RECOVERY TYPE A POSTERIORI ERROR ESTIMATION OF AN ADAPTIVE FINITE ELEMENT METHOD FOR CAHN-HILLIARD EQUATION*

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ABSTRACT. In this paper, we derive a novel recovery type a posteriori error estimation of the Crank-Nicolson finite element method for the Cahn-Hilliard equation. To achieve this, we employ both the elliptic reconstruction technique and a time reconstruction technique based on three time-level approximations, resulting in an optimal a posteriori error estimator. We propose a time-space adaptive algorithm that utilizes the derived a posteriori error estimator as error indicators. Numerical experiments are presented to validate the theoretical findings, including comparing with an adaptive finite element method based on a residual type a posteriori error estimator.

1. Introduction

In this paper, we are interested in an adaptive finite element method for the Cahn–Hilliard equation

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$$\begin{cases} u_{t} + \mathcal{A}\left(\varepsilon \mathcal{A}u + \frac{1}{\varepsilon}f(u)\right) = 0, & \text{in } \Omega \times (0, T], \\ \partial_{\mathbf{n}}u \mid_{\partial\Omega} = 0, & \text{on } \partial\Omega \times [0, T], \\ \partial_{\mathbf{n}}\left(\varepsilon \mathcal{A}u + \frac{1}{\varepsilon}f(u)\right)\mid_{\partial\Omega} = 0, & \text{on } \partial\Omega \times [0, T], \\ u(x, 0) = u_{0}, & \text{in } \Omega \times \{t = 0\}, \end{cases}$$

$$(1.1)$$

where $\Omega \subset R^d(d=2,3)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, \mathbf{n} is the unit outward normal to the boundary $\partial\Omega$, the operator $\mathcal{A}:=-\Delta$, and the interface width $\varepsilon>0$ is a small parameter compared with the characteristic length of the laboratory scale. The nonlinear function $f(u)=F'(u)=u^3-u$ with $F(u)=\frac{1}{4}(u^2-1)^2$, which is a double well potential and drives the solution to two pure states $u=\pm1$.

The Cahn-Hilliard equation, which was introduced by Cahn and Hilliard in the late 1950s to describe the process of phase separation [6], has become a fundamental model in engineering and materials science. It also plays an increasingly important role in many other fields [4, 13]. The Cahn-Hilliard equation can be expressed as the H^{-1} -gradient flow, given by $u_t = \delta_u E(u)$, where $\delta_u E(u)$ is the variational derivative of the total free energy functional

$$E(u) = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} F(u) \right) dx.$$

It is well-known that the Cahn-Hilliard equation (1.1), subject to the prescribed boundary conditions, satisfies an energy dissipative law given by

$$\frac{d}{dt}E(u(t)) = -(u_t, u_t) \le 0.$$

Efficient and easy-to-implement numerical methods for the Cahn-Hilliard equation face several challenges, including the presence of high order derivatives, a nonlinear reaction term f(u), and the smallness of the parameter ε . To overcome these challenges, many spatial discretizations have been studied, including finite difference methods [10], finite element methods [11, 16, 25, 33], discontinuous Galerkin methods [14, 15, 28], and spectral methods [12]. Strategies to address the nonlinearity include convex-splitting methods [18], stabilization methods [32], invariant energy quantization (IEQ) approach [35, 28], and scalar variable auxiliary (SAV) approach [21, 30, 31]. Numerical approximations of the Cahn-Hilliard equation have been extensively investigated, but efficient and accurate methods are still an active area of research.

The smallness of the parameter ε in gradient flow models, including the Cahn–Hilliard equation and the Allen–Cahn equation, results in the interface layer phenomenon. To accurately simulate macroscopic processes described by these equations, it is necessary to use adaptive techniques to adjust the spatial mesh size and time step size according to the interface width ε . In recent years, some works on a posteriori error estimators and adaptive

methods have been proposed. Feng and Wu [17] developed residual-type a posteriori error estimates for conforming and mixed finite element approximations of the Cahn-Hilliard equation. A superconvergent cluster recovery (SCR)-based a posteriori error estimation and a time-space adaptive finite element algorithm was proposed in [8] for the Allen-Cahn equation. The SCR method produces a superconvergent recovered gradient, which leads to an asymptotically exact SCR-based error estimator. The primary focus of [8] was to design an adaptive algorithm based on the SCR-based error estimator, while the time adaptation of the error indicator was simply constructed based on approximations on two time levels.

In this paper, we present a novel SCR-based recovery type a posteriori error estimator for the Crank-Nicolson finite element method applied to the Cahn-Hilliard equation. The a posterior error estimator is derived using both the elliptic reconstruction technique and the time reconstruction technique. Therefore, the a posterior error estimator constructed is of greater precision and efficiency. The elliptic reconstruction technique involves separating the error between the finite element approximation and the exact solution into two categories: elliptic type and parabolic type. The key idea is to leverage pre-existing elliptic a posteriori estimators for the elliptic type error, while controlling the parabolic type error using parabolic energy estimates. In [9], a time reconstruction technique using approximations on two time levels were introduced for the Allen–Cahn equation, which allowed for the construction of a first-order a posteriori error estimator for time discretization. In this work, we utilize the time reconstruction technique involving approximations on three time levels [29], leading to a second-order a posteriori error estimator for time discretization. We employ the derived a posteriori error estimator as error indicators and propose an efficient time-space adaptive algorithm to solve the Cahn-Hilliard equation. Our numerical results show that the proposed recovery type a posteriori error estimator is more effective than a residual type error estimator and a space-only adaptive algorithm. Furthermore, our results demonstrate that the use of time step adaptation is essential in achieving accurate numerical solutions for the Cahn-Hilliard equation.

The paper is organized as follows: In Section 2, we introduce the Crank-Nicolson finite element method for discretizing the Cahn-Hilliard equation, followed by an introduction of the elliptic reconstruction for a nonlinear elliptic problem and its properties. In Section 3, we derive an optimal a posteriori error estimation for the Cahn-Hilliard equation based on the elliptic reconstruction and time reconstruction techniques. Based on the derived error estimator, we propose a time-space adaptive algorithm. In Section 4, we present several numerical examples to verify the accuracy and effectiveness of the proposed error indicators and the corresponding time-space adaptive algorithm. We present concluding remarks in Section 5. Finally, in Appendix A, we provide the proof of Theorem 3.2.

2. The discrete scheme and elliptic reconstruction

For a bounded domain $\Omega \subset \mathbb{R}^d$, we adopt the standard notations for the Sobolev space $W^{m,p}(\Omega)$ equipped with the norm $\|\cdot\|_{m,p,\Omega}$ and the semi-norm $\|\cdot\|_{m,p,\Omega}$. If p=2, we set $W^{m,p}(\Omega)=H^m(\Omega)$, $\|\cdot\|_{m,p,\Omega}=\|\cdot\|_{m,\Omega}$ and $\|\cdot\|_{m,p,\Omega}=\|\cdot\|_{m,\Omega}$. Further, if m=2, we take $\|\cdot\|=\|\cdot\|_{0,\Omega}$.

By introducing the chemical potential

$$w := \varepsilon \mathcal{A}u + \frac{1}{\varepsilon}f(u), \tag{2.1}$$

we can get the equivalent form of (1.1),

$$\begin{cases}
 u_t + \mathcal{A}w = 0, & \text{in } \Omega \times (0, T], \\
 \partial_{\mathbf{n}}w \mid_{\partial\Omega} = 0, & \text{on } \partial\Omega \times [0, T], \\
 \varepsilon \mathcal{A}u + \frac{1}{\varepsilon}f(u) - w = 0, & \text{in } \Omega \times (0, T], \\
 \partial_{\mathbf{n}}u \mid_{\partial\Omega} = 0, & \text{on } \partial\Omega \times [0, T], \\
 u(x, 0) = u_0, & \text{in } \Omega \times \{t = 0\}.
\end{cases}$$
(2.2)

2.1. The Crank-Nicolson Finite Element Scheme. For the homogeneous Neumann boundary conditions, the problem (2.2) is understood in the following weak form: find $(u, w) \in H^1(\Omega) \times H^1(\Omega)$ such that

$$\begin{cases} (u_t, v) + (\nabla w, \nabla v) = 0, & \forall v \in H^1(\Omega), \\ \varepsilon(\nabla u, \nabla \varphi) + \frac{1}{\varepsilon} (f(u), \varphi) - (w, \varphi) = 0, & \forall \varphi \in H^1(\Omega), \\ u(\cdot, 0) = u_0. \end{cases}$$
 (2.3)

Let \mathcal{T}_h be a shape regular triangulation of Ω , and V_h be the corresponding finite element space, which is defined as

$$V_h := \left\{ v \in H^1(\Omega), v |_K \in P_1(K), \forall K \in \mathcal{T}_h \right\},\,$$

where $P_1(K)$ denotes the set of linear polynomials defined in K. The semi-discrete finite element scheme of (2.2) reads: find $(u_h, w_h) \in V_h \times V_h$ such that

$$\begin{cases}
(u_{h,t}, v_h) + (\nabla w_h, \nabla v_h) = 0, & \forall v_h \in V_h, \\
\varepsilon(\nabla u_h, \nabla \varphi_h) + \frac{1}{\varepsilon} (f(u_h), \varphi_h) - (w_h, \varphi_h) = 0, & \forall \varphi_h \in V_h, \\
(u_h(x, 0) - u_0, \phi_h) = 0, & \forall \phi_h \in V_h.
\end{cases}$$
(2.4)

Generally, we rewrite the scheme (2.4) in its pointwise form

$$\begin{cases} u_{h,t} + Aw_h = 0, \\ \varepsilon Au_h + \frac{1}{\varepsilon} Pf(u_h) - w_h = 0, \\ u_h(x,0) = u_h^0 := Pu_0, \end{cases}$$
 (2.5)

where the finite-dimensional space operator $A: V_h \to V_h$ is the discrete Laplacian defined, through the Riesz representation in V_h , by

$$\langle Av, \Phi \rangle = a(v, \Phi), \quad \forall \Phi \in V_h,$$

and $P: L^2(\Omega) \to V_h$ is the $L^2(\Omega)$ -projection operator such that, for each $v \in L^2(\Omega)$, we have

$$\langle Pv, \Phi \rangle = \langle v, \Phi \rangle, \quad \forall \Phi \in V_h.$$

We divide the time interval [0,T] into a partition of N consecutive adjacent subintervals whose endpoints are denoted by $0 = t_0 < t_1 < \cdots < t_N = T$, the n-th time interval $I_n := [t_{n-1}, t_n]$ and the corresponding time step is defined as $\tau_n := t_n - t_{n-1}$. The Crank-Nicolson finite element is to find a sequence of function $(u_h^n, w_h^n) \in V_h^n \times V_h^n$ such that, for each $n = 1, 2, \ldots, N$,

$$\begin{cases}
\left(\frac{u_h^n - u_h^{n-1}}{\tau_n}, v_h\right) + \frac{1}{2} \left(\nabla w_h^n + \nabla w_h^{n-1}, \nabla v_h\right) = 0, & \forall v_h \in V_h^n, \\
\frac{\varepsilon}{2} \left(\nabla u_h^n + \nabla u_h^{n-1}, \nabla \varphi_h\right) + \frac{1}{\varepsilon} \left(\frac{f(u_h^n) + f(u_h^{n-1})}{2}, \varphi_h\right) & \\
-\frac{1}{2} \left(w_h^n + w_h^{n-1}, \varphi_h\right) = 0, & \forall \varphi_h \in V_h^n, \\
u_h(x, 0) = u_h^0.
\end{cases}$$
(2.6)

Similarly to the semi-discrete scheme, the fully discrete scheme can be written in a pointwise form as follows

$$\begin{cases} \frac{u_{h}^{n} - u_{h}^{n-1}}{\tau_{n}} + \frac{1}{2} \left(A^{n} w_{h}^{n} + A^{n-1} w_{h}^{n-1} \right) = 0, \\ \frac{\varepsilon}{2} \left(A^{n} u_{h}^{n} + A^{n-1} u_{h}^{n-1} \right) + \frac{P^{n} f(u_{h}^{n}) + P^{n-1} f(u_{h}^{n-1})}{2\varepsilon} \\ -\frac{1}{2} \left(w_{h}^{n} + w_{h}^{n-1} \right) = 0, \\ u_{h}(x, 0) = u_{h}^{0}, \end{cases}$$

$$(2.7)$$

$$t \to V^{n} \text{ is defined as the discrete Laplacian and } P^{n} : L^{2}(\Omega) \to V^{n} \text{ represents}$$

where $A^n: V_h^n \to V_h^n$ is defined as the discrete Laplacian and $P^n: L^2(\Omega) \to V_h^n$ represents the $L^2(\Omega)$ -projection operator.

2.2. Elliptic Reconstruction. The nonlinear elliptic problem corresponding to a steady state of the nonlinear evolution equation (1.1) is taken as follows: given $g \in L^2(\Omega)$, $r \in L^2(\Omega)$, find $(\mu, \nu) \in H^1(\Omega) \times H^1(\Omega)$ such that

$$\begin{cases}
\mathcal{A}\nu + \nu = g, & \text{in } \Omega, \\
\varepsilon \mathcal{A}\mu + \frac{1}{\varepsilon}h(\mu) - \nu = r, & \text{in } \Omega, \\
\nabla \mu \cdot \mathbf{n} = 0, \, \nabla \nu \cdot \mathbf{n} = 0, & \text{on } \partial \Omega,
\end{cases}$$
(2.8)

with $h(\mu) := \mu^3$. The weak form of the elliptic problem (2.8) reads: find $(\mu, \nu) \in H^1(\Omega) \times H^1(\Omega)$ such that

$$(\nabla \nu, \nabla v) + (\nu, v) = \langle g, v \rangle, \qquad \forall v \in H^1(\Omega), \tag{2.9}$$

$$\varepsilon(\nabla \mu, \nabla \varphi) + \frac{1}{\varepsilon} (h(\mu), \varphi) - (\nu, \varphi) = \langle r, \varphi \rangle, \qquad \forall \varphi \in H^1(\Omega). \tag{2.10}$$

Remark 2.1. The well-posedness of the variational problem (2.9)-(2.10) can be derived as follows. Owing to the variational problem (2.9) is the Euler-Lagrange equation of the functional

$$J(\nu) = \frac{1}{2} \int_{\Omega} |\nabla \nu|^2 + \frac{1}{2} \int_{\Omega} \nu^2 - \int_{\Omega} g\nu,$$
 (2.11)

taking the derivative of the functional $J(\nu)$, it holds that

$$\left(\frac{\delta J(\nu)}{\delta \nu}, v\right) = (\nabla \nu, \nabla v) + (\nu, v) - (g, v) = 0, \quad \forall v \in H^1(\Omega).$$
 (2.12)

Notice that $J(\nu)$ is a convex functional, then the uniqueness of the solution for scheme (2.9) is proved. As for the variational problem (2.10), it is the Euler-Lagrange equation of the functional

$$H(\mu) = \frac{\varepsilon}{2} \int_{\Omega} |\nabla \mu|^2 + \frac{1}{4\varepsilon} \int_{\Omega} \mu^4 - \int_{\Omega} s\mu, \tag{2.13}$$

where $\int_{\Omega} s\mu := (\nu, \mu) + \langle r, \mu \rangle$. Similarly, taking the derivative of the functional $H(\mu)$, it has

$$\left(\frac{\delta H(\mu)}{\delta \mu}, \varphi\right) = \varepsilon \left(\nabla \mu, \nabla \varphi\right) + \frac{1}{\varepsilon} \left(h(\mu), \varphi\right) - (s, \varphi) = 0, \quad \forall \varphi \in H^1(\Omega). \tag{2.14}$$

Due to $H(\mu)$ be a convex functional, then the uniqueness of the solution for scheme (2.10) is proved.

The finite element discretization of the elliptic problem (2.8) reads: find $(\mu_h, \nu_h) \in V_h \times V_h$ such that

$$\begin{cases}
(\nabla \nu_h, \nabla v_h) + (\nu_h, v_h) = \langle g_h, v_h \rangle, & \forall v_h \in V_h, \\
\varepsilon(\nabla \mu_h, \nabla \varphi_h) + \frac{1}{\varepsilon} (h(\mu_h), \varphi_h) - (\nu_h, \varphi_h) = \langle r_h, \varphi_h \rangle, & \forall \varphi_h \in V_h.
\end{cases}$$
(2.15)

Definition 2.1. (Gradient recovery a posteriori estimator function) For the nonlinear elliptic problem (2.8), we define the gradient recovery a posteriori estimator functional

$$\mathcal{E}_v := \mathcal{E}[v, H^1(\Omega), V_h] := ||Gv - \nabla v||, \qquad \forall v \in V_h, \tag{2.16}$$

where G is a gradient recovery operator.

Remark 2.2. As in [26], we utilize $H^1(\Omega)$ to estimate the elliptic a posteriori estimation for the gradient recovery a posteriori estimator functional \mathcal{E}_v . However, it's worth noting that there are alternative methods to compute upper and lower bounds for the error in other functional spaces, such as $L^2(\Omega)$ and $L^{\infty}(\Omega)$. Gradient recovery is a post-processing technique that has gained widespread popularity in the engineering community for its robustness as an a posteriori error estimator, its superconvergence of the recovered derivatives, and its efficiency in implementation. It involves reconstructing gradient approximations from finite element solutions to obtain improved solutions. The practical use of the recovery technique is not only to enhance the quality of the approximation but also to construct a posteriori error estimators in adaptive computation. The gradient of the finite element approximation for the Lagrange element provides a discontinuous approximation to the true gradient. Various techniques have been proposed to recover the gradient, including averaging [5, 22], local or global projections [20, 23], postprocessing interpolation [27, 34], the superconvergent patch recovery (SPR) [37], the polynomial preserving recovery (PPR) [38] and the superconvergent cluster recovery (SCR) [24].

Assumption 2.1. (Elliptic a posteriori error estimators) Assume that (μ, ν) , (μ_h, ν_h) are the exact solution and numerical solution of nonlinear elliptic problem (2.8), respectively, \mathcal{E} defined as Definition 2.1, there exists constants C_0 and C_1 , such that the following bounds hold

$$\|\nabla(\mu_h - \mu)\| \le C_0 \mathcal{E}_{\mu},$$

$$\|\nabla(\nu_h - \nu)\| \le C_1 \mathcal{E}_{\nu}.$$
 (2.17)

In [19], He and Zhou derived both a priori and a posteriori finite element error estimates for the following semilinear elliptic problems

$$\begin{cases}
-\Delta u + b(x, u) = 0, & \text{in } \Omega \times (0, T], \\
u = 0, & \text{on } \partial\Omega \times [0, T],
\end{cases}$$
(2.18)

and if the nonlinear term b satisfies

$$\sup_{x \in \bar{\Omega}} \left| b(x, y) - b(x, y_0) + \frac{\partial b}{\partial y}(x, y_0)(y_0 - y) \right| \lesssim (1 + \max\{|y|^s, |y_0|^s\})|y - y_0|^q, \ \forall y, \ y_0 \in R$$
(2.19)

with $q \in (1,2], s \in [0,5-q]$, then it has the following L^2 promotional property.

Lemma 2.1. [19] (L^2 promotional property) If $h_0 \ll 1$, $h \in (0, h_0]$, then

$$\|\mu_h - \mu\| \le C_0 h \|\mu_h - \mu\|_{1,\Omega},$$

$$\|\nu_h - \nu\| \le C_1 h \|\nu_h - \nu\|_{1,\Omega},$$
 (2.20)

here C_0, C_1 are constants and $h = \max\{h_K, K \in \mathcal{T}_h\}$.

Remark 2.3. In this paper, for the nonlinear elliptic problem (2.8), the nonlinear term $b(u) := h(u) = u^3$ satisfies

$$b(y) - b(y_0) - b'(y_0)(y - y_0) = (2y_0 + y)(y - y_0)^2,$$

which is consistent with the condition (2.19) as q=2. Thus, in view of Lemma 2.1, we have

$$\|\mu_{h} - \mu\|^{2}_{1,\Omega} = \|\mu_{h} - \mu\|^{2} + \|\nabla(\mu_{h} - \mu)\|^{2}$$

$$\leq C_{0}^{2}h^{2}\|\mu_{h} - \mu\|_{1,\Omega}^{2} + \|\nabla(\mu_{h} - \mu)\|^{2},$$

$$\|\nu_{h} - \nu\|^{2}_{1,\Omega} = \|\nu_{h} - \nu\|^{2} + \|\nabla(\nu_{h} - \nu)\|^{2}$$

$$\leq C_{1}^{2}h^{2}\|\nu_{h} - \nu\|_{1,\Omega}^{2} + \|\nabla(\nu_{h} - \nu)\|^{2},$$
(2.21)

thus, if h is small enough, it holds that

$$\|\mu_h - \mu\|_{1,\Omega} \le C_0 \|\nabla(\mu_h - \mu)\| \le C_0 \mathcal{E}_{\mu},$$

$$\|\nu_h - \nu\|_{1,\Omega} \le C_1 \|\nabla(\nu_h - \nu)\| \le C_1 \mathcal{E}_{\nu}.$$
 (2.22)

To link the Cahn–Hilliard equation and the elliptic recovered gradient estimates, we utilize the elliptic reconstruction technique.

Definition 2.2. (Elliptic reconstruction) For $1 \leq n \leq N$ with the discrete elliptic operator A^n defined as (2.7), we define the corresponding elliptic reconstruction operator $R^n: V_h^n \to H^1(\Omega)$, for each $(\chi, \vartheta) \in V_h^n \times V_h^n$, by solving for the elliptic problem

$$\begin{cases} \mathcal{A}R^{n}\vartheta + R^{n}\vartheta = A^{n}\vartheta + \vartheta, \\ \varepsilon \mathcal{A}R^{n}\chi + \frac{1}{\varepsilon}h(R^{n}\chi) - R^{n}\vartheta = \varepsilon A^{n}\chi + \frac{1}{\varepsilon}P^{n}h(\chi) - \vartheta, \end{cases}$$
(2.23)

which can be written in weak form as

$$\begin{cases}
a(R^{n}\vartheta,v) + (R^{n}\vartheta,v) = \langle A^{n}\vartheta,v\rangle + (\vartheta,v), \ \forall v \in H^{1}(\Omega), \\
\varepsilon a(R^{n}\chi,\varphi) + \frac{1}{\varepsilon}(h(R^{n}\chi),\varphi) - (R^{n}\vartheta,\varphi) = \varepsilon \langle A^{n}\chi,\varphi\rangle + \frac{1}{\varepsilon}\langle h(\chi),\varphi\rangle \\
- \langle \vartheta,\varphi\rangle, \ \forall \varphi \in H^{1}(\Omega).
\end{cases} (2.24)$$

By the Definition 2.2, it is obviously that $(R^n u_h^n, R^n w_h^n)$, (u_h^n, w_h^n) are the exact solution and numerical solution of (2.23), respectively. According to Assumption 2.1, we have

$$\|\nabla(u_h^n - R^n u_h^n)\| \le C_0 \mathcal{E}_u^n, \tag{2.25}$$

$$\|\nabla(w_h^n - R^n w_h^n)\| \le C_1 \mathcal{E}_w^n, \tag{2.26}$$

where \mathcal{E}_u^n , \mathcal{E}_w^n are defined following Definition 2.1, respectively, by

$$\mathcal{E}_u^n := \|G^n u_h^n - \nabla u_h^n\|, \qquad \forall u_h^n \in V_h^n, \tag{2.27}$$

$$\mathcal{E}_w^n := \|G^n w_h^n - \nabla w_h^n\|, \qquad \forall w_h^n \in V_h^n, \tag{2.28}$$

with $G^n := G^{V_h^n}$.

3. A POSTERIORI ERROR ESTIMATION AND ADAPTIVE ALGORITHM

In this section, we derive a recovery type a posteriori error estimation for the Cahn–Hilliard equation based on the elliptic reconstruction and time reconstruction techniques, and a time-space adaptive algorithm is also developed based on the proposed a posteriori error estimation.

3.1. A posteriori error estimation. The discrete solution is sequence of finite element functions $u_h^n \in V_h^n$ defined at each discrete time $t_n, 1 \leq n \leq N$. Define the piecewise quadratic extension [29]

$$u_h(t) := \frac{t - t_{n-1}}{\tau_n} u_h^n + \frac{t_n - t}{\tau_n} u_h^{n-1} + \frac{1}{2} (t - t_{n-1}) (t - t_n) \partial_n^2 u_h, \ t \in I_n, \ 1 \le n \le N,$$

$$w_h(t) := \frac{t - t_{n-1}}{\tau_n} w_h^n + \frac{t_n - t}{\tau_n} w_h^{n-1} + \frac{1}{2} (t - t_{n-1}) (t - t_n) \partial_n^2 w_h, \ t \in I_n, \ 1 \le n \le N,$$

$$(3.1)$$

where the term $\partial_n^2 \nu_h$ is defined as

$$\partial_n^2 \nu_h := \frac{\frac{\nu_h^n - \nu_h^{n-1}}{\tau_n} - \frac{\nu_h^{n-1} - \nu_h^{n-2}}{\tau_{n-1}}}{\frac{\underline{\tau_n + \tau_{n-1}}}{2}}$$
(3.2)

with $\nu_h^{-1} = \nu_h^0$ as n = 1.

Then we also define

$$p^n := R^n u_h^n, \qquad q^n := R^n w_h^n, \qquad n = 0, 1, 2, \dots, N,$$
 (3.3)

and denote this sequence's piecewise quadratic reconstruction in time by p(t) and q(t), that is,

$$p(t) := \frac{t - t_{n-1}}{\tau_n} p^n + \frac{t_n - t}{\tau_n} p^{n-1} + \frac{1}{2} (t - t_{n-1}) (t - t_n) \partial_n^2 p, \quad t \in I_n, \ 1 \le n \le N,$$

$$q(t) := \frac{t - t_{n-1}}{\tau_n} q^n + \frac{t_n - t}{\tau_n} q^{n-1} + \frac{1}{2} (t - t_{n-1}) (t - t_n) \partial_n^2 q, \quad t \in I_n, \ 1 \le n \le N.$$

$$(3.4)$$

The corresponding fully discrete error is defined by

$$e_u := u_h(t) - u(t),$$

 $e_w := w_h(t) - w(t),$ (3.5)

and can be split, using the elliptic reconstruction p(t) and q(t), as follows

$$e_u = (p(t) - u(t)) - (p(t) - u_h(t)) := \rho_u - \epsilon_u,$$

$$e_w = (q(t) - w(t)) - (q(t) - w_h(t)) := \rho_w - \epsilon_w.$$
(3.6)

For terms in (3.6), the following result holds.

Theorem 3.1. (Parabolic error identity) For each n = 1, 2, ..., N and each $t \in (t_{n-1}, t_n]$, it holds that

$$\partial_{t}e_{u} + \mathcal{A}\rho_{w} = \frac{A^{n}w_{h}^{n} - A^{n-1}w_{h}^{n-1}}{2} + \mathcal{A}(q(t) - q^{n}) + w_{h}^{n} - R^{n}w_{h}^{n} + (t - t_{n-\frac{1}{2}})\partial_{n}^{2}u_{h},$$

$$\varepsilon \mathcal{A}\rho_{u} - \rho_{w} = \varepsilon \frac{A^{n}u_{h}^{n} - A^{n-1}u_{h}^{n-1}}{2} + \frac{P^{n}f(u_{h}^{n}) - P^{n-1}f(u_{h}^{n-1})}{2\varepsilon} - \frac{w_{h}^{n} - w_{h}^{n-1}}{2}$$

$$+ \varepsilon \mathcal{A}(p(t) - p^{n}) - (q(t) - q^{n}) + \frac{f(u) - f(p^{n})}{\varepsilon} + \frac{1}{\varepsilon}(u_{h}^{n} - p^{n}),$$
(3.7)

where A^n and P^n are defined in (2.7), respectively.

Proof. For n = 1, 2, ..., N and $t \in (t_{n-1}, t_n]$, by the definition of u_h^n , we have

$$\partial_t u_h = \frac{u_h^n - u_h^{n-1}}{\tau_n} + (t - t_{n - \frac{1}{2}}) \partial_n^2 u_h,$$

and using the fully discrete scheme (2.7), we obtain

$$\begin{split} \partial_t u_h + \mathcal{A} q^n + q^n &= \frac{u_h^n - u_h^{n-1}}{\tau_n} + A^n w_h^n + w_h^n + (t - t_{n - \frac{1}{2}}) \partial_n^2 u_h \\ &= \frac{u_h^n - u_h^{n-1}}{\tau_n} + \frac{A^n w_h^n + A^{n-1} w_h^{n-1}}{2} + \frac{A^n w_h^n - A^{n-1} w_h^{n-1}}{2} + w_h^n + (t - t_{n - \frac{1}{2}}) \partial_n^2 u_h \\ &= \frac{A^n w_h^n - A^{n-1} w_h^{n-1}}{2} + w_h^n + (t - t_{n - \frac{1}{2}}) \partial_n^2 u_h, \end{split}$$

$$\begin{split} \varepsilon \mathcal{A}p^{n} &+ \frac{1}{\varepsilon}h(p^{n}) - q^{n} = \varepsilon A^{n}u_{h}^{n} + \frac{1}{\varepsilon}P^{n}h(u_{h}^{n}) - w_{h}^{n} \\ &= \varepsilon \frac{A^{n}u_{h}^{n} + A^{n-1}u_{h}^{n-1}}{2} + \varepsilon \frac{A^{n}u_{h}^{n} - A^{n-1}u_{h}^{n-1}}{2} + \frac{P^{n}h(u_{h}^{n}) + P^{n-1}f(u_{h}^{n-1})}{2\varepsilon} \\ &+ \frac{P^{n}h(u_{h}^{n}) - P^{n-1}f(u_{h}^{n-1})}{2\varepsilon} - \frac{w_{h}^{n} + w_{h}^{n-1}}{2} - \frac{w_{h}^{n} - w_{h}^{n-1}}{2} \\ &= \varepsilon \frac{A^{n}u_{h}^{n} - A^{n-1}u_{h}^{n-1}}{2} + \frac{P^{n}f(u_{h}^{n}) - P^{n-1}f(u_{h}^{n-1})}{2\varepsilon} - \frac{w_{h}^{n} - w_{h}^{n-1}}{2} + \frac{1}{\varepsilon}u_{h}^{n}. \end{split}$$

Hence

$$\begin{split} \partial_t u_h + \mathcal{A} q(t) &= \frac{A^n w_h^n - A^{n-1} w_h^{n-1}}{2} + \mathcal{A} (q(t) - q^n) + w_h^n - R^n w_h^n + (t - t_{n - \frac{1}{2}}) \partial_n^2 u_h, \\ \varepsilon \mathcal{A} p(t) + \frac{1}{\varepsilon} h(p^n) - q(t) &= \varepsilon \frac{A^n u_h^n - A^{n-1} u_h^{n-1}}{2} + \frac{P^n f(u_h^n) - P^{n-1} f(u_h^{n-1})}{2\varepsilon} - \frac{w_h^n - w_h^{n-1}}{2} \\ &+ \varepsilon \mathcal{A} (p(t) - p^n) - (q(t) - q^n) + \frac{1}{\varepsilon} u_h^n, \end{split}$$

and subtracting (2.2) from the above formula, we get

$$\partial_{t}e_{u} + \mathcal{A}\rho_{w} = \frac{A^{n}w_{h}^{n} - A^{n-1}w_{h}^{n-1}}{2} + \mathcal{A}(q(t) - q^{n}) + w_{h}^{n} - R^{n}w_{h}^{n} + (t - t_{n-\frac{1}{2}})\partial_{n}^{2}u_{h},$$

$$\varepsilon \mathcal{A}\rho_{u} - \rho_{w} = \varepsilon \frac{A^{n}u_{h}^{n} - A^{n-1}u_{h}^{n-1}}{2} + \frac{P^{n}f(u_{h}^{n}) - P^{n-1}f(u_{h}^{n-1})}{2\varepsilon} - \frac{w_{h}^{n} - w_{h}^{n-1}}{2}$$

$$+ \varepsilon \mathcal{A}(p(t) - p^{n}) - (q(t) - q^{n}) + \frac{f(u) - f(p^{n})}{\varepsilon} + \frac{1}{\varepsilon}(u_{h}^{n} - p^{n}).$$

Then we have the following result.

Theorem 3.2. Let $(u_h^n, w_h^n)_{n \in [0:N]}$ be the fully discrete solution, defined at each discrete time t_n , its piecewise linear extension u_h , w_h defined as (3.1), and let u, w be the exact solution of the model problem (2.2). Assume that $\overline{\Lambda}_{CH} \in L^1(0,T)$ is a function such that for almost every $t \in (0,T)$, we have

$$-\overline{\Lambda}_{CH}(t) \leq -\Lambda_{CH}(t) := \inf_{v \in \dot{V} \setminus \{0\}} \frac{\varepsilon \|\nabla v\|^2 + \varepsilon^{-1}(f'(u_h)v, v)}{\|\nabla \Delta^{-1}v\|^2},$$

and set

$$a(t) := \left(1 + \frac{5}{2\varepsilon^2} + 2(1 - \varepsilon)\overline{\Lambda}_{CH}(t)\right),$$

$$\mu_g := \sup_{t \in (0,T)} \|\tilde{f}(u_h)\|_{L^{\infty}(\Omega)}.$$

Define

$$\eta^2 := \|\nabla \Delta^{-1} e_u^0\|^2 + \sum_{n=1}^N 4\widetilde{\mathcal{E}_u^n}^2 + \sum_{n=1}^N (\eta_0^2 + \eta_1^2) \, \tau_n,$$

and assume

$$\eta^2 \le \frac{\varepsilon^{2/\sigma}}{(2\mu_a C_S(1+T))^{1/\sigma}} \left(8 \exp\left(\int_0^T a(t) dt \right) \right)^{-1 - \frac{1}{\sigma}},$$

then

$$\sup_{t \in [0,T]} \|\nabla \Delta^{-1} e_u\|^2 + \int_0^T \frac{\varepsilon^2}{2} \|\nabla e_u\|^2 dt \le 8\eta^2 \exp\left(\int_0^T a(t) dt\right),\tag{3.8}$$

where

$$\eta_{0} := \gamma_{w}^{n} + \delta_{w}^{n} + \eta_{w}^{n} + \beta_{u}^{n};
\eta_{1} := \gamma_{u}^{n} + \xi_{u}^{n} + \beta_{w}^{n} + \theta_{u}^{n} + \delta_{u}^{n} + \alpha_{u}^{n} + \zeta_{u}^{n};
\widetilde{\mathcal{E}_{u}}^{n^{2}} := \frac{C_{0}^{2}}{3} \tau_{n} \left((\mathcal{E}_{u}^{n})^{2} + (\mathcal{E}_{u}^{n-1})^{2} + \mathcal{E}_{u}^{n} \mathcal{E}_{u}^{n-1} \right)
+ C_{0}^{2} \frac{\tau_{n}^{2} \tau_{n-1} \left(\mathcal{E}_{u}^{n} + \mathcal{E}_{u}^{n-1} \right) + \tau_{n}^{3} \left(\mathcal{E}_{u-1}^{n} + \mathcal{E}_{u}^{n-2} \right)}{6 \tau_{n-1} (\tau_{n} + \tau_{n-1})} \left(\mathcal{E}_{u}^{n} + \mathcal{E}_{u}^{n-1} \right)$$

$$\begin{split} &+C_{0}^{2}\tau_{n}^{3}\frac{\left(\tau_{n-1}\left(\mathcal{E}_{u}^{n}+\mathcal{E}_{u}^{n-1}\right)+\tau_{n}\left(\mathcal{E}_{u-1}^{n}+\mathcal{E}_{u}^{n-2}\right)\right)^{2}}{30\tau_{n-1}^{2}(\tau_{n}+\tau_{n-1})^{2}};\\ &\gamma_{w}^{n}:=\left\|\frac{A^{n}w_{h}^{n}-A^{n-1}w_{h}^{n-1}}{2}\right\|_{-1};\\ &\beta_{u}^{n}:=\left\|\frac{\tau_{h}^{n}}{8}\cdot\partial_{n}^{2}u_{h}\right\|_{-1};\\ &\eta_{w}^{n}:=\left\|(A^{n-1}w_{h}^{n-1}+w_{h}^{n-1})-(A^{n}w_{h}^{n}+w_{h}^{n})\right\|_{-1}+\left\|\frac{\tau_{h}^{2}}{8}\partial_{n}^{2}(Aw_{h}+w_{h})\right\|_{-1};\\ &\delta_{w}^{n}:=\left\|w_{h}^{n}-w_{h}^{n-1}\right\|_{-1}+\left\|\frac{\tau_{h}^{2}}{8}\partial_{n}^{2}w_{h}\right\|_{-1};\\ &\delta_{w}^{n}:=\left\|w_{h}^{n}-w_{h}^{n-1}\right\|_{-1}+\left\|\frac{\tau_{h}^{2}}{8}\partial_{n}^{2}w_{h}\right\|_{-1};\\ &\delta_{u}^{n}:=\left\|\frac{u_{h}^{n}-u_{h}^{n-1}}{\varepsilon}\right\|;\\ &\delta_{u}^{n}:=\left\|\frac{u_{h}^{n}-u_{h}^{n-1}}{2}\right\|;\\ &\delta_{u}^{n}:=\left\|\frac{P^{n}f(u_{h}^{n})-P^{n-1}f(u_{h}^{n-1})}{2\varepsilon}\right\|;\\ &\theta_{u}^{n}:=2\left\|\left(\varepsilon A^{n-1}u_{h}^{n-1}+\frac{1}{\varepsilon}h(u_{h}^{n-1})-w_{h}^{n-1}\right)-\left(\varepsilon A^{n}u_{h}^{n}+\frac{1}{\varepsilon}h(u_{h}^{n})-w_{h}^{n}\right)\right\|\\ &+\left\|\left(\varepsilon A^{n-1}u_{h}^{n-1}+\frac{1}{\varepsilon}h(u_{h}^{n-1})-w_{h}^{n-1}\right)-\left(\varepsilon A^{n-2}u_{h}^{n-2}+\frac{1}{\varepsilon}h(u_{h}^{n-2})-w_{h}^{n-2}\right)\right\|;\\ &\alpha_{u}^{n}:=\frac{1}{\varepsilon}C\left(\mathcal{E}_{u}^{n}+\mathcal{E}_{u}^{n-1}+\mathcal{E}_{u}^{n-2}\right);\\ &\zeta_{u}^{n}:=\left\|\frac{3(u_{h}^{n})^{2}u_{h}^{n-1}-2(u_{h}^{n})^{3}-(u_{h}^{n-1})^{3}}{\varepsilon}\right\|+\left\|\frac{3u_{h}^{n}(u_{h}^{n-1})^{2}\partial_{u}^{2}u_{h}-3u_{h}^{n}u_{h}^{n-1}\partial_{u}^{2}u_{h}}{8\varepsilon}\\\\ &+\left\|\frac{(3u_{h}^{n}\partial_{u}^{2}u_{h})^{2}-3u_{h}^{n}-\frac{(\alpha_{h}^{n})^{3}}{\varepsilon}-\frac{(\alpha_{h}^{n-1})^{3}-(\alpha_{h}^{n-1})^{3}-(\alpha_{h}^{n})^{3}}{\varepsilon}}\right\|+\left\|\frac{(\partial_{u}^{n}u_{h})^{2}\tau_{h}^{4}}{64\varepsilon}\right\|+\left\|\frac{(\partial_{u}^{n}u_{h})^{3}\tau_{h}^{6}}{512\varepsilon}}\right\|, \end{split}$$

here C_S , C are constants, which are independent of mesh size, and $\mathcal{E}_u^n := \mathcal{E}[u_h^n]$ is defined as (2.27).

The proof of this theorem is provided in Appendix A.

Remark 3.1. The a posteriori error estimator in Theorem 3.2 can be divided into two categories. The terms γ_w^n , β_u^n , η_w^n , δ_w^n , δ_u^n , γ_u^n , ξ_u^n , β_w^n , θ_u^n , ζ_u^n are viewed as the a posteriori error indicators for time discretization, the terms $\widetilde{\mathcal{E}}_u^n$ and α_u^n are the spatial discretization error indicators.

3.2. Adaptive Algorithm. In view of the a posteriori error estimator of Theorem 3.2, we design the algorithms for time-step size control and spatial adaptation in this part.

We adjust the time-step size in view of the error equidistribution strategy, which means that the time discretization error should be evenly distributed to each time interval $(t_{n-1}, t_n]$, n = 1, 2, ..., N. Let TOL_{time} be the tolerance allowed for the part of the a posteriori error estimator in (3.8) related to the time discretization, that is,

$$\sum_{n=1}^{N} \tau_n (\gamma_w^n + \beta_u^n + \eta_w^n + \delta_w^n + \delta_u^n + \gamma_u^n + \xi_u^n + \beta_w^n + \theta_u^n + \zeta_u^n)^2 \le TOL_{time}.$$
 (3.9)

Generally, we can achieve (3.9) by adjusting the time-step size τ_n so as to have the following relations

$$\eta_{time}^{n} := \gamma_{w}^{n} + \beta_{u}^{n} + \eta_{w}^{n} + \delta_{w}^{n} + \delta_{u}^{n} + \gamma_{u}^{n} + \xi_{u}^{n} + \beta_{w}^{n} + \theta_{u}^{n} + \zeta_{u}^{n} \le \sqrt{TOL_{time}/T} := TOL_{t}.$$
(3.10)

We summarize the procedure of time-step size control in Algorithm 1.

Algorithm 1 Time-step size control

- 1: Given tolerances TOL_t , $TOL_{t,m} := \sqrt{TOL_{time,min}/T}$, parameters $\delta_1 \in (0,1)$, $\delta_2 > 1$;
- 2: Set $\tau_n := \tau_{n-1}, t_n := t_{n-1} + \tau_n;$
- 3: Solve the discrete problem and compute the time error estimator η_{time}^n ;
- 4: while $\eta_{time}^n > TOL_t$ or $\eta_{time}^n < TOL_{t,m}$ do
- 5: if $\eta_{time}^n > TOL_t$ then
- 6: Set $\tau_n := \delta_1 \cdot \tau_n$ and $t_n := t_{n-1} + \tau_n$;
- 7: else
- 8: Set $\tau_n := \delta_2 \cdot \tau_n$ and $t_n := t_{n-1} + \tau_n$;
- 9: end if
- 10: Solve the discrete problem and compute the time error estimator η_{time}^n ;
- 11: end while

Let TOL_{space} be the tolerance allowed for the part of the a posteriori error estimator in (3.8) related to the spatial discretization. For the recovery type error estimator, we adopt the SCR-based error estimator. The SCR gradient recovery method was proposed by Huang and Yi in [24, 36], it can produce a superconvergent recovered gradient, which in turn provides the SCR-based error estimator that is asymptotically exact. Similar to time discretization,

we aim to achieve the following relation at each time step n,

$$\eta_{space}^n := \widetilde{\mathcal{E}_u^n} + \alpha_u^n \le \sqrt{TOL_{space}/T} := TOL_s.$$
(3.11)

Given the refinement and the coarsening parameters TOL_r , TOL_c , respectively, we adopt the following Maximum mark strategy to mark the elements for refinement or coarsening. Set

$$\eta_K^n := \|G^n u_h^n - \nabla u_h^n\|_K, \ \eta_{\max}^n := \max\{\eta_K^n, K \in \mathcal{T}_h^n\},$$
(3.12)

choose the elements $\{K: \eta_K^n > TOL_r \times \eta_{\max}^n\}$ for refinement, and choose the elements $\{K: \eta_K^n < TOL_c \times \eta_{\max}^n\}$ for coarsening.

In view of the error indicators above, we design the following time-space adaptive algorithm for Cahn–Hilliard equation, which is outlined in Algorithm 2.

4. Numerical examples

In this section, we present three examples to demonstrate the reliability and effectiveness of the proposed adaptive algorithm based on the a posteriori error estimator of Theorem 3.2. In Example 4.1, we investigate the main part of the space and time discretization error indicators numerically. In Example 4.2, we focus on illustrating the efficiency of the a posteriori error estimator based on the recovery type and the necessity of time-space adaptation by comparing them with the residual type and space-only adaptation, respectively. We provide the corresponding numerical results, including the discrete energy history, the change in the number of nodes and time steps, the numerical solutions, adaptive meshes, and CPU time, to support our conclusions. For the last example, we apply the proposed time-space adaptive algorithm to the three-dimensional Cahn-Hilliard equation.

In all examples, we take the parameters

$$\delta_1 = \frac{1}{2}, \qquad \delta_2 = 2,$$

and the remaining parameters will be specified in each example.

Example 4.1. Consider the Cahn–Hilliard equation (2.2) with the initial condition

$$u_0(x,y) = \tanh\left(\left((x-0.3)^2 + y^2 - 0.25^2\right)/\varepsilon\right) \tanh\left(\left((x+0.3)^2 + y^2 - 0.3^2\right)/\varepsilon\right),$$

where $\Omega = [-1, 1]^2$ and the parameters $\varepsilon = 0.01$, $TOL_t = 50$, $TOL_{t,m} = 5$, $TOL_s = 10$, $TOL_i = 0.002$.

We apply the proposed time-space adaptive algorithm to solve the Cahn–Hilliard equation, the numerical solutions of u and the corresponding adaptive meshes are shown in Figure 1, respectively. From the pictures, we can see that the meshes follow the zeros level set of u as it moves.

Algorithm 2 Time-space adaptive algorithm for the Cahn–Hilliard equation

```
1: Given TOL_t, TOL_{t,m}, TOL_s, TOL_i, \delta_1 \in (0,1), \delta_2 > 1;
 2: Given the initial time step \tau_0, initial mesh \mathcal{T}_h^0, and initial solution u_h^0;
 3: Set n = 0, t_0 = 0, E(u_h^{-1}) = 0;
 4: Compute the initial error estimator \eta_{initial}^0 = ||u_0 - u_h^0||;
 5: Refine \mathcal{T}_h^0 to get a mesh such that \eta_{initial}^0 \leq TOL_i;
 6: Compute the energy E(u_h^0);
 7: while E(u_h^n) - E(u_h^{n-1}) > TOL_e do
       Set n := n + 1, \mathcal{T}_h^n := \mathcal{T}_h^{n-1}, \tau_n := \tau_{n-1}, t_n := t_{n-1} + \tau_n;
       Solve the discrete problem and compute the time error estimator \eta_{time}^n;
 9:
       while \eta_{time}^n > TOL_t or \eta_{time}^n < TOL_{t,m} do
10:
          if \eta_{time}^n > TOL_t then
11:
             Set \tau_n := \delta_1 \cdot \tau_n and t_n := t_{n-1} + \tau_n;
12:
          else
13:
             Set \tau_n := \delta_2 \cdot \tau_n and t_n := t_{n-1} + \tau_n;
14:
15:
          Solve the discrete problem and compute the time error estimator \eta_{time}^n;
16:
       end while
17:
       Compute the space error estimator \eta_{space}^n, \eta_K^n and \eta_{\max}^n;
18:
       while \eta_{space}^n > TOL_s do
19:
           Mark elements for refinement;
20:
          Refine mesh \mathcal{T}_h^n to generate a new mesh \mathcal{T}_h^n;
21:
          Solve the discrete problem for u_h^n on the new mesh \mathcal{T}_h^n using data u_h^{n-1};
22:
           Compute the time error estimator \eta_{time}^n;
23:
          while \eta_{time}^n > TOL_t or \eta_{time}^n < TOL_{t,m} do
24:
             if \eta_{time}^n > TOL_t then
25:
                Set \tau_n := \delta_1 \cdot \tau_n and t_n := t_{n-1} + \tau_n;
26:
27:
             else
                 Set \tau_n := \delta_2 \cdot \tau_n and t_n := t_{n-1} + \tau_n;
28:
29:
             Solve the discrete problem and compute the time error estimator \eta_{time}^n;
30:
          end while
31:
32:
           Compute the space error estimator \eta_{space}^n, \eta_K^n and \eta_{\max}^n;
       end while
33:
34:
       Compute the energy E(u_h^n);
       Mark elements for coarsen and coarsen \mathcal{T}_h^n producing a modified mesh \mathcal{T}_h^n;
35:
36: end while
```

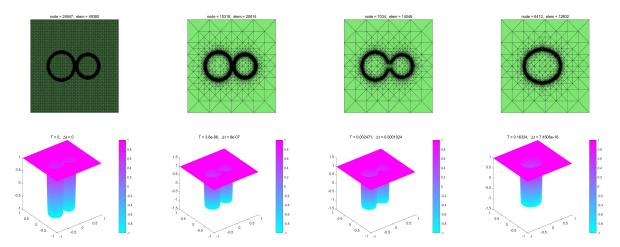


FIGURE 1. Example 4.1, First line: adaptive meshes; Second line: snapshots of numerical solutions for u.

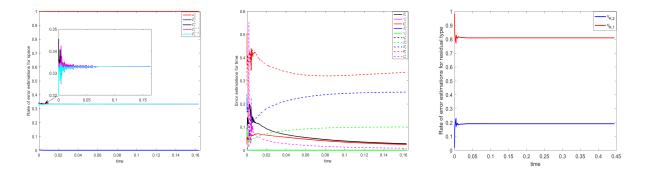


FIGURE 2. Example 4.1. Left: Error indicators for spatial discretization of recovery type; Middle: Error indicators for time discretization of recovery type; Right: Error indicators for spatial discretization of residual type.

From Theorem 3.2, the proposed error estimator contains twelve terms,

$$\begin{split} &\eta_{time}^n = \gamma_w^n + \beta_u^n + \eta_w^n + \delta_w^n + \delta_u^n + \gamma_u^n + \xi_u^n + \beta_w^n + \theta_u^n + \zeta_u^n, \\ &\eta_{space}^n = \widetilde{\mathcal{E}_u^n} + \alpha_u^n. \end{split}$$

We numerically investigate which terms are the main part of the time and space discretization error indicators. We also test the performance of the residual type error estimator provided in [17], in which the local error estimators are defined by

$$\eta_{K,j}(t) = h_K \|R_{K,j}\|_{L^2(K)} + \sum_{\tau \in \partial K} \left(\frac{1}{2} h_\tau \|J_{\tau,j}\|_{L^2(\tau)}^2\right)^{\frac{1}{2}}, \quad j = 1, 2, \tag{4.1}$$

with the element residual

$$R_{K,1} = u_{h,t}|_{K} + \mathcal{A}(w_{h}(t)|_{K}),$$

$$R_{K,2} = \mathcal{A}(u_{h}(t)|_{K}) + \frac{1}{\varepsilon^{2}} f(u_{h}(t)|_{K}) - \frac{1}{\varepsilon} w_{h}(t)|_{K},$$
(4.2)

and the residual jumps across τ

$$J_{\tau,1}(t) = \left(\nabla w_h(t)|_{K_1} - \nabla w_h(t)|_{K_2}\right) \cdot \mathbf{n},$$

$$J_{\tau,2}(t) = \left(\nabla u_h(t)|_{K_1} - \nabla u_h(t)|_{K_2}\right) \cdot \mathbf{n},$$
(4.3)

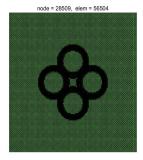
here **n** is the unit normal vector to τ pointing from K_1 to K_2 . The corresponding total spatial discretization error estimator is taken as

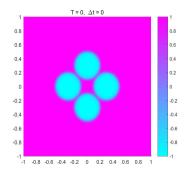
$$\eta(t) = \left(\sum_{K \in \mathcal{T}_L} \left(\eta_{K,1}^2(t) + \eta_{K,2}^2(t)\right)\right)^{\frac{1}{2}}.$$
(4.4)

Figure 2 plots each parts of the error indicators. It shows that: i) for the recovery type error indicator, the time discretization error estiamtor η_{time}^n is dominated by θ_u^n , and the space discretization error estimator η_{space}^n is dominated by \mathcal{E}_u^n ; ii) $\eta_{K,1}(t^n)$ is the main part of the residual type error indicator $\eta(t^n)$. In the following examples, we adopt θ_u^n as the time discretization error indicator, and \mathcal{E}_u^n or $\eta_{K,1}(t^n)$ as the spatial discretization error indicator, respectively.

Example 4.2. Consider the model equation (2.2) with the parameters $\Omega = [-1, 1]^2$, $\varepsilon = 0.01$, $TOL_t = 50$, $TOL_{t,m} = 5$, $TOL_s = 4$, $TOL_i = 0.002$ and the initial condition

$$u_0(x,y) = \tanh\left(\left((x-0.3)^2 + y^2 - 0.2^2\right)/\varepsilon\right) \tanh\left(\left((x+0.3)^2 + y^2 - 0.2^2\right)/\varepsilon\right) \times \tanh\left(\left(x^2 + (y-0.3)^2 - 0.2^2\right)/\varepsilon\right) \tanh\left(\left(x^2 + (y+0.3)^2 - 0.2^2\right)/\varepsilon\right).$$





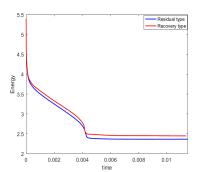


FIGURE 3. Example 4.2, Left: initial mesh; Middle: the contour plot of u_0 ; Right: discrete energy.

In this example, we compare the recovery type a posteriori error estimator with the residual type. Figure 3 displays the initial mesh, contour plot of the initial numerical solution, and

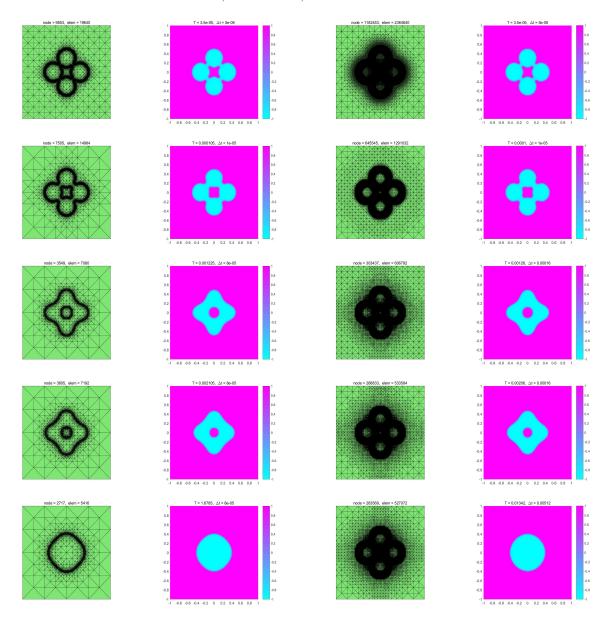


FIGURE 4. Example 4.2, adaptive meshes and snapshots of numerical solutions; First and second column: recovery type; Third and fourth column: residual type.

discrete energy history for the two spatial error estimators based on the proposed timespace adaptive algorithm. We can see clearly that the energy decreases over time. Figure 4 shows the sequences of adaptive meshes and contour plots of the corresponding approximate solutions produced by the time-space adaptive algorithm guided by the recovery and residual type error indicators for the spatial discretization, respectively. The adaptive meshes match the numerical solutions of Algorithm 2 based on the recovery type error indicator better than the residual type. The corresponding time-step and number of nodes are also displayed in

Figure 5. We observe that as the time-step grows, the degree of freedom based on the recovery type is much less than the residual type, indicating that the recovery type a posteriori error estimation is clearly superior to the residual type.

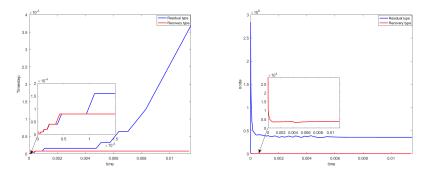


FIGURE 5. Example 4.2, Left: time-steps; Right: number of nodes.

TABLE 1. Example 4.2 (T=0.01), CPU time for two kinds of types by using time-space adaptive algorithm and space-only adaptation, respectively (11th Gen Intel(R) Core(TM) i5-1135G7 @ 2.40GHz 2.42GHz).

CPU time	time-space adaptation	space-only adaptation
Recovery type	389s	1095s
Residual type	106975s	-

Furthermore, we evaluate the efficiency of the adaptive algorithm with time and space adaptation. Table 1 reports the corresponding CPU time. We observe that: i) the time-space adaptive method based on our proposed recovery type error estimator is significantly more efficient than the adaptive method based on the residual type error indicator; ii) the time-space adaptation is more efficient than the adaptive method with space-only adaptation.

Example 4.3. In the last example, we consider the three dimensional Cahn–Hilliard equation (2.2) with the following initial condition

$$u_0(x, y, z) = \varepsilon \cos(1.5\pi x) \cos(1.5\pi y) \left(\sin(\pi z) + \sin(2\pi z)\right),$$

where $\Omega = [-1, 1]^3$ and the parameters $\varepsilon = 0.05$, $TOL_t = 20$, $TOL_{t,m} = 1$, $TOL_s = 1.5$, $TOL_i = 8e - 5$.

Figure 6 displays the contour plots of the discrete energy history, time steps, and the change in the number of nodes with time. It is evident that the energy and the number of nodes both decrease over time, and the time steps change with time. In Figure 7, we show the sequence of adaptive meshes and contour plots of the corresponding approximate solutions. We observe that the meshes adapt around the zero level set, which confirms the effectiveness

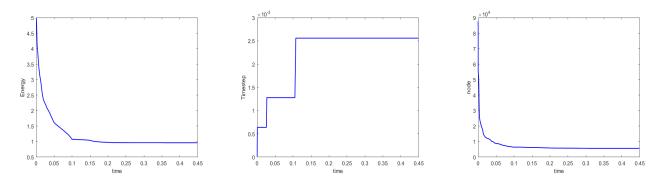


FIGURE 6. Example 4.3, Left: discrete energy; Middle: time-steps; Right: number of nodes.

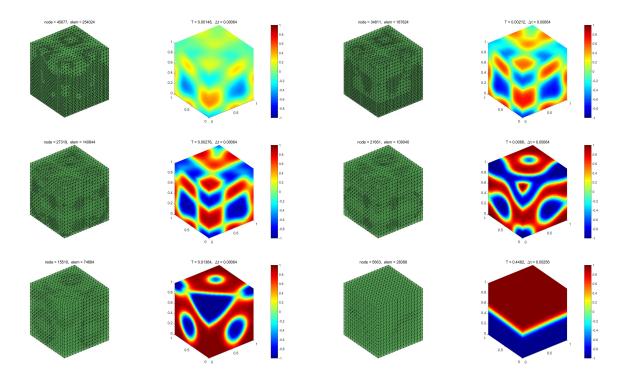


FIGURE 7. Example 4.3, adaptive meshes and snapshots of numerical solutions.

of the derived a posteriori error estimation and adaptive algorithm for the three-dimensional Cahn–Hilliard equation.

5. Conclusions

In this paper, we derived a novel SCR-based recovery type a posteriori error estimator for the Crank-Nicolson finite element method applied to the Cahn-Hilliard equation. The derivation of the error estimator utilized the elliptic reconstruction technique and the time

reconstruction technique, which was based on approximations on three time levels and led to a second order error estimator for time discretization. Based on the derived a posteriori error estimator, we designed an efficient time-space adaptive algorithm. The numerical results indicated that the recovery-type a posteriori error estimator and the time-space adaptive strategy could greatly improve the efficiency of the adaptive algorithm for the Cahn–Hilliard equation. Notably, our proposed time-space adaptive finite element method outperformed the adaptive finite element method based on residual-type a posteriori error estimators, as well as the space-only adaptive finite element method. These results demonstrate the superior efficiency of our method in accurately solving the Chan–Hilliard equation at hand.

APPENDIX A. PROOF OF THEOREM 3.2

In this section, we present the proof of the Theorem 3.2. To begin with, we recall the following results.

Lemma A.1. [2] Let $\dot{V} := \left\{ \phi \in H^1(\Omega), \bar{\phi} := \frac{1}{|\Omega|} \int_{\Omega} \phi dx = 0 \right\}$, there exists $C_I > 0$ such that for all $\phi \in \dot{V}$ if d = 2 and for all $\phi \in \dot{V} \cap L^{\infty}(\Omega)$ if d = 3, we have

$$\|\phi\|_{L^{3}(\Omega)}^{3} \le C_{I} \|\phi\|_{L^{\infty}(\Omega)}^{1-\sigma} \|\nabla \Delta^{-1}\phi\|^{\sigma} \|\nabla \phi\|^{2}, \tag{A.1}$$

where $\sigma = 1$ if d = 2 and $\sigma = \frac{4}{5}$ if d = 3.

Lemma A.2. [3] (Generalized Gronwall's Lemma) Suppose that the nonnegative functions $y_1 \in C([0,T]), y_2, y_3 \in L^1(0,T), a \in L^{\infty}(0,T), and the real number <math>A \geq 0$ satisfy

$$y_1(t) + \int_0^t y_2(s)ds \le A + \int_0^t a(s)y_1(s)ds + \int_0^t y_3(s)ds$$

for all $t \in [0,T]$. Assume that for $B \geq 0$, $\beta \geq 0$ and every $t \in [0,T]$, we have

$$\int_0^t y_3(s)ds \le B \sup_{s \in [0,t]} y_1^{\beta}(s) \int_0^t (y_1(s) + y_2(s)) ds.$$

Setting $E := exp\left(\int_0^T a(s)ds\right)$ and assume that $8AE \le (8B(1+T)E)^{-1/\beta}$, then we obtain

$$\sup_{t \in [0,T]} y_1(t) + \int_0^T y_2(s) ds \le 8A exp\left(\int_0^T a(s) ds\right).$$

Now, we are ready to present the proof of Theorem 3.2.

Proof. According to (3.7), we have that

$$\partial_t e_u + \mathcal{A} e_w = -\mathcal{A} \epsilon_w + \frac{A^n w_h^n - A^{n-1} w_h^{n-1}}{2} + \mathcal{A} (q(t) - q^n) + w_h^n - R^n w_h^n$$

$$+ (t - t_{n-\frac{1}{2}}) \partial_n^2 u_h, \tag{A.2}$$

$$A^n u^n - A^{n-1} u^{n-1} = P^n f(u^n) - P^{n-1} f(u^{n-1}) = w^n - w^{n-1}$$

$$\varepsilon \mathcal{A}e_u - e_w = -\varepsilon \mathcal{A}\epsilon_u + \epsilon_w + \varepsilon \frac{A^n u_h^n - A^{n-1} u_h^{n-1}}{2} + \frac{P^n f(u_h^n) - P^{n-1} f(u_h^{n-1})}{2\varepsilon} - \frac{w_h^n - w_h^{n-1}}{2}$$

$$+ \varepsilon \mathcal{A}(p(t) - p^n) - (q(t) - q^n) + \frac{f(u) - f(p^n)}{\varepsilon} + \frac{1}{\varepsilon} (u_h^n - p^n). \tag{A.3}$$

To make the conclusion clean, we separate the remaining of this proof into eleven steps.

Step 1: Multiplying both sides of (A.2) by $-\Delta^{-1}e_u$ and (A.3) by e_u , respectively, then adding the resulting equations, we obtain

$$\begin{split} &\frac{1}{2}\frac{d}{dt}\|\nabla\Delta^{-1}e_{u}\|^{2}+\varepsilon\|\nabla e_{u}\|^{2}=\left(\frac{A^{n}w_{h}^{n}-A^{n-1}w_{h}^{n-1}}{2},-\Delta^{-1}e_{u}\right)\\ &+\left(\mathcal{A}(q(t)-q^{n}),-\Delta^{-1}e_{u}\right)+\left(w_{h}^{n}-R^{n}w_{h}^{n},-\Delta^{-1}e_{u}\right)+\left((t-t_{n-\frac{1}{2}})\partial_{n}^{2}u_{h},-\Delta^{-1}e_{u}\right)\\ &-\varepsilon a\left(\epsilon_{u},e_{u}\right)+\varepsilon\left(\frac{A^{n}u_{h}^{n}-A^{n-1}u_{h}^{n-1}}{2},e_{u}\right)+\frac{1}{\varepsilon}\left(u_{h}^{n}-p^{n},e_{u}\right)\\ &+\left(\frac{P^{n}f(u_{h}^{n})-P^{n-1}f(u_{h}^{n-1})}{2\varepsilon},e_{u}\right)+\left(-\frac{w_{h}^{n}-w_{h}^{n-1}}{2},e_{u}\right)\\ &+\left(\left(\varepsilon\mathcal{A}\left(p(t)-p^{n}\right)-\frac{1}{\varepsilon}\left(f(p^{n})\frac{t_{n}-t}{\tau_{n}}-f(p^{n-1})\frac{t_{n}-t}{\tau_{n}}\right)-\left(q(t)-q^{n}\right)\right),e_{u}\right)\\ &+\left(\frac{1}{\varepsilon}\left(f(u_{h}^{n})\frac{t-t_{n-1}}{\tau_{n}}+f(u_{h}^{n-1})\frac{t_{n}-t}{\tau_{n}}-f(p^{n})+f(p^{n})\frac{t_{n}-t}{\tau_{n}}-f(p^{n-1})\frac{t_{n}-t}{\tau_{n}}\right),e_{u}\right)\\ &+\left(\frac{1}{\varepsilon}\left(f(u_{h})-f(u_{h}^{n})\frac{t-t_{n-1}}{\tau_{n}}-f(u_{h}^{n-1})\frac{t_{n}-t}{\tau_{n}}\right),e_{u}\right)+\left(\frac{1}{\varepsilon}\left(f(u)-f(u_{h})\right),e_{u}\right),\end{split}$$

for all $t \in (t_{n-1}, t_n]$ and each $n = 1, 2, \dots, N$. Then integrate with respect to t, we get

$$\begin{split} &\frac{1}{2} \|\nabla \Delta^{-1} e_u^N\|^2 + \int_0^T \varepsilon \|\nabla e_u\|^2 dt \\ &= \frac{1}{2} \|\nabla \Delta^{-1} e_u^0\|^2 + \int_0^T \left(\frac{A^n w_h^n - A^{n-1} w_h^{n-1}}{2}, -\Delta^{-1} e_u\right) dt \\ &+ \int_0^T \left(w_h^n - R^n w_h^n, -\Delta^{-1} e_u\right) dt + \int_0^T \left(\left(t - t_{n-\frac{1}{2}}\right) \partial_n^2 u_h, -\Delta^{-1} e_u\right) dt \\ &+ \int_0^T \left(A(q(t) - q^n) + q(t) - q^n, -\Delta^{-1} e_u\right) dt + \int_0^T -\varepsilon a \left(\epsilon_u, e_u\right) dt \\ &+ \int_0^T \left(q^n - q(t) - \left(w_h^n - w_h\right), -\Delta^{-1} e_u\right) dt + \int_0^T \left(w_h^n - w_h, -\Delta^{-1} e_u\right) dt \\ &+ \int_0^T \varepsilon \left(\frac{A^n u_h^n - A^{n-1} u_h^{n-1}}{2}, e_u\right) dt + \int_0^T \frac{1}{\varepsilon} \left(u_h^n - p^n, e_u\right) dt \\ &+ \int_0^T \left(\frac{P^n f(u_h^n) - P^{n-1} f(u_h^{n-1})}{2\varepsilon}, e_u\right) dt + \int_0^T \left(-\frac{w_h^n - w_h^{n-1}}{2}, e_u\right) dt \\ &+ \int_0^T \left(\left(\varepsilon \mathcal{A}\left(p(t) - p^n\right) - \frac{1}{\varepsilon}\left(h(p^n)\frac{t_n - t}{\tau_n} - h(p^{n-1})\frac{t_n - t}{\tau_n}\right)\right) \right) dt \end{split}$$

$$-\frac{1}{2}(t-t_{n-1})(t-t_{n})\frac{h(p^{n})-h(p^{n-1})}{\tau_{n}} - \frac{h(p^{n-1})-h(p^{n-2})}{\tau_{n-1}} - \left(q(t)-q^{n}\right), e_{u}dt$$

$$+\int_{0}^{T} \left(\frac{1}{\varepsilon}\left(p^{n}-p^{n-1}-\left(u_{h}^{n}-u_{h}^{n-1}\right)\right)\frac{t_{n}-t}{\tau_{n}} + \frac{1}{2\varepsilon}(t-t_{n-1})(t-t_{n})\right)$$

$$\left(\frac{p^{n}-p^{n-1}}{\tau_{n}} - \frac{p^{n-1}-p^{n-2}}{\tau_{n-1}} - \frac{u_{h}^{n}-u_{h}^{n-1}}{\tau_{n}} - \frac{u_{h}^{n-1}-u_{h}^{n-2}}{\tau_{n-1}}\right), e_{u}dt + \int_{0}^{T} \left(\frac{1}{\varepsilon}\left(u_{h}^{n}-u_{h}^{n-1}\right)\frac{t_{n}-t}{\tau_{n}}, e_{u}dt\right) dt$$

$$\int_{0}^{T} \left(\frac{1}{\varepsilon}\left(f(u_{h}^{n})\frac{t-t_{n-1}}{\tau_{n}} + f(u_{h}^{n-1})\frac{t_{n}-t}{\tau_{n}} - f(p^{n}) + f(p^{n})\frac{t_{n}-t}{\tau_{n}} - f(p^{n-1})\frac{t_{n}-t}{\tau_{n}}\right) + \frac{1}{2\varepsilon}(t-t_{n-1})(t-t_{n})\left(\frac{f(u_{h}^{n})-f(u_{h}^{n-1})}{\tau_{n}} - \frac{f(u_{h}^{n-1})-f(u_{h}^{n-2})}{\tau_{n-1}} - \frac{f(p^{n-1})-f(p^{n-1})}{\tau_{n}} - \frac{f(p^{n-1})-f(p^{n-2})}{\tau_{n-1}}\right), e_{u}dt$$

$$\int_{0}^{T} \left(\frac{1}{\varepsilon}\left(f(u_{h})-f(u_{h}^{n})\frac{t-t_{n-1}}{\tau_{n}} - f(u_{h}^{n-1})\frac{t_{n}-t}{\tau_{n}} - \frac{1}{2}(t-t_{n-1})(t-t_{n})\right) - \frac{f(u_{h}^{n-1})-f(u_{h}^{n-2})}{\tau_{n}} - \frac{f(u_{h}^{n-1})-f(u_{h}^{n-2})}{\tau_{n}}\right), e_{u}dt$$

$$+\int_{0}^{T} \left(\frac{1}{\varepsilon}\left(f(u)-f(u_{h})\right), e_{u}dt\right) dt$$

$$:=\frac{1}{2}\|\nabla\Delta^{-1}e_{u}^{0}\|^{2} + \mathcal{B}_{1} + \dots + \mathcal{B}_{17}, \tag{A.4}$$

where

$$\mathcal{B}_{1} := \int_{0}^{T} \left(w_{h}^{n} - R^{n} w_{h}^{n}, -\Delta^{-1} e_{u} \right) dt;
\mathcal{B}_{2} := \int_{0}^{T} \left(\frac{A^{n} w_{h}^{n} - A^{n-1} w_{h}^{n-1}}{2}, -\Delta^{-1} e_{u} \right) dt;
\mathcal{B}_{3} := \int_{0}^{T} \left((t - t_{n - \frac{1}{2}}) \partial_{n}^{2} u_{h}, -\Delta^{-1} e_{u} \right) dt;
\mathcal{B}_{4} := \int_{0}^{T} \left(A(q(t) - q^{n}) + q(t) - q^{n}, -\Delta^{-1} e_{u} \right) dt;
\mathcal{B}_{5} := \int_{0}^{T} -\varepsilon a \left(\epsilon_{u}, e_{u} \right) dt;
\mathcal{B}_{6} := \int_{0}^{T} \left(q^{n} - q(t) - (w_{h}^{n} - w_{h}), -\Delta^{-1} e_{u} \right) dt;
\mathcal{B}_{7} := \int_{0}^{T} \left(w_{h}^{n} - w_{h}, -\Delta^{-1} e_{u} \right) dt;
\mathcal{B}_{8} := \int_{0}^{T} \varepsilon \left(\frac{A^{n} u_{h}^{n} - A^{n-1} u_{h}^{n-1}}{2}, e_{u} \right) dt;$$

$$\begin{split} \mathcal{B}_{0} &:= \int_{0}^{t} \frac{1}{\varepsilon} \left(u_{h}^{n} - p^{n}, e_{u} \right) dt; \\ \mathcal{B}_{10} &:= \int_{0}^{T} \left(\frac{P^{n} f(u_{h}^{n}) - P^{n-1} f(u_{h}^{n-1})}{2\varepsilon}, e_{u} \right) dt; \\ \mathcal{B}_{11} &:= -\int_{0}^{T} \left(\frac{w_{h}^{n} - w_{h}^{n-1}}{2\varepsilon}, e_{u} \right) dt; \\ \mathcal{B}_{12} &:= \int_{0}^{T} \left(\left(\varepsilon \mathcal{A} \left(p(t) - p^{n} \right) - \frac{1}{\varepsilon} \left(h(p^{n}) \frac{t_{n} - t}{\tau_{n}} - h(p^{n-1}) \frac{t_{n} - t}{\tau_{n}} \right) - \left(q(t) - q^{n} \right) \right), e_{u} \right) dt; \\ \mathcal{B}_{12} &:= \int_{0}^{T} \left(\frac{1}{\varepsilon} \left(p^{n} - p^{n-1} - \left(u_{h}^{n} - u_{h}^{n-1} \right) \frac{h(p^{n-1}) - h(p^{n-1})}{\tau_{n}} - \frac{h(p^{n-1}) - h(p^{n-2})}{\tau_{n-1}} \right) - \left(q(t) - q^{n} \right) \right), e_{u} \right) dt; \\ \mathcal{B}_{13} &:= \int_{0}^{T} \left(\frac{1}{\varepsilon} \left(p^{n} - p^{n-1} - \left(u_{h}^{n} - u_{h}^{n-1} \right) \right) \frac{t_{n} - t}{\tau_{n}} + \frac{1}{2\varepsilon} (t - t_{n-1}) (t - t_{n}) \right. \\ \left. \left(\frac{p^{n} - p^{n-1}}{\tau_{n}} - \frac{p^{n-1} - p^{n-1}}{\tau_{n-1}} - \frac{w_{h}^{n} - w_{h}^{n-1}}{\tau_{n}} - \frac{w_{h}^{n-1} - w_{h}^{n-2}}{\tau_{n-1}} \right), e_{u} \right) dt; \\ \mathcal{B}_{14} &:= \int_{0}^{T} \left(\frac{1}{\varepsilon} \left(u_{h}^{n} - u_{h}^{n-1} \right) \frac{t_{n} - t}{\tau_{n}}, e_{u} \right) dt; \\ \mathcal{B}_{15} &:= \int_{0}^{T} \left(\frac{1}{\varepsilon} \left(f(u_{h}^{n}) - t_{h}^{n-1} + f(u_{h}^{n-1}) - f(u_{h}^{n-1}) - f(u_{h}^{n-1}) - f(u_{h}^{n-1})}{\tau_{n}} - \frac{f(w_{h}^{n}) - f(w_{h}^{n-1})}{\tau_{n}} - \frac{f(w_{h}^{n}) - f(w_{h}^{n-1})}{\tau_{n}} - \frac{f(w_{h}^{n}) - f(w_{h}^{n-1})}{\tau_{n}} - \frac{f(w_{h}^{n}) - f(w_{h}^{n})}{\tau_{n}} - \frac{f(w_{h}^{n}) - f$$

Next we estimate each of the terms $\{\mathcal{B}_j\}_{j=1,\dots,17}$, separately.

Step 2: First, the term \mathcal{B}_1 , which contains a spatial discretization error term, is bounded by using Schwarz inequality

$$\begin{aligned} |\mathcal{B}_1| &= \left| \int_0^T \left(w_h^n - R^n w_h^n, -\Delta^{-1} e_u \right) dt \right| \\ &= \left| \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left(w_h^n - R^n w_h^n, -\Delta^{-1} e_u \right) dt \right| \end{aligned}$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \|w_{h}^{n} - R^{n} w_{h}^{n}\| \cdot \|\Delta^{-1} e_{u}\| dt
\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} C_{1} h \|w_{h}^{n} - R^{n} w_{h}^{n}\|_{1,\Omega} \|\Delta^{-1} e_{u}\| dt.$$
(A.5)

Owing to Remark 2.1, it can be ignored while h is small enough. In the same way, the term \mathcal{B}_9 can be also ignored.

Similarly, the time discretization terms \mathcal{B}_2 and \mathcal{B}_3 can be estimated as

$$|\mathcal{B}_{2}| = \left| \int_{0}^{T} \left(\frac{A^{n} w_{h}^{n} - A^{n-1} w_{h}^{n-1}}{2}, -\Delta^{-1} e_{u} \right) dt \right|$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left\| \frac{A^{n} w_{h}^{n} - A^{n-1} w_{h}^{n-1}}{2} \right\|_{-1} \cdot \left\| \nabla \Delta^{-1} e_{u} \right\| dt$$

$$:= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \gamma_{w}^{n} \left\| \nabla \Delta^{-1} e_{u} \right\| dt. \tag{A.6}$$

$$|\mathcal{B}_{3}| = \left| \int_{0}^{T} \left((t - t_{n - \frac{1}{2}}) \partial_{n}^{2} u_{h}, -\Delta^{-1} e_{u} \right) dt \right|$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left\| \frac{\tau_{n}^{2}}{8} \cdot \partial_{n}^{2} u_{h} \right\|_{-1} \cdot \left\| \nabla \Delta^{-1} e_{u} \right\| dt$$

$$:= \sum_{n=1}^{N} \int_{0}^{t_{n}} \beta_{n}^{n} \left\| \nabla \Delta^{-1} e_{u} \right\| dt. \tag{A.7}$$

Step 3: For the term \mathcal{B}_4 , based on the definition of elliptic reconstruction, we have

$$\begin{split} |\mathcal{B}_{4}| &= \left| \int_{0}^{T} \left(\mathcal{A}(q(t) - q^{n}) + q(t) - q^{n}, -\Delta^{-1}e_{u} \right) dt \right| \\ &= \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(\mathcal{A}\left(\frac{t - t_{n-1}}{\tau_{n}}q^{n} + \frac{t_{n} - t}{\tau_{n}}q^{n-1} + \frac{1}{2}(t - t_{n-1})(t - t_{n})\partial_{n}^{2}q - q^{n} \right) \right. \\ &+ \left. \left(\frac{t - t_{n-1}}{\tau_{n}}q^{n} + \frac{t_{n} - t}{\tau_{n}}q^{n-1} + \frac{1}{2}(t - t_{n-1})(t - t_{n})\partial_{n}^{2}q - q^{n} \right), -\Delta^{-1}e_{u} \right) dt \right| \\ &= \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(\left(\left((\mathcal{A}R^{n-1}w_{h}^{n-1} + R^{n-1}w_{h}^{n-1}) - (\mathcal{A}R^{n}w_{h}^{n} + R^{n}w_{h}^{n}) \right) \frac{t_{n} - t}{\tau_{n}} \right. \right. \\ &+ \left. \frac{1}{2}(t - t_{n-1})(t - t_{n})(\partial_{n}^{2}\mathcal{A}q + \partial_{n}^{2}q)), -\Delta^{-1}e_{u} \right) dt \right| \\ &\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left\| \left(\mathcal{A}R^{n-1}w_{h}^{n-1} + R^{n-1}w_{h}^{n-1} - (\mathcal{A}R^{n}w_{h}^{n} + R^{n}w_{h}^{n}) \right) \frac{t_{n} - t}{\tau_{n}} + \frac{\tau_{n}^{2}}{8} \partial_{n}^{2} (\mathcal{A}Rw_{h} + Rw_{h}) \right\|_{-1} \cdot \left. \left\| \nabla \Delta^{-1}e_{u} \right\| dt \end{split}$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left(\|A^{n-1}w_h^{n-1} + w_h^{n-1} - (A^n w_h^n + w_h^n)\|_{-1} + \left\| \frac{\tau_n^2}{8} \partial_n^2 (Aw_h + w_h) \right\|_{-1} \right) \cdot \left\| \nabla \Delta^{-1} e_u \| dt \right. \\
: = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \eta_w^n \cdot \left\| \nabla \Delta^{-1} e_u \| dt. \right. \tag{A.8}$$

Step 4: The term \mathcal{B}_5 yields the spatial discretization error, which is bounded as follows

$$|\mathcal{B}_{5}| = \left| -\int_{0}^{T} \varepsilon a(\epsilon_{u}, e_{u}) dt \right|$$

$$= \left| -\sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \varepsilon a(\epsilon_{u}, e_{u}) dt \right|$$

$$\leq \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \varepsilon \|\nabla \epsilon_{u}\| \cdot \|\nabla e_{u}\| dt \right|$$

$$\leq \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(2\|\nabla \epsilon_{u}\|^{2} + \frac{\varepsilon^{2}}{8}\|\nabla e_{u}\|^{2} \right) dt \right|$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(2\|\nabla \epsilon_{u}\|^{2} + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \frac{\varepsilon^{2}}{8}\|\nabla e_{u}\|^{2} dt, \tag{A.9}$$

and in view of the triangle inequality and the linearity of the operators G and ∇ , we get

$$\begin{split} \int_{t_{n-1}}^{t_n} \|\nabla \epsilon_u\|^2 dt &= \int_{t_{n-1}}^{t_n} \|\nabla (p-u_h)\|^2 dt \\ &= \int_{t_{n-1}}^{t_n} \left\|\nabla \left(\left(R^n u_h^n - u_h^n\right) \frac{t-t_{n-1}}{\tau_n} + \left(R^{n-1} u_h^{n-1} - u_h^{n-1}\right) \frac{t_n - t}{\tau_n} \right. \\ &\quad + \frac{1}{2} (t-t_{n-1}) (t-t_n) (\partial_n^2 p - \partial_n^2 u_h) \right) \right\|^2 dt \\ &\leq \int_{t_{n-1}}^{t_n} \left(\left\|\nabla \left(R^n u_h^n - u_h^n\right) \right\|^2 \left(\frac{t-t_{n-1}}{\tau_n}\right)^2 + \left\|\nabla \left(R^{n-1} u_h^{n-1} - u_h^{n-1}\right) \right\|^2 \left(\frac{t_n - t}{\tau_n}\right)^2 \right. \\ &\quad + \frac{(t-t_{n-1})^2 (t-t_n)^2}{4} \|\partial_n^2 \nabla p - \partial_n^2 \nabla u_h\|^2 \\ &\quad + 2 \left\|\nabla \left(R^n u_h^n - u_h^n\right) \right\| \cdot \left\|\nabla \left(R^{n-1} u_h^{n-1} - u_h^{n-1}\right) \right\| \frac{(t-t_{n-1}) (t_n - t)}{\tau_n^2} \\ &\quad + 2 \left\|\nabla \left(R^n u_h^n - u_h^n\right) \right\| \cdot \left\|\partial_n^2 \nabla p - \partial_n^2 \nabla u_h \right\| \frac{(t-t_{n-1})^2 (t-t_n)}{2\tau_n} \\ &\quad + 2 \left\|\nabla \left(R^{n-1} u_h^{n-1} - u_h^{n-1}\right) \right\| \cdot \left\|\partial_n^2 \nabla p - \partial_n^2 \nabla u_h \right\| \frac{(t-t_{n-1}) (t-t_n)^2}{2\tau_n} dt \\ &\leq C_0^2 \left(\left(\mathcal{E}_u^n\right)^2 \frac{(t-t_{n-1})^3}{3(t_n - t_{n-1})^2} \right|_{t_{n-1}}^{t_n} - \left(\mathcal{E}_u^{n-1}\right)^2 \frac{(t_n - t)^3}{3(t_n - t_{n-1})^2} \right|_{t_{n-1}}^{t_n} \end{split}$$

$$+2\mathcal{E}_{u}^{n}\mathcal{E}_{u}^{n-1}\frac{\frac{1}{2}(t_{n}+t_{n-1})t^{2}-\frac{1}{3}t^{3}-t_{n}t_{n-1}\cdot t}{(t_{n}-t_{n-1})^{2}}\Big|_{t_{n-1}}^{t_{n}}\Big)$$

$$+\frac{\tau_{n}^{3}}{12}C_{0}\|\partial_{n}^{2}\nabla p-\partial_{n}^{2}\nabla u_{h}\|\Big(\mathcal{E}_{u}^{n}+\mathcal{E}_{u}^{n-1}\Big)+\frac{\tau_{n}^{5}}{120}\|\partial_{n}^{2}\nabla p-\partial_{n}^{2}\nabla u_{h}\|^{2}$$

$$\leq \frac{C_{0}^{2}}{3}\tau_{n}\left((\mathcal{E}_{u}^{n})^{2}+(\mathcal{E}_{u}^{n-1})^{2}+\mathcal{E}_{u}^{n}\mathcal{E}_{u}^{n-1}\right)$$

$$+C_{0}^{2}\frac{\tau_{n}^{2}\tau_{n-1}\Big(\mathcal{E}_{u}^{n}+\mathcal{E}_{u}^{n-1}\Big)+\tau_{n}^{3}\Big(\mathcal{E}_{u-1}^{n}+\mathcal{E}_{u}^{n-2}\Big)}{6\tau_{n-1}(\tau_{n}+\tau_{n-1})}\Big(\mathcal{E}_{u}^{n}+\mathcal{E}_{u}^{n-1}\Big)$$

$$+C_{0}^{2}\tau_{n}^{3}\frac{\Big(\tau_{n-1}\Big(\mathcal{E}_{u}^{n}+\mathcal{E}_{u}^{n-1}\Big)+\tau_{n}\Big(\mathcal{E}_{u-1}^{n}+\mathcal{E}_{u}^{n-2}\Big)\Big)^{2}}{30\tau_{n-1}^{2}(\tau_{n}+\tau_{n-1})^{2}}$$

$$:=\widetilde{\mathcal{E}}_{u}^{n^{2}},$$
(A.10)

then taking (A.10) into (A.9), we get

$$|\mathcal{B}_6| \le \sum_{n=1}^N 2\widetilde{\mathcal{E}_u^n}^2 + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{\varepsilon^2}{8} \|\nabla e_u\|^2 dt.$$
 (A.11)

Similarly, the spatial discretization error term \mathcal{B}_6 is estimated as follows

$$\begin{split} |\mathcal{B}_{6}| &:= \left| \int_{0}^{T} \left(q^{n} - q(t) - (w_{h}^{n} - w_{h}), -\Delta^{-1} e_{u} \right) dt \right| \\ &= \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(q^{n} - (q^{n} \frac{t - t_{n-1}}{\tau_{n}} + q^{n-1} \frac{t_{n} - t}{\tau_{n}} + \frac{1}{2} (t - t_{n-1}) (t - t_{n}) \partial_{n}^{2} q \right) \right. \\ &- \left. \left(w_{h}^{n} - (w_{h}^{n} \frac{t - t_{n-1}}{\tau_{n}} + w_{h}^{n-1} \frac{t_{n} - t}{\tau_{n}} + \frac{1}{2} (t - t_{n-1}) (t - t_{n}) \partial_{n}^{2} w_{h} \right) \right), -\Delta^{-1} e_{u} \right) dt \Big| \\ &= \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(\left(q^{n} - w_{h}^{n} \right) \frac{t_{n} - t}{\tau_{n}} - \left(q^{n-1} - w_{h}^{n-1} \right) \frac{t_{n} - t}{\tau_{n}} \right. \\ &- \frac{(t - t_{n-1}) (t - t_{n})}{2} (\partial_{n}^{2} q - \partial_{n}^{2} w_{h}), -\Delta^{-1} e_{u} \right) dt \Big| \\ &\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(\| q^{n} - w_{h}^{n} \| \cdot \| \Delta^{-1} e_{u} \| + \| q^{n-1} - w_{h}^{n-1} \| \cdot \| \Delta^{-1} e_{u} \| \right. \\ &+ \left. \left\| \frac{\tau_{n}^{2}}{8} \partial_{n}^{2} (q - w_{h}) \| \cdot \| \Delta^{-1} e_{u} \right\| \right) dt \\ &\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(Ch \| q^{n} - w_{h}^{n} \|_{1,\Omega} + Ch \| q^{n-1} - w_{h}^{n-1} \|_{1,\Omega} \right. \\ &+ Ch \| \frac{\tau_{n}^{2}}{8} \partial_{n}^{2} (q - w_{h}) \|_{1,\Omega} \right) \left\| \Delta^{-1} e_{u} \right\| dt. \end{split} \tag{A.12}$$

According to Remark 2.1, it can be ignored while h is small enough. In the same way, the term \mathcal{B}_{13} can be also ignored.

Step 5: The time discretization term \mathcal{B}_7 is bounded as follows

$$|\mathcal{B}_{7}| = \left| \int_{0}^{T} \left(w_{h}^{n} - w_{h}, -\Delta^{-1} e_{u} \right) dt \right|$$

$$= \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(\left(w_{h}^{n} - w_{h}^{n-1} \right) \frac{t_{n} - t}{\tau_{n}} - \frac{1}{2} (t - t_{n-1}) (t - t_{n}) \partial_{n}^{2} w_{h}, -\Delta^{-1} e_{u} \right) dt \right|$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(\left\| w_{h}^{n} - w_{h}^{n-1} \right\|_{-1} + \left\| \frac{\tau_{n}^{2}}{8} \partial_{n}^{2} w_{h} \right\|_{-1} \right) \left\| \nabla \Delta^{-1} e_{u} \right\| dt$$

$$:= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \delta_{w}^{n} \cdot \left\| \nabla \Delta^{-1} e_{u} \right\| dt. \tag{A.13}$$

Similarly, the terms \mathcal{B}_{14} , \mathcal{B}_{8} , \mathcal{B}_{10} , \mathcal{B}_{11} are also the time discretization terms, and they are estimated as follows

$$|\mathcal{B}_{14}| = \left| \int_{0}^{T} \left(\frac{1}{\varepsilon} \left(u_{h}^{n} - u_{h}^{n-1} \right) \frac{t_{n} - t}{\tau_{n}}, e_{u} \right) \right|$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left\| \frac{u_{h}^{n} - u_{h}^{n-1}}{\varepsilon} \right\| \|e_{u}\| dt$$

$$:= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \delta_{u}^{n} \cdot \|e_{u}\| dt, \tag{A.14}$$

$$|\mathcal{B}_{8}| = \left| \int_{0}^{T} \varepsilon \left(\frac{A^{n} u_{h}^{n} - A^{n-1} u_{h}^{n-1}}{2}, e_{u} \right) dt \right|$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \varepsilon \left\| \frac{A^{n} u_{h}^{n} - A^{n-1} u_{h}^{n-1}}{2} \right\| \cdot \|e_{u}\| dt$$

$$: = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \gamma_{u}^{n} \|e_{u}\| dt, \tag{A.15}$$

$$|\mathcal{B}_{10}| = \left| \int_{0}^{T} \left(\frac{P^{n} f(u_{h}^{n}) - P^{n-1} f(u_{h}^{n-1})}{2\varepsilon}, e_{u} \right) dt \right|$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left\| \frac{P^{n} f(u_{h}^{n}) - P^{n-1} f(u_{h}^{n-1})}{2\varepsilon} \right\| \|e_{u}\| dt$$

$$:= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \xi_{u}^{n} \cdot \|e_{u}\| dt, \qquad (A.16)$$

$$|\mathcal{B}_{11}| = \left| -\int_{0}^{T} \left(\frac{w_{h}^{n} - w_{h}^{n-1}}{2}, e_{u} \right) dt \right|$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left\| \frac{w_h^n - w_h^{n-1}}{2} \right\| \|e_u\| dt
:= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \beta_w^n \cdot \|e_u\| dt.$$
(A.17)

Step 6: The term \mathcal{B}_{12} , which also contains a time discretization term, is estimated by using the definitions of w^n , w and R^n ,

$$\begin{split} |\mathcal{B}_{12}| &= \Big| \int_{0}^{T} \left(\left(\varepsilon \mathcal{A} \Big(p(t) - p^{n} \Big) - \frac{1}{\varepsilon} \Big(h(p^{n}) \frac{t_{n} - t}{\tau_{n}} - h(p^{n-1}) \frac{t_{n} - t}{\tau_{n}} \right) - \frac{1}{\tau_{n}} \right) \\ &- \frac{1}{2} (t - t_{n-1}) (t - t_{n}) \frac{h(p^{n}) - h(p^{n-1})}{\tau_{n}} - \frac{h(p^{n-1}) - h(p^{n-2})}{\tau_{n-1}} \right) - \Big(q(t) - q^{n} \Big) \Big), e_{u} \Big) dt \Big| \\ &= \Big| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(\left(\varepsilon \mathcal{A} \Big(p^{n} \frac{t - t_{n-1}}{\tau_{n}} + p^{n-1} \frac{t_{n} - t}{\tau_{n}} + \frac{1}{2} (t - t_{n-1}) (t - t_{n}) \partial_{n}^{2} p - p^{n} \right) - \frac{1}{\varepsilon} \Big(h(p^{n}) - h(p^{n-1}) \Big) \frac{t_{n} - t}{\tau_{n}} + \frac{1}{2\varepsilon} (t - t_{n-1}) (t - t_{n}) \frac{h(p^{n}) - h(p^{n-1})}{\tau_{n}} - \frac{h(p^{n-1}) - h(p^{n-2})}{\tau_{n-1}} - \frac{1}{\varepsilon} \Big(h(p^{n}) - h(p^{n-1}) \Big) \frac{t_{n} - t}{\tau_{n}} + \frac{1}{2\varepsilon} (t - t_{n-1}) (t - t_{n}) \partial_{n}^{2} q - q^{n} \Big) \Big), e_{u} \Big) dt \Big| \\ &= \Big| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \Big(\Big((\varepsilon \mathcal{A} R^{n-1} u_{h}^{n-1} + \frac{1}{\varepsilon} h(R^{n-1} u_{h}^{n-1}) - q^{n-1} \Big) - \Big(\varepsilon \mathcal{A} R^{n} u_{h}^{n} + \frac{1}{\varepsilon} h(R^{n} u_{h}^{n}) - q^{n} \Big) \Big) \frac{t_{n} - t}{\tau_{n}} + \frac{(\varepsilon \mathcal{A} p^{n} + \frac{h(p^{n})}{\varepsilon} - q^{n}) - (\varepsilon \mathcal{A} p^{n-1} + \frac{h(p^{n-1})}{\varepsilon} - q^{n-1})}{\tau_{n-1}} - \frac{(\varepsilon \mathcal{A} p^{n} + \frac{h(p^{n})}{\varepsilon} - q^{n-1}) - (\varepsilon \mathcal{A} p^{n-2} + \frac{h(p^{n})}{\varepsilon} - q^{n-2})}{\tau_{n-1}} - \frac{(\varepsilon \mathcal{A} p^{n} + \frac{h(p^{n})}{\varepsilon} - q^{n-1}) - (\varepsilon \mathcal{A} p^{n-2} + \frac{h(p^{n})}{\varepsilon} - q^{n-2})}{\tau_{n-1}} - \frac{\tau_{n} + \tau_{n-1}}{\varepsilon}} - \frac{(\varepsilon \mathcal{A} p^{n} + \frac{h(p^{n})}{\varepsilon} - q^{n-1}) - (\varepsilon \mathcal{A} p^{n-2} + \frac{h(p^{n})}{\varepsilon} - q^{n-2})}{\tau_{n-1}} - \frac{\tau_{n} + \tau_{n-1}}{\varepsilon}} - \frac{(\varepsilon \mathcal{A} p^{n} - \frac{h(p^{n})}{\varepsilon} - \frac{h(p^{n})$$

Step 7: The term \mathcal{B}_{15} also yields a time discretization error, which is estimated by using Lagrange mean value theorem and embedding theorem.

$$|\mathcal{B}_{16}| = \left| \int_0^T \left(\frac{1}{\varepsilon} \left(f(u_h^n) \frac{t - t_{n-1}}{\tau_n} + f(u_h^{n-1}) \frac{t_n - t}{\tau_n} - f(p^n) + f(p^n) \frac{t_n - t}{\tau_n} - f(p^{n-1}) \frac{t_n - t}{\tau_n} \right) \right|$$

$$\begin{split} &+\frac{1}{2\varepsilon}(t-t_{n-1})(t-t_n)\left(\frac{f(u_h^n)-f(u_h^{n-1})}{\tau_n} - \frac{f(u_h^{n-1})-f(u_h^{n-2})}{\tau_{n-1}} - \frac{f(p^n)-f(p^{n-1})}{\tau_n} - \frac{f(p^n)-f(p^{n-1})}{\tau_n} - \frac{f(p^{n-1})-f(p^{n-2})}{\tau_{n-1}}\right), e_u\right)dt\Big|\\ &\leq \left|\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left(\frac{t-t_{n-1}}{\tau_n\varepsilon} \left(f(u_h^n)-f(p^n)\right), e_u\right) dt\right| + \left|\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left(\frac{t_{n-1}}{\tau_n\varepsilon} \left(f(u_h^{n-1})-f(p^{n-1})\right), e_u\right) dt\right|\\ &+ \left|\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left(\frac{1}{2\varepsilon} (t-t_{n-1})(t-t_n) \frac{\frac{f(u_h^n)-f(p^n)}{\tau_n}}{\frac{\tau_n}{\tau_{n-1}}}, e_u\right) dt\right|\\ &+ \left|\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left(\frac{1}{\varepsilon} (t-t_{n-1})(t-t_n) \frac{\frac{f(u_h^{n-1})-f(p^{n-1})}{\tau_n}}{\frac{\tau_n}{\tau_{n-1}}}, e_u\right) dt\right|\\ &+ \left|\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left(\frac{1}{2\varepsilon} (t-t_{n-1})(t-t_n) \frac{\frac{f(u_h^{n-1})-f(p^{n-1})}{\tau_n}}{\frac{\tau_n}{\tau_{n-1}}}, e_u\right) dt\right|\\ &+ \left|\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{3}{2\varepsilon} \left\|f'(\xi_1)\right\|_{0,3,\Omega} \cdot \left\|u_h^n-p^n\right\|_{0,6,\Omega} \cdot \left\|e_u\right\| dt\right|\\ &+ \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{1}{2\varepsilon} \left\|f'(\xi_2)\right\|_{0,3,\Omega} \cdot \left\|u_h^{n-1}-p^{n-1}\right\|_{0,6,\Omega} \cdot \left\|e_u\right\| dt\\ &+ \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{3}{2\varepsilon} \left\|f'(\xi_1)\right\|_{1,\Omega} \cdot \left\|u_h^n-p^n\right\|_{1,\Omega} \cdot \left\|e_u\right\| dt\\ &+ \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{3}{2\varepsilon} \left\|f'(\xi_2)\right\|_{1,\Omega} \cdot \left\|u_h^{n-1}-p^{n-1}\right\|_{1,\Omega} \cdot \left\|e_u\right\| dt\\ &+ \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{3}{2\varepsilon} \left\|f'(\xi_2)\right\|_{1,\Omega} \cdot \left\|u_h^{n-1}-p^{n-1}\right\|_{1,\Omega} \cdot \left\|e_u\right\| dt\\ &+ \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{1}{2\varepsilon} \left\|f'(\xi_3)\right\|_{1,3,\Omega} \cdot \left\|u_h^{n-2}-p^{n-2}\right\|_{1,6,\Omega} \cdot \left\|e_u\right\| dt\\ &\leq \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{1}{\varepsilon} C\mathcal{E}_u^n \left\|e_u\right\| dt + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{1}{\varepsilon} C\mathcal{E}_u^{n-1} \left\|e_u\right\| dt + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{1}{\varepsilon} C\mathcal{E}_u^{n-2} \left\|e_u\right\| dt\\ &:= \sum_{n=1}^N \int_{t_n}^{t_n} \frac{1}{\alpha_n} \left\|e_u\right\| dt. \end{split} \tag{A.19}$$

Step 8: In order to estimate the term \mathcal{B}_{16} , which also yields a time discretization error, we first simplify the following formula

$$f(u_h) - f(u_h^n) \frac{t - t_{n-1}}{\tau_n} - f(u_h^{n-1}) \frac{t_n - t}{\tau_n} - \frac{1}{2} (t - t_{n-1}) (t - t_n) \frac{\frac{f(u_h^n) - f(u_h^{n-1})}{\tau_n} - \frac{f(u_h^{n-1}) - f(u_h^{n-2})}{\tau_{n-1}}}{\frac{\tau_n + \tau_{n-1}}{2}}$$

$$\begin{split} &= \left[3(u_{h}^{n})^{2}u_{h}^{n-1} - 2(u_{h}^{n})^{3} - (u_{h}^{n-1})^{3} \right] \cdot \left(\frac{t - t_{n-1}}{\tau_{n}} \right)^{2} \frac{t_{n} - t}{\tau_{n}} \\ &+ \left[3u_{h}^{n}(u_{h}^{n-1})^{2} - (u_{h}^{n})^{3} - 2(u_{h}^{n-1})^{3} \right] \cdot \frac{t - t_{n-1}}{\tau_{n}} \left(\frac{t_{n} - t}{\tau_{n}} \right)^{2} \\ &+ \left[3(u_{h}^{n})^{2}\partial_{n}^{2}u_{h} - 3u_{h}^{n}u_{h}^{n-1}\partial_{n}^{2}u_{h} \right] \cdot \frac{1}{2}(t - t_{n-1})(t - t_{n}) \left(\frac{t - t_{n-1}}{\tau_{n}} \right)^{2} \\ &+ \left[3(u_{h}^{n-1})^{2}\partial_{n}^{2}u_{h} - 3u_{h}^{n}u_{h}^{n-1}\partial_{n}^{2}u_{h} \right] \cdot \frac{1}{2}(t - t_{n-1})(t - t_{n}) \left(\frac{t_{n} - t}{\tau_{n}} \right)^{2} \\ &+ \left[3u_{h}^{n}u_{h}^{n-1}\partial_{n}^{2}u_{h} - \frac{(u_{h}^{n})^{3} - (u_{h}^{n-1})^{3} - (u_{h}^{n-1})^{3} - (u_{h}^{n-2})^{3}}{\frac{\tau_{n} + \tau_{n-1}}{2}} \right] \cdot \frac{1}{2}(t - t_{n-1})(t - t_{n}) \\ &+ \left[\left(3u_{h}^{n}(\partial_{n}^{2}u_{h})^{2} - 3u_{h}^{n-1}(\partial_{n}^{2}u_{h})^{2} \right) \cdot \left(\frac{t - t_{n-1}}{\tau_{n}} \right) + 3u_{h}^{n-1}(\partial_{n}^{2}u_{h})^{2} \right] \cdot \left(\frac{1}{2}(t - t_{n-1})(t - t_{n}) \right)^{2} \\ &+ \left[\left(\frac{1}{2}(t - t_{n-1})(t - t_{n}) \right)^{3}(\partial_{n}^{2}u_{h})^{3} \right], \end{split} \tag{A.20}$$

thus we have

$$\begin{split} |\mathcal{B}_{16}| &= \left| \int_{0}^{T} \left(\frac{1}{\varepsilon} \left(f(u_h) - f(u_h^n) \frac{t - t_{n-1}}{\tau_n} - f(u_h^{n-1}) \frac{t_n - t}{\tau_n} \right. \right. \\ &- \frac{1}{2} (t - t_{n-1}) (t - t_n) \frac{f(u_h^n) - f(u_h^{n-1}) - f(u_h^{n-1}) - f(u_h^{n-1})}{\frac{\tau_n + \tau_{n-1}}{2}} \right), e_u \right) dt \right| \\ &= \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \frac{1}{\varepsilon} \left(\left[3(u_h^n)^2 u_h^{n-1} - 2(u_h^n)^3 - (u_h^{n-1})^3 \right] \cdot \left(\frac{t - t_{n-1}}{\tau_n} \right)^2 \frac{t_n - t}{\tau_n} \right. \\ &+ \left[3u_h^n (u_h^{n-1})^2 - (u_h^n)^3 - 2(u_h^{n-1})^3 \right] \cdot \frac{t - t_{n-1}}{\tau_n} \left(\frac{t_n - t}{\tau_n} \right)^2 \\ &+ \left[3(u_h^n)^2 \partial_n^2 u_h - 3u_h^n u_h^{n-1} \partial_n^2 u_h \right] \cdot \frac{1}{2} (t - t_{n-1}) (t - t_n) \left(\frac{t - t_{n-1}}{\tau_n} \right)^2 \\ &+ \left[3(u_h^{n-1})^2 \partial_n^2 u_h - 3u_h^n u_h^{n-1} \partial_n^2 u_h \right] \cdot \frac{1}{2} (t - t_{n-1}) (t - t_n) \left(\frac{t_n - t}{\tau_n} \right)^2 \\ &+ \left[3u_h^n u_h^{n-1} \partial_n^2 u_h - \frac{(u_h^n)^3 - (u_h^{n-1})^3}{\tau_n} - \frac{(u_h^{n-1})^3 - (u_h^{n-2})^3}{\tau_{n-1}} \right] \cdot \frac{1}{2} (t - t_{n-1}) (t - t_n) \\ &+ \left[\left(3u_h^n (\partial_n^2 u_h)^2 - 3u_h^{n-1} (\partial_n^2 u_h)^2 \right) \cdot \left(\frac{t - t_{n-1}}{\tau_n} \right) + 3u_h^{n-1} (\partial_n^2 u_h)^2 \right] \cdot \left(\frac{1}{2} (t - t_{n-1}) (t - t_n) \right)^2 \\ &+ \left[\left(\frac{1}{2} (t - t_{n-1}) (t - t_n) \right)^3 (\partial_n^2 u_h)^3 \right], e_u \right) dt \\ &\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left\| \frac{3(u_h^n)^2 u_h^{n-1} - 2(u_h^n)^3 - (u_h^{n-1})^3}{\varepsilon} \right\| \cdot \|e_u\| dt \end{split}$$

$$\begin{split} &+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left\| \frac{3u_{h}^{n}(u_{h}^{n-1})^{2} - 2(u_{h}^{n-1})^{3} - (u_{h}^{n})^{3}}{\varepsilon} \right\| \cdot \|e_{u}\| \, dt \\ &+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left\| \frac{(3(u_{h}^{n})^{2} \partial_{n}^{2} u_{h} - 3u_{h}^{n} u_{h}^{n-1} \partial_{n}^{2} u_{h}) \tau_{n}^{2}}{8\varepsilon} \right\| \cdot \|e_{u}\| \, dt \\ &+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left\| \frac{(3(u_{h}^{n-1})^{2} \partial_{n}^{2} u_{h} - 3u_{h}^{n} u_{h}^{n-1} \partial_{n}^{2} u_{h}) \tau_{n}^{2}}{8\varepsilon} \right\| \cdot \|e_{u}\| \, dt \\ &+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left\| \frac{(3u_{h}^{n} u_{h}^{n-1} \partial_{n}^{2} u_{h} - \frac{(u_{h}^{n})^{3} - (u_{h}^{n-1})^{3} - (u_{h}^{n-1})^{3} - (u_{h}^{n-2})^{3}}{7n}}{8\varepsilon} \right\| \cdot \|e_{u}\| \, dt \\ &+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(\left\| \frac{(3u_{h}^{n} u_{h}^{2} \partial_{n}^{2} u_{h} - \frac{(u_{h}^{n})^{3} - (u_{h}^{n-1})^{3} - (u_{h}^{n-2})^{3}}{7n} - \frac{v_{h}^{n-1}}{7n-1}} \right) \tau_{n}^{2}} \right\| + \left\| \frac{(3u_{h}^{n} u_{h}^{2} \partial_{n}^{2} u_{h})^{2} \tau_{n}^{4}}{64\varepsilon} \right\| \cdot \|e_{u}\| \, dt \\ &+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(\left\| \frac{(3u_{h}^{n} (\partial_{n}^{2} u_{h})^{2} - 3u_{h}^{n-1} (\partial_{n}^{2} u_{h})^{2}) \tau_{n}^{4}} \right\| + \left\| \frac{(3u_{h}^{n-1} (\partial_{n}^{2} u_{h})^{2}) \tau_{n}^{4}}{64\varepsilon} \right\| \cdot \|e_{u}\| \, dt \\ &+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \left(\left\| \frac{(\partial_{n}^{2} u_{h})^{3} \tau_{n}^{6}}{512\varepsilon} \right\| \right) \cdot \|e_{u}\| \, dt \\ &:= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \zeta_{u}^{n} \|e_{u}\| \, dt. \end{cases} \tag{A.21}$$

Step 9: Grouping together (A.8), (A.12) and (A.13), we have

$$|\mathcal{B}_{1}| + \dots + |\mathcal{B}_{4}| + |\mathcal{B}_{6}| + |\mathcal{B}_{7}|$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \gamma_{w}^{n} \|\nabla \Delta^{-1} e_{u}\| dt + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \beta_{u}^{n} \|\nabla \Delta^{-1} e_{u}\| dt$$

$$+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \delta_{w}^{n} \cdot \|\nabla \Delta^{-1} e_{u}\| dt + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \eta_{w}^{n} \cdot \|\nabla \Delta^{-1} e_{u}\| dt$$

$$:= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \eta_{0} \|\nabla \Delta^{-1} e_{u}\| dt$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \frac{1}{2} \eta_{0}^{2} dt + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \frac{1}{2} \|\nabla \Delta^{-1} e_{u}\|^{2} dt. \tag{A.22}$$

Summing up (A.14)-(A.21), it holds that

$$|\mathcal{B}_{8}| + \dots + |\mathcal{B}_{16}| \leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \gamma_{u}^{n} \|e_{u}\| dt + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \xi_{u}^{n} \|e_{u}\| dt + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \beta_{w}^{n} \|e_{u}\| dt + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \delta_{u}^{n} \cdot \|e_{u}\| dt + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \delta_{u}^{n} \cdot \|e_{u}\| dt$$

$$+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \alpha_{u}^{n} \|e_{u}\| dt + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \zeta_{u}^{n} \|e_{u}\| dt$$

$$:= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \eta_{1} \|e_{u}\| dt$$

$$\leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \frac{1}{2} \eta_{1}^{2} dt + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_{n}} \frac{1}{2} \|e_{u}\|^{2} dt. \tag{A.23}$$

Step 10: As for estimation of the term \mathcal{B}_{17} , according to the Remark 2.3, the spectrum estimate [1, 7], and the fact that

$$(f(a) - f(b) - f'(b)(a - b)) (a - b) \ge -\tilde{f}(b) |a - b|^3$$

with $\tilde{f}(b) = 3|b|$, we obtain

$$\begin{aligned} |\mathcal{B}_{17}| &= \left| \int_{0}^{T} \left(\frac{1}{\varepsilon} \left(f(u) - f(u_h) \right), e_u \right) dt \right| \\ &\leq \left| \int_{0}^{T} \left(-\frac{1}{\varepsilon} \left(f'(u_h) e_u, e_u \right) + \frac{1}{\varepsilon} \left(\tilde{f}(u_h), |u - u_h|^3 \right) \right) dt \right| \\ &\leq \left| \int_{0}^{T} \left(-\frac{1 - \varepsilon}{\varepsilon} \left(f'(u_h) e_u, e_u \right) - \left(f'(u_h) e_u, e_u \right) + \frac{1}{\varepsilon} \|\tilde{f}(u_h)\|_{L^{\infty}(\Omega)} \|e_u\|_{L^{3}}^{3} \right) dt \right| \\ &\leq \left| \int_{0}^{T} \left((1 - \varepsilon) \overline{\Lambda}_{CH}(t) \|\nabla \Delta^{-1} e_u\|^2 + \varepsilon (1 - \varepsilon) \|\nabla e_u\|^2 + 2\|e_u\|^2 + \frac{1}{\varepsilon} \mu_g \|e_u\|_{L^{3}}^{3} \right) dt \right|. \end{aligned}$$

$$(A.24)$$

Step 11: Taking (A.11), (A.22)-(A.24) into (A.4), we have

$$\frac{1}{2} \|\nabla \Delta^{-1} e_u^N\|^2 + \int_0^T \varepsilon \|\nabla e_u\|^2 dt \leq \frac{1}{2} \|\nabla \Delta^{-1} e_u^0\|^2 + \sum_{n=1}^N 2\widetilde{\mathcal{E}}_u^{n^2} + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{\varepsilon^2}{8} \|\nabla e_u\|^2 dt \\
+ \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{1}{2} \eta_0^2 dt + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{1}{2} \|\nabla \Delta^{-1} e_u\|^2 dt \\
+ \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{1}{2} \eta_1^2 dt + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{1}{2} \|e_u\|^2 dt \\
+ \int_0^T (1 - \varepsilon) \overline{\Lambda}_{CH}(t) \|\nabla \Delta^{-1} e_u\|^2 dt \\
+ \sum_{n=1}^N \int_{t_{n-1}}^{t_n} 2 \|e_u\|^2 dt + \int_0^T \varepsilon (1 - \varepsilon) \|\nabla e_u\|^2 dt \\
+ \int_0^T \frac{1}{\varepsilon} \mu_g \|e_u\|_{L^3}^3 dt. \tag{A.25}$$

Note that

$$\|e_u\|^2 = \left(\nabla(-\Delta^{-1}e_u), \nabla e_u\right)$$

$$\leq \|\nabla \Delta^{-1}e_u\| \|\nabla e_u\|$$

$$\leq \frac{1}{\varepsilon^2} \|\nabla \Delta^{-1}e_u\|^2 + \frac{\varepsilon^2}{4} \|\nabla e_u\|^2, \tag{A.26}$$

plugging (A.26) into (A.25), and further simplification, then we obtain

$$\|\nabla\Delta^{-1}e_{u}^{N}\|^{2} + \int_{0}^{T} \frac{\varepsilon^{2}}{2} \|\nabla e_{u}\|^{2} dt$$

$$\leq \|\nabla\Delta^{-1}e_{u}^{0}\|^{2} + \sum_{n=1}^{N} 4\widetilde{\mathcal{E}_{u}^{n}}^{2} + \sum_{n=1}^{N} \left(\eta_{0}^{2} + \eta_{1}^{2}\right) \tau_{n}$$

$$+ \int_{0}^{T} \left(1 + \frac{5}{2\varepsilon^{2}} + 2\left(1 - \varepsilon\right) \overline{\Lambda}_{CH}(t)\right) \|\nabla\Delta^{-1}e_{u}\|^{2} dt$$

$$+ \int_{0}^{T} \frac{2}{\varepsilon} \mu_{g} \|e_{u}\|_{L^{3}}^{3} dt. \tag{A.27}$$

According to Lemma A.1 and assume that $||e_u||_{L^{\infty}} \leq C$, then it holds that

$$\int_{0}^{T} \|e_{u}\|_{L^{3}}^{3} dt \leq \int_{0}^{T} C_{I} \|e_{u}\|_{L^{\infty}(\Omega)}^{1-\sigma} \|\nabla \Delta^{-1} e_{u}\|^{\sigma} \|\nabla e_{u}\|^{2} dt
\leq \int_{0}^{T} C_{I} \|e_{u}\|_{L^{\infty}(\Omega)}^{1-\sigma} \|\nabla \Delta^{-1} e_{u}\|^{\sigma} \|\nabla e_{u}\|^{2} dt
\leq C_{S} \left(\sup_{t \in (0,T)} \|\nabla \Delta^{-1} e_{u}\|^{\sigma} \right) \int_{0}^{T} \|\nabla e_{u}\|^{2} dt.$$

Setting

$$y_1(t) := \|\nabla \Delta^{-1} e_u\|^2, \qquad y_2(t) := \frac{\varepsilon^2}{2} \|\nabla e_u\|^2, \qquad y_3(t) := 2\varepsilon^{-1} \mu_g \|e_u\|_{L^3}^3,$$

$$B := 2\varepsilon^{-1} \mu_g C_S, \qquad E := \exp\left(\int_0^T a(t) dt\right), \qquad \beta := \sigma,$$

then by Lemma A.2, we have

$$\sup_{t \in [0,T]} \|\nabla \Delta^{-1} e_u\|^2 + \int_0^T \frac{\varepsilon^2}{2} \|\nabla e_u\|^2 dt \le 8\eta^2 \exp\left(\int_0^T a(t) dt\right).$$

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