

More on ordered open end bin packing

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Abstract

We consider the Ordered Open End Bin Packing problem. Items of sizes in $(0, 1]$ are presented one by one, to be assigned to bins in this order. An item can be assigned to any bin for which the current total size strictly below 1. This means also that the bin can be overloaded by its last packed item. We improve lower and upper bounds on the asymptotic competitive ratio in the online case. Specifically, we design the first algorithm whose asymptotic competitive ratio is strictly below 2 and it is close to the lower bound. This is in contrast to the best possible absolute approximation ratio, which is equal to 2. We also study the offline problem where the sequence of items is known in advance, while items are still assigned to bins based on their order in the sequence. For this scenario we design an asymptotic polynomial time approximation scheme.

1 Introduction

We study Ordered Open End Bin Packing (OOEBP). The input for this problem is a sequence of items of positive sizes. An item can be assigned to any bin that has a current total size strictly smaller than 1. In the online problem, items are presented one by one to be packed in this way. In contrast, an offline algorithm knows the sequence of input items in advance. Since the input is ordered, it also has to process the input as a sequence when it creates a packing, where the input is ordered in the same way as it would have been presented to an online algorithm.

We analyze algorithms via worst-case analysis. The absolute competitive ratio (for online algorithms) or absolute approximation ratio (for offline algorithms) is the worst-case ratio between the cost of the algorithm and the optimal (offline) cost (for the same input). The asymptotic measures are the superior limits of these values when we let the optimal cost grow to infinity. The asymptotic measures are known to be the meaningful ones for bin packing problems, and thus, in this paper, we will sometimes omit the word asymptotic. An optimal offline solution is denoted by OPT , and its cost for an input I is denoted by $OPT(I)$.

There are several variants for open end bin packing. The total size of items packed into a bin is called *load*, and in all these variants it is possible to pack sets of items into bins with loads above 1 under certain conditions. In the maximum variant, it is required that every bin has some ordering of items such that the removal of the last item results in load strictly below 1. Thus, in this version, it is required that the load will be below 1 after the removal of the largest item. In the minimum variant, it is required that for every bin, the removal of any item causes the load to be below 1. Thus, in this version, the condition is on the removal of the smallest item. These two variants and ones that are equivalent to them were studied under different names [29, 23, 15, 24, 25, 18]. These variants are generally very different from OOEBP.

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In the maximum variant, an asymptotic fully polynomial time approximation scheme (AFPTAS) was obtained by applying the AFPTAS for standard bin packing [16, 20, 9] and packing the largest items as the last items of their bins [23]. Such a scheme is a family of algorithms, where for every $\varepsilon > 0$ there is an algorithm of asymptotic approximation ratio at most $1 + \varepsilon$ and the running time is polynomial in the input size and in $\frac{1}{\varepsilon}$. In the minimum variant, an AFPTAS was designed as well [25, 15], which required additional ideas, but it cannot be adapted for OOEBP due to the ordered input. More precisely, in the ordered variant, in some cases the last item of a bin may be the item of maximum size, in other cases it can only be the item of minimum size, and typically, the last item (which is just the item of the largest index) is not the maximum or the minimum. Thus, offline algorithms need to be designed carefully such that there is full knowledge on the identity of the last item. Another variant for online algorithms is where the last item is the one whose removal should bring the load below 1, where the online algorithm is compared to an offline algorithm that can reorder the items. We will refer to this variant as the unfair variant.

The problem which we study, OOEBP, was studied Yang and Leung [28]. For this problem, we will use the term 1-items for items of size 1 or larger. As explained in [28], such an item has a special role since once it is packed into a bin, even if the bin is empty, no additional items can be packed into this bin. We assume that all these items have size 1 exactly, since all items of sizes at least 1 are equivalent with respect to the action of any algorithm. Thus, input items have rational sizes in $(0, 1]$. In addition to average-case analysis, Yang and Leung [28] design an algorithm whose asymptotic competitive ratio is strictly below 2 for the case without 1-items, and they show that the algorithm cannot perform much better if there are 1-items. They also proved lower bounds on the asymptotic competitive ratio for the online case: 1.630297 for inputs with 1-items, and 1.415715 for inputs without such items. The lower bound results are proved based on a computer assisted proof, where all packing patterns are enumerated. While the reader has no access to this analysis, we have verified these results independently using a different method (see discussion below on our results). One can observe that upper bounds for the unfair variant are valid for OOEBP, since the actions of the online algorithm and its objective function value are the same for both variants, while an offline algorithm for the unfair variant can perform all actions which it can do for OOEBP, and possibly other actions, so it may have a smaller cost for the unfair variant but not a larger cost. For this variant, tight asymptotic competitive ratio bounds of 2 and 1.5 are known for the cases with and without 1-items respectively [23, 29] (the bound of 2 is tight also for the absolute competitive ratio).

We consider both the offline variant of the problem for which we design an asymptotic approximation scheme and the online case for which we design the current best online algorithms and improved lower bounds on the possible asymptotic competitive ratio that can be achieved by online algorithms.

Note that OOEBP is different from classic online bin packing, for which the current best lower and upper bounds on the asymptotic competitive ratio are 1.542780906 and 1.57828956, respectively [5, 3]. Recall that the absolute competitive ratio is usually seen as a less interesting measure for bin packing problems. It is known that its value for classic online bin packing is $\frac{5}{3}$ [30, 6]. There are several other packing problems where an offline solution still needs to process the input as a sequence [17, 13, 12, 11, 1, 2, 10].

Our results. We design an asymptotic polynomial time approximation scheme (APTAS) for the offline variant. Such a scheme still has an algorithm of asymptotic approximation ratio at most $1 + \varepsilon$ for every $\varepsilon > 0$, but the running time is not necessarily polynomial in $\frac{1}{\varepsilon}$, that is, ε is seen as a constant and for every fixed value of ε the time complexity is polynomial. As explained above, the model for OOEBP is very different from other variants of open end bin packing, and other bin packing problems. The authors are not aware of any asymptotic approximation schemes for packing problems over sequences, and previously known approximation schemes are for the variants where offline solutions can reorder

the input [1, 23, 25, 15].

For the online problem, we briefly discuss the relation between variants, and show that the absolute competitive ratio for OOEBP is exactly 2. Then, we analyze the asymptotic competitive ratio using a combination of new and old methods. We define a new class of algorithms, which allows us to improve the upper bound on the asymptotic competitive ratio from 2 [23] to approximately 1.691561. We show that the obtained ratio is tight for the class of algorithms which we define. We design a similar algorithm for the case without 1-items, which yields an asymptotic competitive ratio of at most 1.44465, improving over the previous bound of 1.5 [29]. As mentioned above, lower bounds on the asymptotic competitive ratio were given with partial proofs [28]. We fill this gap and show that they can be improved slightly using a different input [7]. Thus, the gaps for the asymptotic competitive ratios are now between approximately 1.630483 and approximately 1.691561 for the case with 1-items, and between approximately 1.415752 and 1.44465 for the case without 1-items.

2 An Asymptotic polynomial time approximation scheme (APTAS) for the (offline) OOEBP

Let $\varepsilon > 0$ be such that $\frac{1}{\varepsilon} \geq 3$ is an integer (and in particular, $\varepsilon \leq \frac{1}{3}$). When we will consider multiple instances of OOEBP, it will be useful to denote by $\text{OPT}(I)$ the optimal cost for instance I , but when the instance is clear by context we use OPT to denote this optimal cost.

In order to design an asymptotic polynomial time approximation scheme (APTAS), it suffices to show the existence of a polynomial time algorithm that always returns a solution of cost at most $(1 + \varepsilon)^c \text{OPT} + f(1/\varepsilon)$ for a positive constant $c \geq 1$ and some function f (where OPT is the optimal cost for the same instance). The degree of the polynomial upper bounding the time complexity of this algorithm may depend on ε .

Our scheme applies a guessing step, where this step is followed by a pre-processing step that applies a linear grouping type of rounding [16, 20]. Then, it uses an algorithm for solving a fixed-dimension integer program (IP) based on a configuration IP, in order to create a plan of the output [22, 19]. This last plan is transformed into a feasible solution for OOEBP in the final post-processing step. The guessing step (together with a modification of the solution based on it) is the novel step which allows us to adjust the known methods, which were previously used for problems without sequence-dependent information, for our problem. This step allows us to overcome the complications of designing algorithms for inputs that are sequences rather than sets.

2.1 The guessing step

Let $1, 2, \dots, n$ be the sequence of items that is given as input, and let s_i be the size of item i . We define the exceeding item of a bin in the following way.

Definition 1 *Fix a bin B in a feasible solution. The exceeding item of B , if the total size of items in B is at least 1, is the item of maximum index packed into B . For a bin B with total size of items strictly smaller than 1, its exceeding item is undefined (and it has no exceeding item).*

We note that based on this definition, a 1-item is always an exceeding item of a bin, even if it is the unique item of a bin.

Let OPT be a fixed optimal solution. We next show that we can assume that an exceeding item has size of at least ε^2 , while we still get a maintainable near-optimal solution, denoted by OPT' .

Lemma 2 *Given an optimal solution OPT of cost OPT , there is a feasible solution OPT' of cost at most $(1 + \varepsilon^2)\text{OPT} + 1$ such that every exceeding item has size at least ε^2 .*

Proof. Consider the set of exceeding items of sizes smaller than ε^2 in OPT. We repack these items into new bins such that there are $\frac{1}{\varepsilon^2}$ such items packed into each bin, except perhaps for the last bin that may have a smaller number of items. Let OPT' denote the resulting solution. Observe that by definition, in OPT' all exceeding items are of size at least ε^2 . This holds as all the smaller items that were exceeding items are repacked into bins where no such bin has an exceeding item. Furthermore, the number of new bins is at most $\varepsilon^2 \text{OPT} + 1$, and thus the claim follows. ■

We will establish the existence of a near optimal solution that has a *certificate* which we define as follows.

Definition 3 Let SOL be a feasible solution for OOEBP. We say that SOL has a certificate

$$(e_0, e_1, e_2, \dots, e_{1/\varepsilon})$$

if the following conditions hold:

1. $0 = e_0 < e_1 \leq e_2 \leq \dots \leq e_{1/\varepsilon} = n$ and $e_1, e_2, \dots, e_{1/\varepsilon-1}$ are integers.
2. For every bin B in SOL (exactly) one of the following cases holds:
 - Either the total size of the items in B is strictly smaller than 1, i.e., B does not have an exceeding item,
 - or there is an integer $i(\text{exceed})$ such that the exceeding item has an index strictly larger than $e_{i(\text{exceed})}$ and all other items in B (if there are any such items) have indices at most $e_{i(\text{exceed})}$.

If this holds for every bin, we say that $(e_0, e_1, e_2, \dots, e_{1/\varepsilon})$ is a certificate of SOL.

Every input has at least one solution with a certificate. Specifically, the certificate $(0, n, \dots, n)$ is a certificate of a solution where every item is packed into a different bin. This holds since the only bins with exceeding items are those with 1-items, and for each such item, as $e_0 = 0$, its index is larger.

Definition 4 A solution SOL for OOEBP is called a nice solution if it satisfies the following conditions. First, for every bin B in SOL that has an exceeding item, the size of the exceeding item of B is at least ε^2 , second, SOL has a certain certificate $(e_0, e_1, e_2, \dots, e_{1/\varepsilon})$.

Let OPTN be an optimal nice solution (i.e., a solution of minimal cost among the nice solutions). Next, we show that we can approximate OPTN. With a slight abuse of notation we denote by OPTN both the solution and its cost, and we note that in every case the distinction between the two will be clear by context. Similarly, we will use OPT' to denote the cost of OPT'.

The next lemma also shows in particular that there is at least one nice solution for every input.

Lemma 5 We have $\text{OPTN} \leq (1 + \varepsilon) \cdot \text{OPT}' + \frac{1}{\varepsilon}$.

Proof. Recall that OPT' of cost OPT' satisfies the first condition in the definition of nice solutions. We create a nice solution SOL by modifying OPT' such that the first condition will be maintained while the second condition will be satisfied as well. Then we will show that the cost of SOL is at most $(1 + \varepsilon) \cdot \text{OPT}' + \frac{1}{\varepsilon}$. The claim will follow by the optimality of OPTN among nice solutions.

Consider the solution OPT'. Some bins in this solution have exceeding items while other bins do not have exceeding items. The packing of items that were packed into bins (of OPT') without exceeding items is left without modification. Consider the m bins O_1, O_2, \dots, O_m of OPT' containing exceeding items, such that these bins are sorted according to an increasing order of the indices of the exceeding

items of these bins. Let the integer $\alpha = \lfloor \varepsilon m \rfloor$ be the result of integer division of m by $\frac{1}{\varepsilon}$. The remainder of this division is $m - \frac{\alpha}{\varepsilon}$. Let $\beta = m - (\frac{1}{\varepsilon} - 1) \cdot \alpha$ be the sum of α and the above remainder.

In the case where $m \geq \frac{1}{\varepsilon}$, it holds that $\alpha \geq 1$ and otherwise $\alpha = 0$. In both cases we have $\beta \leq \varepsilon \cdot m + \frac{1}{\varepsilon}$ and $\beta \geq \varepsilon \cdot m \geq \alpha$.

In order to modify OPT' we do the following. The exceeding items of O_1, \dots, O_β are packed into new bins, one item per bin. If the unique item packed into the bin is a 1-item, then it is of size at least ε^2 , and it is an exceeding item but no other item is packed there so both conditions hold by using $i(\text{exceed}) = 0$ for these bins. Otherwise, the bin does not contain an exceeding item so both conditions hold trivially. Thus, the bins obtained in this way satisfy both conditions no matter which certificate vector we consider. In the case $m < \frac{1}{\varepsilon}$ we are done as $\alpha = 0$ and $\beta = m$ hold in this case. In particular, in the last case SOL is indeed nice and its cost is at most $\text{OPT}' + \frac{1}{\varepsilon}$.

Otherwise, $\beta < m$, and we deal with the remaining $m - \beta$ bins with exceeding items. Prior to the last step all bins O_1, \dots, O_β had exceeding items as well, where after this step only bins $O_{\beta+1}, O_{\beta+2}, \dots, O_m$ have exceeding items.

For every bin of O_p such that $\beta + 1 \leq p \leq m$, we repack the exceeding item of O_p into the bin $O_{p-\alpha}$, where we apply this for every such p . The smallest index of any bin receiving an exceeding item is $\beta + 1 - \alpha \geq 1$. Observe that the index of the new item joining a bin is larger than the index of its original exceeding item due to the sorting of these bins, so the packing remains valid. Thus, every repacked item is the new exceeding item of its new bin, if this new bin indeed has an exceeding item after the transformation.

We denote the resulting solution by SOL. Thus, every bin B in SOL satisfies that if B has an exceeding item, then the size of the exceeding item of B is at least ε^2 because every exceeding item of a bin in SOL was an exceeding item of a bin in OPT' . This applies for the bins of OPT' that did not have exceeding items as the packing of items into these bins is the same as well. Note that after the transformation, bins $O_{m-\alpha+1}, O_{m-\alpha+2}, \dots, O_m$ have no exceeding items.

We define the vector $(e_0, e_1, e_2, \dots, e_{1/\varepsilon})$ by letting $e_0 = 0$ and $e_{1/\varepsilon} = n$ as required, and for every $1 \leq i \leq \frac{1}{\varepsilon} - 1$ letting e_i be the index of the exceeding item of $O_{\beta+(i-1)\alpha}$ in the solution OPT' (i.e., the solution before the transformation). Observe that this vector has a monotonically increasing list of components due to the sorting of bins with exceeding items in OPT' .

It suffices to show that for every bin B in SOL, containing both non-exceeding items as well as an exceeding item, we have a value of i , such that all non-exceeding items have indices at most e_i while the exceeding item has index strictly larger than e_i .

We next argue that this vector is a certificate of SOL. We only consider the bins out of

$$O_{\beta+1-\alpha}, O_{\beta+2-\alpha}, \dots, O_{m-\alpha}$$

(where $m - \alpha = \beta + (\frac{1}{\varepsilon} - 2)\alpha$) with exceeding items after the transformation, as other bins with exceeding items were discussed already (those are only bins of SOL with 1-items as their only items). For a given bin B , let i' be such that its index ℓ as a bin O_ℓ is in $(\beta + (i' - 2)\alpha, \beta + (i' - 1)\alpha]$ (where $i' \in \{1, 2, \dots, \frac{1}{\varepsilon} - 1\}$). The item of index $e_{i'}$ was the exceeding item of $O_{\beta+(i'-1)\alpha}$ in OPT' , therefore the index of the original exceeding item of $B = O_\ell$ in OPT' , was at most $e_{i'}$ due to the sorting, and all its other items have smaller indices since the bin was valid. The new exceeding item of B was previously the exceeding item of a bin $O_{\ell'}$ where ℓ' is in $(\beta + (i' - 1)\alpha, \beta + i'\alpha]$, so its index is strictly above $e_{i'}$. Thus, indeed this vector is a certificate of SOL.

Furthermore, the process of transforming OPT' into the new solution, SOL, creates only $\beta \leq \varepsilon \cdot m + \frac{1}{\varepsilon}$ new bins so the cost of SOL is at most $(1 + \varepsilon)\text{OPT}' + \frac{1}{\varepsilon}$ as we argued. ■

The guessing. We guess the certificate vector $(e_0, e_1, e_2, \dots, e_{1/\varepsilon})$ of OPTN where the components of the vectors are integers in $[1, n]$ ($\frac{1}{\varepsilon} - 1$ elements are guessed). Thus, the number of different values of

the guesses is $O(n^{1/\varepsilon})$. Each guess will be examined in an iteration step of a loop in our (guessing) procedure. For every value of the guess, we apply the algorithm in the next step that returns a feasible solution for OOEBP, and among all the solutions obtained in the different iterations of this loop, we pick the cheapest one as the output of the algorithm. In order to analyze our algorithm it suffices to consider the iteration of this loop in which we use the value of the guess corresponding to a certificate of OPTN, and show that for this iteration the cost of the returned solution is at most $(1 + \varepsilon)^c \text{OPTN} + f(1/\varepsilon)$ for a constant $c \geq 1$, and some function f .

2.2 The pre-processing step

The pre-processing step which we apply is described as rounding of large items, but we apply this operation separately for each subsequence of items of indices in $(e_i, e_{i+1}]$ for every i . Similarly to approximation schemes for the bin packing problem, this rounding of large items is carried out using the so-called *linear grouping* rounding method.

In what follows we sometimes introduce (dummy) items in the middle of the input sequence and sometimes we delete items from the sequence. In order to maintain the guessed certificate in these operations, we treat the certificate as a collection of pointers to a (doubly) linked list of the items in the input. Lists are initialized by items of indices in $(e_i, e_{i+1}]$, where items appear in the lists sorted by increasing indices. Now, inserting items means we insert items to the corresponding position in this linked list, and deleting items is done as in linked lists (where deletion can also be of the first or last item).

In the remainder of this step we find an upper bound and a lower bound on $\text{OPTN}(I)$ of an instance I of OOEBP by using two different nice solutions (for the two bounds).

An item is called a *large item of interval* $(e_i, e_{i+1}]$ if its index is in the interval $(e_i, e_{i+1}]$ and its size is at least ε^2 . An item is called a *small item of interval* $(e_i, e_{i+1}]$ if its index is in the interval $(e_i, e_{i+1}]$ and its size is smaller than ε^2 . An item is *large* if it is a large item for some interval, and an item is an *item of interval* $(e_i, e_{i+1}]$ if its index is in the interval $(e_i, e_{i+1}]$, that is, if it is either a large item for this interval or a small item for this interval. Throughout the rest of the scheme, we keep the certificate vector as a requirement of nice solutions in the sense that an exceeding item will belong to a linked list with a larger value of i .

We denote by n_i the number of large items of interval $(e_i, e_{i+1}]$ for input I .

Lemma 6 *Without loss of generality, we assume that for every i , we have that $\varepsilon^3 n_i$ is an integer.*

Proof. For values of i for which the claim does not hold, we add up to $\frac{1}{\varepsilon^3}$ items, each of which has size 1, and they will appear in the sequence of items just before the item e_{i+1} . Note that applying this transformation for all values of i that had not satisfied the claim may add up to $\frac{1}{\varepsilon^4}$ items so it increases the optimal cost of nice solutions by an additive term of at most $\frac{1}{\varepsilon^4}$. Thus, it suffices to approximate the resulting instance after adding these items. ■

By slightly abusing notation, the input with the modification described in the proof of Lemma 6 is still denoted by I .

For every i , let $\ell^i(1), \ell^i(2), \dots, \ell^i(n_i)$ be the large items of interval $(e_i, e_{i+1}]$ sorted in a non-increasing order of their size. That is,

$$s_{\ell^i(1)} \geq s_{\ell^i(2)} \geq \dots \geq s_{\ell^i(n_i)} \geq \varepsilon^2.$$

For every i , and for every $k = 1, 2, \dots, \frac{1}{\varepsilon^3}$, the k -th group of interval $(e_i, e_{i+1}]$ denoted as $G(i, k)$ is the set of $\varepsilon^3 \cdot n_i$ items of indices $\ell^i((k-1) \cdot \varepsilon^3 \cdot n_i + 1), \ell^i((k-1) \cdot \varepsilon^3 \cdot n_i + 2), \dots, \ell^i(k \cdot \varepsilon^3 \cdot n_i)$.

The *rounded instance* is the instance that we obtain by rounding up the size of all large items such that for every i, k , the items of $G(i, k)$ are rounded up to $s_{\ell^i((k-1) \cdot \varepsilon^3 \cdot n_i + 1)}$ (that is the largest size of an item in $G(i, k)$), while small items keep their original size. The small items are not included in $G(i, k)$.

Recall that I denotes the instance prior to this rounding, and let I' be the rounded instance. Furthermore, we denote by I'' the instance obtained from I' by deleting all items in $\bigcup_{i=0}^{1/\varepsilon-1} G(i, 1)$, and observe that I'' can be obtained from I by rounding the size of each item of $G(i, k)$ down to the size of the largest item of $G(i, k+1)$ (for all i and all $k < 1/\varepsilon^3$) and deleting all items of $G(i, 1/\varepsilon^3)$ (for all i). Note that we keep the small items separately, and they are included in I' and I'' .

The use of this rounding is justified by the following lemma.

Lemma 7 *We have $\text{OPTN}(I'') \leq \text{OPTN}(I) \leq \text{OPTN}(I') \leq (1 + 3\varepsilon) \cdot \text{OPTN}(I'')$.*

Proof. The first two inequalities, i.e., $\text{OPTN}(I'') \leq \text{OPTN}(I) \leq \text{OPTN}(I')$ follow as when we decrease the size of some items and perhaps delete some of those items a feasible nice solution with respect to the certificate $(e_0, e_1, e_2, \dots, e_{1/\varepsilon})$ for the instance before the transformations remains feasible nice solution with respect to the same certificate (some bins stop having exceeding items but this does not hurt the property of being nice). Thus the solution $\text{OPTN}(I')$ is a feasible nice solution for I and so $\text{OPTN}(I) \leq \text{OPTN}(I')$, and the solution $\text{OPTN}(I)$ is a feasible nice solution for I'' and so $\text{OPTN}(I'') \leq \text{OPTN}(I)$.

It remains to prove the last inequality. Given the solution $\text{OPTN}(I'')$ we create a solution for I' by packing each item that does not exist in I'' in its dedicated bin. Note that the resulting solution is obviously a feasible nice solution as packing an item into a dedicated bin keeps it feasible and maintain the property of being nice, and in total we added $\varepsilon^3 \sum_{i=0}^{1/\varepsilon-1} n_i$ bins. Therefore,

$$\text{OPTN}(I') \leq \text{OPTN}(I'') + \varepsilon^3 \sum_{i=0}^{1/\varepsilon-1} n_i. \quad (1)$$

However, the instance I'' contains at least $(1 - \varepsilon^3) \sum_{i=0}^{1/\varepsilon-1} n_i$ large items, each of which of size at least ε^2 . Therefore, the optimal cost of a nice solution (or any solution) is at least half the total size of these items, since no bin can contain items of total size above 2.

Thus, $\text{OPTN}(I'') \geq \varepsilon^2/2 \cdot (1 - \varepsilon^3) \sum_{i=0}^{1/\varepsilon-1} n_i \geq \frac{13}{27} \cdot \varepsilon^2 \cdot \sum_{i=0}^{1/\varepsilon-1} n_i$, as by $\varepsilon \leq 1/3$, it holds that $1 - \varepsilon^3 \leq \frac{26}{27}$. From this we obtain that the following holds:

$$\varepsilon^2 \cdot \sum_{i=0}^{1/\varepsilon-1} n_i \leq \frac{27}{13} \cdot \text{OPTN}(I''). \quad (2)$$

Therefore, $\text{OPTN}(I') \leq \text{OPTN}(I'') + \varepsilon^3 \sum_{i=0}^{1/\varepsilon-1} n_i \leq (1 + 3\varepsilon) \text{OPTN}(I'')$, where the first inequality follows by the upper bound on the cost of an optimal nice for I' we derived from $\text{OPTN}(I'')$, i.e. (1), while the second inequality holds by our last bound on the total size of large items, i.e. (2). ■

The last lemma shows that it is sufficient to approximate I' , i.e., it is sufficient to approximate $\text{OPTN}(I')$. This is the goal of the last two steps of the scheme.

2.3 The configuration IP

Throughout this section, we deal with input I' . A configuration is a vector that encodes the packing of one bin (for I'). The intuition is that since we restrict our packings to be nice solutions, the packing of one bin (called its configuration) is characterized by the number of items of each group of each interval (including the exceeding item if it exists), and the total size of small items of each interval. Observe that this information allows us to verify that there is at most one exceeding item by checking that if we delete

an item from the last interval (with respect to the index) for which there is such non-zero component (of the configuration) then the total size is below 1. Second, it allows us to verify for a configuration that if the total size of all items is at least 1, for the last interval where the number of items is positive, this number is 1, and the unique item of the last interval is large. Thus, these components allow us to check that the conditions of nice solutions for the given certificate are satisfied by this configuration. By limiting the number of configurations we will obtain our configuration integer program (IP) that can be solved in polynomial time for fixed values of ε .

For group $G(i, k)$ denote by $s(i, k)$ the common size of the items in this group.

Formally, a configuration of a bin is a vector C consisting of the following components. For every interval $(e_i, e_{i+1}]$ and every group $G(i, k)$ of this interval, we have a component $C_{i,k}$ denoting the number of (exceeding or non-exceeding) items of group $G(i, k)$ in the bin. We note that in the case of an exceeding item, $C_{i,k} = 1$ based on the earlier concepts. Furthermore, for every interval $(e_i, e_{i+1}]$, we have a component C_i denoting the total size of small items of the interval rounded down to the next integer multiple of ε^3 . We will show later that this modification of small items and rounding (down) of the total size of small items works in a sense that it is enough to approximate OPTN well (in a suitable way) by the solution an IP.

Such a vector C is a feasible configuration if one of the following conditions hold:

- the total size of the items (including small items) is strictly smaller than 1, i.e., if

$$\sum_{i=0}^{1/\varepsilon-1} \left(C_i + \sum_{k=1}^{1/\varepsilon^3} C_{i,k} \cdot s(i, k) \right) < 1,$$

- or there is a unique item of the maximum interval (possibly packed into the bin with some items of smaller intervals, and after removing this item, the total size of remaining item is strictly smaller than 1, that is, if there is i_{max} such that

$$\sum_{i=i_{max}}^{1/\varepsilon-1} \sum_{k=1}^{1/\varepsilon^3} C_{i,k} = 1,$$

where i_{max} can be seen as the index of the interval of the exceeding item (but it is possible that there are other smaller indices satisfying this property, but they will not satisfy the second property). By the definition of nice solutions, there may or may not be other items of smaller intervals packed into the bin. It is also required that $C_i = 0$ for all $i \geq i_{max}$, and

$$\sum_{i=0}^{i_{max}-1} \left(C_i + \sum_{k=1}^{1/\varepsilon^3} C_{i,k} \cdot s(i, k) \right) < 1.$$

Denote by \mathcal{C} the set of feasible configurations.

Lemma 8 *The number of feasible configurations is at most $O((1/\varepsilon)^{O(1/\varepsilon^4)})$.*

Proof. A configuration is a vector with $O(1/\varepsilon^4)$ components, due to the following. There are $O(\frac{1}{\varepsilon})$ intervals, and for each one there are $O(\frac{1}{\varepsilon^3})$ different sizes in I' , each having a separate component, plus one component for small items. The components for small items are integers in $[0, 1/\varepsilon^3]$, and other components are integers in $[0, 1/\varepsilon^2]$. ■

Our IP uses the integral decision variables x_C for all $C \in \mathcal{C}$. For a fixed solution, every variable x_C represents the number of bins for each configuration $C \in \mathcal{C}$. There will be no additional variables, so

the dimension of the IP will be a fixed constant and we will be able to solve it in polynomial time. For interval $(e_i, e_{i+1}]$, denote by σ_i the total size of small items of the interval.

The objective is to minimize the number of bins and by grouping the bins according to configurations this is equivalent to $\min \sum_{C \in \mathcal{C}} x_C$. We have two types of constraints, the first family is that we need to pack all large items, so for every interval $(e_i, e_{i+1}]$ and every group $G(i, k)$ we have the constraint that all items of $G(i, k)$ are indeed packed, so $\sum_{C \in \mathcal{C}} C_{i,k} \cdot x_C = \varepsilon^3 n_i$ since the number of items in the group is $\varepsilon^3 n_i$. The second family of constraints is that the total size of small items of each interval which we pack is approximately the total size of small items of the interval. Here, the constraint that we introduce will create some slack in the right hand side, but we will be able to bound its impact. Thus, for every interval $(e_i, e_{i+1}]$ we have the constraint $\sum_{C \in \mathcal{C}} C_i \cdot x_C \geq \sigma_i$. The description of the IP is completed by the non-negativity constraints on the variables. Thus, we solve the following configuration IP denoted as (ConfIP):

$$\begin{aligned} \min \quad & \sum_{C \in \mathcal{C}} x_C && (\text{ConfIP}) \\ \text{s.t.} \quad & \sum_{C \in \mathcal{C}} C_{i,k} \cdot x_C = \varepsilon^3 n_i && \forall i, k, \\ & \sum_{C \in \mathcal{C}} C_i \cdot x_C \geq \sigma_i && \forall i \\ & x_C \geq 0 && \forall C \in \mathcal{C}. \end{aligned}$$

Before presenting our post-processing step that receives an optimal solution for (ConfIP) and constructs a feasible packing of the items into bins, we first find an upper bound on the cost of an optimal solution for (ConfIP) using the cost of $\text{OPTN}(I')$.

Lemma 9 *There is a feasible solution for (ConfIP) whose cost as a solution for this program is at most $(1 + \varepsilon)\text{OPTN}(I')$.*

Proof. Consider the solution $\text{OPTN}(I')$ which is an optimal nice solution for the rounded instance I' . We show that this nice optimal solution of I' induces a feasible solution to (ConfIP). We first define a configuration for each bin B in $\text{OPTN}(I')$. The configuration $C(B)$ corresponding to B is defined as follows. $C(B)_{i,k}$ is the number of items of $G(i, k)$ packed into B , and to compute $C(B)_i$ we first compute the total size of small items of interval $(e_i, e_{i+1}]$ that are packed into B and then round down this value to an integer multiple of ε^3 . Our solution for the IP will be based on the configurations corresponding to the bins with some additional configurations. For every interval $(e_i, e_{i+1}]$ we add to the collection of configurations (one for each bin of $\text{OPTN}(I')$) $2\varepsilon^3 \cdot \text{OPTN}(I')$ configurations that are copies of the configuration with all components being zero except for the unique component C_i that equals $\frac{1}{\varepsilon^3} - 1$. In total we add at most $2\varepsilon^2 \cdot \text{OPTN}(I') \leq \varepsilon \text{OPTN}(I')$ configurations. Next we define the vector x^{OPTN} by setting for every $C \in \mathcal{C}$, the value x_C^{OPTN} to be the number of times C appears in the collection of configurations we defined (the collection with the configurations corresponding to the bins and the additional configurations).

Clearly, the resulting vector is non-negative, and its cost as a solution for the IP is at most $(1 + \varepsilon) \cdot \text{OPTN}(I')$. Since every item of each group of large items is indeed packed in $\text{OPTN}(I')$ we conclude that $\sum_{C \in \mathcal{C}} C_{i,k} \cdot x_C^{\text{OPTN}} = \varepsilon^3 n_i$. Furthermore, for every interval $(e_i, e_{i+1}]$, every bin B may contain a total size of small items of the interval that is larger than $C(B)_i$ by at most ε^3 . Thus, by adding configurations of total size of the small items of this interval of at least $(\frac{1}{\varepsilon^3} - 1) \cdot 2\varepsilon^3 \text{OPTN}(I')$, we guarantee that the constraints $\sum_{C \in \mathcal{C}} C_i \cdot x_C^{\text{OPTN}} \geq \sigma_i$ are also satisfied. ■

2.4 The post-processing step

Let x^* denote an optimal solution for (ConfIP). It remains to show that we are able to construct a feasible packing of the items with cost at most $(1 + \varepsilon) \cdot \sum_{C \in \mathcal{C}} x_C^* + \frac{1}{\varepsilon}$.

We first open $\sum_{C \in \mathcal{C}} x_C^*$ bins where we associate x_C^* bins with configuration C for every $C \in \mathcal{C}$. Furthermore, we have additional set of bins where for every i we open additional $\lceil \varepsilon^2 \cdot \sum_{C \in \mathcal{C}} x_C^* \rceil$ bins associated with the interval, each of which with up to $\frac{1}{\varepsilon^2}$ small items of the interval $(e_i, e_{i+1}]$. In this way we open at most $\varepsilon \cdot \sum_{C \in \mathcal{C}} x_C^* + \frac{1}{\varepsilon}$ additional bins. Let us consider the packing of items into these bins.

For every $C \in \mathcal{C}$ the packing of large items into bins associated with C is carried out such that each such bin is allocated $C_{i,k}$ items of group $G(i, k)$ of interval $(e_i, e_{i+1}]$. Observe that since x^* satisfies the constraint $\sum_{C \in \mathcal{C}} C_{i,k} \cdot x_C^* = \varepsilon^3 n_i$, this allocation of large items into bins associated with configurations allocates all large items.

Next, consider the small items, and for every interval $(e_i, e_{i+1}]$ we allocate small items of this interval to the bins associated with the interval as well as to every bin associated with configurations $C \in \mathcal{C}$ in the following Next Fit type approach where we pack the small items of the interval, one by one in increasing order of their indices. We iterate over the bins associated with C , and pack the small items of the interval $(e_i, e_{i+1}]$ one by one to the current bin associated with C as long as adding the next item does not exceed the upper bound of C_i on the total size of small items of the interval that are allocated to this bin. Recall that C_i was the total size of small items for interval i , round down to an integer multiple of ε^3 . When we are about to pack an item in a way that exceeds the total size upper bound, we allocate this small item to one of the bins associated with the interval, and move to the next bin associated with C (if there is one) or to the next configuration. We note that if $C_i = 0$ then we allocated no small items of the interval $(e_i, e_{i+1}]$ to bins associated with C . This is done for every interval and we guarantee that the total size of small items that are packed into a bin does not exceed the bound defined by the configuration so the packing of such a bin is feasible. Furthermore, the packing of the bins associated with intervals is feasible as every bin among the bins associated with configurations causes at most one small item (with size below ε^2) of the interval to be packed into a bin associated with the interval. Since each bin associated with the interval has room for $\frac{1}{\varepsilon^2}$ such items, we have enough room for all the small items. We conclude the following.

Corollary 10 *There exists a linear time ($O(n)$ time) algorithm that given a solution x^* for (ConfIP), returns a feasible packing of the items into at most $(1 + \varepsilon) \cdot \sum_{C \in \mathcal{C}} x_C^* + \frac{1}{\varepsilon}$ bins.*

Thus, we established our main result in the offline setting as follows.

Theorem 11 *Problem OOEBP admits an asymptotic approximation scheme.*

3 Online OEBP

We recall that OEBP is simply the unfair version of OOEBP. We will briefly discuss the unfair variant where the algorithm processes a sequence but an optimal solution can reorder. As mentioned earlier, any upper bound for this model is also an upper bound for OOEBP, since an offline algorithm has more power while an online algorithm has the same power.

For the case with 1-items, it is known that the tight asymptotic bound is 2 [23] (and this is in fact also an absolute bound). The algorithm is simply Next Fit (NF), that moves to the next bin when the current one has load of at least 1 and cannot receive additional items. In fact this ratio of 2 is tight also for the ratio between OOEBP and the unfair model. To see this fact that is implied by examples in previous work [23, 29], consider an input that starts with N items of size 1 followed by $N(M - 1)$ items of size $\frac{1}{M}$ (for integer $N, M > 1$). When we consider an optimal solution that can reorder the items, we have a solution with N bins each of which with $M - 1$ items of size $1/M$ followed by an item of size 1. When we consider an optimal solution that cannot reorder the items it consists of N bins each of which with one 1-item and another $\lceil N(M - 1)/M \rceil$ bins each of which with at most M items of size $1/M$. The

ratio between these costs approaches 2 when M grows to infinity, and thus an algorithm for OOEBP that is analyzed with respect to an optimal algorithm that can reorder the items cannot have (asymptotic or absolute) approximation ratio smaller than 2 (this applies not only for online algorithms but also e.g. for exponential time offline algorithms).

For the case without 1-items, this algorithm still has asymptotic competitive ratio 2 for OOEBP. Let $N > 0$ be a large integer, let $M = 2N$, and consider the following input. There are $4N$ items in total, where these items have sizes of $1 - \frac{1}{M}$ (large items) and of $\frac{1}{M}$ (smaller items), and the sizes are alternating (so there are $M = 2N$ items of each size). NF creates $2N$ bins, each with two items of different sizes. An optimal solution (which does not even need to reorder the input) has N bins with two large items, and one bin with all small items.

We define a different algorithm as follows. Apply NF separately on items of sizes in $(0, \frac{1}{2})$ and on items of sizes in $[\frac{1}{2}, 1)$. This is a variant of Harmonic, and we call it NF2. It was studied by Zhang [29] and we provide a short alternative proof.

Proposition 12 *NF2 has an asymptotic competitive ratio of at most $\frac{3}{2}$ for inputs without 1-items.*

Proof. We use the standard approach of a weight based analysis for the proof (see e.g. [21]). For every item of size below $\frac{1}{2}$, its weight is equal to its size. For every item of size $\frac{1}{2}$ or more, its weight is $\frac{1}{2}$. Thus, the weight of an item never exceeds its size and never exceeds $\frac{1}{2}$.

For a bin of OPT, the weight is at most $\frac{3}{2}$. This holds since the weight of the last item is at most $\frac{1}{2}$, and the other items have total size below 1.

For every bin of the algorithm, except for possibly two bins, the total size is at least 1 and so is the total weight. ■

Once again for OOEBP if we compare ourselves to an offline optimal solution that can reorder the items this ratio of $\frac{3}{2}$ is tight (for inputs that do not contain 1-items) as shown by Zhang [29]. For a large integer $N > 0$, let $M' = 2N^2$. Consider an input with $2N$ items of size $\frac{1}{2}$ (large items) followed by $2(M' - 1)N$ items of size $\frac{1}{M'}$ (small items). An optimal solution that can reorder the items consider reordering where the small items appear before the large items. It packs $M' - 1$ small items and one large item into every bin, and it has $2N$ bins. For an algorithm that is constrained to consider the input sequence and cannot reorder the items, no matter how many of the large items are packed in pairs and how many are packed alone, as M' is divisible by 2, no bin will have a load above 1, so it has at least $\frac{2(M'-1)N}{M'} + N = 3N - \frac{1}{N}$ bins.

4 Absolute competitive ratio for online OOEBP

In this section we provide a short discussion regarding the absolute competitive ratio for OOEBP. It is easily seen that NF has an absolute competitive ratio of at most 2 since every bin has a load of at least 1 (except for possibly the last bin, which still has a positive load), while an optimal solution has load smaller than 2 for every bin. We show that this is the best possible ratio.

Proposition 13 *The absolute competitive ratio of any online algorithm for OOEBP with 1-items is at least 2.*

Proof. Assume that there is an algorithm with absolute competitive ratio $1 \leq q < 2$. In particular, when an optimal solution has k bins, the algorithm cannot have more than $2k - 1$ bins.

The input is as follows. Let $N \geq 3$ be an integer and let $\varepsilon = \frac{1}{2^N}$. For $i = 1, 2, \dots, N$, there are four items in the i th batch, arriving in the order they are stated here, where their sizes are: $\varepsilon, 2 \cdot i \cdot \varepsilon, 1 - 2 \cdot i \cdot \varepsilon, 1$.

An optimal solution has the following properties. Its cost just before the arrival of the 1-item of the i th batch is at most i . This holds for any i due to the following packing. For $i = 1$, the first three items of the first batch can be packed into one bin. Consider the case $i > 1$. In this case, the first bin contains the first, third, and fourth items of the first batch. For $1 < j < i$, there is a bin containing the second item of batch $j - 1$ and three items from the j th batch, which are the first, third, and fourth items. The total size of the second item of batch $j - 1$ and the first and third items of batch j is $2 \cdot (j - 1) \cdot \varepsilon + \varepsilon + 1 - 2 \cdot j \cdot \varepsilon = 1 - \varepsilon$, and therefore such a bin is valid. For batch i , the second item of batch $i - 1$ and the first three items of batch i are packed into one bin. The total size of the second item of batch $i - 1$ and the first two items of batch i is $2 \cdot (i - 1) \cdot \varepsilon + \varepsilon + 2 \cdot i \cdot \varepsilon = (4 \cdot i - 1) \cdot \varepsilon < 4N\varepsilon < 1$, so this bin is valid as well.

Next, we prove that the items of each batch are packed into two bins by the algorithm, and these bins cannot receive additional items later. Thus, for every batch j , we assume that $2j - 2$ bins were already created, and these bins have loads at least 1, so batch j must be packed into new bins, and we show that two bins are created. Since an optimal solution has at most j bins when the first three items are presented, and there are already $2j - 2$ bins used by the algorithm, only one bin can be used by the algorithm for these three items. After these three items are packed, their bin has load above 1 (the load is $1 + \varepsilon$), and the fourth item is packed into another bin, which will have load 1 as a result. Thus, after all four items of batch j arrived, there are two bins created for this batch, both with loads of at least 1.

After all items have arrived, an optimal solution has at most $N + 1$ bins (since the last item which is a 1-item has to be packed too), while the algorithm has $2N$ bins. Letting N grow without bound shows that the absolute competitive ratio cannot be q . ■

5 An online algorithm for OOEBP for the case with 1-items

We will use a method which was used in the past for classic online bin packing. In this approach, one partitions items to types, and tries to combine some types of relatively large items with other types [21, 26]. The novelty in our method lies in the adaptation of this idea to inputs where the order of items matters. In our analysis, we will split the input at a point where the behavior of the algorithm changes as a result of a different input type. More specifically, the packing is different when those large items stop arriving, and the analysis is different too. While in work for classic bin packing there are two or more scenarios [21, 26, 3], where each of them may happen for some input, here the two scenarios frequently happen for one input.

We start with defining the algorithm (or actually a class or kind of algorithms), and elaborate on the analysis later. The algorithm uses an integer parameter $M \geq 2$. The algorithm classify items into classes based on the size of the items. The item classes are as follows.

- Items of size 1, also called 1-items are class 0.
- For $1 \leq i \leq M - 1$, class i consists of items of sizes in $[\frac{1}{i+1}, \frac{1}{i})$. Such items are called *regular items*, or regular items of class i if all of them belong to this class.
- Class M consists of items of sizes in $(0, \frac{1}{M})$. Such items are called *tiny items*.

For every class $1 \leq i \leq M$, bins for this class will contain items of the class and possibly also a 1-item. There may be two kinds of bins, called *large* and *small*. For $1 \leq i \leq M - 1$, a large bin is planned to have $i + 1$ items of the class, which is always a feasible bin as the total size of i items of this class is below 1. A small bin is planned to have i items of the class, and possibly also a 1-item that arrives after the i items of the class have already been packed into this bin. For tiny items, a large bin will have a total size of at least 1, and a small bin will have a total size in $[1 - \frac{1}{M}, 1)$.

For every class $i \geq 1$, there will be at most one large bin and at most one small bin that did not receive the required total size or total number of items of class i . These bins will be called *active*, and every class may have an active small bin and an active large bin. Other bins are called *inactive*, and there may be an arbitrary number of inactive small bins and inactive large bins. The small inactive bins are partitioned into ready bins, which are bins that did not receive 1-items, and used bins, which are bins that each one of them received a 1-item as its last item (this was done after the bin became inactive).

There is a parameter $0 \leq \beta_i \leq 1$, which is the approximate fraction of bins for class $i \geq 1$ (active and inactive) that are small (and the fraction of large bins is approximately $1 - \beta_i$).

The algorithm is defined as follows, and its action is based on item classes.

- When a 1-item arrives, act as follows. If there is a ready bin of some class $i \geq 1$, pack the new item into one such bin (and the bin becomes used). Otherwise, pack it into an empty bin of class 0.
- When a regular item of class i arrives, act as follows. If there is an active bin for this class, pack it into such a bin. Otherwise, let n_i be the current number of bins for this class (large and small, all inactive). Let n_i^ℓ and n_i^s (where $n_i = n_i^\ell + n_i^s$) be the numbers of large and small bins for class i , respectively (all numbers are calculated excluding the new bin that will be opened). If $n_i^s \leq \beta_i \cdot n_i$, open a new small active bin for class i , and otherwise (in which case $n_i^s > \beta_i \cdot n_i$ and therefore $n_i^\ell = n_i - n_i^s < n_i - \beta_i \cdot n_i = (1 - \beta_i)n_i$) open a new large active bin for class i . The new item is packed into the new bin.

No matter which bin received the item (new or not), if the bin has its planned number of items ($i + 1$ items if it is large, i items if it is small), define the bin to be inactive, and if it is small, additionally define it to be ready.

- When a tiny item arrives, act as follows. If there is an active bin for this class, pack it into such a bin. A large active bin has a total size below 1 so it can receive a new item, and a small active bin has a total size below $1 - \frac{1}{M}$ so it can also receive a new item and remain small.

Otherwise, let n_M the current number of bins for this class (large and small, all inactive). Let n_M^ℓ and n_M^s (where $n_M = n_M^\ell + n_M^s$) be the numbers of large and small bins for class M , respectively. If $n_M^s \leq \beta_M \cdot n_M$, open a new small active bin for class M , and otherwise (in which case $n_M^\ell < (1 - \beta_M) \cdot n_M$) open a new large active bin for class M . The new item is packed into the new bin.

No matter which bin received the item, if the bin has total size of at least 1 and it is large, or if it has total size above $1 - \frac{1}{M}$ and it is small, define the bin to be inactive, and if it is small, additionally define it to be ready.

In what follows, we consider only algorithms defined in this way. For the analysis, we would like to split the input I into two parts I_1 and I_2 by removing items packed by the algorithm into a constant number of bins and partitioning the remaining items. After this removal of some items and partitioning, the remaining bins of the algorithm will not contain items of both sub-inputs simultaneously, and as we will base our weights on this partition, the partitioning property will also be used in the analysis.

Let x denote the last 1-item that is packed into a new bin. If there is no such item, the first part I_1 of the input is defined to be empty. The removal of bins which defines a removal of items from I is defined as follows. For all bins that are active at the time of arrival of x , remove their items from I (including items packed after the arrival of x). Similarly, remove all bins that are active at termination. For the remaining items I' define a partition as follows: I_1 consists of all remaining items arriving before x and including x , and I_2 consists of all other items that were not removed (those arriving strictly after x).

Properties of the partition of the input. Since the order of items in I' is the same as their order in I , and packing of I can be used as a packing for I' , we conclude that $\text{OPT}(I') \leq \text{OPT}(I)$. On the other hand, since there are at most $2M$ active bins at each time, the cost of the algorithm is at most $4M$ plus the number of remaining bins.

Claim 14 *Items of I_1 and items of I_2 are packed into different bins by the algorithm. Bins containing these items are not active at termination.*

Proof. We start with the first part. Assume by contradiction that this is not the case, and there is an item y of I_2 packed with an item z of I_1 . As all items of I_2 arrive after all items of I_1 , y is packed after z . Thus, at the time of arrival of x the bin of z has to be active. However, all active bins of this time were removed to obtain I' , a contradiction.

Assume by contradiction that there is an active bin. It cannot be of I_2 as bins that are active at termination were removed. It cannot be of I_1 since at the time of arrival of x all items of I_1 already arrived, and active bins were removed then too. ■

Claim 15 *At the time of arrival of x , there are no ready bins.*

Proof. This holds since x is packed into a new bin. ■

Claim 16 *Let $1 \leq i \leq M$. Let $n_i(1)$, $n_i(2)$, $n_i^\ell(1)$, $n_i^\ell(2)$, $n_i^s(1)$, and $n_i^s(2)$ denote the total number of bins for class i out of bins of I_1 , the total number of bins for class i out of bins of I_2 , the number of large bins for class i out of bins of I_1 , the number of large bins for class i out of bins of I_2 , the number of small bins for class i out of bins of I_1 , and the number of small bins for class i out of bins of I_2 , respectively.*

Then, it holds that

$$(1 - \beta_i) \cdot n_i(1) - 1 \leq n_i^\ell(1) \leq (1 - \beta_i) \cdot n_i(1) + 1, \quad \beta_i \cdot n_i(1) - 1 \leq n_i^s(1) \leq \beta_i \cdot n_i(1) + 1,$$

$$(1 - \beta_i) \cdot n_i(2) - 3 \leq n_i^\ell(2) \leq (1 - \beta_i) \cdot n_i(2) + 3, \quad \text{and} \quad \beta_i \cdot n_i(2) - 3 \leq n_i^s(2) \leq \beta_i \cdot n_i(2) + 3.$$

Proof. We will prove the properties for the cases where $\beta_i \in (0, 1)$, since the cases $\beta_i \in \{0, 1\}$ are trivial, since $\beta_i = 0$ means that there are no small bins, while $\beta_i = 1$ means that there are no large bins.

Consider the opening time of the last large bin for class i in I_1 , and let N_1 be the number of bins of class i before it is opened. Since at this moment, a new bin is being opened for class i , at this time there are no active bins for class i , and none of the existing bins for this class will be removed in the process of moving from I to I' . Therefore before the bin is opened, the number of large bins for class i is exactly $n_i^\ell(1) - 1$, and $n_i(1) \geq N_1 + 1$, since there will be exactly one additional large bin for class i for I_1 , and possibly small bins. By $n_i^\ell(1) - 1 \leq (1 - \beta_i)N_1$ (which is the opening rule), we have $n_i^\ell(1) \leq (1 - \beta_i) \cdot N_1 + 1 \leq (1 - \beta_i) \cdot (n_i(1) - 1) + 1 \leq (1 - \beta_i) \cdot n_i(1) + 1$, where the second inequality is because of the value of N_1 stated above. Similarly, we can get $n_i^s(1) \leq \beta_i \cdot n_i(1) + 1$. If there are no large bins, or no small bins for class i and I_1 , the inequalities hold trivially. As $n_i(1) = n_i^s(1) + n_i^\ell(1)$, we have $n_i^\ell(1) \geq n_i(1) - (\beta_i \cdot n_i(1) + 1) = (1 - \beta_i)n_i(1) - 1$ and similarly, $n_i^s(1) \geq n_i(1) - ((1 - \beta_i) \cdot n_i(1) + 1) = \beta_i \cdot n_i(1) - 1$.

Now, consider the last large bin for class i that is opened for I_2 and let N_2 be the number of bins of class i before it is opened (including bins of I_1 and removed bins). We still assume here that $0 < \beta_i < 1$, and we assume that there is at least one large bin for proving an upper bound on such bins, and that there is at least one small bin for proving an upper bound on such bins, since the bounds hold trivially otherwise. There are no active bins for class i at this time, and out of such bins at most two existing bins for class i will be removed (those that were active when x arrived) to obtain $n_i(1) + n_i(2)$ bins of class i at termination (active bins that might be removed at termination do not exist yet). On the other

hand, one large bin for class i will be created. Thus, $n_i(1) + n_i(2) \geq N_2 - 1$. By the opening rule, the number of large class i bins at this time is at most $(1 - \beta_i) \cdot N_2$, and the final number is at most $(1 - \beta_i)(n_i(1) + n_i(2) + 1) + 1 \leq (1 - \beta_i)(n_i(1) + n_i(2)) + 2$. Similarly, for small class i bins, the final number is at most $\beta_i \cdot (n_i(1) + n_i(2)) + 2$. Given the lower bounds on the number of bins for I_1 , the bin numbers for I_2 are at most $(1 - \beta_i) \cdot n_i(2) + 3$ and $\beta_i \cdot n_i(2) + 3$, respectively. Thus, for I_2 , since the sum of numbers is $n_i(2)$, the numbers are at least $(1 - \beta_i) \cdot n_i(2) - 3$ and $\beta_i \cdot n_i(2) - 3$, respectively. ■

Claim 17 *For every small bin of I_1 for some class i , the bin is used.*

Proof. When the last item x of I_1 arrives, it is a 1-item packed into a new bin, and it is not removed. Since x cannot be packed into a ready bin, there are no ready bins for I_1 , at the termination of I_1 . ■

Claim 18 *There are no bins of I_2 with a single item that is a 1-item.*

Proof. All items of I_2 arrived after x , which is the last 1-item to be packed into a new bin. This holds if x does not exist as well, because in that case no 1-item of I is packed into a new bin. ■

The weight functions. We define two sets of weights, denoted by w and v , where $v, w : (0, 1] \rightarrow \mathbb{R}$, one for I_1 and one for I_2 . Letting W be the total weight of items of I_1 according to w and letting V be the total weight of items of I_2 according to v , we will show that the cost of the algorithm is at most $W + V + 16 \cdot M$. We will show this claim by considering the non-active bins of I' and show that (almost) every bin has total weight at least 1 according to the suitable weight function. For an optimal solution, we will find a value R such that no bin has total weight above R where we define a weight function $f : I' \rightarrow \mathbb{R}$ for which $f(a) = w(a)$ if $a \in I_1$ and $f(a) = v(a)$ if $a \in I_2$. Since $V + W \leq R \cdot OPT(I')$, the upper bound on the asymptotic competitive ratio will follow.

Function w is defined as follows.

- The weight of a 1-item is 1.
- The weight of any item of class $1 \leq i \leq M - 1$ is $\frac{1 - \beta_i}{i + 1 - \beta_i}$.
- The weight of any item of class M of size ρ is $\frac{1 - \beta_M}{1 - \beta_M/M} \cdot \rho$.

Function v is defined as follows.

- The weight of a 1-item is 0.
- The weight of any item of class $1 \leq i \leq M - 1$ is $\frac{1}{i + 1 - \beta_i}$.
- The weight of any item of class M of size ρ is $\frac{1}{1 - \beta_M/M} \cdot \rho$.

We have $ALG(I) \leq n_1 + n_2 + 4M$, where n_j is the number of bins for I_j for $j = 1, 2$, and we show $n_1 \leq W + 3M$ and $n_2 \leq V + 9M$.

Claim 19 *The total weight of bins of the algorithm for I_1 with respect to w is at least $n_1 - 3M$.*

Proof. First, consider bins with 1-items packed into new bins. Every such bin has weight 1. Every remaining bin belong to a class i , where $0 < i \leq M$.

Consider a class $1 \leq i \leq M - 1$. Recall that by Claim 18 there are no ready bins, so every small bin has a 1-item. The total weight including 1-items and the complete numbers of items of class i (as there are no active bins) is $\frac{1 - \beta_i}{i + 1 - \beta_i}((i + 1)n_i^\ell(1) + i \cdot n_i^s(1)) + n_i^s(1) = \frac{(i + 1)(1 - \beta_i)}{i + 1 - \beta_i}n_i^\ell(1) + \frac{i(1 - \beta_i) + i + 1 - \beta_i}{i + 1 - \beta_i}n_i^s(1) \geq$

$$\frac{(i+1)(1-\beta_i)}{i+1-\beta_i}((1-\beta_i) \cdot n_i(1) - 1) + \frac{i(1-\beta_i)+i+1-\beta_i}{i+1-\beta_i}(\beta_i \cdot n_i(1) - 1) \geq \frac{(i+1)(1-\beta_i)}{i+1-\beta_i}n_i(1) + \frac{i\beta_i}{i+1-\beta_i}n_i(1) - \frac{2(i+1)(1-\beta_i)+i}{i+1-\beta_i} \geq n_i(1) - 3, \text{ since } \frac{2(i+1)(1-\beta_i)+i}{i+1-\beta_i} = 2 + \frac{i(1-2\beta_i)}{i+1-\beta_i} \leq 3.$$

Finally, consider class M . The total weight is at least

$$\begin{aligned} & \frac{1-\beta_M}{1-\beta_M/M}((n_M^\ell(1) + (1-1/M) \cdot n_M^s(1)) + n_M^s(1)) \\ &= \frac{1-\beta_M}{1-\beta_M/M} \cdot n_M^\ell(1) + \frac{(1-\beta_M)(1-1/M) + 1-\beta_M/M}{1-\beta_M/M} n_M^s(1) \\ &\geq \frac{1-\beta_M}{1-\beta_M/M} \cdot ((1-\beta_M)n_M(1) - 1) + \frac{(1-\beta_M)(1-1/M) + 1-\beta_M/M}{1-\beta_M/M} (\beta_M \cdot n_M(1) - 1) \\ &= \frac{1-\beta_M}{1-\beta_M/M} \cdot n_M(1) + \frac{\beta_M(1-1/M)}{1-\beta_M/M} \cdot n_M(1) - \frac{1-\beta_M + (1-\beta_M)(1-1/M) + 1-\beta_M/M}{1-\beta_M/M} \\ &\geq n_M(1) - 3. \blacksquare \end{aligned}$$

Claim 20 *The total weight of bins of the algorithm for I_2 with respect to v is at least $n_2 - 9M$.*

Proof. By definition of x , the bins of I_2 do not include bins where there is only a 1-item. Other bins may contain such items, but their weights are equal to 0, so we do not discuss such items.

Consider a class $1 \leq i \leq M-1$. The total weight is $\frac{1}{i+1-\beta_i}((i+1)n_i^\ell(2) + i \cdot n_i^s(2)) \geq \frac{i+1}{i+1-\beta_i}((1-\beta_i) \cdot n_i(2) - 3) + \frac{i}{i+1-\beta_i} \cdot (\beta_i \cdot n_i(2) - 3) \geq n_i(2) - 9$.

Finally, consider class M . Recall that $M \geq 2$ and $\beta_M \leq 1$, and thus $\frac{1}{1-\frac{\beta_M}{M}} \leq 2$. The total weight is at least $\frac{1}{1-\beta_M/M}((n_M^\ell(1) + (1-1/M) \cdot n_M^s(1)) \geq \frac{1}{1-\beta_M/M} \cdot ((1-\beta_M)n_M(2) - 3) + (1-1/M) \cdot (\beta_M \cdot n_M(2) - 3) \geq n_M(2) - 9$. \blacksquare

The next claim holds using the definition of weights directly.

Claim 21 *For any $1 \leq i \leq M-1$, both weights are in $[0, \frac{1}{i}]$.*

The next claim show that a bin does not contain an item of I_1 and an item of I_2 at the same time.

Claim 22 *It is sufficient to analyze bins of OPT containing items of only one of the sets I_1 and I_2 .*

Proof. For every bin of OPT, if the bin has a 1-item of weight 0, we can remove it from the bin for the calculation without changing the total weight for the bin. Now all 1-items included in bins are those that arrived not later than x .

We consider two cases. If the bin has a 1-item, then this item has to be the last item of the bin, based on the definition of OOEBP. Therefore, the bin only has items that arrived not later than x .

If the bin does not have a 1-item, we can analyze it according the weight function v , as for any item that is not a 1-item, the weight according to v is not smaller than that of w , and this result in the same or a larger value of R for the algorithm, so it can be assumed that it only has items arriving strictly after x . That is, it is analyzed as if only items of I_2 were packed into it originally. \blacksquare

A bad example for algorithms of this class

A simple example showing that an algorithm of the type we consider cannot have a very small asymptotic competitive ratio, i.e., a lower bound is given for this kind of algorithms. Let $M > 500$. There will be no items smaller than $\frac{1}{M}$ and for simplicity we use parameters γ_i . The value γ_i denotes for class i the fraction of items of small bins so $\gamma_i = \frac{i \cdot \beta_i}{(i+1) \cdot (1-\beta_i) + i \cdot \beta_i}$. We ignore rounding issues in this example that can be assumed by using large enough value of N where $N > 0$ be a very large integer.

The input is as follows, consisting of 11 batches.

- $\frac{N}{462}$ items of size $\frac{1}{463}$.
- $\frac{N}{21}$ items of size $\frac{1}{22}$.
- $\frac{N}{3}$ items of size $\frac{1}{7}$.
- N items of size $\frac{1}{3}$.
- N items of size $\frac{1}{2}$.
- N 1-items.
- N items of size $\frac{1}{463}$.
- N items of size $\frac{1}{22}$.
- $2N$ items of size $\frac{1}{7}$.
- $2N$ items of size $\frac{1}{3}$.
- N items of size $\frac{1}{2}$.

To obtain an offline packing, pack items of the first three batches into different bins (i.e., one item from the union of these three batches into each bin), and add one item of each of the next three batches into these bins. As not all items of batches 4 and 5 were packed (only $\frac{N}{462} + \frac{N}{21} + \frac{N}{3}$ bins were used), the remaining items are packed into bins with one item of each size. Before packing the 1-item, every bin has a total size below 1.

For the five last batches, every bin will have one item of size $\frac{1}{463}$, one item of size $\frac{1}{22}$, two items of size $\frac{1}{7}$, two items of size $\frac{1}{3}$, and one item of size $\frac{1}{2}$.

We find $\text{OPT} \leq 2N$.

Consider the action of the algorithm. For the five first batches, it has the following bins:

- $(\gamma_{462} \cdot \frac{N}{462})/462$ small bins of class 462 and $((1 - \gamma_{462}) \cdot \frac{N}{462})/463$ large bins of this class.
- $(\gamma_{21} \cdot \frac{N}{21})/21$ small bins of class 21 and $((1 - \gamma_{21}) \cdot \frac{N}{21})/22$ large bins of this class.
- $(\gamma_6 \cdot \frac{N}{3})/6$ small bins of class 6 and $((1 - \gamma_6) \cdot \frac{N}{3})/7$ large bins of this class.
- $(\gamma_2 \cdot N)/2$ small bins of class 2 and $((1 - \gamma_2) \cdot N)/3$ large bins of this class.
- $(\gamma_1 \cdot N)$ small bins of class 1 and $((1 - \gamma_1) \cdot N)/2$ large bins of this class.

The number of bins after the first six batches are presented is $\max\{N, (\gamma_{462} \cdot \frac{N}{462})/462 + \gamma_{21} \cdot \frac{N}{21} + (\gamma_6 \cdot \frac{N}{3})/6 + (\gamma_2 \cdot N)/2 + (\gamma_1 \cdot N)\} + ((1 - \gamma_{462}) \cdot \frac{N}{462})/463 + ((1 - \gamma_{21}) \cdot \frac{N}{21})/22 + ((1 - \gamma_6) \cdot \frac{N}{3})/7 + ((1 - \gamma_2) \cdot N)/3 + ((1 - \gamma_1) \cdot N)/2 \geq N + ((1 - \gamma_{462}) \cdot \frac{N}{462})/463 + ((1 - \gamma_{21}) \cdot \frac{N}{21})/22 + ((1 - \gamma_6) \cdot \frac{N}{3})/7 + ((1 - \gamma_2) \cdot N)/3 + ((1 - \gamma_1) \cdot N)/2$.

For the second part of the input, the next bins are built:

- $(\gamma_{462} \cdot N)/462$ small bins of class 462 and $((1 - \gamma_{462}) \cdot N)/463$ large bins of this class.
- $(\gamma_{21} \cdot N)/21$ small bins of class 21 and $((1 - \gamma_{21}) \cdot N)/22$ large bins of this class.
- $(\gamma_6 \cdot 2N)/6$ small bins of class 6 and $((1 - \gamma_6) \cdot 2N)/7$ large bins of this class.
- $(\gamma_2 \cdot 2N)/2$ small bins of class 2 and $((1 - \gamma_2) \cdot 2N)/3$ large bins of this class.
- $(\gamma_1 \cdot N)$ small bins of class 1 and $((1 - \gamma_1) \cdot N)/2$ large bins of this class.

The numbers of large bins for the different classes are:

- For class 462, $((1 - \gamma_{462}) \cdot N)/463 + ((1 - \gamma_{462}) \cdot \frac{N}{462})/463 = ((1 - \gamma_{462}) \cdot N)/462$.
- For class 21, $((1 - \gamma_{21}) \cdot N)/22 + ((1 - \gamma_{21}) \cdot \frac{N}{21})/22 = ((1 - \gamma_{21}) \cdot N)/21$.
- For class 6, $((1 - \gamma_6) \cdot 2N)/7 + ((1 - \gamma_6) \cdot \frac{N}{3})/7 = ((1 - \gamma_6) \cdot N)/3$.
- For class 2, $((1 - \gamma_2) \cdot 2N)/3 + ((1 - \gamma_2) \cdot N)/3 = (1 - \gamma_2) \cdot N$.
- For class 1, $((1 - \gamma_1) \cdot N)/2 + ((1 - \gamma_1) \cdot N)/2 = (1 - \gamma_1) \cdot N$.

The numbers of bins for the different classes are (excluding small bins of the first part of the input), in addition to the N bins containing 1-item.

- For class 462, $((1 - \gamma_{462}) \cdot N)/462 + (\gamma_{462} \cdot N)/462 = N/462$.
- For class 21, $((1 - \gamma_{21}) \cdot N)/21 + (\gamma_{21} \cdot N)/21 = N/21$.
- For class 6, $((1 - \gamma_6) \cdot N)/3 + (\gamma_6 \cdot 2N)/6 = N/3$.
- For class 2, $(1 - \gamma_2) \cdot N + (\gamma_2 \cdot 2N)/2 = N$.
- For class 1, $(1 - \gamma_1) \cdot N + (\gamma_1 \cdot N) = N$.

In total we have $3N + N/3 + N/21 + N/462 = 1563N/462$.

The ratio for the algorithms in the examined class is at least $\frac{1563}{924} \approx 1.691558441558442$.

To obtain a set of parameters and a tight example, we define the following sequence: $t_1 = 22$, $t_{i+1} = t_i(t_i - 1) + 1$ (so $t_2 = 463$, $t_3 = 213907$ etc.). We have $\sum_{i=1}^{\infty} \frac{1}{t_i} < \frac{1}{21}$, and we let $C = \sum_{i=1}^{\infty} \frac{1}{t_i - 1} \approx 0.04978822$.

We also let $R = \frac{5}{3} + \frac{C}{2} \approx 1.691560779$ (where $R > 1.691560779$ and $R < 1.69156078$).

The example above can be modified to give a lower bound of R . In both parts of the input, instead of items of sizes $\frac{1}{463} = \frac{1}{t_2}$ and $\frac{1}{22} = \frac{1}{t_1}$, there will be items of sizes $\frac{1}{t_g}, \frac{1}{t_{g-1}}, \dots, \frac{1}{t_1}$, for a fixed integer $g \geq 3$, where $M > \frac{1}{t_g}$. The numbers of these items are N in the second part of the input (after the 1-items arrive), and $\frac{N}{t_i - 1}$ for items of size $\frac{1}{t_i}$ in the beginning of the input. Since $C < 0.05$, it is still possible to pack N bins by assigning the items of the first $g + 1$ batches of sizes $\frac{1}{t_g}, \frac{1}{t_{g-1}}, \dots, \frac{1}{t_1}, \frac{1}{7}$ into different bins, and pack a triple of items of sizes $\frac{1}{3}, \frac{1}{2}$, and 1 into these bins and new bins, such that N bins are created. Since $\sum_{i=1}^{\infty} \frac{1}{t_i} < \frac{1}{21}$, the second part of the input can be still packed into N bins by an offline solution.

For items of size $\frac{1}{t_i}$, the number of large bins for the first part of the input is $(1 - \gamma_{t_i-1}) \frac{N}{t_i - 1} / t_i$. The number of bins for the second part of the input is $\gamma_{t_i-1} \cdot \frac{N}{t_i - 1} + (1 - \gamma_{t_i-1}) \frac{N}{t_i}$. In total we have

$$\frac{N(1 - \gamma_{t_i-1} + t_i \gamma_{t_i-1} + (t_i - 1)(1 - \gamma_{t_i-1}))}{t_i(t_i - 1)} = \frac{N}{t_i - 1}.$$

Thus, while the bound on the optimal cost is unchanged, we can replace $3N + N/3 + N/21 + N/462$ with $\frac{10N}{3} + \sum_{i=1}^g \frac{N}{t_i - 1}$ in the cost of the algorithm. Letting g grow without bound, we get $N(\frac{10}{3} + C) = N \cdot 2R$, for a lower bound of R on the asymptotic competitive ratio of the above class of algorithms.

The best possible algorithms of this class

The set of parameters we will use for our algorithm is as follows.

We let M be a large integer. We exhibit an algorithm for every such value of M whose asymptotic competitive ratio tends to R as M grows unbounded. Here R is the same as in the previous section, i.e., in the bad example. We also use $\beta_1 = \frac{58}{529}$, and $\beta_2 = 3 - \frac{4-2\beta_1}{(3-R)(2-\beta_1)-1} \approx 0.434052654632836$, $\beta_3 = \frac{148}{287}$, $\beta_4 = \frac{15}{23}$, $\beta_5 = \frac{13}{23}$, and $\beta_i = 1$ for $i \geq 6$.

We have $\frac{1-\beta_1}{2-\beta_1} + \frac{1-\beta_2}{3-\beta_2} = 2 - \frac{1}{2-\beta_1} - \frac{2}{3-\beta_2} = 2 - \frac{1}{2-\beta_1} - 2 \cdot \frac{(3-R)(2-\beta_1)-1}{4-2\beta_1} = R - 1$.

Recall that $w_i = (1 - \beta_i)/(1 + i - \beta_i)$. The weights according to w for classes 1, 2, 3, 4, 5 are not larger than 0.471, 0.22056078, 0.139, 0.08, 0.08, and for $6 \leq i \leq M$, the weight is 0.

The weights according to v for classes 1, 2, 3, 4, 5 are not larger than: 0.529, 0.389719611, 0.287, 0.23, 0.184, respectively. For class $6 \leq i \leq M - 1$, the weight according to v is $\frac{1}{i}$, and for class M it is $\frac{M}{M-1}$ times the size.

For $1 \leq i \leq M - 1$, let v_i and w_i denote the weight of an item of class i , based on the weight functions v and w , respectively. We will analyze the possible total weights of packed bins.

Claim 23 *Every bin of OPT with items of I_1 has weight not larger than R according to the weight function w .*

Proof. A given bin can contain a total size below 1, and one additional item. The largest possible weight of the additional item is 1.

Items of sizes in $(0, 1)$ having positive weights according to w are in fact items of sizes in $[\frac{1}{6}, 1)$.

For any item of size $u \in (0, \frac{1}{2})$, we have $w(u) < \frac{2u}{3}$, and therefore if there is no item of size in $[\frac{1}{2}, 1)$, the total weight is at most $\frac{5}{3}$.

We are left with the case that there is an item of size in $[\frac{1}{2}, 1)$ packed into the bin. There can be at most two other items with positive weights packed into the bin, where if there are two such items, at least one of them has size strictly below $\frac{1}{4}$, and if there is an item of size above $\frac{1}{3}$, there is no second item with a positive weight.

Therefore, if there is also an item of size in $[\frac{1}{3}, \frac{1}{2})$, there cannot be another item of size above $\frac{1}{6}$, and the total weight is at most $1 + w_1 + w_2 = 1 + \frac{1-\beta_1}{2-\beta_1} + \frac{1-\beta_2}{3-\beta_2} = R$.

We are left with the case that there is no item of size in $[\frac{1}{3}, \frac{1}{2})$. In this case the total weight of additional items is at most $0.139 + 0.08$, and the total weight for the bin is at most $1.69 < R$. ■

Claim 24 *Every bin of OPT with items of I_2 has weight not larger than R according to weight function v .*

Proof. For an item of size u that belongs to one of the classes $i = 2, 3, 4, 5$, we have $\frac{v(u)}{u} < 1.169158832$, 1.148, 1.15, and 1.104, respectively. For other values of i , this ratio does not exceed the value $\frac{7}{6}$. For $i \geq 6$, and for any item of size u , of class $j \geq i$, it holds that $\frac{v(u)}{u} \leq \frac{i+1}{i}$. We call the ratio between weight and size *the density of the item*.

A given bin can contain a total size below 1, and one additional item (which is an exceeding item). The largest possible weight of the additional item is at most $v_1 \leq 0.529$. If there is an additional such item (where there can be at most one additional item of this class), the total weight will not exceed $2 \cdot 0.529 + 1.169158832 \cdot \frac{1}{2} < 1.6426$.

If no other item except for the additional item has size of at least $\frac{1}{7}$, all items are of classes whose indices are 7 or larger, the densities are no larger than $\frac{8}{7}$, and the total weight is at most $\frac{8}{7} + 0.529 < 1.672$. Thus, we are left with the case that there is one item (of weight at most v_1), there is at least one item of classes 2, 3, 4, 5, 6, and the remaining items are of classes 2, 3, \dots . For classes 2, 3, \dots , $M - 1$ the weights are equal for all items of one class, and therefore we can assume that an item of class i has size $\frac{1}{i+1}$ (the total weight may only increase by decreasing item sizes like this and possibly filling up the possible remaining space by possible other items).

Note that the only classes for which the density may be larger than 1.15 are 2 and 6, and the density for these classes is below 1.169158832 and $\frac{7}{6}$, respectively. The bin can contain at most two items of class 2. We will use the properties $\frac{2}{3} + \frac{3}{7} > 1$, $\frac{2}{3} + \frac{1}{7} > 0.8$, $\frac{1}{3} + \frac{5}{7} > 1$, $\frac{1}{3} + \frac{2}{7} < 0.62$, and $\frac{4}{7} < 0.58$.

If the total size of items of classes 2 and 6 is not larger than 0.62, the total weight is at most $v_1 + 0.62 \cdot 1.169158832 + 0.38 \cdot 1.15 < 1.69088 < R$.

We consider vectors of length 5 to denote multisets of items of classes 2, 3, 4, 5, 6 that may be packed into a common bin. A vector $(z_2, z_3, z_4, z_5, z_6)$ means that there are z_j items of class j . It is valid if $\sum_{j=2}^6 \frac{z_j}{j+1} < 1$. The vector $(0, 0, 0, 0, 0)$ was already considered. We consider all valid vectors for which the total size of items of classes 2, 6 is above 0.62.

For each vector we find the remaining space excluding the already existing items, and we find an integer i such that all remaining items are of classes with indices i or larger. The first component of any valid vector is at most 2. By the lower bound of 0.62 on the total size of items of classes 2 and 6, we conclude the following.

1. If it is equal to 2, the last component is 0, 1 or 2.
2. If it is equal to 1, the last component is 3 or 4.
3. If it is equal to 0, the last component is 5 or 6 (since this component is at most 6 and $\frac{4}{7} < 0.62$).

Taking into account the other three components and the property that the total size is strictly below 1, there are 16 suitable vectors, and we analyze each one of them separately.

- $(0, 1, 0, 0, 5)$. In this case the remaining size is below $\frac{1}{28}$. Thus, all remaining items have sizes below $\frac{1}{28}$ and their density is at most $\frac{29}{28}$. We get a total weight of at most $v_1 + 0.287 + \frac{5}{6} + \frac{29}{28} \cdot \frac{1}{28} < 1.687$.
- $(0, 0, 1, 0, 5)$. In this case the remaining size is $\frac{3}{35} < \frac{1}{11}$. Thus, all remaining items have sizes below $\frac{1}{11}$ and their density is at most $\frac{12}{11}$. We get a total weight of at most $v_1 + 0.23 + \frac{5}{6} + \frac{12}{11} \cdot \frac{3}{35} < 1.686$.
- $(0, 0, 0, 1, 5)$. In this case the remaining size is $\frac{5}{42} < \frac{1}{8}$. Thus, all remaining items have sizes below $\frac{1}{8}$ and their density is at most $\frac{9}{8}$. We get a total weight of at most $v_1 + 0.184 + \frac{5}{6} + \frac{9}{8} \cdot \frac{5}{42} < 1.681$.
- $(0, 0, 0, 0, 5)$. In this case the remaining size is below $\frac{2}{7}$. All remaining items have sizes below $\frac{1}{7}$ and their density is at most $\frac{8}{7}$. We get a total weight of at most $v_1 + \frac{5}{6} + \frac{8}{7} \cdot \frac{2}{7} < 1.689$.
- $(0, 0, 0, 0, 6)$. In this case the remaining size is below $\frac{1}{7}$. All remaining items have sizes below $\frac{1}{7}$ and their density is at most $\frac{8}{7}$. If they are in fact smaller than $\frac{1}{8}$, the densities are at most $\frac{9}{8}$, and we get a total weight of at most $v_1 + 1 + \frac{9}{8} \cdot \frac{1}{7} < 1.69$.

Otherwise, there is an item of class 7 and weight $\frac{1}{7}$, and since its size is $\frac{1}{8}$, the remaining items have total size below $\frac{1}{56}$ and density at most $\frac{57}{56}$. If the maximum density is in fact at most $\frac{58}{57}$, we get a total weight of at most $v_1 + 1 + \frac{1}{7} + \frac{58}{57} \cdot \frac{1}{56} < 1.6900276$.

Finally, if there is an item of size $\frac{1}{57}$ and weight $\frac{1}{56}$, the remaining items have a total size below $\frac{1}{3192}$ and density at most $\frac{3193}{3192}$, and the total weight is at most $v_1 + 1 + \frac{1}{7} + \frac{1}{56} + \frac{3193}{3192} \cdot \frac{1}{3192} < 1.6900277$.

In the next cases, we will use the property $v_1 + v_2 \leq 0.529 + 0.389719611 = 0.918719611$.

- $(1, 0, 0, 1, 3)$. In this case the remaining size is below $\frac{1}{14}$. Thus, all remaining items have sizes below $\frac{1}{14}$ and their density is at most $\frac{15}{14}$. We get a total weight of at most $v_1 + v_2 + 0.184 + \frac{3}{6} + \frac{15}{14} \cdot \frac{1}{14} < 1.679250223245$.
- $(1, 0, 1, 0, 3)$. In this case the remaining size is below $\frac{4}{105}$. Thus, all remaining items have sizes below $\frac{1}{26}$ and their density is at most $\frac{27}{26}$. We get a total weight of at most $v_1 + v_2 + 0.23 + \frac{3}{6} + \frac{27}{26} \cdot \frac{4}{105} < 1.6882800505605$.
- $(1, 0, 0, 0, 3)$. In this case the remaining size is below $\frac{5}{21}$. First, consider the case where all remaining items have sizes below $\frac{1}{8}$. Their density is at most $\frac{9}{8}$. We get a total weight of at most $v_1 + v_2 + \frac{3}{6} + \frac{9}{8} \cdot \frac{5}{21} < 1.6865767538572$.

Otherwise, there is one item of class 7 (there can be at most one such item). Excluding this last item, the remaining total size is at most $\frac{19}{168}$, and the density for it is at most $\frac{9}{8}$. We get a total weight of at most $v_1 + v_2 + \frac{3}{6} + \frac{1}{7} + \frac{9}{8} \cdot \frac{19}{168} < 1.6888088967143$.

- $(1, 0, 0, 0, 4)$. In this case the remaining size is below $\frac{2}{21}$. Thus, all remaining items have sizes below $\frac{1}{10}$ and their density is at most $\frac{11}{10}$.

If there is no item of size $\frac{1}{11}$, the density is at most $\frac{12}{11}$, and we get a total weight of at most $v_1 + v_2 + \frac{4}{6} + \frac{12}{11} \cdot \frac{2}{21} < 1.6893$.

Otherwise, the total size of remaining items in this case is below $\frac{1}{231}$ and the density is at most $\frac{232}{231}$, and we get a total weight of at most $v_1 + v_2 + \frac{4}{6} + \frac{1}{10} + \frac{232}{231} \cdot \frac{1}{231} < 1.68974$.

In the remaining cases we use the property that there is one item of class 1 and two items of class 2. Their total weight is $v_1 + 2v_2 = \frac{1}{2-\beta_1} + \frac{2}{3-\beta_2} = 3 - R \leq 1.308439221$. In the last case we use the definition of R but in the other cases we show a slightly better upper bound.

- $(2, 0, 0, 0, 0)$. In this case the remaining size is below $\frac{1}{3}$. All remaining items have density below $\frac{8}{7}$. We get a total weight of at most $v_1 + 2 \cdot v_2 + \frac{8}{7} \cdot \frac{1}{3} < 1.6893916019524$.
- $(2, 0, 0, 1, 0)$. In this case the remaining size is below $\frac{1}{6}$. All remaining items have density at most $\frac{8}{7}$. We get a total weight of at most $v_1 + 2 \cdot v_2 + 0.184 + \frac{8}{7} \cdot \frac{1}{6} < 1.6829154114762$.
- $(2, 0, 1, 0, 0)$. In this case the remaining size is below $\frac{2}{15}$. All remaining items have density at most $\frac{8}{7}$.
If there is no item of size $\frac{1}{8}$, the densities are not larger than $\frac{9}{8}$, and we get a total weight of at most $v_1 + 2 \cdot v_2 + 0.23 + \frac{9}{8} \cdot \frac{2}{15} < 1.688439222$.
If there is an item of size $\frac{1}{8}$, the remaining items have total size below $\frac{1}{120}$ and density at most $\frac{121}{120}$, and we get a total weight of at most $v_1 + 2 \cdot v_2 + 0.23 + \frac{1}{7} + \frac{121}{120} \cdot \frac{1}{120} < 1.689699141635$.
- $(2, 1, 0, 0, 0)$. In this case the remaining size is below $\frac{1}{12}$. All remaining items have density at most $\frac{13}{12}$. We get a total weight of at most $v_1 + 2 \cdot v_2 + 0.287 + \frac{13}{12} \cdot \frac{1}{12} < 1.6857169988$.
- $(2, 0, 0, 1, 1)$. In this case the remaining size is below $\frac{1}{42}$. All remaining items have density at most $\frac{43}{42}$. We get a total weight of at most $v_1 + 2 \cdot v_2 + 0.184 + \frac{1}{6} + \frac{43}{42} \cdot \frac{1}{42} < 1.6834823049003$.

- $(2, 0, 0, 0, 1)$. In this case the remaining size is below $\frac{4}{21}$. All remaining items have density at most $\frac{8}{7}$.

If all other items have densities of at most $\frac{9}{8}$, we get a total weight of at most $v_1 + 2 \cdot v_2 + \frac{1}{6} + \frac{9}{8} \cdot \frac{4}{21} < 1.6893916019524$.

Otherwise, there is also an item of size $\frac{1}{8}$ and weight $\frac{1}{7}$ and the total size of other items is at most $\frac{11}{168}$, and their densities are at most $\frac{16}{15}$, and we get a total weight of at most $v_1 + 2 \cdot v_2 + \frac{1}{6} + \frac{1}{7} + \frac{16}{15} \cdot \frac{11}{168} < 1.6878043003651$.

- $(2, 0, 0, 0, 2)$.

In this case, we will find the total weight of the largest five items of the analyzed bin.

We have $\frac{1}{2-\beta_1} + \frac{2}{3-\beta_2} + \frac{2}{6} = 3 - R + \frac{2}{6}$.

The analysis of the remaining items is similar to the algorithm Harmonic Fit [21], and we get from these items a weight of C as shown below. Since $R = \frac{5}{3} + \frac{C}{2}$, the total weight of a bin is at most $\frac{10}{3} - R + C = R$.

More specifically, let $\delta = \frac{1}{M(M-1)}$. We show that the total weight of remaining items is at most $C + \delta$, so as M grows to infinity, the bounds tends to R .

The total size of these items is at most $\frac{1}{21}$. Consider a non-increasing sorted list of items of sizes not smaller than $\frac{1}{M}$. Assume that all these items have sizes that are reciprocals of integers by rounding them down (and keeping the weights unchanged). Comparing this list to the list $\frac{1}{t_1}, \frac{1}{t_2}, \dots, \frac{1}{t_f}$, where f is the largest integer such that $t_f \leq M$, consider the first item that is different. We will show and use a certain greedy reciprocal sum property. If no item is different and the lengths of the two lists are equal, we have a total weight of at most $\sum_{i=1}^f \frac{1}{t_i-1} + \frac{1}{t_{f+1}-1} \cdot \frac{M}{M-1} = \sum_{i=1}^{f+1} \frac{1}{t_i-1} + \frac{1}{t_{f+1}-1} \cdot \frac{1}{M-1} < C + \frac{1}{M(M-1)}$. Otherwise, the list may be shorter, or it is possible that there is a different item instead of $\frac{1}{t_j}$ for some j . Since the remaining total size excluding two items of size $\frac{1}{3}$, two items of size $\frac{1}{7}$, and one item of every size $\frac{1}{t_1}, \frac{1}{t_2}, \dots, \frac{1}{t_{j-1}}$ (where this list could be empty if $j = 1$) satisfies that the remaining space is below $\frac{1}{t_j-1}$, so the largest reciprocal of an integers that is next in the list can be $\frac{1}{t_j}$. If the next element is different, it must be smaller.

No matter whether the list is shorter or whether the j th item is smaller than $\frac{1}{t_j}$, the weight of items of the space smaller than $\frac{1}{t_j-1}$ is at most $\frac{t_j+1}{t_j}$ times their total size. We have a total weight of at most $\frac{t_j+1}{t_j(t_j-1)} = \frac{1}{t_j-1} + \frac{1}{t_{j+1}-1}$, since $t_{j+1} = t_j(t_j - 1) + 1$, and therefore the total weight does not exceed $\sum_{i=1}^{j+1} \frac{1}{t_i-1} < C$.

■

We conclude with the following theorem.

Theorem 25 *Our algorithm with the above discussed type parameter set has an asymptotic competitive ratio of at most R for online OEBP (with 1-items).*

6 An online algorithm without 1-items

The algorithm works similarly to the previous one, but for items of class 1 the packing is different and they act as the 1-items of the previous algorithm. However, if there is no ready bin, and there is a bin with exactly one such item (packed there because there was no ready bin at the time of its arrival), the new item of class 1 is packed with another such item and not into an empty bin. Thus, except for

possibly one bin, bins containing only items of class 1 contain pairs of items, and every such item can have a weight of $\frac{1}{2}$. Here, item x is defined to be last item of class 1 assigned into either a new bin or a bin already containing one such item (i.e., not into a ready bin). We stress that here, a bin containing the one item of class 1 is not defined as ready.

We will use two parameters for the algorithm, denoted by θ and Δ , and we let $Q = \frac{1}{1-2\theta}$. We fix the exact values later, but we will ensure that the following properties will hold: $0.319 \leq \Delta \leq 0.3195$, $0.03852 \leq \theta \leq 0.03853$, and $1.0834 < Q < 1.0835$.

Consider the next values. Let $M = 100$. Let $\beta_2 = \frac{2-6\Delta}{1-2\Delta}$ (and therefore $0.22 \leq \beta_2 \leq 0.24$), $\beta_i = 2\theta(i+1) \leq 1$ for any integer $3 \leq i \leq \frac{1}{2\theta} - 1$ (in this case, we have $i+1 < 2\theta$, i.e. $\frac{1}{i+1} > \frac{1}{2\theta}$). Otherwise (for $i \leq M$) $\beta_i = 1$, where $11 < \frac{1}{2\theta} - 1 < 12$. We will prove that the (asymptotic) competitive ratio for an algorithm of the type defined above with these parameters is at most 1.44465.

The formulas for the weights for the input part I_2 (denoted by v) as functions of β_i are the same as before for all classes except for class 1, for which the weights are equal to 0 (and class 0 does not exist at all). Recall that class 0 does not exist for the inputs studied in this section, since there are no 1-items.

For I_1 , the weights w of items of class 1 are $\frac{1}{2}$. For classes $2 \leq i \leq M-1$, instead of the weight $\frac{1-\beta_i}{i+1-\beta_i}$, the weight is $\frac{1-\beta_i/2}{i+1-\beta_i}$. For items of class M , the weight $\frac{1-\beta_M}{1-\beta_M/M}$ times the size is replaced with $\frac{1-\beta_M/2}{1-\beta_M/M}$ times the size. The difference is based on the property that the weight of class 1 items is just $\frac{1}{2}$ so a bin that received such an item still requires a weight $\frac{1}{2}$ from the other items.

We get $w_2 = \frac{1-\beta_2/2}{3-\beta_2} = \frac{1-\frac{1-3\Delta}{1-2\Delta}}{3-\frac{2-6\Delta}{1-2\Delta}} = \Delta$ and $v_2 = \frac{1}{3-\frac{2-6\Delta}{1-2\Delta}} = 1-2\Delta$, where $0.361 \leq 1-2\Delta \leq 0.362$.

For $3 \leq i \leq \frac{1}{2\theta} - 1$ we have $v_i = \frac{1}{i+1-\beta_i} = \frac{1}{i+1-2\theta(i+1)} = \frac{1}{i+1} \cdot \frac{1}{1-2\theta} = \frac{Q}{i+1}$ (and the density is at most Q), and $w_i = \frac{1-\beta_i/2}{i+1-\beta_i} = \frac{1-\theta(i+1)}{i+1-2\theta(i+1)} = \frac{Q}{i+1} - \frac{\theta}{1-2\theta} = \frac{1}{i+1} \cdot (Q - (i+1)\theta/(1-2\theta)) = Q \frac{1-(i+1)\theta}{i+1}$ (and the density is at most $Q(1 - (i+1)\theta)$).

For $M-1 \geq i > \frac{1}{2\theta} - 1$ we have $v_i = \frac{1}{i+1-\beta_i} = \frac{1}{i} = \frac{1}{i+1} \cdot \frac{i+1}{i} = \frac{1}{i+1} \cdot (1 + \frac{1}{i}) \leq \frac{1}{i+1} \cdot (1 + \frac{1}{\frac{1}{2\theta}-1}) = \frac{1}{i+1} \cdot \frac{1}{1-2\theta} = \frac{Q}{i+1}$ (for a density of at most $(\frac{i+1}{i} \leq Q)$) and $w_i = \frac{1-\beta_i/2}{i+1-\beta_i} \leq \frac{1/2}{i+1} \cdot \frac{1}{1-2\theta} = \frac{Q/2}{i+1} \leq \frac{0.54175}{i+1}$ (for a density of at most $\frac{Q}{2} \leq 0.54175$).

For class M , we have $\frac{1-\beta_M/2}{1-\beta_M/M} = \frac{50}{99} < 0.5051 < Q/2$ and $\frac{1}{1-\beta_M/M} = \frac{100}{99} < 1.0102 < Q$, for the weights w and v , respectively.

For a set of items of classes in $\{j, j+1, \dots, M\}$, such that $\frac{1}{2\theta} - 1 < j < M$, the density is at most $\frac{j+1}{2j}$ for w and at most $\frac{j+1}{j}$ for v , due to properties discussed earlier. Similarly, for a set of items of classes in $\{j, j+1, \dots, M\}$, such that $3 \leq j \leq \frac{1}{2\theta} - 1$, the density is at most Q for v and at most $Q(1 - (j+1)\theta)$ for w , because of earlier properties as well, and since the function $Q(1 - (i+1)\theta) = \frac{1-(i+1)\theta}{1-2\theta}$ is a monotonically decreasing function of i (for a fixed value of θ).

For I_1 , the largest weight of any item is $\frac{1}{2}$, according to w and for I_2 , the largest weight of any item according to v is $1-2\Delta$. These values will be used as upper bounds for the last items of bins.

We bound the total weight of items of a fixed bin that are not the last item of the bin (that is, without the exceeding item of the bin). For I_2 , the ratio between the weight and the size never exceeds Q except for items of class 2. The number of items of class 2 is at most two, and we use multiplier Q for other items, so the largest weight is below $(1-2\Delta) + \max\{\frac{Q}{3} + 2(1-2\Delta), 2 \cdot \frac{Q}{3} + (1-2\Delta), Q\}$. This value is calculated later using the exact values of θ (and Q) and Δ .

For I_1 , we consider several cases.

We start with the case where there is no item of size $\frac{1}{2}$ (that is not the last item). In this case there can be at most two items of size $\frac{1}{3}$. We use the property that the density for all other items is at most $\max\{Q/2, Q(1-4\theta)\} \leq \max\{0.55, \frac{1-4\theta}{1-2\theta}\}$. The function $\frac{1-4\theta}{1-2\theta}$ is a monotonically decreasing function of θ and therefore the density is at most 0.91653. For items of size $\frac{1}{3}$, the density is at most $3\Delta \leq 0.9585$.

The total weight is at most $0.5 + 0.9585 \cdot \frac{2}{3} + 0.91653 \cdot \frac{1}{3} \leq 1.44451$.

Otherwise, there are two items of size $\frac{1}{2}$ (the last item and another item), and the remaining items have total size below $\frac{1}{2}$. If there are no items of size $\frac{1}{3}$ and $\frac{1}{4}$, the density of remaining items is at most $Q(1 - 5\theta) < 0.875$ for a total weight of at most $1 + 0.875 \cdot \frac{1}{2} < 1.44$. There can be just one item of size $\frac{1}{3}$ or $\frac{1}{4}$. If its size is $\frac{1}{4}$, the total size of other items is below $\frac{1}{4}$. If all other items have sizes of at most $\frac{1}{6}$, the total weight is at most $1 + 0.91653 \cdot \frac{1}{4} + 0.8331 \cdot \frac{1}{4} < 1.44$. Otherwise, there is also an item of size $\frac{1}{5}$, and the remaining items have total size below $\frac{1}{20}$, and sizes of at most $\frac{1}{21}$. Their densities do not exceed $\frac{21}{40}$. The total weight in this case is at most $1 + 0.91653 \cdot \frac{1}{4} + 0.875 \cdot \frac{1}{5} + \frac{21}{40} \cdot \frac{1}{20} < 1.44$.

We are left with the case that there is also an item of size $\frac{1}{3}$, and the total size of other items is below $\frac{1}{6}$. If there is no item whose size is in the set $\{\frac{1}{7}, \frac{1}{8}\}$ (i.e., of class 6 or 7, where there is at most one such item), the density of other items is at most 0.707853, and the total weight is at most $1 + 0.3195 + 0.707853 \cdot \frac{1}{6} < 1.44$.

If there is an item of size $\frac{1}{8}$, the remaining items have total size below $\frac{1}{24}$, so their sizes do not exceed $\frac{1}{25}$, and the densities are at most $\frac{25}{48}$, for a total weight at most $1 + \Delta + Q \cdot \frac{1-8\theta}{8} + \frac{25}{48} \cdot \frac{1}{24} < 1.435$.

Finally, we have the case where there is an item of size $\frac{1}{7}$, the remaining total size is below $\frac{1}{42}$, the sizes do not exceed $\frac{1}{43}$, and densities are at most $\frac{43}{84}$. The total weight is at most $1 + \Delta + Q \cdot \frac{1-7\theta}{7} + \frac{43}{84} \cdot \frac{1}{42}$.

The specific values we use are $\theta = 0.038526551295994$, $\Delta = 0.319418991002646$, and therefore $1.0834859544 < Q < 1.08348595441$. We have $1 + \Delta + Q \cdot \frac{1-8\theta}{8} + \frac{25}{48} \cdot \frac{1}{24} < 1.435$, and $1 + \Delta + Q \cdot \frac{1-7\theta}{7} + \frac{43}{84} \cdot \frac{1}{42} < 1.44465$. We have $\frac{Q}{3} < 0.3611619848005$ and $1 - 2\Delta < 0.361162018$, so $(1 - 2\Delta) + \max\{\frac{Q}{3} + 2(1 - 2\Delta), 2 \cdot \frac{Q}{3} + (1 - 2\Delta), Q\} < 1.44465$.

A simple bad example for algorithms of this class

Next, we show that our analysis of algorithms of this class, without 1-items, is almost tight. Let $M > 100000$. There will be no items smaller than $\frac{1}{M}$ and for simplicity we use parameters γ_i . Similarly to the example for algorithms with 1-items, the value γ_i denotes for class i the fraction of items of small bins. We ignore rounding issues in this example. Let $N > 0$ be a very large integer.

The input is as follows, consisting of 13 batches.

- $\frac{12N}{90901}$ items of size $\frac{1}{90903}$.
- $\frac{2N}{450}$ items of size $\frac{1}{452}$.
- $\frac{12N}{300}$ items of size $\frac{1}{302}$.
- $\frac{80N}{41}$ items of size $\frac{1}{43}$.
- $16N$ items of size $\frac{1}{7}$.
- $16N$ items of size $\frac{1}{3}$.
- $32N$ items of size $\frac{1}{2}$.
- $6N$ items of size $\frac{1}{90903}$.
- N items of size $\frac{1}{452}$.
- $6N$ items of size $\frac{1}{302}$.
- $40N$ items of size $\frac{1}{43}$.
- $40N$ items of size $\frac{1}{7}$.

- $8N$ items of size $\frac{1}{3}$.

To obtain an offline packing, pack $16N$ bins, where every bin has at most one item of the first four batches (such that all items are packed), one item of size $\frac{1}{7}$, one item of size $\frac{1}{3}$, and one item of size $\frac{1}{2}$. All these bins have total sizes strictly below 1, and each one receives another item of size $\frac{1}{2}$. Afterwards, N bins receive (each) one item of size $\frac{1}{452}$, four items of size $\frac{1}{43}$, four items of size $\frac{1}{7}$, and one item of size $\frac{1}{3}$, for a total size below 1. Another $6N$ bins receive (each) one item of size $\frac{1}{90903}$, one item of size $\frac{1}{302}$, six items of size $\frac{1}{43}$, and six items of size $\frac{1}{7}$, for a total size below 1. Each of these $7N$ bins also receives one item of size $\frac{1}{3}$ as the last item.

We find $\text{OPT} \leq 23N$.

Consider the action of the algorithm. For the first seven batches, it has the following bins:

- $(\gamma_{90902} \cdot \frac{12N}{90901})/90902$ small bins of class 90902 and $((1 - \gamma_{90902}) \cdot \frac{12N}{90901})/90903$ large bins of this class.
- $(\gamma_{451} \cdot \frac{N}{225})/451$ small bins of class 451 and $((1 - \gamma_{451}) \cdot \frac{N}{225})/452$ large bins of this class.
- $\gamma_{301} \cdot \frac{N}{25}/301$ small bins of class 301 and $(1 - \gamma_{301}) \cdot \frac{N}{25}/302$ large bins of this class.
- $\gamma_{42} \cdot \frac{80N}{41}/42$ small bins of class 42 and $(1 - \gamma_{42}) \cdot \frac{80N}{41}/43$ large bins of this class.
- $\gamma_6 \cdot 16N/6$ small bins of class 6 and $(1 - \gamma_6) \cdot 16N/7$ large bins of this class.
- $\gamma_2 \cdot 16N/2$ small bins of class 2 and $(1 - \gamma_2) \cdot 16N/3$ large bins of this class.

Items of size $\frac{1}{2}$ are packed into the existing small bins, one per bin, and the remaining items are packed in pairs. The total number of bins created before these items are packed is below $11N$, and every small bin has items of total size above $\frac{1}{2}$ but below 1.

After rearranging, we see that the number of bins after the first seven batches are presented is:

$$\begin{aligned} & \frac{1}{2} \left(\frac{\gamma_{90902} \cdot 12N}{90901 \cdot 90902} + \frac{\gamma_{451} \cdot N}{225 \cdot 451} + \frac{\gamma_{301} \cdot N}{25 \cdot 301} + \frac{\gamma_{42} \cdot 80N}{41 \cdot 42} + \frac{\gamma_6 \cdot 16N}{6} + \frac{\gamma_2 \cdot 16N}{2} \right) + 16N \\ & + \frac{(1 - \gamma_{90902}) \cdot 12N}{90901 \cdot 90903} + \frac{(1 - \gamma_{451}) \cdot N}{225 \cdot 452} + \frac{(1 - \gamma_{301}) \cdot N}{25 \cdot 302} + \frac{(1 - \gamma_{42}) \cdot 80N}{41 \cdot 43} \\ & + \frac{(1 - \gamma_6) \cdot 16N}{7} + \frac{(1 - \gamma_2) \cdot 16N}{3}. \end{aligned}$$

For the second part of the input, the next bins are built:

- $\gamma_{90902} \cdot 6N/90902$ small bins of class 90902 and $(1 - \gamma_{90902}) \cdot 6N/90903$ large bins of this class.
- $\gamma_{451}N/451$ small bins of class 451 and $(1 - \gamma_{451})N/452$ large bins of this class.
- $\gamma_{301} \cdot 6N/301$ small bins of class 301 and $(1 - \gamma_{301}) \cdot 6N/302$ large bins of this class.
- $\gamma_{42} \cdot 40N/42$ small bins of class 42 and $(1 - \gamma_{42}) \cdot 40N/43$ large bins of this class.
- $\gamma_6 \cdot 40N/6$ small bins of class 6 and $(1 - \gamma_6) \cdot 40N/7$ large bins of this class.
- $\gamma_2 \cdot 8N/2$ small bins of class 2 and $(1 - \gamma_2) \cdot 8N/3$ large bins of this class.

When we compute the total number of bins, we can write it down as a weighted sum of the parameters γ_i plus some constant term (independent of these parameters). In this weighted sum the multipliers of γ_i 's are as follows.

- The multiplier of $\gamma_{90902}N$ is $\frac{6}{90901 \cdot 90902} - \frac{12}{90901 \cdot 90903} + \frac{6}{90902} - \frac{6}{90903} = 0$.
- The multiplier of $\gamma_{451}N$ is $\frac{1}{2 \cdot 225 \cdot 451} - \frac{1}{225 \cdot 452} + \frac{1}{451} - \frac{1}{452} = 0$.
- The multiplier of $\gamma_{301}N$ is $\frac{1}{50 \cdot 301} - \frac{1}{25 \cdot 302} + \frac{6}{301} - \frac{6}{302} = 0$.
- The multiplier of $\gamma_{42}N$ is $\frac{40}{41 \cdot 42} - \frac{80}{41 \cdot 43} + \frac{20}{21} - \frac{40}{43} = 0$.
- The multiplier of γ_6N is $\frac{4}{3} - \frac{16}{7} + \frac{20}{3} - \frac{40}{7} = 0$.
- The multiplier of γ_2N is $4 - \frac{16}{3} + 4 - \frac{8}{3} = 0$.

Thus, we are left with the constant term where we used that it can be seen after rearranging that the above values are all zeroes. That is, there are $N(16 + \frac{12}{90901 \cdot 90903} + \frac{1}{225 \cdot 452} + \frac{1}{25 \cdot 302} + \frac{80}{41 \cdot 43} + \frac{16}{7} + \frac{16}{3} + \frac{6}{90903} + \frac{1}{452} + \frac{6}{302} + \frac{40}{43} + \frac{40}{7} + \frac{8}{3})$ bins. The approximate number of bins is 32.9978979841943, and the resulting ratio is 1.4346912167.

7 Lower bounds on the asymptotic competitive ratio for OOEBP

In this section we improve the known lower bounds on the asymptotic competitive ratio slightly. The main goal of this section is to provide complete analytic proofs for these bounds, as previous work stated them without proving them analytically. Specifically, the analysis was done using packing patterns, and the number of such patterns can be large. Since we provide analytic proof, we can use arbitrarily long sequences. The improvement results also from using a modified input sequence, as in [7] instead of using a sequence similar to that of [27].

Let $M, N \geq 3$ be integers, where M is divisible by $(7^{N+1})!$.

An item of type $(i, 7)$ for $1 \leq i \leq N$ has size $\theta_i = \frac{1}{7^i}$. An item of type j for $j = 1, 2, 3$ has size $\phi_j = \frac{1}{j}$. In other parts of this work items of type 1 are called 1-items (and class 0), but for consistency of the current part we call them items of type 1 here.

Items are always presented sorted by non-decreasing size and in batches of M identical items (except for one case where there is a batch of $2M$ identical items, which can be seen as two batches with M items each, but we analyze it as a single batch). For $1 \leq k \leq N$, input I_k consists of the first $N - k + 1$ batches, where every batch has M items, and these items are of types $(N, 7), (N - 1, 7), \dots, (k, 7)$. Input J_3 consists of I_1 followed by a batch of M items of type 3. Input J_2 consists of J_3 followed by a batch of M items of type 2. Input J_{22} consists of J_3 followed by $2M$ items of type 2, and this is the unique case where the batch has $2M$ items. Input J_1 consists of J_2 followed by M items of size 1. We say that input I_{k_1} is a (proper) prefix of input I_{k_2} if $k_2 > k_1$, and similarly, for every k , input I_k is a prefix of J_3, J_2, J_{22} , and J_1 , additionally, J_3 is a prefix of J_2, J_{22} , and J_1 , and finally J_2 is a prefix of J_1 .

Input J_1 and its prefixes are used for proving a lower bound on the asymptotic competitive ratio of any algorithm where 1-items are possible, and J_{22} and its prefixes are used for the case without 1-items.

Let $\Theta_k = \sum_{i=k}^N \theta_i$. We use $\mu = \theta_N = \Theta_N = \frac{1}{7^N}$, which is the size of the smallest item. It holds that $\theta_k = 7^{N-k} \mu$ for $k = 1, 2, \dots, N$.

We start with finding upper bounds on optimal costs. Let $M'_k = \frac{M \cdot \Theta_k}{1 - \mu + \theta_k}$ for any $1 \leq k \leq N$.

Lemma 26 *We have $\text{OPT}(I_k) = M'_k$ for any $1 \leq k \leq N$.*

Proof. We have $1 - \mu + \theta_k = 1 - \frac{1}{7^N} + \frac{1}{7^k} = \frac{1}{7^N} \cdot (7^N - 1 + 7^{N-k}) = \mu \cdot (7^N - 1 + 7^{N-k}) \geq 1$ and $\Theta_k = \sum_{i=k}^N \frac{1}{7^i} = \frac{1}{7^N} \cdot \sum_{i=k}^N 7^{N-i} = \mu \cdot \sum_{i=0}^{N-k} 7^i$. We have that M is divisible by $7^N - 1 + 7^{N-k}$,

because it is lower than 7^{N+1} , and we get that $\frac{M \cdot \Theta_k}{1 - \mu + \theta_k}$, which will be a number of bins, is an integer. Since $\frac{1}{\mu} = 7^N$, this integer is divisible by 7^N .

We have $\sum_{i=0}^{N-k} 7^i = \frac{7^{N-k+1} - 1}{6} < \frac{7^N}{6}$. Thus, for any $1 \leq k \leq N$, we get $M' = \frac{M \cdot \Theta_k}{1 - \mu + \theta_k} < \frac{M}{6}$.

An upper bound on $\text{OPT}(I_k)$ follows from the property that every bin can contain item of total size strictly below 1 and one additional item. The size of the additional item is at most θ_k . All items have sizes that are integer multiples of μ , so the total size of a set of items that is strictly below 1 is in fact at most $1 - \mu$. Next, we will show that it is possible to produce such a packing.

In the case $k = N$, the claim is $\text{OPT}(I_N) = \frac{M \cdot \Theta_N}{1 - \mu + \theta_k} = M \cdot \mu = \frac{M}{7^N}$, which is achieved by packing 7^N items (each of size μ) into every bin. To prove the claim for other cases, consider a fixed value $1 \leq k \leq N - 1$.

We start the packing of M' bins as follows. Pack six items of each size θ_i for every $k + 1 \leq i \leq N$ into each bin. We packed $6M'$ items so far, therefore there are still $M - 6M'$ unpacked items for each size θ_i . For the remaining items of these sizes, create a partition into subsets called blocks. For every $i = k + 1, k + 2, \dots, N$, we create blocks from $M - 6M' > 0$ items, so there are separate blocks for each size θ_i . The number of items of such a block is 7^{i-k} . Their total size is $7^{i-k} \cdot \theta_i = 7^{i-k} \cdot \frac{1}{7^i} = \frac{1}{7^k} = \theta_k$. The number of blocks is

$$(M - 6M') \sum_{i=k+1}^N \frac{1}{7^{i-k}} = 7^k (M - 6M') \Theta_{k+1} \quad (3)$$

We add $M - M'$ blocks consisting of a single item of size θ_k , where the remaining M' items of this size will be the last items of the M' bins. Every bin receives $7^k - 1$ blocks of size θ_k . We show that the total size of items excluding the last item is strictly below 1 (it is equal to $1 - \theta_N$), and that the number of packed blocks is exactly the number of blocks.

Indeed, we have $6 \cdot \sum_{i=k+1}^N \frac{1}{7^i} + (7^k - 1) \cdot \frac{1}{7^k} = 6 \cdot \frac{7^{N-k} - 1}{6 \cdot 7^k} + 1 - \frac{1}{7^k} = 1 - \frac{1}{7^N}$, and the total sizes packed into bins are as claimed. The number of blocks is $7^k (M - 6M') \Theta_{k+1} + M - M'$, where the first term is exactly (3), and we show that this number of block is equal to $(7^k - 1)M'$. This property is equivalent to $M'(1 + 6\Theta_{k+1}) = M(\Theta_{k+1} + \frac{1}{7^k})$. Therefore, as $\Theta_{k+1} + \frac{1}{7^k} = \Theta_k$, and by definition $M' = \frac{M \Theta_k}{1 - \mu + \theta_k}$, we will show that $1 + 6\Theta_{k+1} = 1 - \mu + \theta_k$, or alternatively, $6\Theta_{k+1} + \mu = \theta_k$ holds for $k < N$. Indeed we have $6\Theta_{k+1} + \mu = 6(\sum_{i=k+1}^N \frac{1}{7^i}) + \mu = 7(\sum_{i=k+1}^N \frac{1}{7^i}) - (\sum_{i=k+1}^N \frac{1}{7^i}) + \mu = \sum_{i=k+1}^N \frac{1}{7^{i-1}} + \mu - \sum_{i=k+1}^N \frac{1}{7^i} = \sum_{i=k}^N \frac{1}{7^i} - \sum_{i=k+1}^N \frac{1}{7^i} = \frac{1}{7^k} = \theta_k$. ■

Corollary 27 We have $\text{OPT}(I_k) \leq \frac{7M}{6 \cdot (7^k + 1)}$ for any $1 \leq k \leq N$.

Proof. We prove that $\frac{\Theta_k}{1 - \mu + \theta_k} \leq \frac{7}{6 \cdot (7^k + 1)}$ holds.

This is equivalent to $6\Theta_k(7^k + 1) \leq 7(1 - \frac{1}{7^N} + \frac{1}{7^k})$. By $\Theta_k = \frac{1}{7^N} \cdot \frac{7^{N-k+1} - 1}{6}$, which is the sum of a geometrical series with common ratio 7, we find that it is required to prove that $(7^{N-k+1} - 1)(7^k + 1) \leq 7^{N+1}(1 - \frac{1}{7^N} + \frac{1}{7^k})$. This is equivalent to $7^{N+1} - 7^k + 7^{N-k+1} - 1 \leq 7^{N+1} - 7 + 7^{N-k+1}$ and to $7^k \geq 6$, which holds for any $k \geq 1$. ■

Lemma 28 We have $\text{OPT}(J_3) \leq \frac{3M}{8}$, $\text{OPT}(J_2) \leq \frac{2M}{3}$, $\text{OPT}(J_{22}) \leq M$, and $\text{OPT}(J_1) \leq M$.

Proof. Create blocks of items in the following way. A block has one item of every size θ_k for $1 \leq k \leq N$. The total size for a block is $\sum_{i=1}^N \frac{1}{7^i} < \sum_{i=1}^{\infty} \frac{1}{7^i} = \frac{1}{6}$. The number of blocks is M .

For J_3 , create $\frac{M}{8}$ bins with four blocks defined above, and $\frac{M}{4}$ bins with two blocks. Add one item of size $\frac{1}{3}$ to each bin out of the first $\frac{M}{8}$ bins and two such items to the last $\frac{M}{4}$ bins. Every bin has total size below 1. For every $1 \leq k \leq N$, the number of packed items of type $(k, 7)$ is $4 \cdot \frac{M}{8} + 2 \cdot \frac{M}{4} = M$. For

type 3 items, the number of packed items is $\frac{M}{8} + 2 \cdot \frac{M}{4} = \frac{5M}{8}$. The remaining type 3 items ($\frac{3M}{8}$ items) are added as last items into the bins. This results in a valid solution.

For J_2 , create $\frac{M}{3}$ bins with one block and $\frac{M}{3}$ bins with two blocks. The first kind of bins will also have one item of type 2 and one item of type 3. The second kind of bins will also have two items of type 3. Every bin has total size below 1. All items are packed except for $\frac{2M}{3}$ items of type 2, and these items are packed as the last items for the bins.

For J_{22} , create M bins, each containing one block, one item of type 3 and one item of type 2. Every bin has total size below 1. All items are packed except for M items of type 2, and these items are packed as the last items for the bins. For J_1 , the packing is the same as for J_{22} with the difference that the last item of each bin is of type 1. ■

We assign weights to items. An item of type j for $j = 1, 2, 3$ has weight 2. An item of type $(i, 7)$ has weight $\frac{1}{7^{i-1}}$. Let W_i and V_i denote the maximum total weights of bins containing items of types presented not earlier than type $(i, 7)$ items for $1 \leq i \leq N$, for the case with and without items of type 1 (that is, where the complete inputs are J_1 and J_{22} , respectively). Let A_j and B_j be the maximum total weights for bins containing items arriving not earlier than items of type j for the two cases (B_1 is undefined).

Lemma 29 *It holds that $A_1 = 2$, $A_2 = B_2 = 4$, and $A_3 = B_3 = 6$. Additionally, $W_k = V_k = 9 - \frac{1}{7^{k-1}}$.*

Proof. Consider a bin B . The last item never has weight above 2, and we find an upper bound on the weight of other items of the bin, whose total size is strictly below 1. Let S be such a subset of items with total size strictly smaller than 1 that maximizes the total weight of its items.

We have $A_1 = 2$, since in this case S must be empty. Additionally, $A_2 = B_2 = 4$, since S has at most one item of type 2 and one such item is a valid S . Finally $A_3 = B_3 = 6$, since S has at most two items and a pair of such items is a valid S .

Next, assume that B can have items arriving not earlier than type $(i, 7)$ items. Consider the subset S . Every item of size $\frac{1}{3}$ or larger can be replaced with two items of size $\frac{1}{7}$ without decreasing the total weight. Every item of size $\frac{1}{7^{i'}}$ with $i' < i$ can be replaced with $7^{i-i'}$ items of size $\frac{1}{7^i}$, having the same total weight. Thus, we can assume that the subset has only items of size $\frac{1}{7^i}$. Their number is at most $7^i - 1$ and therefore their weight is at most $\frac{7^i - 1}{7^{i-1}} = 7 - \frac{1}{7^{i-1}}$ and we note that such number of items of size $\frac{1}{7^i}$ is indeed a valid S . The claim follows from adding the last item of weight at most 2. ■

In the recent work on proving lower bounds for online bin packing type of problems, it was shown [8, 5, 4, 14] that if we can assign weights to items as it is done here, then a lower bound on the asymptotic competitive ratio as follows holds. This lower bound is defined as a ratio between a given pair of a numerator and a denominator. The numerator is the total weight of all items while the denominator is a valid upper bound on the value $\sum_{i=1}^{N-1} (W_{i+1} - W_i)OPT(I_{i+1}) + (W_1 - A_3)OPT(I_1) + (A_3 - A_2)OPT(J_3) + (A_2 - A_1)OPT(J_2) + A_1 \cdot OPT(J_1)$.

We now compute the total weight of all items in the inputs J_1 and J_{22} , where these values are equal, i.e., we consider the numerator of the above ratio. The total weight of items of types 1, 2, 3 is $6M$. The weight of other items is $M \cdot \frac{7^N - 1}{6 \cdot 7^{N-1}}$ (using the sum of a geometrical series with common ratio 7).

Next, we consider the denominator. We have $\sum_{i=1}^{N-1} (W_{i+1} - W_i)OPT(I_{i+1}) + (W_1 - A_3)OPT(I_1) + (A_3 - A_2)OPT(J_3) + (A_2 - A_1)OPT(J_2) + A_1 \cdot OPT(J_1) = \sum_{i=1}^{N-1} (\frac{1}{7^{i-1}} - \frac{1}{7^i})OPT(I_{i+1}) + 2 \cdot (OPT(I_1) + OPT(J_3) + OPT(J_2) + OPT(J_1)) \leq \sum_{i=1}^{N-1} (\frac{1}{7^{i-1}} - \frac{1}{7^i}) \frac{7M}{6 \cdot (7^{i+1} + 1)} + 2M \cdot (\frac{7}{48} + \frac{3}{8} + \frac{2}{3} + 1) = M((\sum_{i=1}^{N-1} \frac{1}{7^{2i+7^{i-1}}}) + 4.375)$.

We find an upper bound on the series $\sum_{i=1}^{N-1} \frac{1}{7^{2i+7^{i-1}}}$. This is done by calculating the first six elements and bounding the other ones. We have $\frac{1}{7^{2i+7^{i-1}}} \leq \frac{1}{7^{2i}} = \frac{1}{49^i}$ and therefore $\sum_{i=7}^{N-1} \frac{1}{7^{2i+7^{i-1}}} \leq \sum_{i=7}^{N-1} \frac{1}{49^i} = \frac{49^{N-7} - 1}{48 \cdot 49^{N-1}}$. The first six elements are $\frac{1}{50} + \frac{1}{2408} + \frac{1}{177698} + \frac{1}{5765144} + \frac{1}{282477650} + \frac{1}{13841304008} \approx 0.0204239557816752$.

Letting N grow to infinity both in the numerator and denominator gives us a numerator of $6 + \frac{7}{6} = \frac{43}{6}$ and denominator of at most $\frac{1}{50} + \frac{1}{2408} + \frac{1}{177698} + \frac{1}{5765144} + \frac{1}{282477650} + \frac{1}{13841304008} + \frac{1}{49^6 \cdot 48} + 4.375 \approx 4.39542395578318$. The resulting lower bound on the competitive ratio is 1.6304835981151. The same value is obtained even if the exact sum of the series is found via simulation. The difference is in the fourth digit after the decimal point.

For the case without 1-items, using $W_k = V_k$ for all k , instead of $(A_3 - A_2) \cdot \text{OPT}(J_3) + (A_2 - A_1) \cdot \text{OPT}(J_2) + A_1 \cdot \text{OPT}(J_1)$ we have $(B_3 - B_2) \cdot \text{OPT}(J_3) + B_2 \cdot \text{OPT}(J_{22})$, so instead of $2 \cdot \frac{3}{8} + 2 \cdot \frac{2}{3} + 2 \cdot 1$ we have $2 \cdot \frac{3}{8} + 4 \cdot 1$. The denominator is larger by $\frac{2}{3}$, and the resulting lower bound is approximately 1.41575234447271. The difference due to using the exact sum of the series is in the fifth digit after the decimal point.

References

- [1] J. Balogh, J. Békési, G. Dósa, L. Epstein, H. Kellerer, A. Levin, and Z. Tuza. Offline black and white bin packing. *Theoretical Computer Science*, 596:92–101, 2015.
- [2] J. Balogh, J. Békési, G. Dósa, L. Epstein, H. Kellerer, and Z. Tuza. Online results for black and white bin packing. *Theory of Computer Systems*, 56(1):137–155, 2015.
- [3] J. Balogh, J. Békési, G. Dósa, L. Epstein, and A. Levin. A new and improved algorithm for online bin packing. In *Proc. of the 26th European Symposium on Algorithms (ESA2018)*, pages 5:1–5:14, 2018.
- [4] J. Balogh, J. Békési, G. Dósa, L. Epstein, and A. Levin. Lower bounds for several online variants of bin packing. *Theory of Computing Systems*, 63(8):1757–1780, 2019.
- [5] J. Balogh, J. Békési, G. Dósa, L. Epstein, and A. Levin. A new lower bound for classic online bin packing. In *Proceedings of the 17th Workshop on Approximation and Online Algorithms (WAOA2019)*, pages 18–28, 2019.
- [6] J. Balogh, J. Békési, G. Dósa, J. Sgall, and R. van Stee. The optimal absolute ratio for online bin packing. *Journal of Computer and System Sciences*, 102:1–17, 2019.
- [7] J. Balogh, J. Békési, and G. Galambos. New lower bounds for certain classes of bin packing algorithms. *Theoretical Computer Science*, 440:1–13, 2012.
- [8] J. Békési, G. Dósa, and L. Epstein. Bounds for online bin packing with cardinality constraints. *Information and Computation*, 249:190–204, 2016.
- [9] S. Berndt, K. Jansen, and K.-M. Klein. Fully dynamic bin packing revisited. *Mathematical Programming*, 179(1):109–155, 2020.
- [10] M. Böhm, G. Dósa, L. Epstein, J. Sgall, and P. Veselý. Colored bin packing: online algorithms and lower bounds. *Algorithmica*, 80(1):155–184, 2018.
- [11] M. Chrobak, J. Sgall, and G. J. Woeginger. Two-bounded-space bin packing revisited. In *Proc. of the 19th Annual European Symposium on Algorithms (ESA2011)*, pages 263–274, 2011.
- [12] G. Dosa, Z. Tuza, and D. Ye. Bin packing with “largest in bottom” constraint: tighter bounds and generalizations. *Journal of Combinatorial Optimization*, 26(3):416–436, 2013.
- [13] L. Epstein. On online bin packing with LIB constraints. *Naval Research Logistics*, 56(8):780–786, 2009.

- [14] L. Epstein. A lower bound for online rectangle packing. *Journal of Combinatorial Optimization*, 38(3):846–866, 2019.
- [15] L. Epstein and A. Levin. Asymptotic fully polynomial approximation schemes for variants of open-end bin packing. *Information Processing Letters*, 109(1):32–37, 2008.
- [16] W. Fernandez de la Vega and G. S. Lueker. Bin packing can be solved within $1 + \varepsilon$ in linear time. *Combinatorica*, 1(4):349–355, 1981.
- [17] L. Finlay and P. Manyem. Online LIB problems: Heuristics for bin covering and lower bounds for bin packing. *RAIRO Operations Research*, 39(3):163–183, 2005.
- [18] L. Gai and G. Zhang. Hardness of lazy packing and covering. *Operations Research Letters*, 37(2):89–92, 2009.
- [19] R. Kannan. Improved algorithms for integer programming and related lattice problems. In *Proceedings of the 15th Annual ACM Symposium on Theory of Computing (STOC1983)*, pages 193–206, 1983.
- [20] N. Karmarkar and R. M. Karp. An efficient approximation scheme for the one-dimensional bin-packing problem. In *Proceedings of the 23rd Annual Symposium on Foundations of Computer Science (FOCS1982)*, pages 312–320, 1982.
- [21] C. C. Lee and D. T. Lee. A simple online bin packing algorithm. *Journal of the ACM*, 32(3):562–572, 1985.
- [22] H. W. Lenstra Jr. Integer programming with a fixed number of variables. *Mathematics of Operations Research*, 8(4):538–548, 1983.
- [23] J. Y.-T. Leung, M. Dror, and G. H. Young. A note on an open-end bin packing problem. *Journal of Scheduling*, 4(4):201–207, 2001.
- [24] M. Lin, Y. Yang, and J. Xu. Improved approximation algorithms for maximum resource bin packing and lazy bin covering problems. *Algorithmica*, 57(2):232–251, 2010.
- [25] M. Lin, Y. Yang, and J. Xu. On lazy bin covering and packing problems. *Theoretical Computer Science*, 411(1):277–284, 2010.
- [26] P. Ramanan, D. J. Brown, C. C. Lee, and D. T. Lee. Online bin packing in linear time. *Journal of Algorithms*, 10:305–326, 1989.
- [27] A. van Vliet. An improved lower bound for online bin packing algorithms. *Information Processing Letters*, 43(5):277–284, 1992.
- [28] J. Yang and J. Y. Leung. The ordered open-end bin packing problem. *Operations Research*, 51(5):759–770, 2003.
- [29] G. Zhang. Parameterized on-line open-end bin packing. *Computing*, 60(3):267–274, 1998.
- [30] G. Zhang. Private communication, 2002.