GENERALIZED TRUNCATED MOMENT PROBLEMS WITH UNBOUNDED SETS

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ABSTRACT. This paper studies generalized truncated moment problems with unbounded sets. First, we study geometric properties of the truncated moment cone and its dual cone of nonnegative polynomials. By the technique of homogenization, we give a convergent hierarchy of Moment-SOS relaxations for approximating these cones. With them, we give a Moment-SOS method for solving generalized truncated moment problems with unbounded sets. Finitely atomic representing measures, or certificates for their nonexistence, can be obtained by the proposed method. Numerical experiments and applications are also given.

1. Introduction

The generalized truncated moment problem (GTMP) concerns whether or not there exists a positive Borel measure which is supported in a prescribed set and whose moments satisfy some linear equations or inequalities. For a Borel measure μ on \mathbb{R}^n , its support is the smallest closed set $T \subseteq \mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus T) = 0$. The support of μ is denoted as $\text{supp}(\mu)$. For a power $\alpha := (\alpha_1, \ldots, \alpha_n)$, the α th moment of μ is the integral

$$\int x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mathrm{d}\mu$$

if it exists. For convenience, we denote that

$$x := (x_1, \dots, x_n), \quad x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n.$$

The sum $|\alpha|$ is called the order of the moment $\int x^{\alpha} d\mu$. In applications, the support of μ is often required to be contained in a set $K \subseteq \mathbb{R}^n$ such that

(1.1)
$$K := \left\{ x \in \mathbb{R}^n \middle| \begin{array}{c} c_i(x) = 0 \ (i \in \mathcal{E}), \\ c_j(x) \ge 0 \ (j \in \mathcal{I}) \end{array} \right\},$$

where all c_i, c_j are polynomials in x. The \mathcal{E} and \mathcal{I} are label sets for the constraints. Let $\mathcal{A} \subset \mathbb{N}^n$ be a finite set of powers. The generalized truncated moment problem is often given as: Does there exists a Borel measure μ such that

$$\operatorname{supp}(\mu) \subseteq K, \quad y_{\alpha} = \int x^{\alpha} d\mu$$

²⁰²⁰ Mathematics Subject Classification. 90C23, 90C22, 44A60, 47A57.

Key words and phrases. moment, polynomial, truncated moment problem, homogenization, Moment-SOS relaxation.

for every $\alpha \in \mathcal{A}$ and the moments $y_{\alpha}(\alpha \in \mathcal{A})$ satisfy

(1.2)
$$\begin{cases} \sum_{\alpha \in \mathcal{A}} a_{i,\alpha} y_{\alpha} &= b_i \ (i = 1, \dots, m_1), \\ \sum_{\alpha \in \mathcal{A}} a_{i,\alpha} y_{\alpha} &\geq b_i \ (i = m_1 + 1, \dots, m)? \end{cases}$$

In the above, $a_{i,\alpha}$, b_i are given constants. For each i, let

$$a_i(x) = \sum_{\alpha \in A} a_{i,\alpha} x^{\alpha}.$$

Then (1.2) is equivalent to the existence of a Borel measure μ supported in K such that

(1.3)
$$\begin{cases} \int a_i(x) d\mu &= b_i \quad (i = 1, \dots, m_1), \\ \int a_i(x) d\mu &\geq b_i \quad (i = m_1 + 1, \dots, m). \end{cases}$$

How do we determine the existence of a Borel measure μ supported in K and satisfying (1.3)? The GTMP is a fundamental question in optimization [8, 9, 23, 24]. It has broad applications (see [28, 30, 37]). The generalized truncated moment optimization problem is studied in [22, 30]. In applications, we are often interested in finitely atomic measures. The measure μ is called finitely atomic if the support supp(μ) is a finite set. It is called r-atomic if supp(μ) consists of r distinct points, i.e., the cardinality $|\sup(\mu)| = r$.

1.1. The \mathcal{A} -truncated K-moment problem. A special case of the GTMP is the \mathcal{A} -truncated K-moment problem (\mathcal{A} -TKMP). For a finite power set $\mathcal{A} \subseteq \mathbb{N}^n$, we denote by $\mathbb{R}^{\mathcal{A}}$ the set of all real vectors y labelled as

$$y = (y_{\alpha})_{\alpha \in \mathcal{A}}.$$

A vector $y \in \mathbb{R}^{\mathcal{A}}$ is called an \mathcal{A} -truncated multi-sequence (\mathcal{A} -tms). The \mathcal{A} -tms y is said to admit a Borel measure μ on \mathbb{R}^n if it holds that

$$(1.4) y_{\alpha} = \int x^{\alpha} d\mu \quad \text{for all} \quad \alpha \in \mathcal{A}.$$

If it satisfies (1.4), the measure μ is called a representing measure for y and y is called an \mathcal{A} -truncated moment sequence. The \mathcal{A} -truncated moment problem concerns the existence and computation of a representing measure μ . If $\operatorname{supp}(\mu) \subseteq K$, the measure μ satisfying (1.4) is called a K-representing measure for y. Denote by $\operatorname{meas}(y,K)$ the set of all K-representing measures for y. This gives the \mathcal{A} -truncated moment cone

(1.5)
$$\mathscr{R}_{\mathcal{A}}(K) := \{ y \in \mathbb{R}^{\mathcal{A}} : meas(y, K) \neq \emptyset \}.$$

For the case y=0, its representing measure is the identically zero measure and the support is the empty set. Let $\mathbb{R}[x]_{\mathcal{A}}$ be the subspace spanned by monomials x^{α} ($\alpha \in \mathcal{A}$). The dual cone of $\mathscr{R}_{\mathcal{A}}(K)$ is the cone of polynomials in $\mathbb{R}[x]_{\mathcal{A}}$ that are nonnegative on K, i.e.,

$$\mathscr{P}_{\mathcal{A}}(K) = \{ p \in \mathbb{R}[x]_{\mathcal{A}} : p(x) \ge 0, \forall x \in K \}.$$

When the polynomials a_i are the monomials x^{α} ($\alpha \in \mathcal{A}$), the GTMP is reduced to the \mathcal{A} -TKMP (see [28]). When \mathcal{A} is the power set (for a degree d)

$$\mathbb{N}_d^n := \{ \alpha \in \mathbb{N}^n : |\alpha| \le d \},$$

the GTMP is reduced to the classical truncated K-moment problem (see [5]).

1.2. **PSOP** and **SCP** tensors. Truncated moment problems have broad applications in tensor computation (see [31, 32, 33]). For positive integers m, n, let $T^m(\mathbb{R}^n)$ denote the space of all order-m tensors over the space \mathbb{R}^n . A tensor $\mathcal{B} \in T^m(\mathbb{R}^n)$ is represented by an m-way array $\mathcal{B} = (\mathcal{B}_{i_1...i_m})_{1 \leq i_1,...,i_m \leq n}$, with real entries $\mathcal{B}_{i_1...i_m}$. The tensor \mathcal{B} is said to be symmetric if the entry $\mathcal{B}_{i_1...i_m}$ is invariant under all permutations of (i_1,\ldots,i_m) . Let $S^m(\mathbb{R}^n)$ be the subspace of symmetric tensors in $T^m(\mathbb{R}^n)$. For $u \in \mathbb{R}^n$, the outer product $u^{\otimes m}$ denotes the tensor in $S^m(\mathbb{R}^n)$ such that for every (i_1,\ldots,i_m) ,

$$(u^{\otimes m})_{i_1...i_m} = u_{i_1} \cdots u_{i_m}.$$

Definition 1.1. A symmetric tensor $\mathcal{B} \in S^m(\mathbb{R}^{n+1})$ is said to be a positive sum of powers (PSOP) tensor if there exist vectors $v_1, \ldots, v_r \in \mathbb{R}^n$ and scalars $\lambda_1, \ldots, \lambda_r \geq 0$ such that

(1.6)
$$\mathcal{B} = \sum_{k=1}^{r} \lambda_k \begin{bmatrix} 1 \\ v_k \end{bmatrix}^{\otimes m}.$$

If all $v_1, \ldots v_r$ are nonnegative vectors (i.e., all their entries are nonnegative), then the tensor \mathcal{B} as in (1.6) is said to be strongly completely positive (SCP).

Denote by PS_m^{n+1} the set of all PSOP tensors in $\mathrm{S}^m(\mathbb{R}^{n+1})$. Similarly, the set of all SCP tensors in $\mathrm{S}^m(\mathbb{R}^{n+1})$ is denoted as SCP_m^{n+1} . Every symmetric tensor in $\mathrm{S}^m(\mathbb{R}^{n+1})$ is uniquely determined by an \mathcal{A} -tms with $\mathcal{A} = \mathbb{N}_m^n$. More specifically, each symmetric tensor $\mathcal{B} = (\mathcal{B}_{i_1...i_m})_{0 \leq i_1,...,i_m \leq n}$ is uniquely determined by $\mathbf{b} = (b_{\alpha}) \in \mathbb{R}^{\mathcal{A}}$ such that

$$(1.7) b_{\alpha} = \mathcal{B}_{i_1 i_2 \dots i_m}$$

for every $\alpha \in \mathbb{N}_m^n$ with $x_0^{m-|\alpha|} \cdot x^{\alpha} = x_{i_1} \cdots x_{i_m}$. Then, one can see that the decomposition (1.6) is equivalent to

$$(1.8) b_{\alpha} = \int x^{\alpha} d\mu \ (\forall \alpha \in \mathbb{N}_{m}^{n}),$$

where $\mu = \sum_{k=1}^{r} \lambda_k \delta_{v_k}$ and δ_{v_k} denotes the unit Dirac measure supported at v_k . This shows that the tensor \mathcal{B} is PSOP if and only if **b** admits a representing measure on \mathbb{R}^n . Similarly, the tensor \mathcal{B} is SCP if and only if **b** admits a representing measure supported in the nonnegative orthant \mathbb{R}^n_+ .

When the order m is even, the tensor \mathcal{B} as in (1.6) is a sum of even powers (SOEP) tensor (see [31]). When every v_k is nonnegative, \mathcal{B} is a completely positive (CP) tensor. For nonzero PSOP and SCP tensors, the first entry of each decomposing vector is required to be strictly positive. Detecting SOEP and CP tensors is equivalent to a homogeneous truncated moment problem with the set K being the unit sphere [28]. However, this is not the case for PSOP and SCP tensor. We remark that every SCP tensor is a CP tensor, but not every CP tensor is SCP. We refer to [34] for introductions to CP tensors.

1.3. Existing work. To solve truncated moment problems, a typical method is to use flat extension (see Section 2.1 for the introduction). We refer to [3, 4, 7, 12] for related work. However, this approach has difficulty for general cases, especially when the moment matrix is positive definite. When the set K is compact, Moment-SOS relaxations can be used to solve truncated moment problems (see [22, 28, 30]). When a tms y admits a K-representing measure, it can be obtained by finding a

flat extension. When it admits no K-representing measures, a certificate for the nonexistence can also be obtained by solving Moment-SOS relaxations. There are other alternate approaches for studying GTMP. For instance, the core variety can be used to characterize existence of representing measures (see [2, 14]).

When $K = \mathbb{R}^n$, the homogenization trick is proposed in [15] to solve the truncated moment problem for the case $\mathcal{A} = \mathbb{N}_d^n$. Homogenization is a useful trick for solving optimization with unbounded sets [20, 25]. When K is an unbounded semialgebraic set, there exists little work for solving truncated moment problems.

Contributions. In this paper, we use homogenization to solve the GTMP when K is an unbounded set. The main approach is lifting K to a compact semialgebraic set in \mathbb{R}^{n+1} via homogenization. Then a hierarchy of Moment-SOS relaxations is proposed to solve the GTMP. Throughout the paper, we assume the feasible set K is closed at ∞ (see Definition 3.2). Our major contributions are:

- I. We study geometric properties of the truncated moment cone $\mathscr{R}_{\mathcal{A}}(K)$ and its dual cone $\mathscr{P}_{\mathcal{A}}(K)$ of nonnegative polynomials. A convergent hierarchy of Moment-SOS approximations is given for $\mathscr{P}_{\mathcal{A}}(K)$ and $\mathscr{R}_{\mathcal{A}}(K)$. The convergence is shown under some general assumptions.
- II. We propose a hierarchy of Moment-SOS relaxations to solve the GTMP when K is an unbounded semialgebraic set. When the closure of the moment system (see (4.2)) admits no K-representing measures, we show that the moment relaxations must be infeasible for all high enough relaxation orders. When the closure of the moment system is feasible, the method can obtain a finitely atomic representing measure. Depending whether or not the first entries of points in the support are zero, we can either obtain a finitely atomic K-representing measure for the original GTMP or obtain arbitrarily accurate approximations.

This paper is organized as follows. Section 2 gives some preliminaries. In Section 3, we give geometric properties of the cones $\mathscr{P}_{\mathcal{A}}(K)$ and $\mathscr{R}_{\mathcal{A}}(K)$. Moment-SOS relaxations are also given for them. In Section 4, a Moment-SOS algorithm is given to solve the GTMP. Its convergence is also shown. Some numerical experiments and applications are presented in Section 5. Some conclusions are given in Section 6.

2. Preliminaries

Notation. Let \mathbb{N} (resp., \mathbb{R}) denote the set of nonnegative integers (resp., real numbers). For $t \in \mathbb{R}$, $\lceil t \rceil$ denotes the smallest integer greater than or equal to t. For a positive integer m, denote $\lceil m \rceil \coloneqq \{1,\ldots,m\}$. Denote by $\mathbb{R}[x]$ the ring of real polynomials in x. Let $\mathbb{R}[x]_d$ be the set of all polynomials with degrees $\leq d$. For a set $S \subseteq \mathbb{R}^n$, cl(S) and int(S) denote its closure and interior in the Euclidean topology. The dual cone of S is the set

$$S^* := \{ y \in \mathbb{R}^n \mid \langle x, y \rangle \ge 0, \ \forall x \in S \}.$$

When S is a convex set, ri(S) denotes its relative interior. For a set V of vectors, Span V stands for the subspace spanned by V and Cone V stands for the conic hull generated by vectors in V. The notation dim V denotes the dimension of affine hull of V. We refer to [1] for these notions.

For a polynomial $p \in \mathbb{R}[x]$, $\deg(p)$ denotes its total degree in x and p^{hom} denotes the homogeneous part of the highest degree terms of p. Denote by \tilde{p} the homogenization of p, i.e., $\tilde{p} := x_0^{\deg(p)} p(\frac{x}{x_0})$. For a finite power set \mathcal{A} of \mathbb{N}^n , the degree of \mathcal{A} , denoted by $\deg(\mathcal{A})$, is $\max\{|\alpha| : \alpha \in \mathcal{A}\}$. Let $[x]_{\mathcal{A}}$ be the column vector of monomials $(x^{\alpha})_{\alpha \in \mathcal{A}}$. For a set $K \subseteq \mathbb{R}^n$, denote by $I(K)_{\mathcal{A}}$ the set of all polynomials in $\mathbb{R}[x]_{\mathcal{A}}$ which identically vanish on K, i.e.,

(2.1)
$$I(K)_{\mathcal{A}} = \{ p \in \mathbb{R}[x]_{\mathcal{A}} : p \mid_{K} \equiv 0 \}.$$

Clearly, $I(K)_{\mathcal{A}}$ is a subspace of $\mathbb{R}[x]_{\mathcal{A}}$. When $\mathcal{A} = \mathbb{N}_d^n$, we denote that

$$\mathscr{P}_d(K) := \mathscr{P}_{\mathcal{A}}(K), \quad \mathscr{R}_d(K) := \mathscr{R}_{\mathcal{A}}(K).$$

2.1. Riesz functionals and moment matrices. An \mathcal{A} -tms $y \in \mathbb{R}^{\mathcal{A}}$ determines the Riesz functional \mathcal{L}_y acting on $\mathbb{R}[x]_{\mathcal{A}}$ as

$$\mathscr{L}_y(\sum_{\alpha\in\mathcal{A}}p_\alpha x^\alpha) := \sum_{\alpha\in\mathcal{A}}p_\alpha y_\alpha.$$

For convenience of notation, we also write that

$$\langle \sum_{\alpha \in \mathcal{A}} p_{\alpha} x^{\alpha}, y \rangle = \sum_{\alpha \in \mathcal{A}} p_{\alpha} y_{\alpha}.$$

The functional \mathcal{L}_y is said to be K-positive if

$$\mathscr{L}_y(p) \ge 0, \ \forall \ p \in \mathscr{P}_{\mathcal{A}}(K).$$

Furthermore, \mathcal{L}_y is said to be strictly K-positive if

$$\mathcal{L}_{y}(p) > 0, \ \forall \ p \in \mathscr{P}_{\mathcal{A}}(K), \ p|_{K} \not\equiv 0.$$

Clearly, if y admits a K-representing measure μ , then \mathcal{L}_y must be K-positive, since $\mathcal{L}_y(p) = \int p(x) d\mu \geq 0$ for all $p \in \mathcal{P}_{\mathcal{A}}(K)$. When K is compact and $\mathcal{A} = \mathbb{N}_t^n$, it is known that $y \in \mathbb{R}^{\mathbb{N}_t^n}$ admits a K-representing measure if and only if the Riesz functional \mathcal{L}_y is K-positive (see [24, 38]). This conclusion can be generalized to any finite-dimensional subspace of $\mathbb{R}[x]$.

Theorem 2.1 ([15]). Let $K \subseteq \mathbb{R}^n$ be a compact set and let H be a finite dimensional subspace of $\mathbb{R}[x]$. Suppose there exists $f \in H$ such that f > 0 on K. For a linear functional $\mathcal{L}: H \to \mathbb{R}$, if \mathcal{L} is K-positive, i.e.,

$$p \in H$$
, $p|_{K} \geqslant 0 \Rightarrow \mathscr{L}(p) \geqslant 0$,

then there exist $m \leq \dim H$, $u_1, \ldots, u_m \in K$, and $\lambda_1, \ldots, \lambda_m > 0$, such that

(2.2)
$$\mathscr{L}(p) = \sum_{i=1}^{m} \lambda_i p(u_i) \quad (\forall p \in H).$$

In Theorem 2.1, the equation (2.2) is equivalent to that \mathcal{L} admits a finitely atomic measure supported in K. When K is compact and $H = \mathbb{R}[x]_{\mathcal{A}}$, solving \mathcal{A} -TKMP essentially requires to check whether \mathcal{L}_y is K-positive or not. However, checking the K-positivity is typically a hard question [13].

To solve truncated moment problems, we review moment and localizing matrices. For $y \in \mathbb{R}^{\mathbb{N}_{2k}^n}$ and $q \in \mathbb{R}[x]_{2k}$, the kth order localizing matrix of y generated by q is the symmetric matrix $L_q^{(k)}$ such that

$$\mathscr{L}_y\left(qp^2\right) = vec(p)^T\left(L_q^{(k)}[y]\right)vec(p),$$

for all $p \in \mathbb{R}[x]$ with $\deg(qp^2) \leq 2k$. Here, vec(p) denotes the coefficient vector of p. In particular, when q = 1, $M_k[y] := L_q^{(k)}[y]$ is called the kth order moment matrix of y, which satisfies

$$\mathcal{L}_y(p^2) = vec(p)^T M_k[y] vec(p)$$

for all $p \in \mathbb{R}[x]_k$. Let K be as in (1.1). Denote the degree

(2.3)
$$d_{K} \coloneqq \max_{i \in \mathcal{E} \cup \mathcal{I}} \left\{ 1, \lceil \deg\left(c_{i}\right)/2 \rceil \right\}.$$

If y admits a K-representing measure μ , then $\mathcal{L}_y(qp^2) = \int qp^2 d\mu$ for all p, q with the degree $\deg(qp^2) \leq 2k$. This implies that

(2.4)
$$M_k[y] \succeq 0, \ L_{c_i}^{(k)}[y] = 0 \ (i \in \mathcal{E}), \ L_{c_i}^{(k)}[y] \succeq 0 \ (j \in \mathcal{I}).$$

The condition (2.4) is necessary for y to admit a K-representing measure, but it may not be sufficient. In computational practice, a convenient condition is the flat extension. The tms $y \in \mathbb{R}^{\mathbb{N}^n_{2k}}$ is said to be flat if it satisfies the rank condition

(2.5)
$$\operatorname{rank} M_{k-d_K}[y] = \operatorname{rank} M_k[y].$$

The following is the classical result about flat extension by Curto and Fialkow [5].

Theorem 2.2 ([5]). Let K be as in (1.1). If $y \in \mathbb{R}^{\mathbb{N}^n_{2k}}$ satisfies (2.4) and it is flat, then y admits a unique K-representing measure, which is rank $M_k[y]$ -atomic.

For the case $y \in \mathbb{R}^{\mathcal{A}}$, $z \in \mathbb{R}^{\mathbb{N}^n_{2k}}$ and $2k \geq \deg(\mathcal{A})$, the z is said to be an extension of y if y is the subvector of z consisting of entries labelled by $\alpha \in \mathcal{A}$. For such a case, we write that $y = z|_{\mathcal{A}}$. Indeed, it is shown in [5] that $y \in \mathbb{R}^{\mathbb{N}^n_t}$ admits a K-representing measure if and only if it has a flat extension $z \in \mathbb{R}^{\mathbb{N}^n_{2k}}$ that satisfies (2.4) for some $2k \geq t$.

2.2. **Positive polynomials.** A set $I \subseteq \mathbb{R}[x]$ is called an ideal if $I \cdot \mathbb{R}[x] \subseteq \mathbb{R}[x]$ and $I + I \subseteq I$. For a polynomial tuple $h = (h_1, \ldots, h_t)$, its real variety is the set

$$V_{\mathbb{R}}(h) := \{x \in \mathbb{R}^n \mid h_1(x) = \dots = h_t(x) = 0\}.$$

The ideal generated by h is denoted by Ideal[h], i.e.,

$$Ideal[h] := h_1 \cdot \mathbb{R}[x] + \dots + h_t \cdot \mathbb{R}[x].$$

For a degree k, the kth truncated ideal of Ideal[h] is

(2.6)
$$\operatorname{Ideal}[h]_k := h_1 \cdot \mathbb{R}[x]_{k-\deg(h_1)} + \dots + h_t \cdot \mathbb{R}[x]_{k-\deg(h_t)}.$$

A polynomial is said to be SOS if $p = p_1^2 + \dots + p_r^2$ for $p_1, \dots, p_r \in \mathbb{R}[x]$. Denote the set of all SOS polynomials by $\Sigma[x]$. For an even degree $k \in \mathbb{N}$, denote the truncation

$$\Sigma[x]_k := \Sigma[x] \cap \mathbb{R}[x]_k$$
.

We refer to [19, 21, 26, 27] for SOS polynomials and matrices. For a polynomial tuple $g = (g_1, \ldots, g_{\ell})$, the quadratic module generated by g is

(2.7)
$$QM[g] := \Sigma[x] + g_1 \cdot \Sigma[x] + \dots + g_{\ell} \cdot \Sigma[x].$$

For an even degree k, its kth truncated quadratic module is defined as

$$(2.8) \quad \mathrm{QM}[g]_k := \Sigma[x]_k + g_1 \cdot \Sigma[x]_{k-2\lceil \deg(g_1)/2\rceil} + \dots + g_\ell \cdot \Sigma[x]_{k-2\lceil \deg(g_\ell)/2\rceil}.$$

The sum Ideal[h] + QM[g] is said to be archimedean if there exists N > 0 such that $N - ||x||^2 \in \text{Ideal}[h] + \text{QM}[g]$. Let

$$S = \{ x \in \mathbb{R}^n : h(x) = 0, \ g(x) \ge 0 \}.$$

Clearly, if Ideal[h] + QM[g] is archimedean, then S is a compact set. Note that if $p \in Ideal[h] + QM[g]$, then $p \ge 0$ on S. However, the inverse may not be true. The following conclusion is usually referenced as Putinar's Positivstellensatz.

Theorem 2.3 ([35]). Suppose Ideal[h] + QM[g] is archimedean. If p is positive on S, then $p \in Ideal[h] + QM[g]$.

We refer to [21, 23, 24, 29] for more introductions to moment and polynomial optimization.

3. Geometric properties of $\mathscr{R}_{\mathcal{A}}(K)$ and $\mathscr{P}_{\mathcal{A}}(K)$

This section gives geometric properties of the moment cone $\mathscr{R}_{\mathcal{A}}(K)$ and the nonnegative polynomial cone $\mathscr{P}_{\mathcal{A}}(K)$. These properties include relative interiors, closedness and duality. Moreover, we give convergent Moment-SOS relaxations for $\mathscr{R}_{\mathcal{A}}(K)$ and $\mathscr{P}_{\mathcal{A}}(K)$. The set K is not assumed to be compact.

The followings are basic properties of the cones $\mathscr{P}_{\mathcal{A}}(K)$ and $\mathscr{R}_{\mathcal{A}}(K)$. Recall that $I(K)_{\mathcal{A}}$ denotes the vanishing ideal as in (2.1).

Theorem 3.1. Let $A \subseteq \mathbb{N}^n$ be a finite power set, $d = \deg(A)$ and let K be the semialgebraic set as in (1.1). Then, we have

(i) It holds the duality relation

(3.1)
$$\mathscr{R}_{\mathcal{A}}(K)^* = \mathscr{P}_{\mathcal{A}}(K), \quad \mathscr{P}_{\mathcal{A}}(K)^* = cl(\mathscr{R}_{\mathcal{A}}(K)).$$

If K is compact and there exists $q \in \mathbb{R}[x]$ such that q > 0 on K, then

$$int(P_A(K)) = \{ p \in \mathbb{R}[x]_A : p|_K > 0 \},$$

and the moment cone $\mathcal{R}_{\mathcal{A}}(K)$ is closed.

(ii) The subspace spanned by the cone $\mathcal{R}_{\mathcal{A}}(K)$ is

(3.2)
$$\operatorname{Span}_{\mathcal{A}}(K) = \{ y \in \mathbb{R}^{\mathcal{A}} \mid \langle p, y \rangle = 0, \ \forall \, p \in I(K)_{\mathcal{A}} \},$$

or equivalently, $Span \mathscr{R}_{\mathcal{A}}(K) = I(K)^{\perp}_{\mathcal{A}}$. This implies that

$$\dim \mathscr{R}_{\mathcal{A}}(K) = |\mathcal{A}| - \dim I(K)_{\mathcal{A}}.$$

Furthermore, the relative interior of $\mathcal{R}_{\mathcal{A}}(K)$ is given as

$$(3.3) ri(\mathscr{R}_{\mathcal{A}}(K)) = \{ y \in \mathbb{R}^{\mathcal{A}} \mid \mathscr{L}_y \text{ is strictly } K\text{-positive} \}.$$

In particular, if $I(K)_{\mathcal{A}} = \{0\}$, the moment cone $\mathscr{R}_{\mathcal{A}}(K)$ has nonempty interior

Proof. (i) By the definition, one can directly check that $\mathscr{R}_{\mathcal{A}}(K)^* = \mathscr{P}_{\mathcal{A}}(K)$. Note that $\mathscr{R}_{\mathcal{A}}(K)$ is a convex cone. Clearly, its closure is a closed convex cone. By the bi-duality theorem (see [1]), it holds that

$$cl(\mathscr{R}_{\mathcal{A}}(K)) = \left(cl(\mathscr{R}_{\mathcal{A}}(K))^*\right)^* = (\mathscr{R}_{\mathcal{A}}(K)^*)^* = \mathscr{P}_{\mathcal{A}}(K)^*.$$

Therefore, the duality relation (3.1) holds.

When K is compact and there exists $q \in \mathbb{R}[x]_A$ such that q > 0 on K, the closedness of $\mathcal{R}_A(K)$ follows from Theorem 2.1. For each p > 0 on K, we have p + h > 0 on K for every $h \in \mathbb{R}[x]_A$ with sufficiently small coefficients. This

means that $p \in int(\mathscr{P}_{\mathcal{A}}(K))$. Conversely, for each $p \in int(\mathscr{P}_{\mathcal{A}}(K))$, we have $p - \epsilon q \in \mathscr{P}_{\mathcal{A}}(K)$ for some $\epsilon > 0$. Then $p \ge \epsilon q > 0$ on K.

(ii) First, we prove (3.2). Note that $\operatorname{Span} \mathscr{R}_{\mathcal{A}}(K)$ is the subspace spanned by all vectors $[u]_{\mathcal{A}}$, with $u \in K$. For every $u \in K$ and for every $p \in I(K)_{\mathcal{A}}$, it is clear that $\langle [u]_{\mathcal{A}}, p \rangle = 0$, so

$$\operatorname{Span} \mathscr{R}_{\mathcal{A}}(K) \subseteq \{ y \in \mathbb{R}^{\mathcal{A}} \mid \langle p, y \rangle = 0, \ \forall \, p \in I(K)_{\mathcal{A}} \}.$$

We prove the reverse containment " \supseteq " also holds. For each $\xi \in \mathbb{R}^A$ such that $\xi \notin \operatorname{Span} \mathscr{R}_A(K)$, there must exist a polynomial $q \in \mathbb{R}[x]_A$ such that

$$\langle q, \xi \rangle = 1, \quad \langle q, [u]_{\mathcal{A}} \rangle = 0, \ \forall \ u \in K.$$

This implies that

$$\operatorname{Span} \mathscr{R}_{\mathcal{A}}(K) \supseteq \{ y \in \mathbb{R}^{\mathcal{A}} \mid \langle p, y \rangle = 0, \ \forall \, p \in I(K)_{\mathcal{A}} \}.$$

So, the equality (3.2) holds and hence

$$\dim \mathcal{R}_{\mathcal{A}}(K) = |\mathcal{A}| - \dim I(K)_{\mathcal{A}}.$$

Second, we prove (3.3). Since K is closed, there exists a positive Borel measure μ whose support is exactly equal to K. Let z be the \mathcal{A} -tms such that $z_{\alpha} = \int_{K} x^{\alpha} d\mu$ for every $\alpha \in \mathcal{A}$. Clearly, \mathcal{L}_{z} is strictly K-positive and $z \in \operatorname{Span} \mathscr{R}_{\mathcal{A}}(K)$. The equality (3.3) is implied by the following two facts.

1) For each $y \in ri(\mathcal{R}_{\mathcal{A}}(K))$, there exists $\epsilon > 0$ such that $y - \epsilon z \in \mathcal{R}_{\mathcal{A}}(K)$, so

$$\mathcal{L}_{y}(p) = \mathcal{L}_{y-\epsilon z}(p) + \epsilon \mathcal{L}_{z}(p) \ge \epsilon \mathcal{L}_{z}(p) > 0$$

for every $p \in \mathscr{P}_{\mathcal{A}}(K)$, $p|_{K} \not\equiv 0$. This means that \mathscr{L}_{y} is strictly K-positive.

2) Consider $y \in \mathbb{R}^{\mathcal{A}}$ such that \mathcal{L}_y is strictly K-positive. Let $P := \mathbb{R}[x]_{\mathcal{A}}/I(K)_{\mathcal{A}}$ be the quotient space. The equivalent class of $p \in \mathbb{R}[x]_{\mathcal{A}}$ in P is denoted as [p]. Its quotient norm is

$$||[p]||_P = \inf_{a \in I(K)_A} ||p - a||_2.$$

Define the Riesz functional $\tilde{\mathcal{L}}_y$ on P such that

$$\tilde{\mathscr{L}}_y([p]) := \mathscr{L}_y(p).$$

For $q \in I(K)_{\mathcal{A}}$, we have $\mathscr{L}_y(q) \geq 0$ and $\mathscr{L}_y(-q) \geq 0$, so $\mathscr{L}_y(q) = 0$. This means that $\tilde{\mathscr{L}}_y$ is well-defined. Denote the set

$$T := \{ [p] \in \mathscr{P}_{\mathcal{A}}(K)/I(K)_{\mathcal{A}} : ||[p]||_{P} = 1 \}.$$

Clearly, T is a compact subset of P. Denote the constants

$$\delta_1 := \min \left\{ \tilde{\mathcal{L}}_y([p]) : [p] \in T \right\},$$

$$\delta_2 := \max \left\{ \tilde{\mathcal{L}}_w([p]) \middle| \begin{array}{l} w \in \operatorname{Span} \mathcal{R}_{\mathcal{A}}(K), \\ \|w\| = 1, [p] \in T \end{array} \right\}.$$

Note that \mathscr{L}_y is strictly K-positive if and only if $\delta_1 > 0$. For every $w \in \operatorname{Span} \mathscr{R}_{\mathcal{A}}(K)$ with ||w|| = 1 and for every $0 \le \epsilon \le \frac{\delta_1}{2\delta_2}$, it holds that

$$\tilde{\mathcal{L}}_{y+\epsilon w}([p]) = \tilde{\mathcal{L}}_y([p]) + \epsilon \tilde{\mathcal{L}}_w([p]) \ge \delta_1 - \frac{\delta_1}{2} > 0$$

for all $[p] \in T$. Thus, $\mathcal{L}_{y+\epsilon w}$ is K-positive and $y + \epsilon w \in cl(\mathcal{R}_{\mathcal{A}}(K))$. Note that $ri(cl(\mathcal{R}_{\mathcal{A}}(K))) = ri(\mathcal{R}_{\mathcal{A}}(K))$. This means that $y \in ri(\mathcal{R}_{\mathcal{A}}(K))$.

Last, if $I(K)_{\mathcal{A}} = \{0\}$, then $\dim \mathcal{R}_{\mathcal{A}}(K) = |\mathcal{A}|$, i.e., $\mathcal{R}_{\mathcal{A}}(K)$ is full-dimensional and it must have nonempty interior.

As an exposition, consider the PSOP tensor cone PS_m^{n+1} (see Definition 1.1). Observe that $\mathrm{PS}_m^{n+1} = \mathscr{R}_m(K)$ with $K = \mathbb{R}^n$. Since $I(\mathbb{R}^n)_{\mathbb{N}_m^n} = \{0\}$, we know the cone PS_m^{n+1} is full-dimensional and has nonempty interior, by Theorem 3.1. When K is unbounded, the moment cone $\mathscr{R}_A(K)$ is generally not closed, while $\mathscr{P}_A(K)$ is always closed. For instance, when $K = \mathbb{R}$ and $A = \mathbb{N}_4^1$, for y = (1, 1, 1, 1, 2), the Riesz functional \mathscr{L}_y is K-positive but y admits no K-representing measures (see [6]). In [11], Easwaran and Fialkow exhibited a family of positive linear functionals that admit no representing measures. However, if K is compact and there exists $q \in \mathbb{R}[x]_A$ such that q > 0 on K, the cone $\mathscr{R}_A(K)$ is closed (see item (i) of Theorem 3.1). When K is compact, convergent semidefinite relaxations for $\mathscr{P}_A(K)$ and $\mathscr{R}_A(K)$ are given in [30]. When K is unbounded, these relaxations typically do not converge. In the following, we use the trick of homogenization to construct convergent semidefinite relaxations for $\mathscr{P}_A(K)$ and $\mathscr{R}_A(K)$ when K is unbounded.

3.1. **Homogenization.** Let K be the set as in (1.1) and let $\tilde{x} := (x_0, x)$. For a polynomial p, recall that $\tilde{p} = x_0^{\deg(p)} p(x/x_0)$ denotes the homogenization of p. Define the sets

(3.4)
$$\widetilde{K}^{h} := \left\{ \tilde{x} \in \mathbb{R}^{n+1} \middle| \begin{array}{l} \tilde{c}_{i}(\tilde{x}) = 0 \ (i \in \mathcal{E}), \\ \tilde{c}_{j}(\tilde{x}) \geq 0 \ (j \in \mathcal{I}), \ x_{0} \geq 0 \end{array} \right\}, \\ \widetilde{K}^{c} := \widetilde{K}^{h} \cap \{\tilde{x} \in \mathbb{R}^{n+1} : x_{0} > 0\}, \\ \widetilde{K} := \widetilde{K}^{h} \cap \{\tilde{x} \in \mathbb{R}^{n+1} : x_{0}^{2} + x^{T}x = 1\}$$

Definition 3.2 ([25]). The set K is said to be closed at ∞ if $cl(\widetilde{K}^c) = \widetilde{K}^h$.

We remark that the above definition of closedness at ∞ depends on the describing polynomials for K. Throughout the paper, when the closedness at ∞ is mentioned, the describing polynomials as in (1.1) are clear in the context. It is interesting to note that the closedness at ∞ is a genericity property (see [17]). The following conclusion is obvious.

Lemma 3.3 ([25]). A polynomial p > 0 (resp., $p \ge 0$) on K if and only if $\tilde{p} > 0$ (resp., $\tilde{p} \ge 0$) on \tilde{K}^c . Moreover, if K is closed at ∞ , then $p \ge 0$ on K if and only if $\tilde{p} > 0$ on \tilde{K} .

Recall that p^{hom} denotes the homogeneous part of the highest degree terms of p. Denote

(3.5)
$$K^{e} := \left\{ x \in \mathbb{R}^{n} \middle| \begin{array}{l} c_{i}^{hom}(x) = 0 \ (i \in \mathcal{E}), \\ c_{j}^{hom}(x) \ge 0 \ (j \in \mathcal{I}), \\ \|x\|^{2} - 1 = 0 \end{array} \right\}.$$

When K is compact, the interior of $\mathscr{P}_d(K)$ is precisely the set of polynomials in $\mathbb{R}[x]_d$ that are positive on K. When K is unbounded, the interior of $\mathscr{P}_d(K)$ is more tricky. The interiors of $\mathscr{P}_d(K)$ and $\mathscr{P}_A(K)$ can be characterized as follows.

Theorem 3.4. Let K be as in (1.1). Suppose K is closed at ∞ . Then, we have that

(i) For a degree d > 0, the interior of $\mathscr{P}_d(K)$ can be given as

(3.6)
$$int(\mathscr{P}_d(K)) = \{ p \in \mathbb{R}[x]_d \mid p|_K > 0, p^{(d)}|_{K^e} > 0 \},$$

where $p^{(d)}$ denotes homogeneous part of degree-d terms of p.

(ii) For a finite set $A \subseteq \mathbb{N}^n$, let $d = \deg(A)$. If $\mathbb{R}[x]_A \cap int(\mathscr{P}_d(K)) \neq \emptyset$, then

(3.7)
$$int(\mathscr{P}_{\mathcal{A}}(K)) = \left\{ p \in \mathbb{R}[x]_{\mathcal{A}} \mid p|_{K} > 0, p^{(d)}|_{K^{e}} > 0 \right\}.$$

Proof. When K is bounded, the conclusions follow directly from item (i) of Theorem 3.1 and the fact that $K^e = \emptyset$. We only consider that K is unbounded in the following. Note that for the unbounded K, the positivity conditions $p|_K > 0$, $p^{(d)}|_{K^e} > 0$ are equivalent to that $p|_K > 0$, $\deg(p) = d$, $p^{hom}|_{K^e} > 0$.

(i) First, consider an arbitrary $p \in int(\mathscr{P}_d(K))$. Then $p - \epsilon \cdot 1 \in \mathscr{P}_d(K)$ for some $\epsilon > 0$, so $p \mid_K > 0$. For every power α with $|\alpha| = d$, we have $p \pm \epsilon x^{\alpha} \in \mathscr{P}_d(K)$ for some $\epsilon > 0$, which is impossible if $\deg(p) < d$. Hence, $\deg(p) = d$. Next, we show that $p^{hom}|_{K^e} > 0$. For every point $u \in K^e$, since ||u|| = 1, we must have $u_i \neq 0$ for some $i = 1, \ldots, n$. There exists $\epsilon > 0$ such that $p - \epsilon \cdot sign(u_i)^d x_i^d \in \mathscr{P}_d(K)$. By Lemma 3.3, we can get

$$(p - \epsilon \cdot sign(u_i)^d x_i^d)^{hom} = p^{hom} - \epsilon \cdot sign(u_i)^d x_i^d \ge 0$$
 on K^e .

Hence, $p^{hom}(u) \ge \epsilon \cdot |u_i|^d > 0$. Since u is arbitrary in K^e , we have $p^{hom}|_{K^e} > 0$.

Second, consider $p \in \mathbb{R}[x]_d$ such that $p|_K > 0$, $p^{(d)}|_{K^e} > 0$. Note that $p \in \mathscr{P}_d(K)$. Suppose otherwise that $p \notin int(\mathscr{P}_d(K))$. Then there exists a supporting hyperplane \mathcal{H} for $\mathscr{P}_d(K)$ through the point p. Let $q \in \mathbb{R}[x]_d$ be the normal direction for \mathcal{H} . Then $p - \frac{1}{k}q \notin \mathscr{P}_d(K)$ for all $k \in \mathbb{N}$. For each k, there exists $x^{(k)} \in K$ such that

$$p\left(x^{(k)}\right) - \frac{1}{k}q\left(x^{(k)}\right) < 0.$$

The sequence $\{x^{(k)}\}_{k=1}^{\infty}$ must be unbounded. If otherwise it is bounded, then $\{x^{(k)}\}$ has an accumulation point and one can assume $x^{(k)} \to x^* \in K$, without loss of generality. Hence, we get

$$p(x^*) = \lim_{k \to \infty} p\left(x^{(k)}\right) - \lim_{k \to \infty} \frac{1}{k} q\left(x^{(k)}\right) \le 0,$$

which contradicts that $p|_K > 0$. Let $y^{(k)} = \frac{x^{(k)}}{\|x^{(k)}\|}$, then the sequence $\{y^{(k)}\}_{k=1}^{\infty}$ is bounded. Similarly, we can assume that $y^{(k)} \to y^*$. Clearly, $\|y^*\| = 1$. Note that

$$c_i^{hom}(y^*) = \lim_{k \to \infty} \frac{c_i(x^{(k)})}{\|x^{(k)}\|^{\deg(c_i)}}$$

for every i. Therefore,

$$c_i^{hom}(y^*) = 0 \ (i \in \mathcal{E}), \quad c_j^{hom}(y^*) \geq 0 \ (j \in \mathcal{I}).$$

This means that $y^* \in K^e$. Furthermore, we have

$$p^{(d)}(y^*) = \lim_{k \to \infty} \frac{p(x^{(k)}) - \frac{1}{k}q(x^{(k)})}{\|x^{(k)}\|^d} \le 0.$$

It contracts that $p^{(d)}|_{K^e} > 0$. Hence, we have $p \in int(\mathscr{P}_d(K))$.

(ii) Suppose $p|_{K} > 0$, $p^{(d)}|_{K^e} > 0$. By the item (i), we have $p \in int(\mathscr{P}_d(K))$, and hence $p \in int(\mathscr{P}_A(K))$.

On the another hand, suppose $p \in int(\mathscr{P}_{\mathcal{A}}(K))$. Let q be a polynomial in $\mathscr{P}_{\mathcal{A}}(K) \cap int(\mathscr{P}_d(K))$. For $\epsilon > 0$ small enough, we have $p - \epsilon q \in \mathscr{P}_{\mathcal{A}}(K)$. Clearly, for $x \in K$, $p(x) \geq \epsilon q(x) > 0$. Moreover, the polynomial $p^{hom} - \epsilon q^{hom} \geq 0$ on K^e , which implies $p^{hom}|_{K^e} > 0$. If $\deg(p) < d$, let $x^{(k)} \in K$ be a sequence such

that $||x^{(k)}|| \to \infty$. Denote $y^{(k)} = \frac{x^{(k)}}{||x^{(k)}||}$. Without loss of generality, assume that $y^{(k)} \to y^*$. Then we have $y^* \in K^e$ and the following holds

$$-\epsilon q^{(d)}(y^*) = \lim_{k \to \infty} \frac{p(x^{(k)}) - \epsilon q(x^{(k)})}{\|x^{(k)}\|^d} \ge 0,$$

which contradicts that $q^{(d)}|_{K^e} > 0$. Thus, we have $\deg(p) = d$ and $p^{(d)}|_{K^e} > 0$. \square

Remark 3.5. It is possible that $\mathscr{P}_d(K)$ has empty interior. For instance, this is the case for $K = \mathbb{R}^n$ and d = 2k+1. For such cases, we should consider the relative interior of $\mathscr{P}_{d}(K)$. Clearly, the relative interior of $\mathscr{P}_{2k+1}(\mathbb{R}^n)$ is the interior of $\mathscr{P}_{2k}(\mathbb{R}^n)$. When K is a general set, it is typically difficult to character the relative interior of $\mathscr{P}_d(K)$. Moreover, when $\mathbb{R}[x]_A \cap int(\mathscr{P}_d(K)) = \emptyset$, the conclusion of Theorem 3.4 may not hold. For instance, when $K = \mathbb{R}^2$ and $A = \{(2,0), (0,2), (2,2)\}$, the polynomial $p = x_1^2 + x_2^2 + x_1^2 x_2^2 \in int(\mathscr{P}_A(K))$ but $p^{hom}(x) = x_1^2 x_2^2$ is not positive on K^e .

For the power set $\mathcal{A} \subseteq \mathbb{N}^n$, let $d = \deg(\mathcal{A})$. Its homogenization is the set

$$\tilde{\mathcal{A}} := \{ \beta \in \mathbb{N}^{n+1} \mid \beta = (d - |\alpha|, \alpha), \ \alpha \in \mathcal{A} \}.$$

Proposition 3.6. Suppose K is closed at ∞ and $\mathbb{R}[x]_{\mathcal{A}} \cap int(\mathscr{P}_d(K)) \neq \emptyset$. Then, we have $p \in int(\mathscr{P}_{\mathcal{A}}(K))$ if and only if $\hat{p}|_{\widetilde{K}} > 0$, where \hat{p} denotes the degree-d homogenization of p.

Proof. By the assumptions, $p \in int(\mathscr{P}_{\mathcal{A}}(K))$ is equivalent to that $p|_{K} > 0, p^{(d)}|_{K^{e}} > 0$ (see Theorem 3.4). This holds if and only if $\hat{p}|_{\widetilde{K}} > 0$.

3.2. Semidefinite relaxations of $\mathcal{R}_{\mathcal{A}}(K)$ and $\mathcal{P}_{\mathcal{A}}(K)$. Each \mathcal{A} -tms $y \in \mathbb{R}^{\mathcal{A}}$ is uniquely determined by its homogenization $\tilde{y} \in \mathbb{R}^{\tilde{\mathcal{A}}}$ such that

$$\tilde{y}_{(d-|\alpha|,\alpha)} = y_{\alpha} (\alpha \in \mathcal{A}),$$

where $d = \deg(\mathcal{A})$. We define the Riesz functional $\mathcal{L}_{\tilde{y}}$ acting on $\mathbb{R}[\tilde{x}]_{\tilde{\mathcal{A}}}$ as

$$\mathscr{L}_{\tilde{y}}(\sum_{\alpha \in \mathcal{A}} p_{\alpha} x_0^{d-|\alpha|} x^{\alpha}) = \sum_{\alpha \in \mathcal{A}} p_{\alpha} y_{\alpha}.$$

Theorem 3.7. Suppose K is closed at ∞ and $\mathbb{R}[x]_{\mathcal{A}} \cap int(\mathscr{P}_d(K)) \neq \emptyset$. Then, we have $y \in cl(\mathscr{R}_{\mathcal{A}}(K))$ if and only if $\tilde{y} \in \mathscr{R}_{\tilde{\mathcal{A}}}(\tilde{K})$.

Proof. Suppose $y \in cl(\mathscr{R}_{\mathcal{A}}(K))$, then the Reisz functional \mathscr{L}_y is K-positive (see Theorem 3.1). If $\ell \in \mathscr{P}_{\tilde{A}}(\widetilde{K})$, then $\ell(1,x) \geq 0$ for $x \in K$. Hence, we have

$$\mathscr{L}_{\tilde{y}}(\ell(\tilde{x})) = \mathscr{L}_{y}(\ell(1,x)) \ge 0,$$

which implies $\mathscr{L}_{\tilde{y}}$ is \widetilde{K} -positive. Note that \widetilde{K} is compact and there exists a polynomial $\eta \in \mathbb{R}[\tilde{x}]_{\tilde{A}}$ such that $\eta \mid_{\widetilde{K}} > 0$ (see Proposition 3.6), thus conditions of Theorem 2.1 are satisfied and $\tilde{y} \in \mathscr{R}_{\tilde{A}}(\widetilde{K})$. On the another hand, if $\tilde{y} \in \mathscr{R}_{\tilde{A}}(\widetilde{K})$, then $\mathscr{L}_{\tilde{y}}$ is \widetilde{K} -positive. For $p \in \mathscr{P}_{\mathcal{A}}(K)$, the following holds

$$\sum_{\alpha \in \mathcal{A}} p_{\alpha} x_0^{d-|\alpha|} x^{\alpha} = x_0^{d-\deg(p)} \tilde{p} \ge 0, \ \forall \tilde{x} \in \widetilde{K}.$$

Hence, we have

$$\mathscr{L}_{y}(p) = \mathscr{L}_{\tilde{y}}(\sum_{\alpha \in A} p_{\alpha} x_{0}^{d-|\alpha|} x^{\alpha}) \ge 0.$$

By Theorem 3.1, it implies that $y \in cl(\mathcal{R}_{\mathcal{A}}(K))$.

Remark 3.8. It is possible that $y \in cl(\mathscr{R}_{\mathcal{A}}(K))$ while y admits no K-representing measures. In such case, y is the limit of a sequence of \mathcal{A} -truncated multi-sequences that admit finitely atomic K-representing measures. When $y \in cl(\mathscr{R}_{\mathcal{A}}(K))$, we have $\tilde{y} \in \mathscr{R}_{\tilde{\mathcal{A}}}(\tilde{K})$. Suppose the measure $\nu = \tilde{\lambda}_1 \delta_{(\tau_1, v_1)} + \cdots + \tilde{\lambda}_r \delta_{(\tau_r, v_r)}$ is a finitely atomic \tilde{K} -representing measure of \tilde{y} , where $(\tau_i, v_i) \in \tilde{K}$, $\tilde{\lambda}_i > 0$. If $y \notin \mathscr{R}_{\mathcal{A}}(K)$, then there exist $1 \leq i_1, \ldots, i_t \leq r$ such that $\tau_{i_1} = \cdots = \tau_{i_t} = 0$. When K is closed at ∞ , we have $\tilde{u}^{(i_1,k)} \to (0, v_{i_1}), \ldots, \tilde{u}^{(i_t,k)} \to (0, v_{i_t})$ for

$$\tilde{u}^{(i_1,k)} = (u_0^{(i_1,k)}, u^{(i_1,k)}), \dots, \tilde{u}^{(i_t,k)} = (u_0^{(i_t,k)}, u^{(i_t,k)}) \in \tilde{K}^c.$$

Let $\mu^{(k)} := \lambda_1^{(k)} \delta_{u_r^{(k)}} + \dots + \lambda_r^{(k)} \delta_{u_r^{(k)}}$, where

$$\lambda_i^{(k)} = \begin{cases} \tilde{\lambda}_i \tau_i^d & \text{if } i \neq i_1, \dots, i_t, \\ \tilde{\lambda}_i (u_0^{(i,k)})^d & \text{if } i = i_1, \dots, i_t, \end{cases}$$

$$u_i^{(k)} = \begin{cases} \frac{\nu_i}{\tau_i} & \text{if } i \neq i_1, \dots, i_t, \\ \frac{u^{(i,k)}}{u_0^{(i,k)}} & \text{if } i = i_1, \dots, i_t. \end{cases}$$

Let $z^{(k)} = \int_K [x]_{\mathcal{A}} d\mu^{(k)}$ for each k, then $z^{(k)} \in \mathscr{R}_{\mathcal{A}}(K)$ and $z^{(k)} \to y$ as $k \to \infty$.

For the semialgebraic set K as in (1.1), denote the constraining polynomial tuples

(3.8)
$$\begin{aligned}
\tilde{c}_{eq} &:= (\tilde{c}_i(\tilde{x}))_{i \in \mathcal{E}} \cup \{\|\tilde{x}\|^2 - 1\}, \\
\tilde{c}_{in} &:= (\tilde{c}_j(\tilde{x}))_{j \in \mathcal{I}} \cup \{x_0\}
\end{aligned}$$

In the following, we give a certificate about nonexistence of K-representing measures

Lemma 3.9. Suppose K is closed at ∞ and $\mathbb{R}[x]_{\mathcal{A}} \cap int(\mathscr{P}_d(K)) \neq \emptyset$. Then $y \notin cl(\mathscr{R}_{\mathcal{A}}(K))$ if and only if there exists $q \in \mathbb{R}[\tilde{x}]_{\tilde{\mathcal{A}}}$ such that

(3.9)
$$\mathscr{L}_{\tilde{q}}(q) < 0, \quad q \in Ideal[\tilde{c}_{eq}] + QM[\tilde{c}_{in}].$$

Proof. If $y \notin cl(\mathscr{R}_{\mathcal{A}}(K))$, then $\tilde{y} \notin \mathscr{R}_{\tilde{\mathcal{A}}}(\widetilde{K})$ and there exists a polynomial $q_1 \in \mathscr{P}_{\tilde{\mathcal{A}}}(\widetilde{K})$ such that $\mathscr{L}_{\tilde{y}}(q_1) < 0$ (cf. Theorem 3.7). By Proposition 3.6, there exists a polynomial $\eta \in \mathbb{R}[\tilde{x}]_{\tilde{\mathcal{A}}}$ such that $\eta \mid_{\tilde{K}} > 0$. For $\epsilon > 0$ small enough, we have

$$q_1 + \epsilon \eta|_{\widetilde{K}} > 0, \ \mathscr{L}_{\widetilde{y}}(q_1 + \epsilon \eta) < 0.$$

Note that \widetilde{K} is archimedean. By Theorem 2.3, the following holds

$$q_1 + \epsilon \eta \in \text{Ideal}[\tilde{c}_{eq}] + \text{QM}[\tilde{c}_{in}].$$

Hence, the polynomial $q := q_1 + \epsilon \eta$ satisfies (3.9). For the contrary, suppose otherwise $y \in cl(\mathcal{R}_{n,d}(K))$. Clearly, if $q \in \text{Ideal}[\tilde{c}_{eq}] + \text{QM}[\tilde{c}_{in}]$, then $q \geq 0$ on \widetilde{K} . It implies that $\mathcal{L}_{\tilde{y}}(q) \geq 0$, which is a contradiction.

In the following, we give convergent semidefinite relaxations for $\mathcal{R}_{\mathcal{A}}(K)$ and $\mathcal{P}_{\mathcal{A}}(K)$. For each $k \in \mathbb{N}$, we consider the following kth order relaxation for the

cone $\mathcal{R}_{\mathcal{A}}(K)$

(3.10)
$$\mathscr{R}^{(k)}(K) := \left\{ y \in \mathbb{R}^{\mathcal{A}} \middle| \begin{array}{l} \tilde{y} = w|_{\tilde{\mathcal{A}}}, \ w \in \mathbb{R}^{\mathbb{N}_{2k}^{n+1}}, \\ L_{\tilde{c}_{i}}^{(k)}[w] = 0 \ (i \in \mathcal{E}), \\ L_{\|\tilde{x}\|^{2}-1}^{(k)}[w] = 0, \\ L_{\tilde{c}_{j}}^{(k)}[w] \succeq 0 \ (j \in \mathcal{I}), \\ L_{x_{0}}^{(k)}[w] \succeq 0, \ M_{k}[w] \succeq 0 \end{array} \right\}.$$

Recall that the homogenization of a polynomial p(x) is $\tilde{p} := x_0^{\deg(p)} p(x/x_0)$. The kth order SOS approximation for the cone $\mathscr{P}_{\mathcal{A}}(K)$ is

$$(3.11) \mathscr{P}^{(k)}(K) := \left\{ p \in \mathbb{R}[x]_{\mathcal{A}} : \tilde{p} \in \text{Ideal}[\tilde{c}_{eq}]_{2k} + \text{QM}[\tilde{c}_{in}]_{2k} \right\}.$$

Theorem 3.10. Suppose K is closed at ∞ and $\mathbb{R}[x]_{\mathcal{A}} \cap int(\mathscr{P}_d(K)) \neq \emptyset$. Then, we have that

(3.12)
$$\bigcap_{k=1}^{\infty} \mathscr{R}^{(k)}(K) = cl(\mathscr{R}_{\mathcal{A}}(K)),$$

(3.13)
$$int(\mathscr{P}_{\mathcal{A}}(K)) \subseteq \bigcup_{k=1}^{\infty} \mathscr{P}^{(k)}(K) \subseteq \mathscr{P}_{\mathcal{A}}(K).$$

Proof. In (3.12), the containment " \supseteq " follows from

$$cl(\mathscr{R}_{\mathcal{A}}(K)) = \mathscr{R}_{\tilde{\mathcal{A}}}(\widetilde{K}) \subseteq \bigcap_{k=1}^{\infty} \mathscr{R}^{(k)}(K).$$

To prove the reverse containment " \subseteq ", it is enough to show that if $y \notin cl(\mathscr{R}_{\mathcal{A}}(K))$, then $y \notin \mathscr{R}^{(k)}(K)$ for some k big enough. Suppose $y \notin cl(\mathscr{R}_{\mathcal{A}}(K))$, then there exists $q \in \mathbb{R}[\tilde{x}]_{\tilde{\mathcal{A}}}$ satisfying

$$\langle q, \tilde{y} \rangle < 0, \ q \in \text{Ideal}[\tilde{c}_{eq}]_{2k_1} + \text{QM}[\tilde{c}_{in}]_{2k_1},$$

for some integer k_1 (see Lemma 3.9). If $y \in \mathcal{R}^{(k_1)}(K)$, then $\tilde{y} = w|_{\tilde{\mathcal{A}}}$ for some w satisfying (3.10). Since w belongs to the dual cone of Ideal $[\tilde{c}_{eq}]_{2k_1} + \mathrm{QM}[\tilde{c}_{in}]_{2k_1}$, we get the contradiction

$$0 > \langle q, \tilde{y} \rangle = \langle q, w \rangle \ge 0.$$

Thus, $y \notin \mathcal{R}^{(k)}(K)$ for all $k \geq k_1$. So, the relation (3.12) holds.

Now, we prove (3.13). If $p \in \mathscr{P}^{(k)}(K)$, we have $\tilde{p} \geq 0$ on \tilde{K} . Then for $x \in K$, $p(x) = \tilde{p}(1,x) \geq 0$, which implies $p \in \mathscr{P}_{\mathcal{A}}(K)$. If $p \in int(\mathscr{P}_{\mathcal{A}}(K))$, then $\tilde{p} \mid_{\tilde{K}} > 0$. By Theorem 2.3, we have $\tilde{p} \in Ideal[\tilde{c}_{eq}] + QM[\tilde{c}_{in}]$, i.e., $p \in \bigcup_{k=1}^{\infty} \mathscr{P}^{(k)}(K)$.

4. Solving GTMPs

In this section, we give a Moment-SOS approach for solving generalized truncated moment problems with unbounded sets, based on homogenization. Let K be the semialgebraic set as in (1.1). Let $\mathcal{A} \subseteq \mathbb{N}^n$ be a finite power set. We have seen that the truncated moment system (1.2) is equivalent to the existence of an \mathcal{A} -tms y satisfying

(4.1)
$$\begin{cases} \langle a_i, y \rangle &= b_i \ (1 \le i \le m_1), \\ \langle a_i, y \rangle &\ge b_i \ (m_1 < i \le m), \\ y &\in \mathscr{R}_{\mathcal{A}}(K) \end{cases}$$

for some given polynomials $a_1, \ldots, a_m \in \mathbb{R}[x]_{\mathcal{A}}$.

When K is unbounded, a difficulty for solving (4.1) is that the cone $\mathcal{R}_{\mathcal{A}}(K)$ is typically not closed. In computational practice, it is usually more convenient for considering its closure moment system

(4.2)
$$\begin{cases} \langle a_i, y \rangle &= b_i \ (1 \le i \le m_1), \\ \langle a_i, y \rangle &\ge b_i \ (m_1 < i \le m), \\ y &\in cl(\mathscr{R}_{\mathcal{A}}(K)) \end{cases} .$$

It is important to observe the following: if the moment system (4.1) is feasible, then the closure system (4.2) must also be feasible; if (4.2) is infeasible, then (4.1) must also be infeasible. There may exist boundary cases that (4.1) is infeasible while (4.2) is feasible, which is a difficult case in computational practice.

Let $d = \deg(\mathcal{A})$. Recall that each $y \in \mathbb{R}^{\mathcal{A}}$ can be viewed as a homogeneous tms $\tilde{y} \in \mathbb{R}^{\tilde{\mathcal{A}}}$ labelled such that

$$\tilde{y}_{(d-|\alpha|,\alpha)} = y_{\alpha}$$
 for every $\alpha \in \mathcal{A}$.

For each $i = 1, \ldots, m$, let

$$\hat{a}_i \coloneqq x_0^d a_i(x/x_0)$$

be the degree-d homogenization of a_i . Note that $d > \deg(a_i)$ is possible. For a degree $d_1 > d$, select a generic polynomial $f \in \mathbb{R}[\tilde{x}]_{d_1}$ and consider the following linear conic optimization problem

(4.3)
$$\begin{cases} \min & \langle f, w \rangle \\ s.t. & \langle \hat{a}_i, w \rangle = b_i \ (1 \le i \le m_1), \\ & \langle \hat{a}_i, w \rangle \ge b_i \ (m_1 < i \le m), \\ & w \in \mathcal{R}_{d_1}(\widetilde{K}). \end{cases}$$

The dual optimization problem of (4.1) is

(4.4)
$$\begin{cases} \max & \sum_{i=1}^{m} b_{i}\theta_{i} \\ s.t. & f - \sum_{i=1}^{m} \theta_{i}\hat{a}_{i} \in \mathscr{P}_{d_{1}}(\widetilde{K}), \\ \theta_{i} \geq 0 & (m_{1} < i \leq m). \end{cases}$$

For convenience, denote the set

(4.5)
$$H_0 = \operatorname{Span}\{a_1, \dots, a_{m_1}\} - \operatorname{Cone}\{a_{m_1+1}, \dots, a_m\}.$$

The following are some properties of the pair (4.3)-(4.4).

Proposition 4.1. Let $d_1 > d$ be even. Suppose the system (4.1) is feasible.

- (i) If $f \in int(\Sigma[\tilde{x}]_{d_1})$, then strong duality holds and the minimum value of (4.3) is achievable.
- (ii) Suppose f is generic in $\Sigma[\tilde{x}]_{d_1}$, then the optimization problem (4.3) has a unique minimizer w^* . Moreover, every \widetilde{K} -representing measure μ of w^* is r-atomic with $r \leq m$.

Proof. (i) Since \widetilde{K} is compact, the cone $\mathscr{R}_{d_1}(\widetilde{K})$ is closed (see Theorem 3.1). The feasible set of (4.3) is nonempty, since (4.1) is feasible. When $f \in int(\Sigma[\tilde{x}]_{d_1})$, the dual optimization (4.4) has an interior point. Hence the strong duality holds and the optimization (4.3) must achieve its minimum value (see [1]).

(ii) If f is generic in $\Sigma[\tilde{x}]_{d_1}$, the objective $\langle f, w \rangle$ is a generic linear functional. Note that \tilde{K} is compact, the uniqueness can be implied by Proposition 5.2 of [28]. Since the minimizer w^* is unique, it must be an extreme point of the convex set

$$(4.6) E = \left\{ w \in \mathscr{R}_{d_1}(\widetilde{K}) \middle| \begin{array}{l} \langle \hat{a}_i, w \rangle = b_i \, (1 \le i \le m_1), \\ \langle \hat{a}_i, w \rangle \ge b_i \, (m_1 < i \le m) \end{array} \right\}.$$

Suppose μ is a \widetilde{K} -representing measure of w^* . Let $J \subseteq \{m_1 + 1, \ldots, m\}$ be the index set of active inequality constraints for w^* . In the following, we prove that μ is r-atomic with $r \leq m$.

First, we consider the case that μ is finitely atomic, say, $\mu = \lambda_1 \delta_{u_1} + \dots + \lambda_r \delta_{u_r}$, where $u_i \in \mathbb{R}^{n+1}$ are distinct and $\lambda_i > 0$. We need to show that $r \leq m$. Suppose otherwise that r > m. We claim that there exist $\lambda_1^*, \dots, \lambda_r^* \in \mathbb{R}$ and $p^* \in \mathbb{R}[\tilde{x}]_{d_1} \setminus \operatorname{Span}\{\hat{a}_1, \dots, \hat{a}_m\}$ such that

(4.7)
$$\sum_{k=1}^{r} \lambda_k^* \hat{a}_i(u_k) = 0 \ (i \in [m_1] \cup J), \\ \sum_{k=1}^{r} \lambda_k^* p^*(u_k) \neq 0$$

If such p^* and λ_k^* do not exist, then for every $p \in \mathbb{R}[\tilde{x}]_{d_1} \setminus \operatorname{Span}\{\hat{a}_1, \dots, \hat{a}_m\}$, the vector $(p(u_1), \dots, p(u_r))$ is a linear combination of the vectors

$$(\hat{a}_i(u_1),\ldots,\hat{a}_i(u_r)), i\in[m_1]\cup J.$$

We can extend $\{\hat{a}_i : i \in [m_1] \cup J\}$ to a basis of $\mathbb{R}[\tilde{x}]_{d_1}$ by choosing

$$p_1, \ldots, p_t \in \mathbb{R}[\tilde{x}]_{d_1} \setminus \operatorname{Span}\{\hat{a}_1, \ldots, \hat{a}_m\}.$$

For j = 1, ..., t and k = 1, ..., r, there exist scalars $s_{i,i}$ such that

$$\sum_{i \in [m_1] \cup J} s_{j,i} \hat{a}_i(u_k) = p_j(u_k).$$

Let $q_j := p_j - \sum_{i \in [m_1] \cup J} s_{j,i} \hat{a}_i$ for each $j = 1, \dots, t$. Let I^* be the ideal generated by

 q_1, \ldots, q_t . We show that the vectors $[\hat{a}_i]$, with $i \in [m_1] \cup J$, span the quotient space $\mathbb{R}[\tilde{x}]/I^*$. For each $p \in \mathbb{R}[\tilde{x}]_{d_1}$, we write that

$$p = \sum_{i \in [m_1] \cup J} c_i \hat{a}_i + \sum_{j=1}^t d_j p_j,$$

for some scalars $c_i, d_i \in \mathbb{R}$. Note that

$$p \equiv \sum_{i \in [m_1] \cup J} c_i \hat{a}_i + \sum_{j=1}^t d_j p_j \equiv \sum_{i \in [m_1] \cup J} c_i \hat{a}_i + \sum_{j=1}^t d_j \left(\sum_{i \in [m_1] \cup J} s_{j,i} \hat{a}_i \right) \mod I^*.$$

Thus, the equivalent class [p] of p can be spanned by vectors $[\hat{a}_i]$ $(i \in [m_1] \cup J)$. For $|\beta| = d_1 + 1$, we have $x^{\beta} = x_j x^{\alpha}$ for some $j \in [n]$ and $|\alpha| = d_1$. Since $d_1 > d$ and $\deg(x_j \hat{a}_i) \leq d_1$, it follows that $[x^{\beta}]$ can be spanned by $[\hat{a}_i]$ $(i \in [m_1] \cup J)$. By induction, we know that the vectors $[\hat{a}_i]$ $(i \in [m_1] \cup J)$ span $\mathbb{R}[\tilde{x}]/I^*$. By Theorem 2.6 of [24], we have dim $\mathbb{R}[\tilde{x}]/I^* \leq m_1 + |J|$, and the number of common zeros of polynomials q_j is no more than $m_1 + |J|$. However, for $k = 1, \ldots, r$, we have

$$q_j(u_k) = p_j(u_k) - \sum_{i \in [m_1] \cup J} s_{j,i}(u_k)\hat{a}_i(u_k) = 0,$$

which contradicts with $r > m \ge m_1 + |J|$. Hence, the equation (4.7) holds. Let $\epsilon > 0$ be small enough such that $\lambda_k \pm \epsilon \lambda_k^* > 0$ (k = 1, ..., r), and denote

$$\mu_1 = (\lambda_1 + \epsilon \lambda_1^*) \delta_{u_1} + \dots + (\lambda_r + \epsilon \lambda_r^*) \delta_{u_r},$$

$$\mu_2 = (\lambda_1 - \epsilon \lambda_1^*) \delta_{u_1} + \dots + (\lambda_r - \epsilon \lambda_r^*) \delta_{u_r}.$$

Let

$$w_1 = \int_{\widetilde{K}} [\widetilde{x}]_{d_1} d\mu_1, \quad w_2 = \int_{\widetilde{K}} [\widetilde{x}]_{d_1} d\mu_2.$$

Note that for $j \in \{m_1 + 1, ..., m\} \setminus J$, we have $\langle \hat{a}_j, w \rangle > b_j$. Thus if ϵ is small enough, we can easily check $w_1, w_2 \in E$, $w^* = \frac{1}{2}(w_1 + w_2)$, and $w_1 \neq w^*$, $w_2 \neq w^*$, which contradicts that z is an extreme point of E.

Second, consider the case that μ is not finitely atomic. It can be reduced to the case of finitely atomic measures. We refer to [28, Lemma 3.5]. Thus, every \widetilde{K} -representing measure μ of w^* must be r-atomic with $r \leq m$.

For an order $k \geq \lceil d_1/2 \rceil$, the kth order moment relaxation of (4.3) is

(4.8)
$$\begin{cases} \min & \langle f, w \rangle \\ s.t. & \langle \hat{a}_i, w \rangle = b_i \ (1 \le i \le m_1), \\ & \langle \hat{a}_i, w \rangle \ge b_i \ (m_1 < i \le m), \\ & L_{\bar{c}_i}^{(i)}[w] = 0 \ (i \in \mathcal{E}), \\ & L_{\|\bar{x}\|^2 - 1}^{(k)}[w] = 0, \\ & L_{\bar{c}_i}^{(k)}[w] \succeq 0 \ (j \in \mathcal{I}), \\ & L_{x_0}^{(k)}[w] \succeq 0, \\ & M_k[w] \succeq 0, \ w \in \mathbb{R}^{\mathbb{N}_{2k}^{n+1}}. \end{cases}$$

The dual optimization problem of (4.8) is

(4.9)
$$\begin{cases} \max & b_1 \theta_1 + \dots + b_m \theta_m \\ s.t. & f - \sum_{i=1}^m \theta_i \hat{a}_i \in \text{Ideal}[\tilde{c}_{eq}]_{2k} + \text{QM}[\tilde{c}_{in}]_{2k}, \\ \theta_{m_1+1} \ge 0, \dots, \theta_m \ge 0. \end{cases}$$

In the above, \tilde{c}_{eq} and \tilde{c}_{in} are from (3.8). The following are some properties of the relaxations (4.8)-(4.9).

Theorem 4.2. Let $d_1 > d$ be even. Then we have:

- (i) If the moment relaxation (4.8) is infeasible for some order k, then the truncated moment system (4.1) is infeasible.
- (ii) Suppose K is closed at ∞ and $H_0 \cap int(\mathscr{P}_d(K)) \neq \emptyset$. Then, the closure moment system (4.2) is infeasible if and only if the moment relaxation (4.8) is infeasible for some order k.
- (iii) Suppose K is closed at ∞ and w^* is a minimizer of (4.8) such that the truncation $w^*|_{2t}$ (2t \geq d) is flat. Then, the closure moment system (4.2) is feasible. Moreover, if ν is the \widetilde{K} -representing measure for $w^*|_{2t}$ and

$$(4.10) supp(\nu) \subset \{x_0 > 0\},$$

then the moment system (4.1) is also feasible.

Proof. (i) This conclusion is obvious, because (4.8) is a relaxation of (4.3). If (4.1) is feasible, then (4.3) and (4.8) are also feasible.

(ii) The "if" part follows from Theorem 3.7, so we only need to show the "only if" part. Suppose the system (4.2) is infeasible, i.e., the following optimization

(4.11)
$$\begin{cases} \max & 0 \\ s.t. & \langle \hat{a}_i, w \rangle = b_i \ (1 \le i \le m_1), \\ & \langle \hat{a}_i, w \rangle \ge b_i \ (m_1 < i \le m), \\ & w \in \mathcal{R}_{\mathcal{A}}(\widetilde{K}) \end{cases}$$

is infeasible. The dual problem of (4.11) is

(4.12)
$$\begin{cases} \min & \sum_{i \in [m_1]} b_i t_i - \sum_{m_1 < j \le m} b_j t_j \\ s.t. & \sum_{i \in [m_1]} t_i \hat{a}_i - \sum_{m_1 < j \le m} t_j \hat{a}_j \in \mathscr{P}_{\mathcal{A}}(\widetilde{K}), \\ t_{m_1+1} \ge 0, \dots, t_m \ge 0. \end{cases}$$

Since $H_0 \cap int(\mathscr{P}_d(K)) \neq \emptyset$, we know (4.12) has an interior point. By the strong duality, the optimization (4.12) must be unbounded below, i.e., there exists $\lambda_1^*, \dots, \lambda_{m_1}^*$, $\lambda_{m_1+1}^* \geq 0, \dots, \lambda_m^* \geq 0$ such that

$$b(\lambda^*) := \sum_{i \in [m_1]} b_i \lambda_i^* - \sum_{m_1 < j \le m} b_j \lambda_j^* < 0,$$
$$\hat{a}(\lambda^*) := \sum_{i \in [m_1]} \lambda_i^* \hat{a}_i - \sum_{m_1 < j \le m} \lambda_j^* \hat{a}_j \in \mathscr{P}_{\mathcal{A}}(\widetilde{K}).$$

Since $H_0 \cap int(\mathscr{P}_d(K)) \neq \emptyset$, let $\bar{\lambda}$ be such that $\hat{a}(\bar{\lambda}) > 0$ on K. Then for $\epsilon > 0$ small enough, $\hat{a}(\epsilon \bar{\lambda} + \lambda^*) > 0$ on K. By Putinar's Positivstellensatz (see Theorem 2.3), the following holds

$$\hat{a}(\epsilon \bar{\lambda} + \lambda^*) \in \text{Ideal}[\tilde{c}_{eq}]_{2k_1} + \text{QM}[\tilde{c}_{in}]_{2k_1}, \quad b(\epsilon \bar{\lambda} + \lambda^*) < 0,$$

for some k_1 and $\epsilon > 0$ small enough. This means that (4.9) has an improving ray, so (4.8) must be infeasible.

(iii) Since the truncation $w^*|_{2t}$ is flat, it admits a unique \widetilde{K} -representing measure ν . Moreover, the measure ν is finitely atomic (see Theorem 2.2), say,

$$\nu = \tilde{\lambda}_1 \delta_{(\tau_1, v_1)} + \dots + \tilde{\lambda}_r \delta_{(\tau_r, v_r)},$$

with each point $(\tau_i, v_i) \in \widetilde{K}$. Hence, the dehomogenized tms of $w^*|_{\widetilde{A}}$ is a feasible solution of the closure moment system (4.2). Moreover, when the assumption (4.10) holds, we have $\tau_i > 0$ for every i. Let $u_i = v_i/\tau_i$, then $u_i \in K$. So the measure

$$\mu = \lambda_1 \delta_{u_1} + \dots + \lambda_r \delta_{u_r},$$

where $\lambda_i = \tilde{\lambda}_1 \tau_i^d$, is a K-representing measure for the dehomogenized tms of $w^*|_{2t}$. This implies that the system (4.1) is feasible.

In view of the above properties, we get the following algorithm.

Algorithm 4.3. (A Moment-SOS algorithm for solving the GTMP) For the semi-algebraic set K as in (1.1), let \widetilde{K} be as in (3.4) and let d_K be as in (2.3).

Input: The polynomials $a_1, \ldots, a_{m_1}, \ldots, a_m$ and constants $b_1, \ldots, b_{m_1}, \ldots, b_m$, and a finite power set $A \subseteq \mathbb{N}^n$.

- Step 0 Let $d = \deg(A)$ and let $\hat{a}_i = x_0^d a_i(x/x_0)$. Choose an even integer $d_1 > d$ and a generic $f \in \Sigma[\tilde{x}]_{d_1}$. Let $k := \frac{d_1}{2}$.
- Step 2 Solve the Moment-SOS relaxation (4.8) (4.9). If (4.8) is infeasible, output that (4.1) is infeasible. If (4.8) is feasible, solve it for a minimizer w^* . Let $t = \max\{d_K, \lceil d/2 \rceil\}$.
- Step 3 Check whether the rank condition

(4.13)
$$\operatorname{rank} M_{t-d_K}[w^*|_{2t}] = \operatorname{rank} M_t[w^*|_{2t}].$$

is satisfied or not. If yes, compute the finitely atomic measure ν for $w^*|_{2t}$:

$$\nu = \rho_1 \delta_{\tilde{u}_1} + \dots + \rho_r \delta_{\tilde{u}_r},$$

where $r = \operatorname{rank} M_t(w^*|_{2t})$, $\tilde{u}_i = (\tau_i, v_i) \in \widetilde{K}$ and $\rho_i > 0$. If all $\tau_i > 0$, let $\mu := \lambda_1 \delta_{u_1} + \cdots + \lambda_r \delta_{u_r}$, where each $\lambda_i = \rho_i \tau_i^d$, $u_i = v_i / \tau_i$, and stop. Otherwise, go to Step 4.

Step 4 If t < k, let t := t + 1 and go to Step 3. Otherwise, let k := k + 1 and go to Step 2.

Output The finitely atomic K-representing measure μ for y satisfying (4.1), or a detection that (4.1) is infeasible.

Remark 4.4. There are the cases that $w^*|_{2t}$ is flat while its representing measure ν has an atom $\tilde{u}_l = (\tau_l, v_l)$ with the x_0 -coordinate $\tau_l = 0$. For such a case, we cannot obtain a K-representing measure satisfying (4.1), although (4.2) is feasible. When this happens, it is possible that (4.3) has other feasible points that have a finitely atomic \tilde{K} -representing measure whose support has positive x_0 -coordinates. There are two ways to get a feasible point for (4.1):

- (i) Choose a different generic f to get a new feasible point of (4.3).
- (ii) Strength the constraint $x_0 \ge 0$ in K to $x_0 \ge \epsilon$ for some $\epsilon > 0$ small.

However, no matter if a finitely atomic K-representing measure whose support has positive x_0 -coordinates can be obtained, we can always approximately get a feasible point of (4.1) by a sequence of finitely atomic K-representing measures. We refer to Remark 3.8 for how to do this.

The following is the convergence result for Algorithm 4.3.

Theorem 4.5. Let $d_1 > d$ be even. Suppose the moment system (4.1) is feasible and f is generic in $\Sigma[\tilde{x}]_{d_1}$. Then, the moment relaxation (4.8) has an optimizer w^k for every order k. Moreover, we also have:

- (i) For all t big enough, the sequence $\{w^k|_{2t}\}$ is bounded and all its accumulation points are flat. Moreover, each of its accumulation points admits a r-atomic K-representing measure with $r \leq m$.
- (ii) Assume that (4.4) has a maximizer θ^* such that

(4.14)
$$f - \sum_{i=1}^{m} \theta_i^* \hat{a}_i \in Ideal[\tilde{c}_{eq}]_{2k_1} + QM[\tilde{c}_{in}]_{2k_1}$$

and the optimization problem

(4.15)
$$\begin{cases} \min & f(\tilde{x}) - \sum_{i=1}^{m} \theta_{i}^{*} \hat{a}_{i}(\tilde{x}) \\ s.t. & \tilde{x} \in \widetilde{K} \end{cases}$$

has finitely many KKT points. Then, there exists t > 0 such that the truncation $w^k|_{2t}$ is flat for all k big enough.

Proof. Since f is generic in $\Sigma[\tilde{x}]_{d_1}$, we can assume $f \in int(\Sigma[\tilde{x}]_{d_1})$. Hence, we know (4.9) has an interior point, so the optimal values of (4.8) and (4.9) are equal, and (4.8) has an optimal minimizer w^k , by the strong duality theorem (see [1]).

(4.8) has an optimal minimizer w^k , by the strong duality theorem (see [1]). (i) First, we show the sequence $\{w^k|_{2t}\}_{k=t}^{\infty}$ is bounded. Note that $f \in int(\Sigma[\tilde{x}]_{d_1})$ and $M_k[w^k] \succeq 0$. We have that $f - \epsilon \in \Sigma[\tilde{x}]_{d_1}$ for $\epsilon > 0$ small enough and $\langle f - \epsilon, w^k \rangle \geq 0$. Let OPT(f) be the optimal value of (4.3). Since (4.8) is a relaxation of (4.3), we have

$$(4.16) (w^k)_0 \le \langle f, w^k \rangle / \epsilon \le OPT(f) / \epsilon.$$

Hence, the sequence $\{(w^k)_0\}$ is bounded. Moreover, the constraint $L_{\|\tilde{x}\|^2-1}^{(k)}[w^k]=0$ gives that

$$\langle \|\tilde{x}\|^{2t}, w^k \rangle = \langle \|\tilde{x}\|^{2(t-1)}, w^k \rangle = \dots = \langle 1, w^k \rangle = (w^k)_0,$$

which implies that the diagonal elements of $M_t[w^k|_{2t}]$ are all bounded by $(w^k)_0$. Combining with (4.16), it implies that $\{w^k|_{2t}\}_{k=t}^{\infty}$ is bounded. Suppose w^* is an accumulation point of $\{w^k|_{2t}\}_{k=t}^{\infty}$ for $t \geq \frac{d_1}{2}$. It is easy to see that w^* is a minimizer of (4.3). When t is sufficiently big, the flatness of w^* can be proved as in [28, Theorem 5.3]. We omit it for cleanness. Note that for generic f, the optimization problem (4.3) has a unique minimizer. Hence, the truncation of each accumulation point at degree d must be an extreme point of (4.6). Combining with Proposition 4.1, we have that each of its accumulation points admits a r-atomic K-representing measure with $r \leq m$.

(ii) Since f is generic in $\Sigma[\tilde{x}]_{d_1}$, the optimal values of (4.3) and (4.4) are equal. The condition (4.14) implies that the relaxation (4.8) – (4.9) is tight for all $k \geq k_1$. For $k \geq k_1$, the following holds

$$\langle f - \sum_{i=1}^m \theta_i^* \hat{a}_i, w^k \rangle = \langle f, w^k \rangle - \sum_{i=1}^m \theta_i^* \langle \hat{a}_i, w^k \rangle \le \langle f, w^k \rangle - \sum_{i=1}^m b_i \theta_i^* = 0.$$

Note that $f - \sum_{i=1}^{m} \theta_i^* \hat{a}_i \in \mathscr{P}_{d_1}(\widetilde{K})$. It implies that the optimal value of (4.15) is zero. The ℓ th order SOS relaxation for (4.15) is

(4.17)
$$\begin{cases} \max & \gamma \\ s.t. & f - \sum_{i=1}^{m} \theta_i^* \hat{a}_i - \gamma \in \text{Ideal}[\tilde{c}_{eq}]_{2\ell} + \text{QM}[\tilde{c}_{in}]_{2\ell}. \end{cases}$$

The above implies that the relaxation (4.17) is tight for all $\ell \geq k_1$. Hence, the assumptions in [30] for the problem (4.15) are satisfied and the truncation $w^k|_{2t}$ is flat for all t, k big enough.

5. Numerical Experiments

In this section, we give examples to solve truncated moment problems with unbounded sets by Algorithm 4.3. In our computation, we set

$$d_0 = \lceil \frac{d+1}{2} \rceil, \quad f = [x]_{d_0}^T R^T R[x]_{d_0},$$

where R is a random square matrix obeying standard Gaussian distribution. Algorithm 4.3 is implemented in the software GloptiPoly 3 [18] by calling the semidefinite program package Sedumi [36]. The computation is implemented in MATLAB R2019a on a Lenovo Laptop with CPU@1.10GHz and RAM 16.0G. For cleanness of the paper, only four digits are displayed for computational results.

Recall that a symmetric tensor $\mathcal{B} = (\mathcal{B}_{i_1...i_m})_{0 \leq i_1,...,i_m \leq n}$ is uniquely determined by the tms $\mathbf{b} := (b_{\alpha})_{\alpha \in \mathbb{N}_{m}^{n}}$ such that

$$b_{\alpha} = \mathcal{B}_{i_1 i_2 \dots i_m}$$

for every $\alpha \in \mathbb{N}_m^n$ with $x_0^{m-|\alpha|}x^{\alpha} = x_{i_1}\cdots x_{i_m}$. First, we give some examples of PSOP tensor decompositions.

Example 5.1. (i) Let $\mathcal{B} \in S^3(\mathbb{R}^5)$ be the symmetric tensor such that

$$\mathcal{B}_{ijk} = \begin{cases} 1 \text{ if } ijk = 0, \\ 3 \text{ otherwise.} \end{cases}$$

We can see that $\mathcal{B} \in cl(\mathrm{PS}_3^5)\backslash \mathrm{PS}_3^5$, i.e., the tms $\mathbf{b} \in cl(\mathscr{R}_3(\mathbb{R}^4))\backslash \mathscr{R}_3(\mathbb{R}^4)$. For each $\epsilon > 0$, let

$$\mathbf{b}^{\epsilon} \coloneqq 2\epsilon^{3} [(\frac{1}{\epsilon}, \frac{1}{\epsilon}, \frac{1}{\epsilon}, \frac{1}{\epsilon})]_{\mathbb{N}_{3}^{4}} + [(1, 1, 1, 1)]_{\mathbb{N}_{3}^{4}}.$$

The membership $\mathbf{b} \in cl(\mathcal{R}_3(\mathbb{R}^4))$ follows from $\mathbf{b}^\epsilon \to \mathbf{b}$ as $\epsilon \to 0$. To see $\mathbf{b} \notin \mathcal{R}_3(\mathbb{R}^4)$, suppose otherwise that **b** admits a representing measure μ . Then,

$$\int (1-x_1)^2 d\mu = b_{(0,0,0,0)} - 2b_{(1,0,0,0)} + b_{(2,0,0,0)} = 0.$$

This implies that $supp(\mu) \subseteq \{x_1 = 1\}$, so

$$0 = \int x_1 (1 - x_1)^2 d\mu = b_{(1,0,0,0)} - 2b_{(2,0,0,0)} + b_{(3,0,0,0)} = 2,$$

which is a contradiction. We apply the Algorithm 4.3 to check $\mathbf{b} \in cl(\mathcal{R}_3(\mathbb{R}^4))$. For each random instance of f, we get rank $M_1(w^k) = \operatorname{rank} M_2(w^k) = 2$ at the relaxation order k=2 and obtain a 2-atomic representing measure whose support consists of the points

$$(0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}),\quad (\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}}).$$

Note that x_0 -coordinate of the atom $(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is zero. This implies that $\mathcal{B} \in$ $cl(PS_3^5)$ by Remark 3.8. The computation took about 0.48 second.

(ii) Let $\mathcal{B} \in S^4(\mathbb{R}^6)$ be the symmetric tensor such that

$$\mathcal{B}_{ijks} = i + k + j + s.$$

By applying Algorithm 4.3, the relaxation (4.8) is infeasible at the order k=3. Therefore, we get $\mathcal{B} \notin cl(PS_4^6)$ and \mathcal{B} is not PSOP. The computation took about 48.12 seconds.

(iii) Let $\mathcal{B} \in S^3(\mathbb{R}^{n+1})$ be the symmetric tensor such that

$$\mathcal{B}_{ijk} = n - \max\{i, j, k\}.$$

For the case $n = 1, 2, \dots, 9$, Algorithm 4.3 always produces a PSOP decomposition for \mathcal{B} at the order k=2. For instance, when n=5, Algorithm 4.3 produces the

Table 5.1. Example 5.1 (iii)

n	time	n	time	n	time
1	0.08	2	0.11	3	0.21
4	0.43	5	0.85	6	1.93
7	4.48	8	10.81	9	25.95

following PSOP decomposition

$$\mathcal{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The computing time is presented in Table 5.1. Time consumption in the table is measured in seconds. The computational time increases as the dimension increases. This is because the matrix size of the relaxation (4.8) for the order k is around $O(n^k)$.

(iv) Consider the tensor $\mathcal{B} \in S^6(\mathbb{R}^3)$ such that $\mathbf{b} := (b_\beta)_{\beta \leq \mathbb{N}_6^2}$ is (listed in degree-lexicographic order)

$$(1,1,0,1,0,1,1,0,1,c,1,0,1,c,1+c^2,1,0,1,c,1+c^2,2c+c^3,1,0,1,c,1+c^2,2c+c^3,1+3c^2+c^4+t),$$

for two parameters $c, t \in \mathbb{R}$. This example is a variation of Example 5.2 of [16]. As pointed in [16], the tms **b** is flat and admits a representing measure when t = 0 and $\mathbf{b} \in cl(\mathscr{R}_6(\mathbb{R}^2)) \backslash \mathscr{R}_6(\mathbb{R}^2)$ when t > 0. For the case t = c = 0, Algorithm 4.3 produces the PSOP decomposition

$$\mathcal{B} = \frac{1}{2} \begin{bmatrix} 1\\1\\-1 \end{bmatrix}^{\otimes 6} + \frac{1}{2} \begin{bmatrix} 1\\1\\1 \end{bmatrix}^{\otimes 6}$$

at the relaxation order k=4. It took around 0.97 second. Next we consider the case t=1, c=0. For each random instance of f, we get rank $M_1(w^k) = \operatorname{rank} M_2(w^k) = 3$ at the order k=4 and it produces a 3-atomic measure μ for the homogenization $\tilde{\mathbf{b}}$. For some random instances of f, the support $\sup(\mu)$ consists of the points

$$(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (0, 0, -1).$$

For other instances of f, the support $supp(\mu)$ consists of the points

$$(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (0, 0, 1).$$

Since x_0 -coordinates of (0,0,1) and (0,0,-1) are zero, we get $\mathcal{B} \in cl(PS_6^3)$. The computation took about 0.96 second.

In the following, we give some examples of SCP tensor decompositions. Recall that the tensor $\mathcal B$ is SCP if and only if $\mathbf b$ admits a representing measure supported in the nonnegative orthant.

Example 5.2. Consider the symmetric tensor $\mathcal{B}(t) \in S^3(\mathbb{R}^6)$ such that

$$\mathcal{B}(t) = \begin{bmatrix} 1\\1\\0\\0\\0\\0\\0 \end{bmatrix}^{\otimes 3} + \begin{bmatrix} 1\\0\\1\\0\\0\\0 \end{bmatrix}^{\otimes 3} + \begin{bmatrix} 1\\0\\0\\1\\0\\0 \end{bmatrix}^{\otimes 3} + \begin{bmatrix} 1\\0\\0\\0\\1\\0 \end{bmatrix}^{\otimes 3} + \begin{bmatrix} t\\0\\0\\0\\1\\0 \end{bmatrix}^{\otimes 3}$$

for a parameter $t \in \mathbb{R}$. For t > 0, we clearly have $\mathcal{B}(t) \in SCP_3^6$. However, for t = 0, we have $\mathcal{B}(0) \in cl(SCP_3^6) \backslash SCP_3^6$. To see this, note that

$$\mathcal{B}(t)_{555} = 1$$
, $\mathcal{B}(t)_{055} = t$.

If $\mathcal{B}(0)$ was otherwise SCP, then there would exist a Borel measure μ such that $\operatorname{supp}(\mu) \subseteq \mathbb{R}^6$ and

$$\int x_5^3 d\mu = \mathcal{B}(0)_{555} = 1, \int x_5^2 d\mu = \mathcal{B}(0)_{055} = 0.$$

But such a Borel measure μ cannot exist, so we get $\mathcal{B}(0) \notin SCP_3^6$. Moreover, since $\mathcal{B}(t) \to \mathcal{B}(0)$ as $t \to 0$ and t > 0, we get $\mathcal{B}(0) \in cl(SCP_3^6)$. Similarly, we can show that $\mathcal{B}(t) \notin SCP_3^6$ for t < 0. The following is the performance of Algorithm 4.3 for checking SCP tensors.

(i) Consider the case t=1. By applying Algorithm 4.3, it produces the decomposition $\mathcal{B}(1) = \sum_{k=1}^{5} \rho_k \tilde{u}_k^{\otimes 3}$ at the relaxation order k=2, where

$$\begin{split} \tilde{u}_1 &= \frac{1}{\sqrt{2}}(1,1,0,0,0,0), & \rho_1 &= 2\sqrt{2}; \\ \tilde{u}_2 &= \frac{1}{\sqrt{2}}(1,0,1,0,0,0), & \rho_2 &= 2\sqrt{2}; \\ \tilde{u}_3 &= \frac{1}{\sqrt{2}}(1,0,0,1,0,0), & \rho_3 &= 2\sqrt{2}; \\ \tilde{u}_4 &= \frac{1}{\sqrt{2}}(1,0,0,0,1,0), & \rho_4 &= 2\sqrt{2}; \\ \tilde{u}_5 &= \frac{1}{\sqrt{2}}(1,0,0,0,0,1), & \rho_5 &= 2\sqrt{2}. \end{split}$$

It took about 0.97 second. Since all the x_0 -coordinates of \tilde{u}_k are positive, we know that $\mathcal{B}(1)$ is an SCP tensor.

(ii) Consider the case t=0. By applying Algorithm 4.3, it produces the decomposition $\mathcal{B}(0)=\sum_{k=1}^5 \rho_k \tilde{u}_k^{\otimes 3}$ at the order k=2, where

$$\begin{array}{lll} \tilde{u}_1 = & (0,0,0,0,0,1), & \rho_1 = 1; \\ \tilde{u}_2 = \frac{1}{\sqrt{2}}(1,1,0,0,0,0), & \rho_2 = 2\sqrt{2}; \\ \tilde{u}_3 = \frac{1}{\sqrt{2}}(1,0,1,0,0,0), & \rho_3 = 2\sqrt{2}; \\ \tilde{u}_4 = \frac{1}{\sqrt{2}}(1,0,0,1,0,0), & \rho_4 = 2\sqrt{2}; \\ \tilde{u}_5 = \frac{1}{\sqrt{2}}(1,0,0,0,1,0), & \rho_5 = 2\sqrt{2}. \end{array}$$

Since x_0 -coordinate of \tilde{u}_1 is zero, we get $\mathcal{B}(0) \in cl(\mathrm{SCP}_3^6)$. The computation took about 1.09 seconds.

(iii) Consider the case t = -1. By applying Algorithm 4.3, the relaxation (4.8) is infeasible at the relaxation order k = 2, so we get $\mathcal{B}(-1) \notin SCP_3^6$. The computation took about 0.82 second.

In the following, we present some examples on the generalized truncated moment problems with unbounded sets.

Example 5.3. (i) Consider $K = \{x \in \mathbb{R}^6 : ||x||^2 \ge 1\}, y \in \mathbb{R}^{\mathcal{A}}$ is listed as follows

$$y_{(0,0,0,0,0,0)} = t, \ y_{(2,0,0,0,0,0)} = \dots = y_{(0,0,0,0,0,0)} = 1.$$

When t > 6, y admits no K-representing measures. This is because if μ is such a measure, we have

$$6 = \int_K x_1^2 + \dots + x_6^2 d\mu \ge \int_K 1 d\mu = t,$$

which is a contradiction. And if t=6, the measure $\nu=6\delta_{(\frac{1}{\sqrt{6}},\cdots,\frac{1}{\sqrt{6}})}$ is a K-representing measure for y. Hence, $y\in \mathscr{R}_{\mathcal{A}}(K)$ when t=6.

Consider t=6, and we apply the Algorithm 4.3. At the order k=2, we have that $\operatorname{rank} M_1(w^k)=\operatorname{rank} M_2(w^k)=1$ and obtain a K-representing measure $6\delta_{(\frac{1}{\sqrt{6}},\cdots,\frac{1}{\sqrt{6}})}$ of y. The computation took about 5.93 seconds. Next, we consider t=7. At the order k=2, the relaxation (4.8) is infeasible, which gives a certificate that y admits no K-representing measures when t=7. It took about 4.23 seconds. (ii) Consider $K=\{x\in\mathbb{R}^3\mid x_1x_2x_3(x_1+x_2+x_3-6)=0,\,x_i\geq 0,\,i=1,2,3\}$, the index set \mathcal{A} and $y\in\mathbb{R}^{\mathcal{A}}$ are listed as follows

By applying Algorithm 4.3, we get rank $M_1(w^k) = \operatorname{rank} M_3(w^k) = 5$ at the order k = 3. We get a 5-atomic K-representing measure $\mu = \sum_{i=1}^{5} \lambda_i \delta_{u_i}$ of y as follows

$$\begin{array}{ll} u_1 = (4.8367, 0.0000, -8.6952), & \lambda_1 = 0.0535; \\ u_2 = (0.0000, -0.4954, 0.6272), & \lambda_2 = 4.4833; \\ u_3 = (0.0898, 0.0000, 0.7193), & \lambda_3 = 11.5641; \\ u_4 = (1.4278, 2.2915, 2.2807), & \lambda_4 = 3.2163; \\ u_5 = (0.3530, 1.1177, 0.0000), & \lambda_5 = 8.8139. \end{array}$$

The computation took about 0.24 second.

(iii) Consider $K = \{x \in \mathbb{R}^6 \mid x_i^2 - x_i x_{i+1} + 1 = 0, i = 1, \dots, 5\}$, the index set \mathcal{A} and $y \in \mathbb{R}^{\mathcal{A}}$ are listed as follows

Note that y admits no K-representing measures. Suppose otherwise μ is a K-representing measure of y, then we have

$$3 = \int_{K} x_2^4 d\mu = \int_{K} (x_1 + \frac{1}{x_1})^4 d\mu \ge \int_{K} x_1^4 + 6 d\mu = 7,$$

which is a contradiction. By applying Algorithm 4.3, we have rank $M_2(w^k) = \operatorname{rank} M_3(w^k) = 6$ at the order k = 3. The atoms of the finitely atomic measure ν

for $w^*|_6$ are

$$\begin{array}{ll} \frac{1}{\sqrt{6}}(0,1,1,1,1,1,1), & \frac{1}{\sqrt{5}}(0,0,1,1,1,1,1), \\ \frac{1}{2}(0,0,0,1,1,1,1), & \frac{1}{\sqrt{3}}(0,0,0,0,1,1,1), \\ \frac{1}{\sqrt{2}}(0,0,0,0,0,1,1), & (0,0,0,0,0,0,1). \end{array}$$

Thus, we know that $y \in cl(\mathcal{R}_{\mathcal{A}}(K))$. The computation took about 13.88 seconds. (iv) Let $K = \mathbb{R}^6$. One wonders whether there exists a measure μ supported in K such that

$$\int x_i^2 d\mu = i (i = 1, \dots, 6), \int x_i x_{i+1} d\mu = i^2 (i = 1, \dots, 5), \int x_1 x_6 d\mu = 1.$$

By applying Algorithm 4.3, the relaxation (4.8) is infeasible at the order k=2, which gives a certificate that the system above is infeasible. The computation took about 1.56 seconds.

(v) Let $K = \{x \in \mathbb{R}^4 \mid ||x||^2 \ge 1\}$. We want to know whether there exists a measure μ supported in K such that

$$\int x_1^3 x_2^3 d\mu = \int x_2^3 x_3^3 d\mu = \int x_3^3 x_4^3 d\mu = \int x_4^3 x_1^3 d\mu,$$

$$\int \left(x_1^4 x_2^2 + x_2^4 x_3^2 + x_3^4 x_4^2 + x_4^4 x_1^2 \right) d\mu \ge 4, \int \left(x_1^6 + x_2^6 + x_3^6 + x_4^6 \right) d\mu \ge 4,$$

$$\int \left(x_1^2 x_2^2 x_3^2 + x_2^2 x_3^2 x_4^2 + x_3^2 x_4^2 x_1^2 + x_4^2 x_1^2 x_2^2 \right) d\mu \le 4.$$

By applying Algorithm 4.3, we have that rank $M_1(w^k) = \operatorname{rank} M_2(w^k) = 4$ at the order k = 4, and the x_0 -coordinates of all atoms are nonzero. Thus, we get a 4-atomic K-representing measure $\mu = \sum_{i=1}^4 \lambda_i \delta_{u_i}$ of y as follows

$$\begin{array}{ll} u_1 = (-0.1401, -1.8084, -1.3964, -0.1014), & \lambda_1 = 0.0460; \\ u_2 = (-1.8723, -1.3077, -0.1776, 0.0536), & \lambda_2 = 0.0504; \\ u_3 = (-1.4048, -0.0263, -0.0682, -1.8086), & \lambda_3 = 0.0452; \\ u_4 = (-0.0154, -0.0324, 1.8546, 1.3451), & \lambda_4 = 0.0477. \end{array}$$

The computation took about 320.71 seconds.

We would like to point out that Algorithm 4.3 can be directly applied to solve moment optimization problems with unbounded sets of type (4.3). This is discussed in the work [22, 30]. We give such an example of rational polynomial optimization. Consider the rational polynomial optimization

(5.1)
$$\begin{cases} \min & \frac{f(x)}{g(x)} \\ s.t. & x \in K, \end{cases}$$

where f, g are polynomials. Here K is as in (1.1) and g(x) is assumed to be nonnegative on K. It can be shown that (5.1) is equivalent to

(5.2)
$$\begin{cases} \min & \langle f, y \rangle \\ s.t. & \langle g, y \rangle = 1, \\ y \in \mathcal{R}_d(K), \end{cases}$$

where $d = \max\{\deg(f), \deg(g)\}$. Note that (5.2) is a generalized problem of moments. Let \hat{f} , \hat{g} be the degree-d homogenization of f, g respectively. When K is closed at ∞ , one can see that (5.2) is equivalent to

(5.3)
$$\begin{cases} \min & \langle \hat{f}, w \rangle \\ s.t. & \langle \hat{g}, w \rangle = 1, \\ w \in \mathcal{R}_d(\widetilde{K}). \end{cases}$$

Algorithm 4.3 can be similarly applied to solve it. The following is such an example.

Example 5.4. Consider the problem

$$\min_{x_1, x_2 \ge 0} \frac{x_1^3 + x_2^3 + 3x_1x_2 + 1}{x_1(x_2^2 + 1) + x_2(x_1^2 + 1) + (x_1^2 + x_2^2)}.$$

In this example.

$$f = x_1^3 + x_2^3 + 3x_1x_2 + 1$$
, $g = x_1(x_2^2 + 1) + x_2(x_1^2 + 1) + (x_1^2 + x_2^2)$.

The optimal value is 1. We apply the Moment-SOS relaxation (4.8)-(4.9) to solve (5.3). At the order k=2, the flat truncation (4.13) is satisfied, and the computed optimal value is 1.

6. Conclusions and discussions

In this paper, we study the geometric properties of the moment cone and the cone of nonnegative polynomials. Semidefinite relaxations are constructed to approximate $\mathscr{R}_{\mathcal{A}}(K)$ and $\mathscr{P}_{\mathcal{A}}(K)$, and the convergence is guaranteed under the assumptions that K is closed at ∞ and $\mathbb{R}[x]_{\mathcal{A}} \cap int(\mathscr{P}_d(K)) \neq \emptyset$. Based on these relaxations, efficient algorithms are given for solving the generalized truncated moment problem. Note that we do not need K to be compact as the previous work, while asymptotic and finite convergence is guaranteed. There are broad applications of generalized moment optimization problems (see [9, 20, 22, 30]).

There is still much interesting future work to do. How can one certificate whether or not a multi-sequence $y \in \mathbb{R}^A$ admits a K-representing measure for the case that y lies on the boundary of $\mathscr{R}_{\mathcal{A}}(K)$? When $\mathscr{P}_{\mathcal{A}}(K) \cap int(\mathscr{P}_d(K)) = \emptyset$, do we still have the same properties for the Moment-SOS relaxations? How can we characterize the relative interior of $\mathscr{P}_{\mathcal{A}}(K)$ when it lies on the boundary of $\mathscr{P}_d(K)$? These questions are mostly open for us. Last, if $y \in \mathbb{R}^A$ admits a K-representing measure, how can we get a finitely atomic measure with minimum length? This problem is quite important for detecting different ranks of matrices and tensors.

Acknowledgements. Lei Huang and Ya-Xiang Yuan are partially supported by the National Natural Science Foundation of China (No. 12288201).

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