Second-order and nonuniform time-stepping schemes for time fractional evolution equations with time-space dependent coefficients

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Abstract

The numerical analysis of time fractional evolution equations with the second-order elliptic operator including general time-space dependent variable coefficients is challenging, especially when the classical weak initial singularities are taken into account. In this paper, we introduce a concise technique to construct efficient time-stepping schemes with variable time step sizes for two-dimensional time fractional sub-diffusion and diffusion-wave equations with general time-space dependent variable coefficients. By means of the novel technique, the nonuniform Alikhanov type schemes are constructed and analyzed for the sub-diffusion and diffusion-wave problems. For the diffusion-wave problem, our scheme is constructed by employing the recently established symmetric fractional-order reduction (SFOR) method. The unconditional stability of proposed schemes is rigorously discussed under mild assumptions on variable coefficients and, based on reasonable regularity assumptions and weak time mesh restrictions, the second-order convergence is obtained with respect to discrete H^1 -norm. Numerical experiments are given to demonstrate the theoretical statements.

Key words: time fractional evolution equations; variable coefficients; weak singularity; nonuniform mesh

AMS subject classifications: 65M06; 65M12; 35B65; 35R11

1 Introduction

Fractional differential equations (FDEs) are extremely powerful mathematical tools for the modeling of diverse processes and phenomena which contain memory and hereditary properties, interested readers may refer to [5,21,22,24] and references therein for some practical applications of FDEs in physics, biology and chemistry, etc. Finding efficient and accurate numerical solutions of FDEs becomes an increasingly hot research topic as it is hard to obtain the reliable analytic solution in general.

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In this work, we consider numerical analysis of the two-dimensional time fractional evolution equations with general time-space dependent variable coefficients:

$$\mathcal{D}_t^{\alpha} u = \mathcal{A}u + f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \ t \in (0, T]; \tag{1.1}$$

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad \text{if} \quad \alpha \in (0, 1);$$
 (1.2)

$$u(\mathbf{x}, 0) = \phi(\mathbf{x}), \ u_t(\mathbf{x}, 0) = \psi(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad \text{if} \quad \alpha \in (1, 2);$$
 (1.3)

subject to the homogeneous boundary condition $u(\mathbf{x},t) = 0$ for $(\mathbf{x},t) \in \partial\Omega \times (0,T]$, where $\Omega = (x_l, x_r) \times (y_l, y_r)$, $\mathbf{x} = (x, y)$ and \mathcal{A} is a linear second-order elliptic operator which is dependent on time and space:

$$\mathcal{A}u := \left\{ a_1(\mathbf{x}, t)\partial_{xx}^2 + a_2(\mathbf{x}, t)\partial_{yy}^2 + b_1(\mathbf{x}, t)\partial_x + b_2(\mathbf{x}, t)\partial_y + b_3(\mathbf{x}, t) \right\} u. \tag{1.4}$$

The fractional derivative \mathcal{D}_t^{α} in (1.1) is defined by the Caputo sense:

$$\mathcal{D}_t^{\alpha} u(t) := \int_0^t \omega_{n-\alpha}(t-s) u^{(n)}(s) \, \mathrm{d}s \quad \text{with} \quad \omega_{n-\alpha}(t) = \frac{t^{n-1-\alpha}}{\Gamma(n-\alpha)}, \quad n = \lceil \alpha \rceil, \quad t > 0.$$

With error analyses basing on sufficient smoothness in time of the analytical solutions, various numerical methods are designed for fractional sub-diffusion or diffusion-wave equations with variable coefficients which are time-space dependent (e.g. [31]) or only space dependent (e.g. [4,26,28,30,32,33]). However, it is well-known that the solution of time fractional initial value problems typically exhibits weak initial singularities. Thus most of the traditional time-stepping methods fail to preserve the desired convergence rates in this general and practical situation. In [10], Kopteva discussed the L1-type discretizations on graded time meshes for fractional parabolic equation with classical weak singular solutions, where the second-order elliptic operator $\mathcal{L}u = \sum_{k=1}^{d} \{-\partial_{x_k} a_k(x) \partial_{x_k} u + b_k(x) \partial_{x_k} u\} + c(x)u \ (d=1,2,3)$ is only space dependent. A secondorder convergent method was studied lately in Wei et al. [29], where the Alikhanov formula [1] on the graded time meshes is considered to deal with the weak initial singularity of the twodimensional time fractional diffusion equations with the elliptic operator $\mathcal{L}u = \operatorname{div}(a(\mathbf{x})\nabla u)$ which is symmetric and space dependent only. Recently, the sub-diffusion problems with timespace dependent coefficients and nonsmooth data were studied in several research works. In [23], Mustapha studied a semidiscrete Galerkin finite element method for the time fractional diffusion equations with time-space dependent diffusivity coefficient:

$$\mathcal{D}_t^{\alpha} u(\mathbf{x}, t) = \operatorname{div}(a(\mathbf{x}, t) \nabla u(\mathbf{x}, t)) + f(\mathbf{x}, t) \quad \text{in } \Omega \times (0, T], \ \alpha \in (0, 1),$$
(1.5)

where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$, and the optimal error bounds in L^2 - and H^1 -norms are obtained for both smooth and nonsmooth initial data. Jin, Li and Zhou [8] then proposed an efficient numerical scheme with the Galerkin finite element method in space and backward Euler convolution quadrature in time for the problem (1.5). The optimal convergence with first-order temporal accuracy is obtained provided a certain regularity of the solutions is proved for both nonsmooth initial data and incompatible source term. The second-order temporal convergence was further achieved for the convolution quadrature generated by second-order backward differentiation formula with proper correction at the first time step [9], where an improved regularity was shown. We remark that, based on some mild and natural assumptions on $a(\mathbf{x},t)$, the time-space dependent elliptic operator in (1.5) is symmetric and is a particular case (for d=2) of \mathcal{A} in (1.4) (or L in [31]) because $\operatorname{div}(a(\mathbf{x},t)\nabla u(\mathbf{x},t)) = a(\mathbf{x},t)\Delta u(\mathbf{x},t) + \nabla a(\mathbf{x},t) \cdot \nabla u(\mathbf{x},t)$. There have been many works on the theoretical and numerical study of classical parabolic and hyperbolic

equations with general time-space dependent elliptic operator, e.g. [2,11,18]. To the best of our knowledge, taking the weak initial singularity into account, there is no study on the efficient numerical methods for time fractional evolution equations (sub-diffusion and diffusion-wave) where the elliptic operators include general time-space dependent coefficients, i.e., the elliptic operators take the form (1.4).

In the past few years, numerical methods on nonuniform time meshes are found to be very efficient and thus are of great interests in resolving the weak initial singularities of the time fractional initial value problems [3, 6, 10, 13, 15–17, 20, 25, 29]. As the operator \mathcal{A} in (1.4) is in general non-symmetric and is substantially different from the one in [12], this brings challenges in the analysis of the standard nonuniform approximations of (1.1)–(1.3). To tackle the problem, we will introduce a novel and concise technique to study highly accurate numerical methods for the time fractional evolution equations with general time-space dependent coefficients on nonuniform time meshes. By the proposed technique, an important estimate, i.e. the inequality (2.2), can be guaranteed in the analysis of corresponding nonuniform algorithms. Our numerical schemes will utilize the Alikhanov formula on possible nonuniform time meshes to approximate the Caputo derivatives. We recall that, for a given positive integer N, the Alikhanov formulas for the Caputo derivative $\mathcal{D}_t^{\beta} g(t_{n-\theta})$ (0 < β < 1) on arbitrary time meshes $0 = t_0 < t_1 < \cdots < t_N = T$ is expressed by the following summation of convolution structure [15]:

$$(\mathcal{D}_{\tau}^{\beta}g)^{n-\theta} := \sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_{\tau} g^{k}, \text{ where } g^{k} = g(t_{k}) \text{ and } \nabla_{\tau} g^{k} = g^{k} - g^{k-1},$$
 (1.6)

where $\theta := \beta/2$. For simplicity of presentation, the precise formulation of the coefficients $A_{n-k}^{(n)}$ and its corresponding properties are given in Appendix (Subsection 7.1). To study the diffusion-wave problem, we further employ a symmetric fractional-order reduction (SFOR) method which was investigated in our very recent work [20].

Remark 1.1. In the rest of this paper, we always take the setting:

$$\beta = \left\{ \begin{array}{ll} \alpha, & \text{if } \alpha \in (0,1), \text{ i.e., while concerning the sub-diffusion problem;} \\ \alpha/2, & \text{if } \alpha \in (1,2), \text{ i.e., while concerning the diffusion-wave problem.} \end{array} \right.$$

In the construction and analysis of our proposed numerical methods, the variable coefficients involved in \mathcal{A} are assumed to satisfy two generic conditions: For $\mathbf{x} \in \Omega$, $t \in [0, T]$,

V1.
$$a_k(\mathbf{x},t) > 0$$
, and $a_k(\cdot,t) \in \mathcal{C}^1([0,T])$ with $|(a_1)_t/a_1| + |(a_2)_t/a_2| \leq C_p$, for $k = 1, 2$;

V2.
$$a_k(\mathbf{x}, \cdot) \in \mathcal{C}^3(\Omega)$$
 for $k = 1, 2$, and $|b_l(\mathbf{x}, t)| \leq C_l$ for $l = 1, 2, 3$,

where C_p and C_l are positive constants. We will obtain the second-order H^1 -norm convergence (in time and space) of the proposed nonuniform schemes for both the sub-diffusion and diffusion-wave problems under the following assumptions on regularity (C_u is a positive constant): For $t \in (0,T]$ and k = 1,2,3,

$$||u||_{H^4(\Omega)} \le C_u$$
, for $\alpha \in (0,1) \cup (1,2)$; (1.7)

$$\|\partial_t^{(k)} u\|_{H^3(\Omega)} \le C_u(1 + t^{\sigma_1 - k}), \quad \text{if } \alpha \in (0, 1);$$
 (1.8)

$$\|\partial_t^{(k)} u\|_{H^3(\Omega)} \le C_u(1+t^{\sigma_2-k}), \quad \|\partial_t^{(k)} v\|_{H^3(\Omega)} \le C_u(1+t^{\sigma_3-k}), \quad \text{if } \alpha \in (1,2),$$

$$(1.9)$$

where $v := \mathcal{D}_t^{\beta} \tilde{u}$ with $\tilde{u} := u - t\psi$, $\sigma_1 \in (0,1) \cup (1,2)$, $\sigma_2 \in (1,2) \cup (2,3)$ and $\sigma_3 \in (1/2,1) \cup (1,2)$; Furthermore, we impose the weak mesh assumption:

MA. There is a constant $C_{\gamma} > 0$ such that $\tau_k \leq C_{\gamma}\tau \min\{1, t_k^{1-1/\gamma}\}$ for $1 \leq k \leq N$, with $t_k \leq C_{\gamma}t_{k-1}$ and $\tau_k/t_k \leq C_{\gamma}\tau_{k-1}/t_{k-1}$ for $2 \leq k \leq N$,

where $\gamma \geq 1$ is the mesh parameter, $\tau_k := t_k - t_{k-1}$ denotes the k-th time step size and $\tau := \max_{1 \leq k \leq N} \{\tau_k\}.$

The rest of the paper is organized as follows. In Section 2, we will introduce a concise technique which is used to construct and analyze nonuniform schemes for the governing problems. In Section 3, the numerical scheme which based on the nonuniform Alikhanov formula is proposed for sub-diffusion equation with general variable coefficients, and its stability and second-order convergence are rigorously discussed with respect to discrete H^1 -norm. In Section 4, by applying the SFOR method, the nonuniform Alikhanov type scheme is constructed for the diffusion-wave equation with general variable coefficients. We also show that the scheme is stable and second-order convergent in the discrete H^1 -norm. Numerical examples are given in Section 5 to demonstrate the theoretical statements. As an appendix, in Section 7, the precise definitions of the coefficients of Alikhanov formula, the proof of inequality (2.2) and the analysis of truncation errors are given.

2 A technique for numerical analysis

In this section, we will present a concise technique to study numerical schemes with variable time step sizes for time fractional evolution equations with general time-space dependent variable coefficients.

Firstly, we show an important lemma which extends the one in [14, Lemma 4.1].

Lemma 2.1. For a continuous (w.r.t. x and t) function q(x,t) > 0, $x \in (x_l, x_r) \subset \mathbb{R}$, $t \in [0,T]$, we define a diagonal matrix

$$\mathbf{Q}^{(k)} := \operatorname{diag}(q(x_1, t_k), q(x_2, t_k), \cdots, q(x_m, t_k)), \quad m \ge 1, \ k \ge 0,$$

where $t_k \in [0,T]$ with $t_j < t_{j+1}$, and $x_i \in (x_l, x_r)$. Let $\mathbf{z}^k := (z_1^k, z_2^k, \cdots, z_m^k)^T$ be a real vector, and $\mathbf{z}^{n-\theta} := (1-\theta)\mathbf{z}^n + \theta\mathbf{z}^{n-1}$. Then

$$(\mathbf{z}^{n-\theta})^T \mathbf{Q}^{(n)} (\mathcal{D}_{\tau}^{\beta} \mathbf{z})^{n-\theta} \ge \frac{1}{2} \sum_{k=1}^n A_{n-k}^{(n)} \nabla_{\tau} [(\mathbf{z}^k)^T \mathbf{Q}^{(n)} \mathbf{z}^k]. \tag{2.1}$$

Moreover, if q(x,t) is non-increasing w.r.t. t for every fixed x, it holds that

$$(\mathbf{z}^{n-\theta})^T \mathbf{Q}^{(n)} (\mathcal{D}_{\tau}^{\beta} \mathbf{z})^{n-\theta} \ge \frac{1}{2} \sum_{k=1}^n A_{n-k}^{(n)} \nabla_{\tau} [(\mathbf{z}^k)^T \mathbf{Q}^{(k)} \mathbf{z}^k]. \tag{2.2}$$

Proof. The inequality (2.1) can be verified according to [14, Lemma 4.1], we move its derivation to the Appendix (Subsection 7.2).

If q(x,t) is non-increasing w.r.t. t for every fixed x, we have $(\mathbf{z}^k)^T \mathbf{Q}^{(n)} \mathbf{z}^k \leq (\mathbf{z}^k)^T \mathbf{Q}^{(k)} \mathbf{z}^k$

while $k \leq n$. Then

$$\begin{split} & (\mathbf{z}^{n-\theta})^T \mathbf{Q}^{(n)} (\mathcal{D}_{\tau}^{\beta} \mathbf{z})^{n-\theta} \\ & \geq \frac{1}{2} \sum_{k=1}^n A_{n-k}^{(n)} \nabla_{\tau} [(\mathbf{z}^k)^T \mathbf{Q}^{(n)} \mathbf{z}^k] \\ & = \frac{1}{2} \left[A_0^{(n)} (\mathbf{z}^n)^T \mathbf{Q}^{(n)} \mathbf{z}^n - \sum_{k=1}^{n-1} (A_{n-k-1}^{(n)} - A_{n-k}^{(n)}) (\mathbf{z}^k)^T \mathbf{Q}^{(n)} \mathbf{z}^k - A_{n-1}^{(n)} (\mathbf{z}^0)^T \mathbf{Q}^{(n)} \mathbf{z}^0 \right] \\ & \geq \frac{1}{2} \left[A_0^{(n)} (\mathbf{z}^n)^T \mathbf{Q}^{(n)} \mathbf{z}^n - \sum_{k=1}^{n-1} (A_{n-k-1}^{(n)} - A_{n-k}^{(n)}) (\mathbf{z}^k)^T \mathbf{Q}^{(k)} \mathbf{z}^k - A_{n-1}^{(n)} (\mathbf{z}^0)^T \mathbf{Q}^{(0)} \mathbf{z}^0 \right] \\ & = \frac{1}{2} \sum_{k=1}^n A_{n-k}^{(n)} \nabla_{\tau} [(\mathbf{z}^k)^T \mathbf{Q}^{(k)} \mathbf{z}^k]. \end{split}$$

Next, we will utilize Lemma 2.1 to obtain some properties for our numerical analysis. For a continuous (w.r.t \mathbf{x} and t) function $p(\mathbf{x},t) > 0$, suppose $p_1 = pa_1$ and $p_2 = pa_2$, we have

$$\mathcal{A}u = p^{-1}(p\mathcal{A}u) = p^{-1} \left\{ p_1 \partial_{xx}^2 + p_2 \partial_{yy}^2 + pb_1 \partial_x + pb_2 \partial_y + pb_3 \right\} u$$

$$= p^{-1} \left\{ \partial_x (p_1 \partial_x u) + \partial_y (p_2 \partial_y u) + [pb_1 - (p_1)_x] \partial_x u + [pb_2 - (p_2)_y] \partial_y u + pb_3 u \right\}$$

$$= p^{-1} \left[\partial_x (p_1 \partial_x u) + \partial_y (p_2 \partial_y u) \right] + [b_1 - p^{-1}(p_1)_x] \partial_x u + [b_2 - p^{-1}(p_2)_y] \partial_y u + b_3 u.$$

Some spatial notations are required. For two positive integers M_x and M_y , denote $h_x := (x_r - x_l)/M_x$ and $h_y := (y_r - y_l)/M_y$. Define the mesh space $\Omega_h := \{\mathbf{x}_h = (x_l + ih_x, y_l + jh_y)|1 \le i \le M_x - 1, 1 \le j \le M_y - 1\}$ and $\bar{\Omega}_h := \Omega_h \cup \partial \Omega$. For any grid functions $u_h := \{u_{i,j} = u(x_i, y_j) | (x_i, y_j) \in \bar{\Omega}_h\}$, the central difference operators are given by

$$\delta_x u_{i+\frac{1}{2},j} := (u_{i+1,j} - u_{i,j})/h_x, \ 0 \le i \le M_x - 1; \quad \delta_{\hat{x}} u_{i,j} := (u_{i+1,j} - u_{i-1,j})/(2h_x), \ 1 \le i \le M_x - 1;$$

and $\delta_y u_{i,j+\frac{1}{2}}$, $\delta_{\hat{y}} u_{i,j}$ are defined similarly.

Denote $p_3 := b_1 - p^{-1}(p_1)_x$, $p_4 := b_2 - p^{-1}(p_2)_y$, the discrete function $p_h^{n-\theta} := p(\mathbf{x}_h, t_{n-\theta})$ $(0 \le n \le N)$ with $p_h^{-\theta} := p(\mathbf{x}_h, t_0)$, and we use similar notations for $(p_k)_h^{n-\theta}$ (k = 1, 2, 3, 4) and $(b_3)_h^{n-\theta}$. Then we define a discrete operator corresponding to \mathcal{A} :

$$\mathcal{A}_h^{n-\theta} := (p_h^{n-\theta})^{-1} \left\{ \delta_x [(p_1)_h^{n-\theta} \delta_x] + \delta_y [(p_2)_h^{n-\theta} \delta_y] \right\} + (p_3)_h^{n-\theta} \delta_{\hat{x}} + (p_4)_h^{n-\theta} \delta_{\hat{y}} + (b_3)_h^{n-\theta}.$$

Since the numerical schemes and corresponding analysis in the next two sections will be done in

matrix form, we define the following matrices (the symbol '&' denotes the Kronecker product)

$$\begin{split} \mathbf{P}^{n-\theta} &:= \mathrm{diag}(p_{1,1}^{n-\theta}, \cdots, p_{M_{x-1,1}}^{n-\theta}, p_{1,2}^{n-\theta}, \cdots, p_{M_{x-1,2}}^{n-\theta}, \cdots, p_{1,M_{y-1}}^{n-\theta}, \cdots, p_{M_{x-1},M_{y-1}}^{n-\theta}), \\ \mathbf{P}_{1}^{n-\theta} &:= \mathrm{diag}((p_{1})_{1/2,1}^{n-\theta}, \cdots, (p_{1})_{M_{x-1}/2,1}^{n-\theta}, (p_{1})_{1/2,2}^{n-\theta}, \cdots, (p_{1})_{M_{x-1}/2,2}^{n-\theta}, \\ & \qquad \cdots, (p_{1})_{1/2,M_{y-1}}^{n-\theta}, \cdots, (p_{1})_{M_{x-1}/2,M_{y-1}}^{n-\theta}), \\ \mathbf{P}_{2}^{n-\theta} &:= \mathrm{diag}((p_{2})_{1,1/2}^{n-\theta}, \cdots, (p_{2})_{M_{x-1,1/2}}^{n-\theta}, (p_{2})_{1,3/2}^{n-\theta}, \cdots, (p_{2})_{M_{x-1},3/2}^{n-\theta}, \\ & \qquad \cdots, (p_{2})_{1,M_{y-1/2}}^{n-\theta}, \cdots, (p_{2})_{M_{x-1},M_{y-1/2}}^{n-\theta}), \\ \mathbf{A}^{n-\theta} &:= (I_{y} \otimes S_{x})^{T} \mathbf{P}_{1}^{n-\theta}(I_{y} \otimes S_{x}) + (S_{y} \otimes I_{x})^{T} \mathbf{P}_{2}^{n-\theta}(S_{y} \otimes I_{x}), \\ \mathbf{B}^{n-\theta} &:= \mathbf{P}_{3}^{n-\theta}[I_{y} \otimes (\hat{S}_{x} - \hat{S}_{x}^{T})] + \mathbf{P}_{4}^{n-\theta}[(\hat{S}_{y} - \hat{S}_{y}^{T}) \otimes I_{x}], \\ \mathbf{C}^{n-\theta} &:= \mathrm{diag}((b_{3})_{1,1}^{n-\theta}, \cdots, (b_{3})_{M_{x-1,1}}^{n-\theta}, (b_{3})_{1,2}^{n-\theta}, \cdots, (b_{3})_{M_{x-1,2}}^{n-\theta}, \\ & \qquad \cdots, (b_{3})_{1,M_{y}-1}^{n-\theta}, \cdots, (b_{3})_{M_{x-1},M_{y}-1}^{n-\theta}), \\ \mathbf{u}^{n} &:= (u_{1,1}^{n}, \cdots, u_{M_{x-1,1}}^{n}, u_{1,2}^{n}, \cdots, u_{M_{x-1,1}}^{n}, u_{1,2}^{n}, \cdots, u_{M_{x-1},M_{y}-1}^{n-\theta})^{T}, \end{split}$$

where $\mathbf{P}_3^{n-\theta}$ and $\mathbf{P}_4^{n-\theta}$, with entries coming from $(p_3)_{i,j}^{n-\theta}$ and $(p_4)_{i,j}^{n-\theta}$ respectively, are all $(M_x-1)(M_y-1)\times (M_x-1)(M_y-1)$ diagonal matrices defined similarly to $\mathbf{P}^{n-\theta}$, while I_x and I_y are (M_x-1) and (M_y-1) dimensional identity matrices respectively. Furthermore, we have used the notations

$$S_x := \frac{1}{h_x} \begin{bmatrix} -1 & & & & \\ 1 & -1 & & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}_{M_x \times (M_x - 1)}, \ \hat{S}_x := \frac{1}{2h_x} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix}_{(M_x - 1) \times (M_x - 1)}$$

and S_y , \hat{S}_y are defined in a similar way.

Therefore, if p is non-increasing w.r.t. t for every fixed \mathbf{x} , according to Lemma 2.1, we have

$$2(\mathbf{u}^{n-\theta})^T \mathbf{P}^{n-\theta} (\mathcal{D}_{\tau}^{\beta} \mathbf{u})^{n-\theta} \ge \sum_{k=1}^n A_{n-k}^{(n)} \nabla_{\tau} [(\mathbf{u}^k)^T \mathbf{P}^{k-\theta} \mathbf{u}^k].$$
 (2.3)

Moreover, taking

$$\mathbf{u}_x^{n-\vartheta} := (I_y \otimes S_x)\mathbf{u}^{n-\vartheta}, \quad \mathbf{u}_y^{n-\vartheta} := (S_y \otimes I_x)\mathbf{u}^{n-\vartheta}, \quad \text{where} \quad \vartheta = \theta \text{ or } 0, \tag{2.4}$$

if p_1 and p_2 are all non-increasing w.r.t. t for every fixed \mathbf{x} , we have

$$\begin{split} & 2(\mathbf{u}^{n-\theta})^T \mathbf{A}^{n-\theta} (\mathcal{D}_{\tau}^{\beta} \mathbf{u})^{n-\theta} \\ & \geq A_0^{(n)} (\mathbf{u}^n)^T \mathbf{A}^{n-\theta} \mathbf{u}^n - \sum_{k=1}^{n-1} (A_{n-k-1}^{(n)} - A_{n-k}^{(n)}) (\mathbf{u}^k)^T \mathbf{A}^{n-\theta} \mathbf{u}^k - A_{n-1}^{(n)} (\mathbf{u}^0)^T \mathbf{A}^{n-\theta} \mathbf{u}^0 \\ & = A_0^{(n)} \left[(\mathbf{u}_x^n)^T \mathbf{P}_1^{n-\theta} \mathbf{u}_x^n + (\mathbf{u}_y^n)^T \mathbf{P}_2^{n-\theta} \mathbf{u}_y^n \right] - \sum_{k=1}^{n-1} (A_{n-k-1}^{(n)} - A_{n-k}^{(n)}) \left[(\mathbf{u}_x^k)^T \mathbf{P}_1^{n-\theta} \mathbf{u}_x^k + (\mathbf{u}_y^k)^T \mathbf{P}_2^{n-\theta} \mathbf{u}_y^k \right] \\ & - A_{n-1}^{(n)} \left[(\mathbf{u}_x^n)^T \mathbf{P}_1^{n-\theta} \mathbf{u}_x^n + (\mathbf{u}_y^n)^T \mathbf{P}_2^{n-\theta} \mathbf{u}_y^n \right] - \sum_{k=1}^{n-1} (A_{n-k-1}^{(n)} - A_{n-k}^{(n)}) \left[(\mathbf{u}_x^k)^T \mathbf{P}_1^{k-\theta} \mathbf{u}_x^k + (\mathbf{u}_y^k)^T \mathbf{P}_2^{k-\theta} \mathbf{u}_y^k \right] \\ & - A_{n-1}^{(n)} \left[(\mathbf{u}_x^n)^T \mathbf{P}_1^{n-\theta} \mathbf{u}_x^n + (\mathbf{u}_y^n)^T \mathbf{P}_2^{n-\theta} \mathbf{u}_y^n \right]. \end{split} \tag{2.5}$$

The two inequalities (2.3) and (2.5) play critical roles in the analysis of our methods. Therefore, we must fulfill the following:

• find a continuous and positive function $p(\mathbf{x},t)$ which is non-increasing w.r.t. t for every fixed $\mathbf{x} \in \Omega$ such that $p_1(\mathbf{x},t) = pa_1$ and $p_2(\mathbf{x},t) = pa_2$ are all non-increasing w.r.t. $t \in [0,T]$ for every fixed $\mathbf{x} \in \Omega$.

The above task can be completed by choosing the candidates presented in the following lemma.

Lemma 2.2. For the positive variable coefficients a_1 and a_2 , consider

$$p(\mathbf{x},t) := \frac{d(\mathbf{x})e^{-C_p t}}{a_1(\mathbf{x},t)a_2(\mathbf{x},t)}, \quad p_1(\mathbf{x},t) := \frac{d(\mathbf{x})e^{-C_p t}}{a_2(\mathbf{x},t)}, \quad p_2(\mathbf{x},t) := \frac{d(\mathbf{x})e^{-C_p t}}{a_1(\mathbf{x},t)}, \quad (2.6)$$

for $x \in \Omega$, $t \in [0,T]$; where C_p is the constant in $\mathbf{V1}$ and $d(\mathbf{x})$ is a positive and continuous function. If a_1 and a_2 satisfy $\mathbf{V1}$, then the functions p, p_1 and p_2 are positive and continuous. Furthermore, they are all non-increasing w.r.t. t for every fixed $\mathbf{x} \in \Omega$.

Proof. It is obvious that p, p_1 and p_2 are all positive and continues.

By taking the partial derivative w.r.t. t, we have

$$p_{t} = d(\mathbf{x})e^{-C_{p}t} \left[\frac{-C_{p}a_{1}a_{2} - (a_{1}a_{2})_{t}}{(a_{1}a_{2})^{2}} \right],$$

$$(p_{1})_{t} = d(\mathbf{x})e^{-C_{p}t} \left[\frac{-C_{p}a_{2} - (a_{2})_{t}}{(a_{2})^{2}} \right],$$

$$(p_{2})_{t} = d(\mathbf{x})e^{-C_{p}t} \left[\frac{-C_{p}a_{1} - (a_{1})_{t}}{(a_{1})^{2}} \right].$$

Then it is easy to reach the desired result provided the assumptions in V1 hold.

Remark 2.3. Since the numerical methods proposed later depend on precise choices of p, p_1 and p_2 , here we list some simple candidates for $d(\mathbf{x})$ and C_p . In fact, the pool for choices is large. One may take $d(\mathbf{x}) = 1$, $e^{\sin(x+y)}$, $e^{\cos(x+y)}$ and $C_p = \sup\{|(a_1)_t/a_1| + |(a_2)_t/a_2|\}$, etc.

In the rest of this paper, we always take functions p, p_1 and p_2 as those given in (2.6). Consequently the two inequalities (2.3) and (2.5) are true basing on **V1**.

3 The sub-diffusion equation with time-space dependent coefficients

3.1 The numerical scheme

Let u_h^k be the numerical approximations of $u(\mathbf{x}_h, t_k)$, $\mathbf{x}_h \in \Omega_h, 0 \le k \le N$. Denote $u_h^{n-\theta} := (1-\theta)u_h^n + \theta u_h^{n-1}$, $f_h^{n-\theta} := f(\mathbf{x}_h, t_{n-\theta})$ for $n \ge 1$ and $\varphi_h := \varphi(\mathbf{x}_h)$.

From Section 2 (noting that $\beta = \alpha$ here), it is natural to construct an implicit scheme to solve the sub-diffusion problem (1.1)–(1.2) in the following form:

$$(\mathcal{D}_{\tau}^{\alpha}u_h)^{n-\theta} = \mathcal{A}_h^{n-\theta}u_h^{n-\theta} + f_h^{n-\theta}, \quad \mathbf{x}_h \in \Omega_h, 1 \le n \le N; \tag{3.1}$$

$$u_h^0 = \varphi_h, \quad \mathbf{x}_h \in \Omega_h, \tag{3.2}$$

subject to the zero boundary conditions.

To perform the numerical analysis, we rewrite the scheme (3.1)–(3.2) in the following matrix representation:

$$(\mathcal{D}_{\tau}^{\alpha}\mathbf{u})^{n-\theta} = \left[-(\mathbf{P}^{n-\theta})^{-1}\mathbf{A}^{n-\theta} + \mathbf{B}^{n-\theta} + \mathbf{C}^{n-\theta} \right] \mathbf{u}^{n-\theta} + \mathbf{f}^{n-\theta}, \quad 1 \le n \le N;$$
(3.3)

$$\mathbf{u}^0 = \Phi; \tag{3.4}$$

where $\mathbf{u}^{n-\theta} := (1-\theta)\mathbf{u}^n + \theta\mathbf{u}^{n-1}$ and

$$\mathbf{f}^{n-\theta} := (f_{1,1}^{n-\theta}, \cdots, f_{M_x-1,1}^{n-\theta}, f_{1,2}^{n-\theta}, \cdots, f_{M_x-1,2}^{n-\theta}, \cdots, f_{1,M_y-1}^{n-\theta}, \cdots, f_{M_x-1,M_y-1}^{n-\theta})^T,$$

$$\Phi := (\varphi_{1,1}, \cdots, \varphi_{M_x-1,1}, \varphi_{1,2}, \cdots, \varphi_{M_x-1,2}, \cdots, \varphi_{1,M_y-1}, \cdots, \varphi_{M_x-1,M_y-1})^T.$$

3.2 Stability and convergence

The next lemma shows a discrete fractional Grönwall inequality which is a slightly modified version of [14, Theorem 3.1] (noting that $\pi_A = 11/4$ and ρ is the maximum time-step ratio (see Appendix)).

Lemma 3.1. [20, Lemma 3.2] Let $(g^n)_{n=1}^N$ and $(\lambda_l)_{l=0}^{N-1}$ be given nonnegative sequences. Assume that there exists a constant Λ (independent of the step sizes) such that $\Lambda \geq \sum_{l=0}^{N-1} \lambda_l$, and that the maximum step size satisfies

$$\max_{1 \le n \le N} \tau_n \le \frac{1}{\alpha \sqrt{4\pi_A \Gamma(2 - \alpha)\Lambda}}.$$

Then, for any nonnegative sequences $(u^k)_{k=0}^N$ and $(v^k)_{k=0}^N$ satisfying

$$\sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_{\tau} \left[(u^{k})^{2} + (v^{k})^{2} \right] \leq \sum_{k=1}^{n} \lambda_{n-k} \left(u^{k-\theta} + v^{k-\theta} \right)^{2} + (u^{n-\theta} + v^{n-\theta}) g^{n}, \quad 1 \leq n \leq N,$$

it holds that

$$u^{n} + v^{n} \le 4E_{\alpha}(4\max(1,\rho)\pi_{A}\Lambda t_{n}^{\alpha}) \left(u^{0} + v^{0} + \max_{1 \le k \le n} \sum_{j=1}^{k} P_{k-j}^{(k)} g^{j}\right) \quad \text{for } 1 \le n \le N, \quad (3.5)$$

where $E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}$ is the Mittag-Leffler function.

The coefficients $P_{n-j}^{(n)}$ in (3.5) are called the discrete complementary convolution kernels (see more details in [14]), and they satisfy ([14, Lemmma 2.1])

$$0 \le P_{n-j}^{(n)} \le \pi_A \Gamma(2-\alpha) \tau_j^{\alpha}, \quad \sum_{j=1}^n P_{n-j}^{(n)} \omega_{1-\alpha}(t_j) \le \pi_A, \quad 1 \le j \le n \le N.$$
 (3.6)

For u_h, v_h belonging to the space of grid functions which vanish on $\partial \Omega_h$, we introduce the discrete inner product $\langle u, v \rangle := h_x h_y \sum_{\mathbf{x}_h \in \Omega_h} u_h v_h$, the discrete L^2 -norm $||u|| := \sqrt{\langle u, u \rangle}$, the discrete H^1 seminorms $||\delta_x u||$ and $||\delta_y u||$, and $||\nabla_h u|| := \sqrt{||\delta_x u||^2 + ||\delta_y u||^2}$. Suppose \tilde{C}_0 , \hat{C}_0 , \tilde{C}_l and \hat{C}_l are positive constants such that

$$\tilde{C}_0 \le |p(\mathbf{x}, t)| \le \hat{C}_0, \quad \tilde{C}_l \le |p_l(\mathbf{x}, t)| \le \hat{C}_l \quad \text{for} \quad l = 1, 2, 3, 4.$$

Now we are going to show the stability and convergence for the proposed scheme (3.1)–(3.2).

Theorem 3.2. If V1 is valid, the numerical scheme (3.1)–(3.2) is unconditionally stable and the discrete solutions u_h^n ($\mathbf{x}_h \in \Omega_h, 1 \le n \le N$) satisfy

$$\|\nabla_h u^n\| \le C \left(\|\nabla_h u^0\| + \max_{1 \le k \le n} \sum_{j=1}^k P_{k-j}^{(k)} \|\nabla_h f^{j-\theta}\| \right) \le C \left(\|\nabla_h u^0\| + \max_{1 \le k \le n} \{t_k^\alpha \|\nabla_h f^{k-\theta}\}\| \right).$$

Proof. Multiplying both sides of (3.3) by $(\mathbf{A}^{n-\theta}\mathbf{u}^{n-\theta})^T$ gives:

$$(\mathbf{u}^{n-\theta})^T \mathbf{A}^{n-\theta} (\mathcal{D}_{\tau}^{\alpha} \mathbf{u})^{n-\theta} + (\mathbf{A}^{n-\theta} \mathbf{u}^{n-\theta})^T (\mathbf{P}^{n-\theta})^{-1} (\mathbf{A}^{n-\theta} \mathbf{u}^{n-\theta})$$

$$= (\mathbf{u}^{n-\theta})^T \left[\mathbf{A}^{n-\theta} \mathbf{B}^{n-\theta} + \mathbf{A}^{n-\theta} \mathbf{C}^{n-\theta} \right] \mathbf{u}^{n-\theta} + (\mathbf{u}^{n-\theta})^T \mathbf{A}^{n-\theta} \mathbf{f}^{n-\theta}, \quad 1 \le n \le N.$$
(3.7)

The first term on the left-hand side of (3.7) is evaluated by (2.5).

For the terms on the right-hand side, we first notice that for a real vector $\mathbf{z} = (z_1, z_2, \dots, z_{M_x-1})^T$,

$$4h_x^2(\hat{S}_x\mathbf{z})^T(\hat{S}_x\mathbf{z}) = \sum_{i=1}^{M_x-2} (u_{i+1} - u_i)^2 + u_{M_x-1}^2 \le u_1^2 + \sum_{i=1}^{M_x-2} (u_{i+1} - u_i)^2 + u_{M_x-1}^2 = h_x^2(S_x\mathbf{z})^T(S_x\mathbf{z}),$$

$$4h_x^2(\hat{S}_x^T\mathbf{z})^T(\hat{S}_x^T\mathbf{z}) = u_1^2 + \sum_{i=1}^{M_x-2} (u_{i+1} - u_i)^2 \le h_x^2(S_x\mathbf{z})^T(S_x\mathbf{z}).$$

Then it further holds that

$$\max \left\{ [(I_y \otimes \hat{S}_x) \mathbf{u}^k]^T [(I_y \otimes \hat{S}_x) \mathbf{u}^k], [(I_y \otimes \hat{S}_x^T) \mathbf{u}^k]^T [(I_y \otimes \hat{S}_x^T) \mathbf{u}^k] \right\} \leq \frac{1}{4} [(I_y \otimes S_x) \mathbf{u}^k]^T [(I_y \otimes S_x) \mathbf{u}^k],$$

$$\max \left\{ [(\hat{S}_y \otimes I_x) \mathbf{u}^k]^T [(\hat{S}_y \otimes I_x) \mathbf{u}^k], [(\hat{S}_y^T \otimes I_x) \mathbf{u}^k]^T [(\hat{S}_y^T \otimes I_x) \mathbf{u}^k] \right\} \leq \frac{1}{4} [(S_y \otimes I_x) \mathbf{u}^k]^T [(S_y \otimes I_x) \mathbf{u}^k].$$

Thus the Cauchy-Schwarz inequality leads to

$$(\mathbf{u}^{n-\theta})^{T} \left[(\mathbf{B}^{n-\theta})^{T} \mathbf{B}^{n-\theta} \right] \mathbf{u}^{n-\theta}$$

$$\leq \left[\left(I_{y} \otimes (\hat{S}_{x} - \hat{S}_{x}^{T}) \right) \mathbf{u}^{n-\theta} \right]^{T} (\mathbf{P}_{3}^{n-\theta})^{2} \left[\left(I_{y} \otimes (\hat{S}_{x} - \hat{S}_{x}^{T}) \right) \mathbf{u}^{n-\theta} \right]$$

$$+ \left[\left((\hat{S}_{y} - \hat{S}_{y}^{T}) \otimes I_{x} \right) \mathbf{u}^{n-\theta} \right]^{T} (\mathbf{P}_{4}^{n-\theta})^{2} \left[\left((\hat{S}_{y} - \hat{S}_{y}^{T}) \otimes I_{x} \right) \mathbf{u}^{n-\theta} \right]$$

$$\leq 2\hat{C}_{3}^{2} \left\{ \left[(I_{y} \otimes \hat{S}_{x}) \mathbf{u}^{n-\theta} \right]^{T} \left[(I_{y} \otimes \hat{S}_{x}) \mathbf{u}^{n-\theta} \right] + \left[(I_{y} \otimes \hat{S}_{x}^{T}) \mathbf{u}^{n-\theta} \right]^{T} \left[(I_{y} \otimes \hat{S}_{x}^{T}) \mathbf{u}^{n-\theta} \right] \right\}$$

$$+ 2\hat{C}_{4}^{2} \left\{ \left[(\hat{S}_{y} \otimes I_{x}) \mathbf{u}^{n-\theta} \right]^{T} \left[(\hat{S}_{y} \otimes I_{x}) \mathbf{u}^{n-\theta} \right] + \left[(\hat{S}_{y}^{T} \otimes I_{x}) \mathbf{u}^{n-\theta} \right]^{T} \left[(\hat{S}_{y}^{T} \otimes I_{x}) \mathbf{u}^{n-\theta} \right] \right\}$$

$$\leq \hat{C}_{3}^{2} (\mathbf{u}_{x}^{n-\theta})^{T} \mathbf{u}_{x}^{n-\theta} + \hat{C}_{4}^{2} (\mathbf{u}_{y}^{n-\theta})^{T} \mathbf{u}_{y}^{n-\theta}$$

$$\leq C_{5} \left[(\mathbf{u}_{x}^{n-\theta})^{T} \mathbf{P}_{1}^{n-\theta} \mathbf{u}_{x}^{n-\theta} + (\mathbf{u}_{y}^{n-\theta})^{T} \mathbf{P}_{2}^{n-\theta} \mathbf{u}_{y}^{n-\theta} \right],$$

$$(3.8)$$

where $C_5 := \max\{\hat{C}_3^2, \hat{C}_4^2\} \cdot \max\{1/\tilde{C}_1, 1/\tilde{C}_2\}$. Then the first part of the first term on the right-hand side of (3.7) can be estimated as

$$2(\mathbf{u}^{n-\theta})^{T}\mathbf{A}^{n-\theta}\mathbf{B}^{n-\theta}\mathbf{u}^{n-\theta}$$

$$\leq \frac{1}{\hat{C}_{0}}(\mathbf{A}^{n-\theta}\mathbf{u}^{n-\theta})^{T}(\mathbf{A}^{n-\theta}\mathbf{u}^{n-\theta}) + \hat{C}_{0}(\mathbf{u}^{n-\theta})^{T}\left[(\mathbf{B}^{n-\theta})^{T}\mathbf{B}^{n-\theta}\right]\mathbf{u}^{n-\theta}$$

$$\leq (\mathbf{A}^{n-\theta}\mathbf{u}^{n-\theta})^{T}(\mathbf{P}^{n-\theta})^{-1}(\mathbf{A}^{n-\theta}\mathbf{u}^{n-\theta}) + \hat{C}_{0}C_{5}\left[(\mathbf{u}_{x}^{n-\theta})^{T}\mathbf{P}_{1}^{n-\theta}\mathbf{u}_{x}^{n-\theta} + (\mathbf{u}_{y}^{n-\theta})^{T}\mathbf{P}_{2}^{n-\theta}\mathbf{u}_{y}^{n-\theta}\right]. (3.9)$$

Noticing the embedding inequality $||u^k|| \le C_{\Omega} ||\nabla u^k||$, $k \ge 0$, it leads to

$$(\mathbf{u}^{n-\theta})^T \mathbf{u}^{n-\theta} \le C_{\Omega}^2 \left[(\mathbf{u}_x^{n-\theta})^T \mathbf{u}_x^{n-\theta} + (\mathbf{u}_y^{n-\theta})^T \mathbf{u}_y^{n-\theta} \right]. \tag{3.10}$$

Then similar to the derivation of (3.9), one gets

$$2(\mathbf{u}^{n-\theta})^{T}\mathbf{A}^{n-\theta}\mathbf{C}^{n-\theta}\mathbf{u}^{n-\theta}$$

$$\leq \frac{1}{\hat{C}_{0}}(\mathbf{A}^{n-\theta}\mathbf{u}^{n-\theta})^{T}(\mathbf{A}^{n-\theta}\mathbf{u}^{n-\theta}) + \hat{C}_{0}(\mathbf{u}^{n-\theta})^{T}\left[(\mathbf{C}^{n-\theta})^{T}\mathbf{C}^{n-\theta}\right]\mathbf{u}^{n-\theta}$$

$$\leq (\mathbf{A}^{n-\theta}\mathbf{u}^{n-\theta})^{T}(\mathbf{P}^{n-\theta})^{-1}(\mathbf{A}^{n-\theta}\mathbf{u}^{n-\theta}) + \hat{C}_{0}C_{3}^{2}(\mathbf{u}^{n-\theta})^{T}\mathbf{u}^{n-\theta}$$

$$\leq (\mathbf{A}^{n-\theta}\mathbf{u}^{n-\theta})^{T}(\mathbf{P}^{n-\theta})^{-1}(\mathbf{A}^{n-\theta}\mathbf{u}^{n-\theta}) + \hat{C}_{0}C_{6}\left[(\mathbf{u}_{x}^{n-\theta})^{T}\mathbf{P}_{1}^{n-\theta}\mathbf{u}_{x}^{n-\theta} + (\mathbf{u}_{y}^{n-\theta})^{T}\mathbf{P}_{2}^{n-\theta}\mathbf{u}_{y}^{n-\theta}\right],$$
(3.11)

where $C_6 := \max\{1/\tilde{C}_1, 1/\tilde{C}_2\}C_3^2C_{\Omega}^2$.

For the last term on the right-hand side of (3.7), we have

$$(\mathbf{u}^{n-\theta})^{T} \mathbf{A}^{n-\theta} \mathbf{f}^{n-\theta} = \left[(I_{y} \otimes S_{x}) \mathbf{u}^{n-\theta} \right]^{T} \mathbf{P}_{1}^{n-\theta} (I_{y} \otimes S_{x}) \mathbf{f}^{n-\theta}$$

$$+ \left[(S_{y} \otimes I_{x}) \mathbf{u}^{n-\theta} \right]^{T} \mathbf{P}_{2}^{n-\theta} (S_{y} \otimes I_{x}) \mathbf{f}^{n-\theta}$$

$$= (\mathbf{u}_{x}^{n-\theta})^{T} \mathbf{P}_{1}^{n-\theta} (I_{y} \otimes S_{x}) \mathbf{f}^{n-\theta} + (\mathbf{u}_{y}^{n-\theta})^{T} \mathbf{P}_{2}^{n-\theta} (S_{y} \otimes I_{x}) \mathbf{f}^{n-\theta}$$

$$\leq \sqrt{\hat{C}_{1}} \sqrt{(\mathbf{u}_{x}^{n-\theta})^{T} \mathbf{P}_{1}^{n-\theta} \mathbf{u}_{x}^{n-\theta}} \sqrt{\left[(I \otimes S_{x}) \mathbf{f}^{n-\theta} \right]^{T} (I \otimes S_{x}) \mathbf{f}^{n-\theta}}$$

$$+ \sqrt{\hat{C}_{2}} \sqrt{(\mathbf{u}_{y}^{n-\theta})^{T} \mathbf{P}_{2}^{n-\theta} \mathbf{u}_{y}^{n-\theta}} \sqrt{\left[(S_{y} \otimes I) \mathbf{f}^{n-\theta} \right]^{T} (S_{y} \otimes I) \mathbf{f}^{n-\theta}}.$$
 (3.12)

Therefore, it follows from (3.7)–(3.12) and (2.5) that

$$\sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_{\tau} \left[(\mathbf{u}_{x}^{k})^{T} \mathbf{P}_{1}^{k-\theta} \mathbf{u}_{x}^{k} + (\mathbf{u}_{y}^{k})^{T} \mathbf{P}_{2}^{k-\theta} \mathbf{u}_{y}^{k} \right] \\
\leq \hat{C}_{0} (C_{5} + C_{6}) \left[(\mathbf{u}_{x}^{n-\theta})^{T} \mathbf{P}_{1}^{n-\theta} (\mathbf{u}_{x}^{n-\theta}) + (\mathbf{u}_{y}^{n-\theta})^{T} \mathbf{P}_{2}^{n-\theta} (\mathbf{u}_{y}^{n-\theta}) \right] \\
+ 2\sqrt{\hat{C}_{1}} \sqrt{(\mathbf{u}_{x}^{n-\theta})^{T} \mathbf{P}_{1}^{n-\theta} \mathbf{u}_{x}^{n-\theta}} \sqrt{\left[(I_{y} \otimes S) \mathbf{f}^{n-\theta} \right]^{T} (I_{y} \otimes S) \mathbf{f}^{n-\theta}} \\
+ 2\sqrt{\hat{C}_{2}} \sqrt{(\mathbf{u}_{y}^{n-\theta})^{T} \mathbf{P}_{2}^{n-\theta} \mathbf{u}_{y}^{n-\theta}} \sqrt{\left[(S \otimes I_{x}) \mathbf{f}^{n-\theta} \right]^{T} (S \otimes I_{x}) \mathbf{f}^{n-\theta}}. \tag{3.13}$$

In view of the relationships

$$\|\delta_x u^{k-\theta}\| = \sqrt{h_x h_y (\mathbf{u}_x^{k-\theta})^T (\mathbf{u}_x^{k-\theta})}$$
 and $\|\delta_y u^{k-\theta}\| = \sqrt{h_x h_y (\mathbf{u}_y^{k-\theta})^T (\mathbf{u}_y^{k-\theta})}$

where $k \geq 0$, we define the following norms

$$\|\delta_x u^{k-\theta}\|_{P_1} := \sqrt{h_x h_y (\mathbf{u}_x^{k-\theta})^T \mathbf{P}_1^{k-\theta} (\mathbf{u}_x^{k-\theta})} \quad \text{and} \quad \|\delta_y u^{k-\theta}\|_{P_2} := \sqrt{h_x h_y (\mathbf{u}_y^{k-\theta})^T \mathbf{P}_2^{k-\theta} (\mathbf{u}_y^{k-\theta})}.$$

Moreover, denote

$$||v||_{P_k}^{(n-\theta)} := (1-\theta)||v^n||_{P_k} + \theta||v^{n-1}||_{P_k}$$
 for $v_h \in \Omega_h$ and $k = 1, 2$.

Then the triangle inequality yields $||v^{n-\theta}||_{P_k} \leq ||v^{(n-\theta)}||_{P_k}$. Now, take $C_7 := 2 \max\{\sqrt{\hat{C}_1}, \sqrt{\hat{C}_2}\}$. Multiplying both sides of the inequality (3.13) by $h_x h_y$, it follows

$$\begin{split} &\sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_{\tau} \left[\| \delta_{x} u^{k} \|_{P_{1}}^{2} + \| \delta_{y} u^{k} \|_{P_{1}}^{1} \right] \\ \leq &\hat{C}_{0}(C_{5} + C_{6}) \left[\| \delta_{x} u^{n-\theta} \|_{P_{1}}^{2} + \| \delta_{y} u^{n-\theta} \|_{P_{2}}^{2} \right] + 2 \sqrt{\hat{C}_{1}} \| \delta_{x} u^{n-\theta} \|_{P_{1}} \cdot \| \delta_{x} f^{n-\theta} \| \\ &\quad + 2 \sqrt{\hat{C}_{2}} \| \delta_{y} u^{n-\theta} \|_{P_{2}} \cdot \| \delta_{y} f^{n-\theta} \| \\ \leq &\hat{C}_{0}(C_{5} + C_{6}) \left[\| \delta_{x} u^{n-\theta} \|_{P_{1}}^{2} + \| \delta_{y} u^{n-\theta} \|_{P_{2}}^{2} \right] + C_{7} \left(\| \delta_{x} u^{n-\theta} \|_{P_{1}} + \| \delta_{y} u^{n-\theta} \|_{P_{2}} \right) \| \nabla_{h} f^{n-\theta} \| \\ \leq &\hat{C}_{0}(C_{5} + C_{6}) \left[\left(\| \delta_{x} u \|_{P_{1}}^{(n-\theta)} \right)^{2} + \left(\| \delta_{y} u \|_{P_{2}}^{(n-\theta)} \right)^{2} \right] + C_{7} \left(\| \delta_{x} u \|_{P_{1}}^{(n-\theta)} + \| \delta_{y} u \|_{P_{2}}^{(n-\theta)} \right) \| \nabla_{h} f^{n-\theta} \|. \end{split}$$

Applying Lemma 3.1, we get

$$\|\delta_x u^n\|_{P_1} + \|\delta_y u^n\|_{P_2}$$

$$\leq 4E_{\beta} \left(4 \max(1, \rho) \pi_A \hat{C}_0(C_5 + C_6) t_n^{\beta} \right) \left[\| \delta_x u^0 \|_{P_1} + \| \delta_y u^0 \|_{P_2} + C_7 \max_{1 \leq k \leq n} \sum_{j=1}^k P_{k-j}^{(k)} \| \nabla_h f^{k-\theta} \| \right].$$

Since $\|\delta_x u^0\|_{P_1} \le \sqrt{\hat{C}_1} \|\delta_x u^0\|$, $\|\delta_y u^0\|_{P_2} \le \sqrt{\hat{C}_2} \|\delta_y u^0\|$, and

$$\|\delta_x u^n\| = \sqrt{h_x h_y(\mathbf{u}_x^n)^T \mathbf{u}_x^n} \le \frac{1}{\sqrt{\tilde{C}_1}} \|\delta_x u^n\|_{P_1}, \quad \|\delta_y u^n\| = \sqrt{h_x h_y(\mathbf{u}_y^n)^T \mathbf{u}_y^n} \le \frac{1}{\sqrt{\tilde{C}_2}} \|\delta_y u^n\|_{P_2},$$

we obtain

$$\|\nabla_{h}u^{n}\| \leq \|\delta_{x}u^{n}\| + \|\delta_{y}u^{n}\| \leq \max\{\frac{1}{\sqrt{\tilde{C}_{1}}}, \frac{1}{\sqrt{\tilde{C}_{2}}}\} \cdot (\|\delta_{x}u^{n}\|_{P_{1}} + \|\delta_{y}u^{n}\|_{P_{2}})$$

$$\leq C \left(\|\nabla_{h}u^{0}\| + \max_{1 \leq k \leq n} \sum_{j=1}^{k} P_{k-j}^{(k)} \|\nabla_{h}f^{j-\theta}\|\right)$$

$$\leq C \left(\|\nabla_{h}u^{0}\| + \max_{1 \leq k \leq n} \{t_{k}^{\alpha}\|\nabla_{h}f^{k-\theta}\|\}\right),$$

where (3.6) has been utilized.

Remark 3.3. We remark that one may consider numerical approximations of (1.1) based on a more simplified equivalent equation with

$$Au = \partial_x (a_1 \partial_x u) + \partial_y (a_2 \partial_y u) + [b_1 - (a_1)_x] \partial_x u + [b_2 - (a_2)_y] \partial_y u + b_3 u.$$
 (3.14)

The corresponding numerical approximation of (3.14) will be

$$(\mathcal{D}_{\tau}^{\alpha}\mathbf{u})^{n-\theta} = \left[-\tilde{\mathbf{A}}^{n-\theta} + \tilde{\mathbf{B}}^{n-\theta} + \tilde{\mathbf{C}}^{n-\theta} \right] \mathbf{u}^{n-\theta} + \mathbf{f}^{n-\theta}, \quad 1 \le n \le N;$$
 (3.15)

where $\tilde{\mathbf{A}}^{n-\theta} := (I_y \otimes S_x)^T \mathbf{A}_1^{n-\theta} (I_y \otimes S_x) + (S_y \otimes I_x)^T \mathbf{A}_2^{n-\theta} (S_y \otimes I_x)$, and \mathbf{A}_1 , \mathbf{A}_2 , $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ are diagonal matrices with entries from corresponding variable coefficients in (3.14).

To obtain the unconditional H^1 -norm stability and convergence, one should multiply both sides of (3.15) by $(\tilde{\mathbf{A}}^{n-\theta}\mathbf{u}^{n-\theta})^T$, which leads to a serious difficulty for estimating the term $(\mathbf{u}^{n-\theta})^T \tilde{\mathbf{A}}^{n-\theta} (\mathcal{D}_{\tau}^{\alpha}\mathbf{u})^{n-\theta}$ on the left-hand side. This is the main reason why we introduce the concise technique in Section 2. The advantage of such technique will be more obvious for diffusionwave equation as its numerical approximations have a coupled structure (see also(4.5)–(4.6)). For more details, see the first three steps (4.8)–(4.10) of the proof in the next section.

Next, we show the convergence of the proposed scheme (3.3)–(3.4).

Theorem 3.4. Denote $e_h^k := u(\mathbf{x}_h, t_k) - u_h^k$ ($\mathbf{x}_h \in \bar{\Omega}_h$, $0 \le k \le N$). If **V1**, **V2**, **MA** and the assumptions in (1.7)–(1.8) are valid, the numerical scheme (3.1)–(3.2) is unconditionally convergent with

$$\|\nabla_h e^n\| \le C(\tau^{\{2,\gamma\sigma_1\}} + h_x^2 + h_y^2), \quad \text{for} \quad 1 \le n \le N.$$
 (3.16)

Proof. Denote \mathbf{e}^k the error vector with enteries $e_{i,j}^k$ being arranged similar to those of \mathbf{u}^k . One can easily obtain the error equations

$$(\mathcal{D}_{\tau}^{\alpha} \mathbf{e})^{n-\theta} = \left[-(\mathbf{P}^{n-\theta})^{-1} \mathbf{A}^{n-\theta} + \mathbf{B}^{n-\theta} + \mathbf{C}^{n-\theta} \right] \mathbf{e}^{n-\theta} + \mathbf{R}^{n-\theta}, \quad 1 \le n \le N;$$
(3.17)

$$\mathbf{e}^0 = \mathbf{0},\tag{3.18}$$

where $e^{n-\theta} := (1-\theta)e^n + \theta e^{n-1}$, and

$$\mathbf{R}^{n-\theta} := (R_{1,1}^{n-\theta}, \cdots, R_{M_x-1,1}^{n-\theta}, R_{1,2}^{n-\theta}, \cdots, R_{M_x-1,2}^{n-\theta}, \cdots, R_{1,M_y-1}^{n-\theta}, \cdots, R_{M_x-1,M_y-1}^{n-\theta})$$

with

$$R_h^{n-\theta} = -\mathcal{T}_u(x_h, t_{n-\theta}) + \mathcal{T}_A(x_h, t_{n-\theta}) + \mathcal{S}(x_h, t_{n-\theta}), \quad \mathbf{x}_h \in \Omega_h.$$
 (3.19)

The estimation of the temporal and spatial truncation errors $\mathcal{T}_u(x_h, t_{n-\theta})$, $\mathcal{T}_A(x_h, t_{n-\theta})$ and $S(x_h, t_{n-\theta})$ are given in the Appendix (Subsection 7.3).

Following the proof of Theorem 3.2, we can get

$$\|\nabla_h e^n\| \le C \max_{1 \le k \le n} \sum_{j=1}^k P_{k-j}^{(k)} \|\nabla_h R^{j-\theta}\|, \quad 1 \le n \le N.$$
 (3.20)

Therefore, the claimed result can be verified by combining (3.20), (7.4) and (7.6)–(7.7).

The diffusion-wave equation with time-space dependent coef-4 ficients

The numerical scheme

A novel order reduction method (SFOR) proposed in [20] will be employed to construct efficient numerical scheme on nonuniform time partitions for the diffusion-wave problem (1.1) and (1.3). The underlying idea of the SFOR method is demonstrated in the following lemma.

Lemma 4.1. [20, Lemma 2.1] For $\alpha \in (1,2)$ and $u(t) \in C^2((0,T])$, it holds that

$$\mathcal{D}_t^{\alpha} u(t) = \mathcal{D}_t^{\frac{\alpha}{2}} \left(\mathcal{D}_t^{\frac{\alpha}{2}} u(t) \right) - u'(0) \omega_{2-\alpha}(t).$$

Moreover, if we take $\tilde{u}(t) := u(t) - tu'(0)$, then

$$\mathcal{D}_t^{\alpha} u(t) = \mathcal{D}_t^{\alpha} \tilde{u}(t) = \mathcal{D}_t^{\frac{\alpha}{2}} \left(\mathcal{D}_t^{\frac{\alpha}{2}} \tilde{u}(t) \right).$$

Utilizing Lemma 4.1, the equation (1.1) can be rewritten as $(\beta = \alpha/2 \text{ here})$

$$\mathcal{D}_{t}^{\beta}v = \mathcal{A}\tilde{u} + f(\mathbf{x}, t) + \mathcal{A}(t\psi), \tag{4.1}$$

$$v = \mathcal{D}_t^{\beta} \tilde{u},\tag{4.2}$$

with $\tilde{u} = u - t\psi$, for $\mathbf{x} \in \Omega$ and $t \in (0, T]$.

It is obvious that the problem (4.1)–(4.2) is equivalently to (1.1) and (1.3) provided $u(\cdot,t) \in$ $\mathcal{C}^2((0,T])$ and p is invertible, i.e., they have the same analytical solution. Then we can design the numerical approximation based on the model (4.1)–(4.2) in order to solve the original problem (1.1) and (1.3).

By using the discrete Caputo formula (1.6) and the discrete operator $\mathcal{A}_h^{n-\theta}$ given in Section 2, with $\tilde{u}_h^n = u_h^n - t_n \psi_h$, we propose the following implicit numerical scheme for solving (4.1)–(4.2):

$$(\mathcal{D}_{\tau}^{\beta}v_h)^{n-\theta} = \mathcal{A}_h^{n-\theta}\tilde{u}_h^{n-\theta} + f_h^{n-\theta} + [\mathcal{A}(t\psi)]_h^{n-\theta}, \quad \mathbf{x}_h \in \Omega_h, 1 \le n \le N;$$

$$(4.3)$$

$$v_h^{n-\theta} = (\mathcal{D}_{\tau}^{\beta} \tilde{u}_h)^{n-\theta}, \quad \mathbf{x}_h \in \Omega_h, 1 \le n \le N;$$

$$(4.4)$$

subject to the zero boundary conditions and initial conditions $u_h^0 = \phi_h$ and $v_h^0 = 0$. Denote $\tilde{\psi}_h^{n-\theta} := [\mathcal{A}(t\psi)]_h^{n-\theta}$, and

$$\boldsymbol{\Psi}^{n-\theta} := \operatorname{diag}\left(\tilde{\psi}_{1,1}^{n-\theta}, \cdots, \tilde{\psi}_{M_x-1,1}^{n-\theta}, \tilde{\psi}_{1,2}^{n-\theta}, \cdots, \tilde{\psi}_{M_x-1,2}^{n-\theta}, \cdots, \tilde{\psi}_{1,M_y-1}^{n-\theta}, \cdots, \tilde{\psi}_{M_x-1,M_y-1}^{n-\theta}\right).$$

The matrix form of the numerical scheme (4.3)–(4.4) is:

$$(\mathcal{D}_{\tau}^{\beta}\mathbf{v})^{n-\theta} = \left[-(\mathbf{P}^{n-\theta})^{-1}\mathbf{A}^{n-\theta} + \mathbf{B}^{n-\theta} + \mathbf{C}^{n-\theta} \right] \tilde{\mathbf{u}}^{n-\theta} + \mathbf{f}^{n-\theta} + \mathbf{\Psi}^{n-\theta}; \tag{4.5}$$

$$\mathbf{v}^{n-\theta} = (\mathcal{D}_{\tau}^{\beta} \tilde{\mathbf{u}})^{n-\theta}; \tag{4.6}$$

for $\mathbf{x}_h \in \Omega_h$, $1 \le n \le N$.

4.2 Stability and convergence

In the same way as Lemma 3.1, we can also simply go through the proof of [14, Theorem 3.1] to have an analogy version of the discrete fractional Grönwall inequality (with $\pi_A = 11/4$):

Lemma 4.2. Let $(g^n)_{n=1}^N$ and $(\lambda_l)_{l=0}^{N-1}$ be given nonnegative sequences. Assume that there exists a constant Λ (independent of the step sizes) such that $\Lambda \geq \sum_{l=0}^{N-1} \lambda_l$, and that the maximum step size satisfies

$$\max_{1 \le n \le N} \tau_n \le \frac{1}{\beta \sqrt{4\pi_A \Gamma(2-\beta)\Lambda}}.$$

Then, for any nonnegative sequence $(u^k)_{k=0}^N$, $(v^k)_{k=0}^N$ and $(w^k)_{k=0}^N$ satisfying

$$\sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_{\tau} \left[(u^{k})^{2} + (v^{k})^{2} + (w^{k})^{2} \right] \leq \sum_{k=1}^{n} \lambda_{n-k} \left(u^{k-\theta} + v^{k-\theta} + w^{k-\theta} \right)^{2} + (u^{n-\theta} + v^{n-\theta} + w^{n-\theta}) g^{n}, \quad 1 \leq n \leq N,$$

it holds that

$$u^{n} + v^{n} + w^{n} \le 6E_{\beta}(6\max(1, \rho)\pi_{A}\Lambda t_{n}^{\beta}) \left(u^{0} + v^{0} + w^{0} + \max_{1 \le k \le n} \sum_{j=1}^{k} P_{k-j}^{(k)} g^{j}\right) \quad \text{for } 1 \le n \le N.$$

Similar to (3.6), the discrete complementary convolution kernels $P_{n-j}^{(n)}$ in the above lemma fulfill

$$0 \le P_{n-j}^{(n)} \le \pi_A \Gamma(2-\beta) \tau_j^{\beta}, \quad \sum_{i=1}^n P_{n-j}^{(n)} \omega_{1-\beta}(t_j) \le \pi_A, \quad 1 \le j \le n \le N.$$

Theorem 4.3. If V1 is valid, the numerical scheme (4.3)–(4.4) is unconditionally stable and the discrete solutions u_h^n ($\mathbf{x}_h \in \Omega_h, 1 \le n \le N$) satisfy

$$\|\nabla_{h}u^{n}\| \leq C \left[\|\nabla_{h}\varphi\| + t_{n}\|\nabla_{h}\psi\| + \max_{1\leq k\leq n} \sum_{j=1}^{k} P_{k-j}^{(k)}(\|f^{j-\theta}\| + \|\tilde{\psi}^{j-\theta}\|) \right]$$

$$\leq C \left[\|\nabla_{h}\varphi\| + t_{n}\|\nabla_{h}\psi\| + \max_{1\leq k\leq n} \left\{ t_{k}^{\frac{\alpha}{2}}(\|f^{k-\theta}\| + \|\tilde{\psi}^{k-\theta}\|) \right\} \right]. \tag{4.7}$$

Proof. Multiplying both sides of (4.5) by $(\mathbf{P}^{n-\theta}\mathbf{v}^{n-\theta})^T$ yields

$$(\mathbf{v}^{n-\theta})^T \mathbf{P}^{n-\theta} (\mathcal{D}_{\tau}^{\beta} \mathbf{v})^{n-\theta} + (\mathbf{v}^{n-\theta})^T \mathbf{A}^{n-\theta} \tilde{\mathbf{u}}^{n-\theta}$$

$$= (\mathbf{v}^{n-\theta})^T \mathbf{P}^{n-\theta} \left(\mathbf{B}^{n-\theta} + \mathbf{C}^{n-\theta} \right) \tilde{\mathbf{u}}^{n-\theta} + (\mathbf{v}^{n-\theta})^T \mathbf{P}^{n-\theta} (\mathbf{f}^{n-\theta} + \mathbf{\Psi}^{n-\theta}). \tag{4.8}$$

On the other hand, the multiplication of $(\tilde{\mathbf{u}}^{n-\theta})^T \mathbf{A}^{n-\theta}$ on both sides of (4.6) gives

$$(\tilde{\mathbf{u}}^{n-\theta})^T \mathbf{A}^{n-\theta} \mathbf{v}^{n-\theta} = (\tilde{\mathbf{u}}^{n-\theta})^T \mathbf{A}^{n-\theta} (\mathcal{D}_{\tau}^{\beta} \tilde{\mathbf{u}})^{n-\theta}. \tag{4.9}$$

Thus, it follows from (4.8) and (4.9) that

$$(\mathbf{v}^{n-\theta})^T \mathbf{P}^{n-\theta} (\mathcal{D}_{\tau}^{\beta} \mathbf{v})^{n-\theta} + (\tilde{\mathbf{u}}^{n-\theta})^T \mathbf{A}^{n-\theta} (\mathcal{D}_{\tau}^{\beta} \tilde{\mathbf{u}})^{n-\theta}$$

$$= (\mathbf{v}^{n-\theta})^T \mathbf{P}^{n-\theta} \left(\mathbf{B}^{n-\theta} + \mathbf{C}^{n-\theta} \right) \tilde{\mathbf{u}}^{n-\theta} + (\mathbf{v}^{n-\theta})^T \mathbf{P}^{n-\theta} (\mathbf{f}^{n-\theta} + \mathbf{\Psi}^{n-\theta}). \tag{4.10}$$

The first and second terms on the left-hand side of (4.10) are evaluated by means of (2.3) and (2.5), respectively. Then we consider the terms on the right-hand side. Applying the Cauchy-Schwarz inequality and utilizing (3.8), we have

$$2(\mathbf{v}^{n-\theta})^{T}\mathbf{P}^{n-\theta}\mathbf{B}^{n-\theta}\tilde{\mathbf{u}}^{n-\theta}$$

$$\leq (\mathbf{v}^{n-\theta})^{T}(\mathbf{P}^{n-\theta})^{2}\mathbf{v}^{n-\theta} + (\tilde{\mathbf{u}}^{n-\theta})^{T}\left[(\mathbf{B}^{n-\theta})^{T}\mathbf{B}^{n-\theta}\right]\tilde{\mathbf{u}}^{n-\theta}$$

$$\leq \frac{1}{\tilde{C}_{0}}(\mathbf{v}^{n-\theta})^{T}\mathbf{P}^{n-\theta}\mathbf{v}^{n-\theta} + C_{5}\left[(\tilde{\mathbf{u}}_{x}^{n-\theta})^{T}\mathbf{P}_{1}^{n-\theta}\tilde{\mathbf{u}}_{x}^{n-\theta} + (\tilde{\mathbf{u}}_{y}^{n-\theta})^{T}\mathbf{P}_{2}^{n-\theta}\tilde{\mathbf{u}}_{y}^{n-\theta}\right]. \tag{4.11}$$

With the embedding inequality (3.10), one has

$$2(\mathbf{v}^{n-\theta})^{T}\mathbf{P}^{n-\theta}\mathbf{C}^{n-\theta}\tilde{\mathbf{u}}^{n-\theta}$$

$$\leq (\mathbf{v}^{n-\theta})^{T}(\mathbf{P}^{n-\theta})^{2}\mathbf{v}^{n-\theta} + (\tilde{\mathbf{u}}^{n-\theta})^{T}(\mathbf{C}^{n-\theta})^{2}\tilde{\mathbf{u}}^{n-\theta}$$

$$\leq \frac{1}{\tilde{C}_{0}}(\mathbf{v}^{n-\theta})^{T}\mathbf{P}^{n-\theta}\mathbf{v}^{n-\theta} + C_{3}^{2}C_{\Omega}^{2}\left[(\tilde{\mathbf{u}}_{x}^{n-\theta})^{T}\tilde{\mathbf{u}}_{x}^{n-\theta} + (\tilde{\mathbf{u}}_{y}^{n-\theta})^{T}\tilde{\mathbf{u}}_{y}^{n-\theta}\right]$$

$$\leq \frac{1}{\tilde{C}_{0}}(\mathbf{v}^{n-\theta})^{T}\mathbf{P}^{n-\theta}\mathbf{v}^{n-\theta} + C_{6}\left[(\tilde{\mathbf{u}}_{x}^{n-\theta})^{T}\mathbf{P}_{1}^{n-\theta}\tilde{\mathbf{u}}_{x}^{n-\theta} + (\tilde{\mathbf{u}}_{y}^{n-\theta})^{T}\mathbf{P}_{2}^{n-\theta}\tilde{\mathbf{u}}_{y}^{n-\theta}\right]. \tag{4.12}$$

Hence, from (4.10)–(4.12), (2.3) and (2.5), we obtain

$$\sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_{\tau} \left[(\mathbf{v}^{k})^{T} \mathbf{P}^{k-\theta} \mathbf{v}^{k} + (\tilde{\mathbf{u}}_{x}^{k})^{T} \mathbf{P}_{1}^{k-\theta} \tilde{\mathbf{u}}_{x}^{k} + (\tilde{\mathbf{u}}_{y}^{k})^{T} \mathbf{P}_{2}^{k-\theta} \tilde{\mathbf{u}}_{y}^{k} \right] \\
\leq 2 \max \left\{ \frac{1}{\tilde{C}_{0}}, C_{5}, C_{6} \right\} \left[(\mathbf{v}^{n-\theta})^{T} \mathbf{P}^{n-\theta} \mathbf{v}^{n-\theta} + (\tilde{\mathbf{u}}_{x}^{n-\theta})^{T} \mathbf{P}_{1}^{n-\theta} \tilde{\mathbf{u}}_{x}^{n-\theta} + (\tilde{\mathbf{u}}_{y}^{n-\theta})^{T} \mathbf{P}_{2}^{n-\theta} \tilde{\mathbf{u}}_{y}^{n-\theta} \right] \\
+ 2 \left[(\mathbf{P}^{n-\theta})^{\frac{1}{2}} \mathbf{v}^{n-\theta} \right]^{T} (\mathbf{P}^{n-\theta})^{\frac{1}{2}} (\mathbf{f}^{n-\theta} + \mathbf{\Psi}^{n-\theta}). \tag{4.13}$$

Multiplying both sides of (4.13) by $h_x h_y$ and taking $C_8 := 2 \max\{\frac{1}{\tilde{C}_0}, C_5, C_6\}$, we further get

$$\sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_{\tau} \left[\|v^{k}\|_{P}^{2} + \|\delta_{x}\tilde{u}^{k}\|_{P_{1}}^{2} + \|\delta_{y}\tilde{u}^{k}\|_{P_{2}}^{2} \right]
\leq C_{8} \left(\|v^{n-\theta}\|_{P}^{2} + \|\delta_{x}\tilde{u}^{n-\theta}\|_{P_{1}}^{2} + \|\delta_{y}\tilde{u}^{n-\theta}\|_{P_{2}}^{2} \right) + \sqrt{\hat{C}_{0}} \|v^{n-\theta}\|_{P} \cdot \|f^{n-\theta} + \tilde{\psi}^{n-\theta}\|
\leq C_{8} \left[\left(\|v\|_{P}^{(n-\theta)} \right)^{2} + \left(\|\delta_{x}\tilde{u}\|_{P_{1}}^{(n-\theta)} \right)^{2} + \left(\|\delta_{y}\tilde{u}\|_{P_{2}}^{(n-\theta)} \right)^{2} \right]
+ \sqrt{\hat{C}_{0}} \left(\|v\|_{P}^{(n-\theta)} + \|\delta_{x}\tilde{u}\|_{P_{1}}^{(n-\theta)} + \|\delta_{y}\tilde{u}\|_{P_{2}}^{(n-\theta)} \right) (\|f^{n-\theta}\| + \|\tilde{\psi}^{n-\theta}\|), \tag{4.14}$$

where $||v^k||_P^2 := h_x h_y(\mathbf{v}^k)^T \mathbf{P}^{k-\theta} \mathbf{v}^k$.

Now, combining (4.14) with the fractional Grönwall inequality (Lemma 4.2), it follows

$$||v^{n}||_{P} + ||\delta_{x}\tilde{u}^{n}||_{P_{1}} + ||\delta_{y}\tilde{u}^{n}||_{P_{2}} \leq 6E_{\beta}(6\max(1,\rho)\pi_{A}C_{8}t_{n}^{\beta}) \Big[||v^{0}||_{P} + ||\delta_{x}\tilde{u}^{0}||_{P_{1}} + ||\delta_{y}\tilde{u}^{0}||_{P_{2}} + \sqrt{\hat{C}_{0}} \max_{1\leq k\leq n} \sum_{j=1}^{k} P_{k-j}^{(k)}(||f^{j-\theta}|| + ||\tilde{\psi}^{j-\theta}||)\Big],$$

and hence

$$\begin{split} \|\nabla_{h}\tilde{u}^{n}\| &\leq \|\delta_{x}\tilde{u}^{n}\| + \|\delta_{y}\tilde{u}^{n}\| \leq \max\{\frac{1}{\sqrt{\tilde{C}_{1}}}, \frac{1}{\sqrt{\tilde{C}_{2}}}\} \cdot (\|\delta_{x}\tilde{u}^{n}\|_{P_{1}} + \|\delta_{y}\tilde{u}^{n}\|_{P_{2}}) \\ &\leq C \left[\|v^{0}\|_{P} + \|\nabla_{h}\tilde{u}^{0}\| + \max_{1 \leq k \leq n} \sum_{j=1}^{k} P_{k-j}^{(k)} (\|f^{j-\theta}\| + \|\tilde{\psi}^{j-\theta}\|) \right] \\ &\leq C \left[\|v^{0}\|_{P} + \|\nabla_{h}\tilde{u}^{0}\| + \max_{1 \leq k \leq n} \{t_{k}^{\beta} (\|f^{k-\theta}\| + \|\tilde{\psi}^{k-\theta}\|)\} \right]. \end{split}$$

Then the claimed result (4.7) can be reached by the properties $\|\nabla_h u^n\| \leq \|\nabla_h \tilde{u}^n\| + t_n \|\nabla_h \psi\|$, $\|v^0\|_P = 0$ and $\|\nabla_h \tilde{u}^0\| = \|\nabla_h \varphi\|$.

The next theorem shows the convergence of proposed scheme (4.5)–(4.6).

Theorem 4.4. Denote $e_h^k := u(\mathbf{x}_h, t_k) - u_h^k$ ($\mathbf{x}_h \in \bar{\Omega}_h$, $0 \le k \le N$). If **V1**, **V2**, **MA** and the assumptions in (1.7) and (1.9) are valid, the numerical scheme (4.3)–(4.4) is unconditionally convergent with

$$\|\nabla_h e^n\| \le C(\tau^{\min\{2,\gamma\sigma_2,\gamma\sigma_3\}} + h_x^2 + h_y^2), \quad \text{for} \quad 1 \le n \le N.$$
 (4.15)

Proof. We have

$$e_h^k = u(\mathbf{x}_h, t_k) - u_h^k = \tilde{u}(\mathbf{x}_h, t_k) - \tilde{u}_h^k, \quad \mathbf{x}_h \in \Omega_h, \ 1 \le k \le N.$$

Denote $\check{e}_h^k := v(\mathbf{x}_h, t_k) - v_h^k \ (1 \le k \le N)$ and

$$\tilde{R}_h^{n-\theta} := -\mathcal{T}_{v1}(\mathbf{x}_h, t_{n-\theta}) + \mathcal{T}_A(\mathbf{x}_h, t_{n-\theta}) + \mathcal{S}(\mathbf{x}_h, t_{n-\theta}), \ \hat{R}_h^{n-\theta} := -\mathcal{T}_{v2}(\mathbf{x}_h, t_{n-\theta}) + \mathcal{T}_{\tilde{u}}(\mathbf{x}_h, t_{n-\theta}),$$

$$(4.16)$$

for $\mathbf{x}_h \in \Omega_h$, where the above truncation errors are discussed in the Appendix (Subsection 7.3). Denote the vector

$$\check{\mathbf{e}}^k := (\check{e}^k_{1,1}, \cdots, \check{e}^k_{M_x-1,1}, \cdots, \check{e}^k_{1,2}, \cdots, \check{e}^k_{M_x-1,2}, \cdots \cdots, \check{e}^k_{1,M_y-1}, \cdots, \check{e}^k_{M_x-1,M_y-1}),$$

while $\tilde{\mathbf{R}}^{n-\theta}$, $\hat{\mathbf{R}}^{n-\theta}$ are similarity defined with entries $\tilde{R}_h^{n-\theta}$ and $\hat{R}_h^{n-\theta}$, respectively.

The error equations to scheme (4.5)–(4.6) can be given as

$$(\mathcal{D}_{\tau}^{\beta}\check{\mathbf{e}})^{n-\theta} = \left[-(\mathbf{P}^{n-\theta})^{-1}\mathbf{A}^{n-\theta} + \mathbf{B}^{n-\theta} + \mathbf{C}^{n-\theta} \right] \mathbf{e}^{n-\theta} + \tilde{\mathbf{R}}^{n-\theta}; \tag{4.17}$$

$$\check{\mathbf{e}}^{n-\theta} = (\mathcal{D}_{\tau}^{\beta} \mathbf{e})^{n-\theta} + \hat{\mathbf{R}}^{n-\theta}. \tag{4.18}$$

The proof of convergence is similar to that of Theorem 4.3 with a slight difference only at the step for (4.9). We now have:

$$(\mathbf{e}^{n-\theta})^T\mathbf{A}^{n-\theta}\check{\mathbf{e}}^{n-\theta} = (\mathbf{e}^{n-\theta})^T\mathbf{A}^{n-\theta}(\mathcal{D}_{\tau}^{\beta}\mathbf{e})^{n-\theta} + (\mathbf{e}^{n-\theta})^T\mathbf{A}^{n-\theta}\hat{\mathbf{R}}^{n-\theta}.$$

The term can be estimated like that in (3.12). Going though the remaining part of the proof like that of Theorem 4.3, one should find no difficulty to obtain

$$\|\check{e}^n\| + \|\nabla_h e^n\| \le C \max_{1 \le k \le n} \sum_{j=1}^k P_{k-j}^{(k)}(\|\tilde{R}^{j-\theta}\| + \|\nabla_h \hat{R}^{j-\theta}\|).$$

Thus, combining with (7.3), (7.5) and (7.8)–(7.10), the desired result is true.

5 Numerical Experiments

Numerical examples will be provided in this section to show the accuracy and efficiency of proposed schemes (3.1)–(3.2) and (4.3)–(4.4). The variable coefficients in the two examples in this section are chose as

$$a_1(\mathbf{x}, t) = e^{x+y}(1 + \cos(t)), \quad a_2(\mathbf{x}, t) = e^{(x+y)t}(1 + t^{\frac{3}{2}}),$$

 $b_1(\mathbf{x}, t) = \sin(xyt), \quad b_2(\mathbf{x}, t) = \cos(xyt), \quad b_3(\mathbf{x}, t) = (x^2 + y^2)t.$

The above variable coefficients satisfy V1 and V2 clearly. Then we take the function $d(\mathbf{x})$ and constant C_p in Lemma 2.2 as follows

$$d(\mathbf{x}) = e^{\sin(x+y)}$$
 and $C_p = 3$.

Since the problem we considered in the paper are linear fractional evolution equations, we choose the classical graded mesh $t_k = T(k/N)^{\gamma}$ for the time partition to compensate for the lack of smoothness of the solution near the initial time. The graded mesh is definitely in accordance with the mesh assumption **MA**. In all of the numerical tests, we take $M = M_x = M_y$, the discrete H^1 -norm errors $E_1(M,N) = \max_{1 \le n \le N} \|U^n - u^n\|_{H^1}$ will be recorded in each run, and the temporal and spatial convergence orders are given by

$$\operatorname{Order}_{\tau} = \log_2 \left[\frac{E_1(M, N/2)}{E_1(M, N)} \right] \quad \text{and} \quad \operatorname{Order}_h = \log_2 \left[\frac{E_1(M/2, N)}{E_1(M, N)} \right],$$

respectively.

Moreover, we will always employ the sum-of-exponentials (SOE) technique [7] to the proposed schemes while discretizing the Caputo derivative to save the memory and computational costs, since the SOE method does not bring any additional essential differences to the numerical analysis of the nonuniform schemes. One may refer to [16, Section 5.1] for the details of the fast Alikhanov formula and refer to [7, 19] for the advantage of the SOE approximation in the computational aspect. The absolute tolerance error ϵ and the cut-off time Δt of the fast Alikhanov formula (see [16, Lemma 5.1]) are set as $\epsilon = 10^{-12}$ and $\Delta t = \tau_1$ in all of the following tests.

Example 5.1. We consider the sub-diffusion problem (1.1)–(1.2) with $\Omega = (0,1)^2$, T = 1, $\varphi = \sin(\pi x)\sin(\pi y)$ and

$$f(u, \mathbf{x}, t) = \sin(\pi x)\sin(\pi y) \left[\Gamma(\alpha + 1) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \right] - \mathcal{A}(\sin(\pi x)\sin(\pi y))(1 + t + t^{\alpha}), \quad \alpha \in (0, 1),$$

such that the exact solution is $u = \sin(\pi x)\sin(\pi y)(1+t+t^{\alpha})$.

One may notice that the regularity parameter in (1.7) should be $\sigma_1 = \alpha$ for Example 5.1. Therefore, according to Theorem 3.4, the optimal mesh parameter is $\gamma_{opt} = 2/\alpha$ for the scheme (3.1)–(3.2) on the graded time meshes.

The temporal accuracy by applying the scheme (3.1)–(3.2) with fixed M=1000 and different parameters α, γ for solving Example 5.1 is listed in Tables 1–3, while Table 4 shows the spatial accuracy with fixed N=500. From the four tables, we can clearly observe the optimal second-order accuracy of the proposed scheme, and the optimal choice of the grading parameter ($\gamma_{opt}=2/\alpha$) is well reflected.

Table 1: Numerical accuracy in temporal direction of the scheme (3.1)–(3.2) for solving Example 5.1, where $\alpha = 0.5$.

	$\gamma = 1$		$\gamma_{opt} = 2/$	$\gamma_{opt} = 2/\alpha = 4$		$\gamma = 2.5/\alpha = 5$	
N	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$	
4	1.6124e-01	*	4.4642e-02	*	6.4526 e - 02	*	
8	1.1090e-01	0.54	1.2036e-02	1.89	1.8154e-02	1.83	
16	7.5477e-02	0.56	3.1256e-03	1.95	4.7969e-03	1.92	
32	5.0612e-02	0.58	8.0038e-04	1.97	1.2989e-03	1.88	
Theoretical Order		0.50		2.00		2.00	

Table 2: Numerical accuracy in temporal direction of the scheme (3.1)–(3.2) for solving Example 5.1, where $\alpha = 0.7$.

	$\gamma = 1$		$\gamma_{opt} = 2/\alpha \approx 2.86$		$\gamma = 2.5/\alpha \approx 3.57$	
N	$E_1(M,N)$	Order_{τ}	$E_1(M,N)$	Order_{τ}	$E_1(M,N)$	$\mathrm{Order}_{ au}$
4	1.0592e-01	*	2.5978e-02	*	3.8595 e-02	*
8	6.1506 e - 02	0.78	6.5510 e-03	1.99	1.0043e-02	1.94
16	3.4534e-02	0.83	1.6656e-03	1.98	2.5747e-03	1.96
32	1.8403 e-02	0.91	4.2368e-04	1.98	6.5520 e-04	1.97
Theoretical Order		0.70		2.00		2.00

Table 3: Numerical accuracy in temporal direction of the scheme (3.1)–(3.2) for solving Example 5.1, where $\alpha = 0.9$.

	$\gamma = 1$		$\gamma_{opt} = 2/\alpha$	$\gamma_{opt} = 2/\alpha \approx 2.22$		≈ 2.78
N	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$
4	3.4213e-02	*	8.0750 e-03	*	1.2235e-02	*
8	1.6299e-02	1.07	1.8998e-03	2.09	2.9302e-03	2.06
16	7.0935e-03	1.20	4.7949e-04	1.99	7.4266e-04	1.98
32	2.6405 e-03	1.43	1.2448e-04	1.95	1.9062e-04	1.96
Theoretical Order		0.90		2.00		2.00

Example 5.2. We then consider the diffusion-wave problem (1.1) and (1.3) with $\Omega = (0,1)^2$, T = 1, $\phi = \psi = \sin(\pi x)\sin(\pi y)$ and

$$f(u, \mathbf{x}, t) = \Gamma(\alpha + 1)\sin(\pi x)\sin(\pi y) - \mathcal{A}(\sin(\pi x)\sin(\pi y))(1 + t + t^{\alpha}), \quad \alpha \in (1, 2),$$

such that the exact solution is $u = \sin(\pi x)\sin(\pi y)(1+t+t^{\alpha})$.

For Example 5.2, the regularity parameters in (1.8) are $\sigma_2 = \alpha$ and $\sigma_3 = \alpha/2$. Then, Theorem 4.4 indicates that the optimal mesh parameter is $\gamma_{opt} = 2/\sigma_3 = 4/\alpha$ for the scheme (4.3)–(4.4) on the graded time meshes. One can notice that the grading parameter γ_{opt} is bounded and not

Table 4: Numerical accuracy in spatial direction of the scheme (3.1)–(3.2) for solving Example 5.1, where $\alpha = 0.7$.

	$\gamma = 1$		$\gamma_{opt} = 2/\alpha \approx 2.86$		$\gamma = 2.5/\alpha \approx 3.57$	
M	$E_1(M,N)$	Order_h	$E_1(M,N)$	Order_h	$E_1(M,N)$	$Order_h$
4	3.6943e-01	*	3.6942e-01	*	3.6931e-01	*
8	9.1710e-02	2.01	9.1710e-02	2.01	9.1666e-02	2.01
16	2.2891e-02	2.00	2.2891e-02	2.00	2.2864e-02	2.00
32	5.7205 e-03	2.00	5.7213e-03	2.00	5.6977e-03	2.00
Theoretical Order		2.00		2.00		2.00

large while $\alpha \to 1^+$, this keeps the robustness of the graded scheme in practical applications when the fractional order α is close to one.

Similarly, we display the temporal accuracy, which is obtained by applying the scheme (4.3)–(4.4) with fixed M=1000 and different parameters for solving Example 5.2, in Tables 5–8. The spatial accuracy of the scheme with fixed N=500 is displayed in Table 9. The numerical results show that the proposed scheme (4.3)–(4.4) also works very well with optimal second-order accuracy and is robust for $\alpha \to 1^+$ in solving the diffusion-wave problem with general variable coefficients.

Table 5: Numerical accuracy in temporal direction of scheme (4.3)–(4.4) for Example 5.2, where $\alpha = 1.01$.

	$\gamma = 1$		$\gamma_{opt} = 4/\alpha \approx 3.96$		$\gamma = 4.5/\alpha \approx 4.46$	
N	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$
4	1.2885e-02	*	4.7702e-03	*	4.7959e-03	*
8	1.1231e-02	0.66	1.5632e-03	1.85	1.4205 e-03	1.76
16	9.2424e-03	0.28	4.2372e-04	1.88	4.0423e-04	1.81
32	6.9173e-03	0.42	1.0616e-04	2.00	1.0064e-04	2.01
Theoretical Order		0.505		2.00		2.00

Table 6: Numerical accuracy in temporal direction of scheme (4.3)–(4.4) for Example 5.2, where $\alpha = 1.1$.

	$\gamma = 1$		$\gamma_{opt} = 4/\alpha$	$\gamma_{opt} = 4/\alpha \approx 3.64$		$\gamma = 4.5/\alpha \approx 4.09$	
N	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$	
4	2.4901e-02	*	1.6750 e-02	*	2.0056e-02	*	
8	1.5761e-02	0.66	4.6593e-03	1.85	5.7785e-03	1.80	
16	1.0245 e-02	0.62	1.2195e-03	1.93	1.5306e-03	1.92	
32	6.3732e-03	0.68	3.0860e-04	1.98	3.9009e-04	1.97	
Theoretical Order		0.55		2.00		2.00	

Table 7: Numerical accuracy in temporal direction of scheme (4.3)–(4.4) for Example 5.2, where $\alpha = 1.5$.

	$\gamma = 1$		$\gamma_{opt} = 4/\alpha \approx 2.67$		$\gamma = 4.5/\alpha = 3$	
N	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$
4	5.3444e-02	*	7.8413e-02	*	9.6727e-02	*
8	1.7243e-02	1.63	2.0766e-02	1.92	2.5910e-02	1.90
16	6.1521e-03	1.49	5.3057e-03	1.97	6.6772 e-03	1.96
32	2.3596e-03	1.38	1.3373e-03	1.99	1.6881e-03	1.98
Theoretical Order		0.75		2.00		2.00

Table 8: Numerical accuracy in temporal direction of scheme (4.3)–(4.4) for Example 5.2, where $\alpha = 1.9$.

	$\gamma = 1$		$\gamma_{opt} = 4/\alpha \approx 2.11$		$\gamma = 4.5/\alpha \approx 2.37$	
N	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$	$E_1(M,N)$	$\mathrm{Order}_{ au}$
4	6.1480e-02	*	1.3149e-01	*	1.6643e-01	*
8	1.6140 e-02	1.93	3.0479e-02	2.11	3.8947e-02	2.10
16	4.1813e-03	1.95	7.8883e-03	1.95	9.9179e-03	1.97
32	1.1127e-03	1.91	2.0132e-03	1.97	2.5274e-03	1.97
Theoretical Order		0.95		2.00		2.00

Table 9: Numerical accuracy in spatial direction of scheme (4.3)–(4.4) for Example 5.2, where $\alpha = 1.5$.

	$\gamma = 1$		$\gamma_{opt} = 4/\alpha$	$\gamma_{opt} = 4/\alpha \approx 2.67$		$\alpha = 3$
M	$E_1(M,N)$	Order_h	$E_1(M,N)$	Order_h	$E_1(M,N)$	$Order_h$
4	2.4719e-01	*	2.4718e-01	*	2.4718e-01	*
8	6.1357e-02	2.01	6.1352e-02	2.01	6.1349 e-02	2.01
16	1.5313e-02	2.00	1.5309e-02	2.00	1.5306e-02	2.00
32	3.8262 e-03	2.00	3.8214e-03	2.00	3.8190e-03	2.00
Theoretical Order		2.00		2.00		2.00

6 Conclusion

We introduced a novel and concise technique to study numerical methods on nonuniform time partitions for solving time fractional evolution equations (including the sub-diffusion and diffusion-wave equations) with general time-space dependent variable coefficients. The proposed numerical schemes utilized the Alikhanov formula on nonuniform meshes. Under reasonable assumptions on the solution regularity, the variable coefficients, and weak mesh restrictions, we showed that the nonuniform schemes are unconditionally stable and second-order convergent with respect to discrete H^1 -norm. The efficiency and accuracy of proposed schemes are well verified by some

numerical experiments.

7 Appendix

7.1 The coefficients of Alikhanov formulas

The coefficients $A_{n-k}^{(n)}$ of the Alikhanov formula on general meshes are defined as ([15])

$$A_{n-k}^{(n)} := \begin{cases} a_0^{(n)} + \rho_{n-1}b_1^{(n)}, & k = n, \\ a_{n-k}^{(n)} + \rho_{k-1}b_{n-k+1}^{(n)} - b_{n-k}^{(n)}, & 2 \le k \le n-1, & \text{for } n \ge 2, \\ a_{n-1}^{(n)} - b_{n-1}^{(n)}, & k = 1, \end{cases}$$

where

$$a_{n-k}^{(n)} := \frac{1}{\tau_k} \int_{t_{k-1}}^{\min\{t_k, t_{n-\theta}\}} \omega_{1-\beta}(t_{n-\theta} - s) \, \mathrm{d}s, \ 1 \le k \le n,$$

$$b_{n-k}^{(n)} := \frac{2}{\tau_k(\tau_k + \tau_{k+1})} \int_{t_{k-1}}^{t_k} \omega_{1-\beta}(t_{n-\theta} - s)(s - t_{k-\frac{1}{2}}) \, \mathrm{d}s, \ 1 \le k \le n - 1,$$

with $\rho_k := \tau_k/\tau_{k+1}$ being the local time step-size ratios. It has been proved in [14, 15] that the discrete coefficients of the nonuniform Alikhanov formula (with $\pi_A = 11/4$ and $\rho = 7/4$, where $\rho := \max_k \{\rho_k\}$ is the maximum step-size ratio) satisfy two basic properties:

A1. The discrete kernels are positive and monotone: $A_0^{(n)} \ge A_1^{(n)} \ge \cdots \ge A_{n-1}^{(n)} > 0$;

A2. The discrete kernels fulfill $A_{n-k}^{(n)} \ge \frac{1}{\pi_A} \int_{t_{k-1}}^{t_k} \frac{\omega_{1-\beta}(t_n-s)}{\tau_k} \, \mathrm{d}s$ for $1 \le k \le n \le N$.

7.2 The proof of (2.1)

We will go through the proof of [14, Lemma A.1] to show that

$$2(\mathbf{z}^n)^T \mathbf{Q}^{(n)} (\mathcal{D}_{\tau}^{\beta} \mathbf{z})^{n-\theta} \ge \sum_{k=1}^n A_{n-k}^{(n)} \nabla_{\tau} [(\mathbf{z}^k)^T \mathbf{Q}^{(n)} \mathbf{z}^k] + \frac{((\mathcal{D}_{\tau}^{\beta} \mathbf{z})^{n-\theta})^T \mathbf{Q}^{(n)} (\mathcal{D}_{\tau}^{\beta} \mathbf{z})^{n-\theta}}{A_0^{(n)}}, \tag{7.1}$$

$$2(\mathbf{z}^{n-1})^T \mathbf{Q}^{(n)} (\mathcal{D}_{\tau}^{\beta} \mathbf{z})^{n-\theta} \ge \sum_{k=1}^n A_{n-k}^{(n)} \nabla_{\tau} [(\mathbf{z}^k)^T \mathbf{Q}^{(n)} \mathbf{z}^k] - \frac{((\mathcal{D}_{\tau}^{\beta} \mathbf{z})^{n-\theta})^T \mathbf{Q}^{(n)} (\mathcal{D}_{\tau}^{\beta} \mathbf{z})^{n-\theta}}{A_0^{(n)} - A_1^{(n)}}, \quad (7.2)$$

for $1 \le n \le N$ and $A_1^{(1)} := 0$.

For fix n, denote

$$J_n := 2(\mathbf{z}^n)^T \mathbf{Q}^{(n)} (\mathcal{D}_{\tau}^{\beta} \mathbf{z})^{n-\theta} - \sum_{k=1}^n A_{n-k}^{(n)} \nabla_{\tau} [(\mathbf{z}^k)^T \mathbf{Q}^{(n)} \mathbf{z}^k].$$

Then

$$J_{n} = \sum_{k=1}^{n} A_{n-k}^{(n)} \left[2(\mathbf{z}^{n})^{T} \mathbf{Q}^{(n)} (\mathbf{z}^{k} - \mathbf{z}^{k-1}) - (\mathbf{z}^{k} + \mathbf{z}^{k-1})^{T} \mathbf{Q}^{(n)} (\mathbf{z}^{k} - \mathbf{z}^{k-1}) \right]$$

$$= \sum_{k=1}^{n} A_{n-k}^{(n)} \left[\left(2\mathbf{z}^{n} - (\mathbf{z}^{k} + \mathbf{z}^{k-1}) \right)^{T} \mathbf{Q}^{(n)} (\mathbf{z}^{k} - \mathbf{z}^{k-1}) \right]$$

$$= \sum_{k=1}^{n} A_{n-k}^{(n)} (\mathbf{z}^{k} - \mathbf{z}^{k-1})^{T} \mathbf{Q}^{(n)} (\mathbf{z}^{k} - \mathbf{z}^{k-1}) + 2 \sum_{k=1}^{n} A_{n-k}^{(n)} \sum_{j=k+1}^{n} (\mathbf{z}^{j} - \mathbf{z}^{j-1})^{T} \mathbf{Q}^{(n)} (\mathbf{z}^{k} - \mathbf{z}^{k-1})$$

$$= \sum_{k=1}^{n} A_{n-k}^{(n)} (\mathbf{z}^{k} - \mathbf{z}^{k-1})^{T} \mathbf{Q}^{(n)} (\mathbf{z}^{k} - \mathbf{z}^{k-1}) + 2 \sum_{j=2}^{n} \sum_{k=1}^{j-1} A_{n-k}^{(n)} (\mathbf{z}^{j} - \mathbf{z}^{j-1})^{T} \mathbf{Q}^{(n)} (\mathbf{z}^{k} - \mathbf{z}^{k-1}).$$

where the identity $2\mathbf{z}^n - (\mathbf{z}^k + \mathbf{z}^{k-1}) = \mathbf{z}^k - \mathbf{z}^{k-1} + 2\sum_{j=k+1}^n (\mathbf{z}^j - \mathbf{z}^{j-1})$ has been employed in the third equality.

Next, introduce the quantities

$$\mathbf{w}^j := \sum_{k=1}^j A_{n-k}^{(n)} (\mathbf{z}^k - \mathbf{z}^{k-1})$$
 and $B_j := \frac{1}{A_{n-j}^{(n)}}$ for $1 \le j \le n$.

It holds that $\mathbf{z}^j - \mathbf{z}^{j-1} = B_j(\mathbf{w}^j - \mathbf{w}^{j-1})$ for $2 \leq j \leq n$, and $B_1 \geq B_2 \geq \cdots \geq B_n$ (according to the monotone property in $\mathbf{A1}$). Then

$$J_{n} = B_{1}(\mathbf{w}^{1})^{T} \mathbf{Q}^{(n)} \mathbf{w}^{1} + \sum_{j=2}^{n} B_{j}(\mathbf{w}^{j} - \mathbf{w}^{j-1})^{T} \mathbf{Q}^{(n)} (\mathbf{w}^{j} - \mathbf{w}^{j-1}) + 2 \sum_{j=2}^{n} B_{j} (\mathbf{w}^{j} - \mathbf{w}^{j-1})^{T} \mathbf{Q}^{(n)} \mathbf{w}^{j-1}$$

$$= B_{1}(\mathbf{w}^{1})^{T} \mathbf{Q}^{(n)} \mathbf{w}^{1} + \sum_{j=2}^{n} B_{j} \left[(\mathbf{w}^{j})^{T} \mathbf{Q}^{(n)} \mathbf{w}^{j} - (\mathbf{w}^{j-1})^{T} \mathbf{Q}^{(n)} \mathbf{w}^{j-1} \right]$$

$$= B_{n}(\mathbf{w}^{n})^{T} \mathbf{Q}^{(n)} \mathbf{w}^{n} + \sum_{j=1}^{n-1} (B_{j} - B_{j+1}) (\mathbf{w}^{j})^{T} \mathbf{Q}^{(n)} \mathbf{w}^{j}$$

$$> B_{n}(\mathbf{w}^{n})^{T} \mathbf{Q}^{(n)} \mathbf{w}^{n}.$$

because $\mathbf{Q}^{(n)}$ is a positive definite matrix. Hence, the inequality (7.1) is valid since $\mathbf{w}^n = (\mathcal{D}_{\tau}^{\beta}\mathbf{z})^{n-\theta}$ and $B_n = 1/A_0^{(n)}$. Similarly, it is easy to trace the remaining parts of [14, Lemma A.1] to check inequality (7.2).

According to [14, Lemma 4.1] and [15, Corollary 2.3], with the maximum time-step ratio $\rho = 7/4$, we have

$$\frac{1-\theta}{A_0^{(n)}} - \frac{\theta}{A_0^{(n)} - A_1^{(n)}} \ge 0,$$

which further leads to (2.1) by a simple combination of (7.1) and (7.2).

7.3 Truncation error analysis

The truncation errors in (3.19) and (4.16) are given by

$$\mathcal{T}_{u}(\mathbf{x}_{h}, t_{n-\theta}) := \mathcal{D}_{t}^{\alpha} u(\mathbf{x}_{h}, t_{n-\theta}) - \left(\mathcal{D}_{\tau}^{\alpha} u(\mathbf{x}_{h}, \cdot)\right)^{n-\theta},
\mathcal{T}_{A}(\mathbf{x}_{h}, t_{n-\theta}) := \mathcal{A}_{h}^{n-\theta} \left\{ u(\mathbf{x}_{h}, t_{n-\theta}) - \left[(1 - \theta) u(\mathbf{x}_{h}, t_{n}) + \theta u(\mathbf{x}_{h}, t_{n-1}) \right] \right\},
\mathcal{T}_{\tilde{u}}(\mathbf{x}_{h}, t_{n-\theta}) := \mathcal{D}_{t}^{\beta} \tilde{u}(\mathbf{x}_{h}, t_{n-\theta}) - \left(\mathcal{D}_{\tau}^{\beta} \tilde{u}(\mathbf{x}_{h}, \cdot)\right)^{n-\theta},
\mathcal{T}_{v}(\mathbf{x}_{h}, t_{n-\theta}) := \mathcal{D}_{t}^{\beta} v(\mathbf{x}_{h}, t_{n-\theta}) - \left(\mathcal{D}_{\tau}^{\beta} v(\mathbf{x}_{h}, \cdot)\right)^{n-\theta},
\mathcal{S}(\mathbf{x}_{h}, t_{n-\theta}) := (\mathcal{A}u)(\mathbf{x}_{h}, t_{n-\theta}) - \mathcal{A}_{h}^{n-\theta} u(\mathbf{x}_{h}, t_{n-\theta}),$$

for $\mathbf{x}_h \in \Omega_h$ and $1 \leq n \leq N$.

We first study the spatial error $\mathcal{S}(\mathbf{x}_h, t_{n-\theta})$. By the Taylor expansion (see also [27, eq. (31)]), we can take a continuous function $\xi^n(\mathbf{x})$ such that

$$\xi^{n}(\mathbf{x}_{h}) = \left[\partial_{x}(p_{1}\partial_{x}u) + \partial_{y}(p_{1}\partial_{y}u)\right](\mathbf{x}_{h}, t_{n-\theta}) - \left\{\delta_{x}\left[(p_{1})_{h}^{n-\theta}\delta_{x}\right] + \delta_{y}\left[(p_{1})_{h}^{n-\theta}\delta_{y}\right]\right\}u(\mathbf{x}_{h}, t_{n-\theta}),$$

where $\mathbf{x}_h \in \Omega_h$ and $1 \leq n \leq N$, and $|\xi^n(\mathbf{x}_h)| \leq C(h_x^2 + h_y^2)$ provided that $||u||_{H^4} \leq C$ and $p_k(\mathbf{x},\cdot) \in \mathcal{C}^3(\Omega)$ for k = 1, 2.

Similarly, there is a continuous function $\eta^n(\mathbf{x})$ such that

$$\eta^{n}(\mathbf{x}_{h}) = \left(p_{3}\partial_{x}u + p_{4}\partial_{y}u\right)\left[\left(\mathbf{x}_{h}, t_{n-\theta}\right) - \left[\left(p_{1}\right)_{h}^{n-\theta}\delta_{\hat{x}} + \left(p_{4}\right)_{h}^{n-\theta}\delta_{y}\right]u(\mathbf{x}_{h}, t_{n-\theta}),$$

where $\mathbf{x}_h \in \Omega_h$ and $1 \le n \le N$, and $|\eta^n(\mathbf{x}_h)| \le C(h_x^2 + h_y^2)$ provided $||u||_{H^3} \le C$ and $|p_k(\mathbf{x}, \cdot)| \le C$ for k = 3, 4.

Hence, based on V2 and the regularity assumption (1.7), we have

$$|\mathcal{S}(\mathbf{x}_h, t_{n-\theta})| = \mathcal{O}(h_x^2 + h_y^2). \tag{7.3}$$

By the Taylor expansion with integral remainder, we further get that

$$\delta_x \xi^n(x_{i+\frac{1}{2}}, y_j) = \frac{1}{2} \int_0^1 \left[\xi_x^n \left(x_{i+\frac{1}{2}} + \frac{h_x}{2} s, y_j \right) + \xi_x^n \left(x_{i+\frac{1}{2}} - \frac{h_x}{2} s, y_j \right) \right] (1 - s) \, \mathrm{d}s,$$

for $0 \le i \le M_x$, $1 \le j \le M_y - 1$, and

$$\delta_y \xi^n(x_i, y_{j+\frac{1}{2}}) = \frac{1}{2} \int_0^1 \left[\xi_y^n \left(x_i, y_{j+\frac{1}{2}} + \frac{h_y}{2} s \right) + \xi_y^n \left(x_i, y_{j+\frac{1}{2}} - \frac{h_y}{2} s \right) \right] (1 - s) \, \mathrm{d}s,$$

for $1 \le i \le M_x - 1$, $0 \le j \le M_y$. Similar formulations work for $\delta_x \eta^n(x_{i+\frac{1}{2}}, y_j)$ and $\delta_y \eta^n(x_i, y_{j+\frac{1}{2}})$. Thus, under the assumptions in **V2** and (1.7), it is easy to know that

$$\|\nabla_h \mathcal{S}(\mathbf{x}_h, t_{n-\theta})\| \le C(h_x^2 + h_y^2), \quad \mathbf{x}_h \in \Omega_h, \ 1 \le n \le N.$$
 (7.4)

For the temporal truncation errors, according to [20, Lemma 6.1], we have

$$\sum_{j=1}^{n} P_{n-j}^{(n)} \| (\mathcal{T}_A)^{n-\theta} \| \le C \tau^{\min\{2,\gamma\sigma\}}. \tag{7.5}$$

Referring to [20, eqs. (6.5), (6.6) and (6.8)], similar to the estimation of $\|\nabla_h \mathcal{S}(\mathbf{x}_h, t_{n-\theta})\|$, we have

$$\sum_{j=1}^{n} P_{n-j}^{(n)} \| \nabla_h (\mathcal{T}_A)^{n-\theta} \| \le C \tau^{\min\{2, \gamma\sigma\}}, \tag{7.6}$$

$$\sum_{i=1}^{n} P_{n-j}^{(n)} \|\nabla_h (\mathcal{T}_u)^{n-\theta}\| \le C \tau^{\min\{3-\beta,\gamma\sigma_1\}},\tag{7.7}$$

$$\sum_{j=1}^{n} P_{n-j}^{(n)} \| \nabla_h (\mathcal{T}_{\tilde{u}})^{n-\theta} \| \le C \tau^{\min\{3-\beta,\gamma\sigma_2\}}, \tag{7.8}$$

$$\sum_{i=1}^{n} P_{n-j}^{(n)} \| (\mathcal{T}_{v1})^{n-\theta} \| \le C \tau^{\min\{3-\beta,\gamma\sigma_3\}}, \tag{7.9}$$

$$\sum_{i=1}^{n} P_{n-j}^{(n)} \|\nabla_h (\mathcal{T}_{v2})^{n-\theta}\| \le C \tau^{\min\{2, \gamma \sigma_3\}}, \tag{7.10}$$

for $1 \le n \le N$, provided that assumptions in **V2** and (1.8)–(1.9) are valid.

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