon(\boldsymbol{\phi}))) - \langle [\boldsymbol{\sigma}_h], \{(\boldsymbol{P}_h - I)\boldsymbol{\phi}\}\rangle \\ &+ \langle \check{\boldsymbol{\sigma}}_h + [\boldsymbol{\sigma}_h]\gamma^T, [\boldsymbol{P}_h\boldsymbol{\phi}]\rangle.\end{aligned} \tag{3.12}$$

Proof. Note that the formulation (3.5) is consistant, namely, $(\boldsymbol{\sigma}, 0, \boldsymbol{u}, 0)$ satisfies (3.5). Let

$$\hat{\boldsymbol{\sigma}}_h = \{\boldsymbol{\sigma}_h\} + [\boldsymbol{\sigma}_h]\gamma^T + \check{\boldsymbol{\sigma}}_h, \qquad \hat{\boldsymbol{u}}_h = \{\boldsymbol{u}_h\} - (\gamma \cdot \boldsymbol{n})[\boldsymbol{u}_h]\boldsymbol{n} + \check{\boldsymbol{u}}_h \tag{3.13}$$

with $\gamma \in \mathbb{R}^{n\times 1}$. By the DG identity (3.4), the formulation (3.5) and $\mathrm{div}_h\boldsymbol{\Sigma}_h \subset \boldsymbol{V}_h$,

$$\begin{cases} (A\boldsymbol{d}_h, \boldsymbol{\tau}_h) + (\boldsymbol{e}_{\boldsymbol{u}}, \mathrm{div}_h\boldsymbol{\tau}_h) = \langle \{\boldsymbol{u} - \hat{\boldsymbol{u}}_h\}, [\boldsymbol{\tau}_h]\rangle & \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h \\ -(\mathrm{div}_h\boldsymbol{d}_h, \boldsymbol{v}_h) = \langle \{\boldsymbol{\sigma}_h - \hat{\boldsymbol{\sigma}}_h\}, [\boldsymbol{v}_h]\rangle + \langle [\boldsymbol{\sigma}_h], \{\boldsymbol{v}_h\}\rangle & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{cases} \tag{3.14}$$

For any $\psi \in V_h$, since $e_u \in V_h$, by the commuting diagram (3.10),

$$(e_u, \psi) = (e_u, \operatorname{div}\Pi_h(A^{-1}\epsilon(\phi))). \tag{3.15}$$

Let $\tau_h = \Pi_h(A^{-1}\epsilon(\phi))$ in (3.14). It follows from $[\tau_h] = 0$ that

$$(e_u, \psi) = -(Ad_h, \Pi_h(A^{-1}\epsilon(\phi))) = -(d_h, \epsilon(\phi)) + (Ad_h, (I - \Pi_h)(A^{-1}\epsilon(\phi))). \tag{3.16}$$

Let $v_h = \mathbf{P}_h\phi$ in (3.14). It holds that

$$(\operatorname{div}_h d_h, \phi) = (\operatorname{div}_h d_h, (I - \mathbf{P}_h)\phi) - \langle\{\sigma_h - \hat{\sigma}_h\}, [\mathbf{P}_h\phi]\rangle - \langle[\sigma_h], \{\mathbf{P}_h\phi\}\rangle. \tag{3.17}$$

A combination of (3.16), (3.17) and

$$(d_h, \epsilon(\phi)) = -(\operatorname{div}_h d_h, \phi) + \langle[d_h], \{\phi\}\rangle \tag{3.18}$$

gives

$$\begin{aligned}(e_u, \psi) =& (\operatorname{div}_h d_h, (I - \mathbf{P}_h)\phi) + (Ad_h, (I - \Pi_h)(A^{-1}\epsilon(\phi))) - \langle[\sigma_h], \{\mathbf{P}_h\phi\}\rangle \\ &- \langle[d_h], \{\phi\}\rangle - \langle\{\sigma_h - \hat{\sigma}_h\}, [\mathbf{P}_h\phi]\rangle.\end{aligned} \tag{3.19}$$

According to (3.11) and (3.13),

$$\begin{aligned}\langle[\sigma_h], \{\mathbf{P}_h\phi\}\rangle + \langle[d_h], \{\phi\}\rangle =& \langle[\sigma_h], \{(\mathbf{P}_h - I)\phi\}\rangle, \\ \langle\{\sigma_h - \hat{\sigma}_h\}, [\mathbf{P}_h\phi]\rangle =& -\langle\check{\sigma}_h + [\sigma_h]\gamma^T, [\mathbf{P}_h\phi]\rangle.\end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.19),

$$\begin{aligned}(e_u, \psi) =& (\operatorname{div}_h d_h, (I - \mathbf{P}_h)\phi) + (Ad_h, (I - \Pi_h)(A^{-1}\epsilon(\phi))) - \langle[\sigma_h], \{(\mathbf{P}_h - I)\phi\}\rangle \\ &+ \langle\check{\sigma}_h + [\sigma_h]\gamma^T, [\mathbf{P}_h\phi]\rangle,\end{aligned}$$

which completes the proof. □

It was analyzed in [27] that there exists such a conforming interpolation Π_h with commuting property (3.10) for $k \geq n$, and it holds that

$$\|\sigma - \Pi_h\sigma\|_0 \lesssim h^{k+2}|\sigma|_{k+2}. \tag{3.21}$$

The following theorem shows that $\|e_u\|_0$ converges at the rate $k+3$ if solutions are smooth enough. The accuracy is presented in the form of $h^{\min(2k+2,k+3)}$ to be consistent with the result in Theorem 2.2 for scalar elliptic problems.

Theorem 3.4. *Suppose $\sigma \in H^{k+2}(\Omega, \mathcal{S})$, $u \in H^{k+1}(\Omega, \mathbb{R}^n)(k \geq n)$, and $(\sigma_h, \check{\sigma}_h, u_h, \check{u}_h)$, which is in $\Sigma_h^{k+1} \times \check{\Sigma}_h^k \times V_h^k \times \check{V}_h^{k+1}$, is the solution of the four-field formulation* (3.5) *with $\eta = (\rho h_e)^{-1}$, $\tau \cong \eta^{-1} = \rho h_e$. It holds that*

$$\|P_h^k u - u_h\|_{0,\Omega} \lesssim h^{\min(2k+2,k+3)}(|\sigma|_{k+2} + |u|_{k+1}). \tag{3.22}$$

Proof. Since $\mathbf{P}_h^k$ is the L^2-projection onto V_h^k,

$$\|(I - \mathbf{P}_h^k)v\|_{0,K} \lesssim h^{k+1}|v|_{k+1,K}, \quad \forall v \in H^{k+1}(K, \mathbb{R}^n). \tag{3.23}$$

By the triangle inequality, (3.6) and (3.23),

$$\left|(\mathrm{div}_h \boldsymbol{d}_h, (I-\mathbf{P}_h^k)\phi)\right| \le \|\mathrm{div}_h \boldsymbol{d}_h\|_0 \|(I-\mathbf{P}_h^k)\phi\|_0 \lesssim h^{\min(2k+2,k+3)}|\phi|_2(|\boldsymbol{\sigma}|_{k+2}+|\boldsymbol{u}|_{k+1}). \tag{3.24}$$

The L^2 error estimate of $\|\boldsymbol{d}_h\|_0$ in (3.7), (3.21) and (3.23) indicate that

$$\begin{aligned}\left|(A\boldsymbol{d}_h, (I-\boldsymbol{\Pi}_h)(A^{-1}\epsilon(\phi))\right| &\le \|A\boldsymbol{d}_h\|_0\|(I-\boldsymbol{\Pi}_h)(A^{-1}\epsilon(\phi))\|_0\\ &\lesssim h^{\min(2k+4,k+3)}|\phi|_2(|\boldsymbol{\sigma}|_{k+2}+|\boldsymbol{u}|_{k+1}).\end{aligned} \tag{3.25}$$

It follows from the error estimates (3.6), (3.23) and trace inequality that

$$\begin{aligned}\left|\langle[\boldsymbol{\sigma}_h], \{(I-\mathbf{P}_h^k)\phi\}\rangle\right| &\le (\eta h)^{-1/2}\|\eta^{1/2}[\boldsymbol{\sigma}_h]\|_{\mathcal{E}_h}\|(I-\mathbf{P}_h^k)\phi\|_0\\ &\lesssim h^{\min(2k+2,k+3)}|\phi|_2(|\boldsymbol{\sigma}|_{k+2}+|\boldsymbol{u}|_{k+1}),\end{aligned} \tag{3.26}$$

$$\begin{aligned}\left|\langle\check{\boldsymbol{\sigma}}_h+[\boldsymbol{\sigma}_h]\gamma^T, [\mathbf{P}_h^k\phi]\rangle\right| &\le h^{-\frac{1}{2}}\|\check{\boldsymbol{\sigma}}_h+[\boldsymbol{\sigma}_h]\gamma^T\|_{\mathcal{E}_h}\|(I-\mathbf{P}_h^k)\phi\|_0\\ &\lesssim h^{\min(2k+2,k+3)}|\phi|_2(|\boldsymbol{\sigma}|_{k+2}+|\boldsymbol{u}|_{k+1}).\end{aligned} \tag{3.27}$$

A substitution of (3.24), (3.25), (3.26) and (3.27) into (3.12) leads to

$$|(\boldsymbol{e_u}, \psi)| \lesssim h^{\min(2k+2,k+3)}|\phi|_2(|\boldsymbol{\sigma}|_{k+2}+|\boldsymbol{u}|_{k+1}). \tag{3.28}$$

Since $|\phi|_2 \lesssim \|\psi\|_0$,

$$\|\boldsymbol{e_u}\|_0 = \sup_{0\neq\psi\in L^2(\Omega)} \frac{(\boldsymbol{e_u}, \psi)}{\|\psi\|_0} \lesssim h^{\min(2k+2,k+3)}(|\boldsymbol{\sigma}|_{k+2}+|\boldsymbol{u}|_{k+1}), \tag{3.29}$$

which completes the proof. □

Remark 3.1. *Since the four-field extended Galerkin method recovers most of discontinuous Galerkin methods in literature [21, 25], Theorem 2.2 and Theorem 3.4 imply that most of the $H(\mathrm{div})$-based discontinuous Galerkin methods in literature [13, 21, 24, 36] admit this superclose property.*

3.3 Postprocess technique for linear elasticity problems

Consider the linear elasticity problems (1.2). Denote the rigid motion, the kernel of the symmetric gradient operator $\epsilon(\cdot)$, by

$$\mathrm{RM}(K,\mathbb{R}^2) = \mathrm{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}y\\-x\end{pmatrix}\right\}.$$

For any $\boldsymbol{v}\in L^2(K,\mathbb{R}^2)$, define L^2-projection onto $\mathrm{RM}(K,\mathbb{R}^2)$ by $\mathbf{P}_h^* v$, namely,

$$\int_K (I-\mathbf{P}_h^*)\boldsymbol{v}\cdot\boldsymbol{w}_h dx = 0, \quad \forall\ \boldsymbol{w}_h \in \mathrm{RM}(K,\mathbb{R}^2).$$

Note that for any positive integer $k\ge 1$,

$$\mathbf{P}_h^*\mathbf{P}_h^k\boldsymbol{v} = \mathbf{P}_h^*\boldsymbol{v}. \tag{3.30}$$

Consider the $H(\mathrm{div})$-based four-field formulation (3.5) with $\boldsymbol{\Sigma}_h = \boldsymbol{\Sigma}_h^{k+1}$, $\check{\boldsymbol{\Sigma}}_h = \check{\boldsymbol{\Sigma}}_h^k$, $\boldsymbol{V}_h = \boldsymbol{V}_h^k$ and $\check{\boldsymbol{V}}_h = \check{\boldsymbol{V}}_h^{k+1}$. Lemma 3.1 guarantees the wellposedness of this problem. We introduce a new postprocess procedure for linear elasticity problem. Let $\boldsymbol{u}_h^* \in \boldsymbol{V}_h^{k+2}$ be the solution of the following problem

$$\begin{cases} (\epsilon(\boldsymbol{u}_h^*), \epsilon(\boldsymbol{v}_h))_K = (A\boldsymbol{\sigma}_h, \epsilon(\boldsymbol{v}_h))_K, & \forall \boldsymbol{v}_h \in P_{k+2}(K, \mathbb{R}^2) \\ \mathbf{P}_h^*(\boldsymbol{u}_h^* - \boldsymbol{u}_h) = 0, \end{cases} \tag{3.31}$$

where $(\boldsymbol{\sigma}_h, \check{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h, \check{\boldsymbol{u}}_h)$ is the solution of the mixed discontinuous Galerkin formulation (3.5).

The following theorem illustrates that the postprocessing solution $\boldsymbol{u}_h^*$ admits a higher accuracy compared to the approximation $\boldsymbol{u}_h$.

Theorem 3.5. *Suppose $\boldsymbol{\sigma} \in \boldsymbol{H}^{k+2}(\Omega)$, $\boldsymbol{u} \in H^{k+3}(\Omega)(k \geq n)$, and $(\boldsymbol{\sigma}_h, \check{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h, \check{\boldsymbol{u}}_h) \in \boldsymbol{\Sigma}_h^{k+1} \times \check{\boldsymbol{\Sigma}}_h^k \times \boldsymbol{V}_h^k \times \check{\boldsymbol{V}}_h^{k+1}$ is the solution of the four-field formulation* (3.5) *with $\eta = (\rho h_e)^{-1}$, $\tau \cong \eta^{-1} = \rho h_e$. It holds that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h^*\|_0 \lesssim h^{\min(2k+2, k+3)} |\boldsymbol{u}|_{k+3}.$$

Proof. A combination of (1.2) and (3.31) gives

$$(\epsilon(\boldsymbol{u}_h^*) - \epsilon(\mathbf{P}_h^{k+2}\boldsymbol{u}), \epsilon(\boldsymbol{v}_h)) = (A(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}), \epsilon(\boldsymbol{v}_h)) + (\epsilon(\boldsymbol{u}) - \epsilon(\mathbf{P}_h^{k+2}\boldsymbol{u}), \epsilon(\boldsymbol{v}_h)) \tag{3.32}$$

According to Lemma 3.1 and the definition of $\mathbf{P}_h^{k+2}$,

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_A \lesssim h^{k+2}(|\boldsymbol{\sigma}|_{k+2} + |\boldsymbol{u}|_{k+1}), \qquad \|\epsilon(\boldsymbol{u}) - \epsilon(\mathbf{P}_h^{k+2}\boldsymbol{u})\|_0 \lesssim h^{k+2}|\boldsymbol{u}|_{k+3}.$$

Let $\boldsymbol{v}_h = \boldsymbol{u}_h^* - \mathbf{P}_h^{k+2}\boldsymbol{u}$ in (3.32). It follows from (3.7) that

$$\|\epsilon(\boldsymbol{v}_h)\|_0 \leq \|A(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|_0 + \|\epsilon(\boldsymbol{u}) - \epsilon(P_h^{k+2}\boldsymbol{u})\|_0 \lesssim h^{k+2}|\boldsymbol{u}|_{k+3}. \tag{3.33}$$

Since $k \geq n \geq 1$, by (3.30) and Theorem 3.4,

$$\|\mathbf{P}_h^*\boldsymbol{v}_h\|_0 = \|\mathbf{P}_h^*(\boldsymbol{u}_h - \mathbf{P}_h^k\boldsymbol{u})\|_0 \leq \|\boldsymbol{u}_h - \mathbf{P}_h^k\boldsymbol{u}\|_0 \lesssim h^{k+3}|\boldsymbol{u}|_{k+3}. \tag{3.34}$$

Since $(I - \mathbf{P}_h^*)\boldsymbol{w}_h = 0$ for any $\boldsymbol{w}_h \in \mathrm{RM}(K, \mathbb{R}^2)$, it follows from (3.33) and the scaling technique that

$$\|(I - \mathbf{P}_h^*)\boldsymbol{v}_h\|_0 \lesssim h\|\epsilon(\boldsymbol{v}_h)\|_0 \lesssim h^{k+3}|\boldsymbol{u}|_{k+3}. \tag{3.35}$$

A combination of (3.34) and (3.35) gives

$$\|\boldsymbol{v}_h\|_0 \leq \|\mathbf{P}_h^*\boldsymbol{v}_h\|_0 + \|(I - \mathbf{P}_h^*)\boldsymbol{v}_h\|_0 \lesssim h^{k+3}|\boldsymbol{u}|_{k+3}.$$

Consequently,

$$\|\boldsymbol{u}_h^* - \boldsymbol{u}\|_0 \leq \|\boldsymbol{v}_h\|_0 + \|\mathbf{P}_h^{k+2}\boldsymbol{u} - \boldsymbol{u}\|_0 \lesssim h^{k+3}|\boldsymbol{u}|_{k+3},$$

which completes the proof. □

Remark 3.2. *For the case $k < n$, the analysis in [21] indicates that as long as Assumption 3.1 holds with*

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h\boldsymbol{\tau}\|_0 \lesssim h^{k+2}|\boldsymbol{\tau}|_{k+2}, \ \forall \boldsymbol{\tau} \in H^{k+2}(\Omega, \mathcal{S}), \tag{3.36}$$

there exists

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \lesssim h^{k+2}(|\boldsymbol{\sigma}|_{k+2} + |\boldsymbol{u}|_{k+1}),$$

where $(\boldsymbol{\sigma}_h, \check{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h, \check{\boldsymbol{u}}_h)$ *is the solution of the discontinuous Galerkin formulation* (3.5) *in* $\boldsymbol{\Sigma}_h^{k+1} \times \check{\boldsymbol{\Sigma}}_h^k \times \boldsymbol{V}_h^k \times \check{\boldsymbol{V}}_h^{k+1}$*. This means that Assumption 3.1 guarantees the superclose property* (3.22)*, which implies the superconvergence of the postprocessed approximation* $\boldsymbol{u}_h^*$ *following the analysis of Theorem 3.5. The numerical results in Table 6 and 7 show that* $\boldsymbol{u}_h^*$ *converges at the same rate as* $\boldsymbol{u}_h$ *for* $k < n$*. This implies that Assumption 3.1 is not true for* $k < n$*, namely there exists no such* $H(\mathrm{div})$*-conforming projection for low order discrete spaces.*

Remark 3.3. *For a general mixed discontinuous Galerkin formulation* (3.5) *with the conditions (A1)-(A3), if there holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \lesssim h^r \inf_{\boldsymbol{q}_h \in \boldsymbol{\Sigma}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h} (\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathrm{div,h}} + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{0,h}),$$

and

$$\|(I - \boldsymbol{P}_h)\phi\|_0 \lesssim h^t |\phi|_2, \quad \|(I - \boldsymbol{\Pi}_h)(A^{-1}\epsilon(\phi))\|_0 \lesssim h^s |\phi|_2.$$

Then, a similar analysis proves the superconvergence result

$$\|\boldsymbol{u} - \boldsymbol{u}_h^*\|_0 \lesssim h^{\min(s+r,t)} \inf_{\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h} (\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathrm{div,h}} + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{0,h}).$$

4 Numerical Tests

In this section, some numerical experiments in 2D are presented to verify the estimate provided in Theorem 2.2, 2.3, 3.4 and 3.5.

4.1 Example 1: scalar elliptic problems

We consider the model problem (1.1) on the unit square $\Omega = (0,1)^2$ with

$$u = \sin(\pi x)\sin(\pi y),$$

and set f and g to satisfy the above exact solution of (1.1). The domain is partitioned by uniform triangles. The level one triangulation $\mathcal{T}_1$ consists of two right triangles, obtained by cutting the unit square with a north-east line. Each triangulation $\mathcal{T}_i$ is refined into a half-sized triangulation uniformly, to get a higher level triangulation $\mathcal{T}_{i+1}$.

Consider the four-field formulation (2.5) with $\eta = h_e^{-1}$, $\tau = h_e$, $\gamma = 1$ and

$$\boldsymbol{Q}_h = \boldsymbol{Q}_h^{\alpha_1},\ \check{\boldsymbol{Q}}_h = \check{\boldsymbol{Q}}_h^{\alpha_2},\ V_h = V_h^{\alpha_3},\ \check{V}_h = \check{V}_h^{\alpha_4},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfies $\alpha_1 = \alpha_4 = k+1, \alpha_2 = \alpha_3 = k$ for $k = 0, 1$ and 2. According to Lemma 2.1, these formulations are well posed. Denote the corresponding solution by $(\boldsymbol{p}_h, \check{\boldsymbol{p}}_h, u_h, \check{u}_h)$.

Table 1 - 3 record the errors $\|u - u_h\|_0$, $\|\mathrm{P}_h u - u_h\|_0$, $\|u - u_{1,h}^*\|_0$, $\|u - u_{2,h}^*\|_0$ and the corresponding convergence rates for the aforementioned four-field formulations (2.5) with $\mathrm{P}_h = \mathrm{P}_h^0$ in (2.10) and (2.11). It reveals in these tables that $\|\mathrm{P}_h u - u_h\|_0$, $\|u - u_{1,h}^*\|_0$ and $\|u - u_{2,h}^*\|_0$ converge at the same

	$\|u-u_h\|_0$	rates	$\|u_h-\mathrm{P}_h u\|_0$	rates	$\|u-u_{1,h}^*\|_0$	rates	$\|u-u_{2,h}^*\|_0$	rates
$\mathcal{T}_3$	1.45E-01	0.92	6.81E-02	0.93	6.91E-02	1.00	6.87E-02	1.04
$\mathcal{T}_4$	6.89E-02	1.08	2.26E-02	1.59	2.27E-02	1.61	2.26E-02	1.60
$\mathcal{T}_5$	3.33E-02	1.05	6.24E-03	1.86	6.25E-03	1.86	6.24E-03	1.86
$\mathcal{T}_6$	1.64E-02	1.02	1.62E-03	1.95	1.62E-03	1.95	1.62E-03	1.95
$\mathcal{T}_7$	8.19E-03	1.00	4.11E-04	1.98	4.11E-04	1.98	4.11E-04	1.98
$\mathcal{T}_8$	4.09E-03	1.00	1.03E-04	1.99	1.03E-04	1.99	1.03E-04	1.99

Table 1: Superconvergence for the scalar elliptic problem using $\alpha = (1, 0, 0, 1)$.

	$\|u-u_h\|_0$	rates	$\|u_h-\mathrm{P}_h u\|_0$	rates	$\|u-u_{1,h}^*\|_0$	rates	$\|u-u_{2,h}^*\|_0$	rates
$\mathcal{T}_3$	1.96E-02	1.95	1.99E-03	3.35	2.41E-03	3.42	2.22E-03	3.52
$\mathcal{T}_4$	4.95E-03	1.98	1.43E-04	3.80	1.68E-04	3.84	1.49E-04	3.89
$\mathcal{T}_5$	1.24E-03	1.99	9.40E-06	3.92	1.10E-05	3.94	9.63E-06	3.96
$\mathcal{T}_6$	3.11E-04	2.00	6.02E-07	3.97	7.00E-07	3.97	6.11E-07	3.98
$\mathcal{T}_7$	7.78E-05	2.00	3.80E-08	3.98	4.42E-08	3.99	3.85E-08	3.99

Table 2: Superconvergence for the scalar elliptic problem using $\alpha = (2, 1, 1, 2)$.

	$\|u-u_h\|_0$	rates	$\|u_h-\mathrm{P}_h u\|_0$	rates	$\|u-u_{1,h}^*\|_0$	rates	$\|u-u_{2,h}^*\|_0$	rates
$\mathcal{T}_3$	2.17E-03	2.92	8.89E-05	4.42	3.48E-04	3.91	1.55E-04	3.78
$\mathcal{T}_4$	2.75E-04	2.98	3.20E-06	4.80	1.34E-05	4.70	6.07E-06	4.68
$\mathcal{T}_5$	3.45E-05	2.99	1.06E-07	4.92	4.50E-07	4.90	2.04E-07	4.89
$\mathcal{T}_6$	4.31E-06	3.00	3.39E-09	4.97	1.44E-08	4.96	6.56E-09	4.96
$\mathcal{T}_7$	5.39E-07	3.00	1.07E-10	4.98	4.57E-10	4.98	2.08E-10	4.98

Table 3: Superconvergence for the scalar elliptic problem using $\alpha = (3, 2, 2, 3)$.

rate 2.00 if $\alpha = (1,0,0,1)$, 4.00 if $\alpha = (2,1,1,2)$ and 5.00 if $\alpha = (3,2,2,3)$, which coincides with the analysis in Theorem 2.2 and Theorem 2.3. The comparison between $\|u - u_{1,h}^*\|_0$ and $\|u - u_{2,h}^*\|_0$ shows that the postprocessing approximations $u_{2,h}^*$ admit a slightly higher accuracy than $u_{1,h}^*$. It is analyzed in [25] that the four-field formulation (2.5) with $\eta = \tau^{-1}$ and $\gamma = 0$ is hybridizable. For this formulation, the postprocess technique (2.11) with $\tilde{\boldsymbol{p}}_h = \hat{\boldsymbol{p}}_h$ is better than the other one (2.10) in two aspects, one is the higher accuracy of $u_{2,h}^*$ and the other one is that there is no need to solve $\boldsymbol{p}_h$ from the reduced formulation.

4.2 Example 2: linear elasticity problems

We consider the linear elasticity problem (1.2) on the unit square $\Omega = (0,1)^2$ with the exact displacement

$$\boldsymbol{u} = (\sin(\pi x)\sin(\pi y), \sin(\pi x)\sin(\pi y))^T,$$

and set f and g are chosen corresponding to the above exact solution of (1.2) with $E = 1$ and $\nu = 0.4$. The domain is partitioned by uniform triangles. The level one triangulation $\mathcal{T}_1$ consists of two right triangles, obtained by cutting the unit square with a north-east line. Each triangulation $\mathcal{T}_i$ is refined into a half-sized triangulation uniformly, to get a higher level triangulation $\mathcal{T}_{i+1}$. For this numerical tests, fix the parameters $\rho_1 = \rho_2 = \gamma = 1$.

	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0$	rates	$\|\boldsymbol{u}_h - \mathbf{P}_h\boldsymbol{u}\|_0$	rates	$\|\boldsymbol{u} - \boldsymbol{u}_h^*\|_0$	rates
$\mathcal{T}_1$	9.87E-02	-	2.90E-02	-	3.68E-02	-
$\mathcal{T}_2$	2.37E-02	2.06	5.65E-03	2.36	6.83E-03	2.43
$\mathcal{T}_3$	3.08E-03	2.94	3.71E-04	3.93	3.74E-04	4.19
$\mathcal{T}_4$	3.89E-04	2.99	1.51E-05	4.62	1.43E-05	4.71
$\mathcal{T}_5$	4.87E-05	3.00	5.17E-07	4.87	4.79E-07	4.90
$\mathcal{T}_6$	6.10E-06	3.00	1.67E-08	4.95	1.54E-08	4.96

Table 4: Superconvergence for the elasticity problem with $\alpha = (3,2,2,3)$

	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0$	rates	$\|\boldsymbol{u}_h - \mathbf{P}_h\boldsymbol{u}\|_0$	rates	$\|\boldsymbol{u} - \boldsymbol{u}_h^*\|_0$	rates
$\mathcal{T}_1$	7.17E-02	-	2.51E-02	-	3.47E-02	-
$\mathcal{T}_2$	4.12E-03	4.12	7.52E-04	5.06	8.82E-04	5.30
$\mathcal{T}_3$	2.69E-04	3.94	2.27E-05	5.05	2.28E-05	5.27
$\mathcal{T}_4$	1.70E-05	3.98	5.76E-07	5.30	5.29E-07	5.43
$\mathcal{T}_5$	1.06E-06	4.00	1.13E-08	5.68	1.01E-08	5.71

Table 5: Superconvergence for the elasticity problem with $\alpha = (4,3,3,4)$

Table 4 and 5 list the errors $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$, $\|\mathbf{P}_h\boldsymbol{u}-\boldsymbol{u}_h\|_0$, $\|\boldsymbol{u}-\boldsymbol{u}_h^*\|_0$ and the corresponding convergence rates of the discontinuous Galerkin formulation (3.5) with $k \geq n$ for elasticity problem (1.2). It is shown that both $\|\mathbf{P}_h\boldsymbol{u}-\boldsymbol{u}_h\|_0$ and $\|\boldsymbol{u}-\boldsymbol{u}_h^*\|_0$ of the discontinuous Galerkin formulation (3.5) with $k=2$ and $k=3$ converge at the rates 5.00 and 6.00, respectively. This verifies that error estimates in Theorem 3.5.

We also test the postprocessing scheme (3.31) on the formulation (3.5) with $k<n$, namely,

$$\alpha = (1,0,0,1) \quad \text{and} \quad (2,1,1,2),$$

where the results are listed in Table 6 and 7, respectively. It shows that postprocessing solution $\boldsymbol{u}_h^*$ converges at the same rate as the finite element solution $\boldsymbol{u}_h$, which is $k+1$ for the case $k<n$. This implies that there is no such $H(\mathrm{div})$-conforming projection that admits the commuting diagram (3.10).

	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	rates	$\|\boldsymbol{u}_h-\mathbf{P}_h\boldsymbol{u}\|_0$	rates	$\|\boldsymbol{u}-\boldsymbol{u}_h^*\|_0$	rates
$\mathcal{T}_2$	4.25E-01	-	2.50E-01	-	3.45E-01	-
$\mathcal{T}_3$	2.13E-01	0.99	1.12E-01	1.16	1.29E-01	1.42
$\mathcal{T}_4$	1.00E-01	1.09	3.89E-02	1.53	4.61E-02	1.48
$\mathcal{T}_5$	4.83E-02	1.05	1.41E-02	1.46	1.84E-02	1.33
$\mathcal{T}_6$	2.39E-02	1.02	6.06E-03	1.22	8.39E-03	1.13
$\mathcal{T}_7$	1.19E-02	1.00	2.88E-03	1.07	4.08E-03	1.04
$\mathcal{T}_8$	5.96E-03	1.00	1.42E-03	1.02	2.03E-03	1.01

Table 6: Superconvergence for the elasticity problem with $\alpha = (1,0,0,1)$

	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	rates	$\|\boldsymbol{u}_h-\mathbf{P}_h\boldsymbol{u}\|_0$	rates	$\|\boldsymbol{u}-\boldsymbol{u}_h^*\|_0$	rates
$\mathcal{T}_1$	4.74E-01	-	3.08E-01	-	4.00E-01	-
$\mathcal{T}_2$	1.11E-01	2.09	4.11E-02	2.90	5.27E-02	2.92
$\mathcal{T}_3$	2.85E-02	1.97	7.20E-03	2.51	8.57E-03	2.62
$\mathcal{T}_4$	7.22E-03	1.98	1.77E-03	2.02	1.89E-03	2.18
$\mathcal{T}_5$	1.82E-03	1.99	4.56E-04	1.96	4.64E-04	2.03
$\mathcal{T}_6$	4.55E-04	2.00	1.16E-04	1.98	1.16E-04	2.00
$\mathcal{T}_7$	1.14E-04	2.00	2.91E-05	1.99	2.91E-05	2.00

Table 7: Superconvergence for the elasticity problem with $\alpha = (2,1,1,2)$

Acknowledgements

The author wishes to thank the partial support from the Center for Computational Mathematics and Applications, the Pennsylvania State University. The author would also like to thank Professor Jinchao Xu from the Pennsylvania State University and Professor Jun Hu from the Peking University for their guidance and helpful suggestions pertaining to this work.

References

[1] D. N. Arnold, G. Awanou, and R. Winther, *Finite elements for symmetric tensors in three dimensions*, Mathematics of Computation, 77 (2008), pp. 1229–1251.

[2] D. N. Arnold and F. Brezzi, *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, ESAIM: Mathematical Modelling and Numerical Analysis, 19 (1985), pp. 7–32.

[3] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM Journal on Numerical Analysis, 39 (2002), pp. 1749–1779.

[4] D. N. Arnold and R. Winther, *Mixed finite elements for elasticity*, Numerische Mathematik, 92 (2002), pp. 401–419.

[5] R. E. Bank and Y. Li, *Superconvergent recovery of Raviart–Thomas mixed finite elements on triangular grids*, Journal of Scientific Computing, 81 (2019), pp. 1882–1905.

[6] J. H. Bramble and J. Xu, *A local post-processing technique for improving the accuracy in mixed finite-element approximations*, SIAM journal on numerical analysis, 26 (1989), pp. 1267–1275.

[7] J. H. Brandts, *Superconvergence and a posteriori error estimation for triangular mixed finite elements*, Numerische Mathematik, 68 (1994), pp. 311–324.

[8] J. H. Brandts, *Superconvergence for triangular order k=1 Raviart-Thomas mixed finite elements and for triangular standard quadratic finite element methods*, Applied Numerical Mathematics, 34 (2000), pp. 39–58.

[9] C. Chen and Y. Huang, *High Accuracy Theory of Finite Element Methods*, 1995.

[10] H. Chen and B. Li, *Superconvergence analysis and error expansion for the Wilson nonconforming finite element*, Numerische Mathematik, 69 (2013), pp. 125–140.

[11] B. Cockburn and G. Fu, *Devising superconvergent HDG methods with symmetric approximate stresses for linear elasticity by M-decompositions*, IMA Journal of Numerical Analysis, 38 (2018), pp. 566–604.

[12] B. Cockburn, J. Gopalakrishnan, and J. Guzmán, *A new elasticity element made for enforcing weak stress symmetry*, Mathematics of computation, 79 (2010), pp. 1331–1349.

[13] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, *Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems*, SIAM Journal on Numerical Analysis, 47 (2009), pp. 1319–1365.

[14] B. Cockburn, J. Guzmán, and H. Wang, *Superconvergent discontinuous Galerkin methods for second-order elliptic problems*, Mathematics of Computation, 78 (2009), pp. 1–24.

[15] B. Cockburn, W. Qiu, and K. Shi, *Conditions for superconvergence of HDG methods for second-order elliptic problems*, Mathematics of Computation, 81 (2012), pp. 1327–1353.

[16] B. Cockburn and K. Shi, *Superconvergent HDG methods for linear elasticity with weakly symmetric stresses*, IMA Journal of Numerical Analysis, 33 (2013), pp. 747–770.

[17] J. Douglas and J. Wang, *Superconvergence of mixed finite element methods on rectangular domains*, Calcolo, 26 (1989), pp. 121–133.

[18] R. E. Bank and J. Xu, *Asymptotically exact a posteriori error estimators, part i: Grids with superconvergence*, Journal on Numerical Analysis, 41 (2003), pp. 2294–2312.

[19] L. Gastaldi and R. H. Nochetto, *Sharp maximum norm error estimates for general mixed finite element approximations to second order elliptic equations*, ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique, 23 (1989), pp. 103–128.

[20] J. Gopalakrishnan and J. Guzmán, *A second elasticity element using the matrix bubble*, IMA Journal of Numerical Analysis, 32 (2012), pp. 352–372.

[21] Q. Hong, J. Hu, L. Ma, and J. Xu, *An extended Galerkin analysis for linear elasticity with strongly symmetric stress tensor*, arXiv preprint arXiv:2002.11664, (2020).

[22] Q. Hong, J. Hu, S. Shu, and J. Xu, *A discontinuous Galerkin method for the fourth-order curl problem*, Journal of Computational Mathematics, (2012), pp. 565–578.

[23] Q. Hong, J. Kraus, J. Xu, and L. Zikatanov, *A robust multigrid method for discontinuous Galerkin discretizations of stokes and linear elasticity equations*, Numerische Mathematik, 132 (2016), pp. 23–49.

[24] Q. Hong, F. Wang, S. Wu, and J. Xu, *A unified study of continuous and discontinuous Galerkin methods*, Science China Mathematics, 62 (2019), pp. 1–32.

[25] Q. Hong, S. Wu, and J. Xu, *An Extended Galerkin Analysis for Elliptic Problems*, arXiv preprint arXiv:1908.08205v2, (2019).

[26] Q. Hong and J. Xu, *Uniform stability and error analysis for some discontinuous galerkin methods*, arXiv preprint arXiv:1805.09670, (2018).

[27] J. Hu, *Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^n$: the higher order case*, Journal of Computational Mathematics, 33 (2015), pp. 1–14.

[28] J. Hu, L. Ma, and R. Ma, *Optimal superconvergence analysis for the Crouzeix-Raviart and the Morley elements*, arXiv preprint arXiv:1808.09810, (2018).

[29] J. Hu and R. Ma, *Superconvergence of both the Crouzeix-Raviart and Morley elements*, Numerische Mathematik, 132 (2016), pp. 491–509.

[30] J. Hu and S. Zhang, *A family of conforming mixed finite elements for linear elasticity on triangular grids*, arXiv preprint arXiv:1406.7457, (2014).

[31] J. Hu and S. Zhang, *A family of symmetric mixed finite elements for linear elasticity on tetrahedral grids*, Science China Mathematics, 58 (2015), pp. 297–307.

[32] J. Hu and S. Zhang, *Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^n$: The lower order case*, Mathematical Models and Methods in Applied Sciences, 26 (2016), pp. 1649–1669.

[33] J. Hu and S. Zhang, *Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^n$: the lower order case*, Mathematical Models and Methods in Applied Sciences, 26 (2016), pp. 1649–1669.

[34] R. Stenberg, *A family of mixed finite elements for the elasticity problem*, Numerische Mathematik, 53 (1988), pp. 513–538.

[35] R. Stenberg, *Postprocessing schemes for some mixed finite elements*, ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique, 25 (1991), pp. 151–167.

[36] F. Wang, S. Wu, and J. Xu, *A mixed discontinuous Galerkin method for linear elasticity with strongly imposed symmetry*, arXiv preprint arXiv:1902.08717, (2019).

[37] Z. Xie, Z. Zhang, and Z. Zhang, *A numerical study of uniform superconvergence of LDG method for solving singularly perturbed problems*, Journal of Computational Mathematics, (2009), pp. 280–298.

[38] J. Xu, *Iterative methods by space decomposition and subspace correction*, SIAM review, 34 (1992), pp. 581–613.