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Admittance identification from point-wise sound pressure measurements using reduced-order modelling

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ADMITTANCE IDENTIFICATION FROM POINT-WISE SOUND PRESSURE MEASUREMENTS USING REDUCED-ORDER MODELLING

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Abstract. In this work an acoustic application is studied. The goal is to estimate the complex-valued admittance from given point measurements of the sound pressure. This parameter identification problem is formulated in terms of an infinite-dimensional optimization problem. First- and second-order optimality conditions are discussed. For the numerical realization a reduced-order model based on proper orthogonal decomposition is used. Numerical examples illustrate the efficiency of the proposed approach.

1. Introduction

The acoustical impedance of a component or trim part is one of its most important characteristics. The trim and its absorption behavior contributes significantly to the comfort inside the car. Therefore, correct impedance values are needed when acoustical simulations of car interior noise are carried out.

A generally used methodology to determine the acoustical impedance of relevant acoustic materials is to use cut-out round samples of the material in question and measure the acoustic characteristic in the impedance tube; see Figure 1.1.

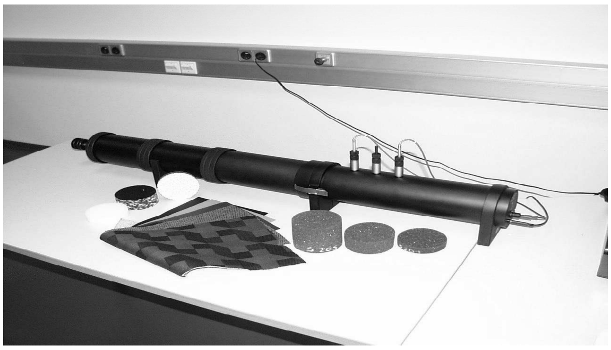

Figure 1.1. Classical impedance tube for the measuremant of normal impedances; see [10].

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As a result, values for the normal impedance and absorption coefficients can be obtained for this material. Typical damping materials are shown in Figure 1.2. Disadvantages of this method are that the measurement considers normal acoustic

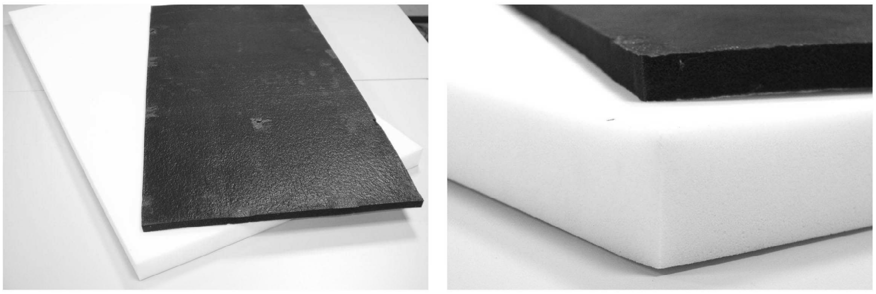

FIGURE 1.2. Damping materials Melamin (white) and C20mm (black); see [10].

waves, only, that some materials are inappropriate for the impedance tube and that the effects of the shape of the whole part have to be neglected. Therefore efforts have been made to develop methods for impedance measurements of entire trim parts, such as carpets, dashboards or seats.

Our approach is based on a mathematical model for the sound pressure distribution (inside the idealized car geometry). More precisely, the sound pressure is governed by the Helmholtz equatin, where the impedance or admittance (the reciprocal value of the impedance) arises as a parameter in the Helmholtz equation.

In this paper we formulate the identification problem as an optimal control problem, where the cost functional contains a regularization term as well as a least-squares term for the difference of the measurements and the sound pressure p computed by solving the Helmholtz equation. The obtained optimal control problem can be treated by methods from infinite dimensional optimization; see, e.g., [12]. In contrast to [10] we identify the admittance $A \in \mathbb{C}$ instead of the impedance $Z = 1/A$. Due to the the term Ap in the Helmholtz equation (see (2.1c)) the obtained optimal control problem has a bilinear structure, whereas in [10] the nonlinearity is of the form p/Z. If the admittance A has been estimated, then $Z = 1/A$ is an estimate for the impedance. The optimal control problem is solved by a globalized quasi-Newton method with BFGS update of the Hessian. Furthermore, a discretization based on proper orthogonal decomposition (POD) is utilized for the solution of the Helmholtz equation. POD is a powerful technique for model reduction of nonlinear systems. It is based on a Galerkin type discretization with basis elements created from solutions to the Helmholtz equation itself.

Compared to [25] the new contribution of this article are the following aspects:

- Since the measurements are usually difficult to get for each single frequency, we identify the admittance from mean values over a frequency band.
- We consider a different cost functional involving a log-function, because the measurements are usually provided in the logarithmic unit decibel (dB).
- First- and second-order optimality conditions for the infinite-dimensional optimization problem are investigated.

- Inequality constraints for the admittance are taken into account in our numerical solution method (positivity of the real and imaginary parts of the admittance).

POD is successfully used in different fields including signal analysis and pattern recognition (see, e.g., [6]), fluid dynamics and coherent structures (see, e.g., [14, 23]) and more recently in control theory (see, e.g., [11, 18]). The relationship between POD and balancing is considered in [17, 22, 26].

Let us mention that in [10] a standard finite element discretization for the Helmholtz equation is applied. Alternatively, the wave based technique (WBT) is used in [4, 9]. A-posteriori analysis is utilized in [24] to determine the number of POD ansatz functions in the POD Galerkin projection for an optimal control problem governed by the Helmholtz equation. We refer to [2], where an optimal control problem for an impedance factor is studied. The frequency is kept fixed and the goal is to obtain the least amount of noise propagation to the far field.

The paper is organized in the following manner: In Section 2 we formulate our parameter identification problem as an infinite dimensional optimization problem and prove existence of optimal solutions. Optimality conditions are derived in Section 3. The reduced-order approach is described in Section 4, where we also explain our numerical optimization method. Numerical experiments are carried out in Section 5. Finally, some of the proofs are given in the Appendix.

2. Admintance estimation as a non-linear optimization problem

In section we formulate the parameter estimation problem for the admittance as an infinite-dimensional optimization problem. Existence of optimal solutions are proved and first-order necessary optimality conditions are derived.

2.1. **Function spaces.** Througout the paper we write $\Re e(z)$ and $\Im m(z)$ for the real and imaginary part, respectively, for $z \in \mathbb{C}$. Moreover, $\overline{z}$ stands for the complex conjugate of z.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open and bounded domain with Lipschitz-continuous boundary $\Gamma = \partial\Omega$. The domain Ω represents the interior of the car vehicle. Recall that for $s \in [1, \infty)$ the Lebesgue space $L^s_{\mathbb{C}}(\Omega)$ is defined as

$$L^s_{\mathbb{C}}(\Omega) = \left\{ \varphi : \Omega \to \mathbb{C} \,\Big|\, \varphi \text{ measurable and } \|\varphi\|_{L^s_{\mathbb{C}}(\Omega)} = \left(\int_\Omega \big|\varphi(\mathbf{x})\big|^s_{\mathbb{C}} \,\mathrm{d}\mathbf{x} \right)^{1/s} < \infty \right\}.$$

In particular, we set $H = L^2_{\mathbb{C}}(\Omega)$, which is a Hilbert space endowed with the inner product

$$\langle \varphi, \psi \rangle_H = \int_\Omega \varphi(\mathbf{x})\overline{\psi(\mathbf{x})} \,\mathrm{d}\mathbf{x} \quad \text{for } \varphi, \psi \in H$$

and the induced norm $\|\varphi\|_H = \langle \varphi, \varphi \rangle_H^{1/2}$ for $\varphi \in H$. The Hilbert space $V = H^1_{\mathbb{C}}(\Omega)$ is supplied with the inner product

$$\langle \varphi, \psi \rangle_V = \langle \varphi, \psi \rangle_H + \sum_{i=1}^{3} \left\langle \frac{\partial \varphi}{\partial \mathbf{x}_i}, \frac{\partial \psi}{\partial \mathbf{x}_i} \right\rangle_H \quad \text{for } \varphi, \psi \in V$$

and its induced norm $\|\varphi\|_V = \langle \varphi, \varphi \rangle_V^{1/2}$ for $\varphi \in V$. For more details on Lebesgue and Sobolev spaces we refer the reader, e.g., to [1, 5].

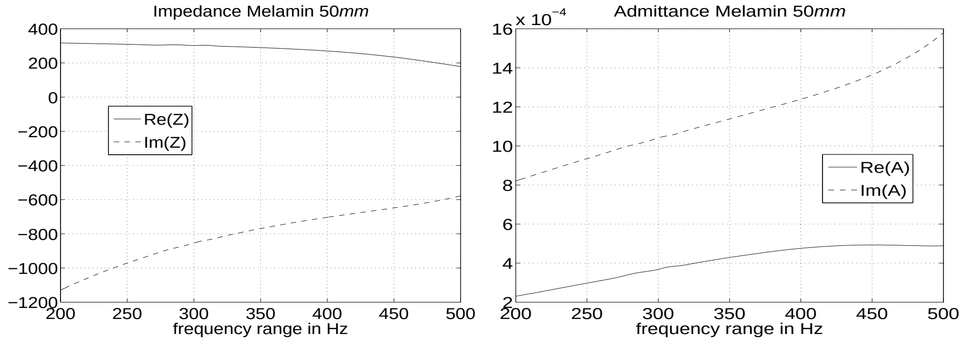

FIGURE 2.1. Impedance (in $\frac{Pa}{m/s}$) and admittance (in $\frac{m/s}{Pa}$) values for Melamin $50mm$ in the frequency range from 200 to $500Hz$.

2.2. **The Helmholtz equation.** For given frequency $f > 0$ the complex-valued sound pressure $p = p(\mathbf{x})$ solves the Helmholtz equation

$$-\Delta p(\mathbf{x}) - k^2 p(\mathbf{x}) = q(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \Omega, \tag{2.1a}$$

$$\frac{\jmath}{\varrho_\circ \omega} \frac{\partial p}{\partial n}(\mathbf{s}) = 0 \qquad \text{for all } \mathbf{s} \in \Gamma_\mathrm{N} \subsetneq \Gamma = \partial\Omega, \tag{2.1b}$$

$$\frac{\jmath}{\varrho_\circ \omega} \frac{\partial p}{\partial n}(\mathbf{s}) = Ap(\mathbf{s}) = \frac{p(\mathbf{s})}{Z} \qquad \text{for all } \mathbf{s} \in \Gamma_\mathrm{R} = \Gamma \setminus \Gamma_\mathrm{N}, \tag{2.1c}$$

where q stands for the complex-valued source term modelling the excitation at the point $\mathbf{x} \in \Omega$ at the frequency f, the parameter $A \in \mathbb{C}$ denotes the admittance and $Z = 1/A$ stands for the impedance. Furthermore, $\jmath$ is the imaginary unit, n denotes the outward normal vector and the constants in (2.1) are

$$c = 343.799\,\frac{\mathrm{m}}{\mathrm{s}},\ k = \frac{2\pi f}{c},\ \varrho_\circ = 1.19985\,\frac{\mathrm{kg}}{\mathrm{m}^3},\ \omega = 2\pi f = ck, \tag{2.2}$$

i.e., both the wave number k and the angular frequency ω depend on f. The admittance and therefore the impedance are frequency-dependent. In Figure 2.1 we plot the impedance and admittance values for the damping material Melamin $50mm$ in the frequency range from 200 to $500Hz$.

Integration by parts and using (2.1c) yield

$$\begin{aligned}
\langle -\Delta p, \varphi \rangle_H &= \int_\Omega -\Delta p \overline{\varphi}\,\mathrm{d}\mathbf{x} = -\int_{\Gamma_\mathrm{R}} \frac{\partial p}{\partial n}\,\overline{\varphi}\,\mathrm{d}\mathbf{s} + \int_\Omega \nabla p \cdot \nabla \overline{\varphi}\,\mathrm{d}\mathbf{x} \\
&= -\frac{1}{\jmath} \int_{\Gamma_\mathrm{R}} \varrho_\circ \omega A p\,\overline{\varphi}\,\mathrm{d}\mathbf{s} + \int_\Omega \nabla p \cdot \nabla \overline{\varphi}\,\mathrm{d}\mathbf{x} \\
&= \int_\Omega \nabla p \cdot \nabla \overline{\varphi}\,\mathrm{d}\mathbf{x} + \jmath \varrho_\circ \omega A \int_{\Gamma_\mathrm{R}} p \overline{\varphi}\,\mathrm{d}\mathbf{s}
\end{aligned}$$

for every $\varphi \in V$, where we have assumed that $-\Delta p \in H$ and $\langle \cdot\,, \cdot \rangle_H$ denotes the inner product in H of square integrable functions $\varphi : \Omega \to \mathbb{C}$. This motivates the next definition.

Definition 2.1. *For given $f > 0$ and $A \in \mathbb{C}$ the function p is called a weak solution to* (2.1) *provided $p \in V$ holds and p satisfies*

$$B(p, \varphi; A) = \int_\Omega q(f)\overline{\varphi}\,\mathrm{d}\mathbf{x} \quad \text{for all } \varphi \in V,$$

where the parameter-dependent bilinear form $B(\cdot\,, \cdot\,; A) : V \times V \to \mathbb{C}$ is defined as

$$B(p, \varphi; A) = \int_\Omega \nabla p \cdot \nabla\overline{\varphi} - k^2 p\overline{\varphi}\,\mathrm{d}\mathbf{x} + \jmath\varrho_\circ\omega A \int_{\Gamma_\mathrm{R}} p\overline{\varphi}\,\mathrm{d}\mathbf{s} \quad \text{for } p,\, \varphi \in V. \tag{2.3}$$

The existence of a weak solution to (2.1) follows from the the next theorem. Its proof is based on the Fredholm alternative; see, e.g., [5, p. 640-644]. For more details we refer to the Appendix A.1.

Theorem 2.2. 1) *One of the following statements hold: either*

$$\left\{\begin{array}{l} \text{for each } q \in H \text{ there exists a unique weak solution } p \text{ to} \\ \text{(2.4a)} \quad \left\{\begin{array}{lll} -\Delta p - k^2 p & = q & \text{in } \Omega, \\ \frac{\jmath}{\varrho_\circ\omega}\frac{\partial p}{\partial n} & = 0 & \text{on } \Gamma_\mathrm{N}, \\ \frac{\jmath}{\varrho_\circ\omega}\frac{\partial p}{\partial n} & = Ap & \text{on } \Gamma_\mathrm{R} \end{array}\right. \end{array}\right. \tag{2.4}$$

or else

$$\left\{\begin{array}{l} \text{there exists a unique weak solution } \psi \neq 0 \text{ to} \\ \text{(2.5a)} \quad \left\{\begin{array}{lll} -\Delta \psi - k^2 \psi & = 0 & \text{in } \Omega, \\ \frac{\jmath}{\varrho_\circ\omega}\frac{\partial \psi}{\partial n} & = 0 & \text{on } \Gamma_\mathrm{N}, \\ \frac{\jmath}{\varrho_\circ\omega}\frac{\partial \psi}{\partial n} & = A\psi & \text{on } \Gamma_\mathrm{R}. \end{array}\right. \end{array}\right. \tag{2.5}$$

2) *If* (2.5) *is satisfied, the dimension of the subspace $\mathcal{N} \subset V$ of weak solutions to* (2.5a) *is finite and equals the dimension of the subspace $\mathcal{N}^* \subset V$ of weak solutions λ to the adjoint (or dual) problem*

$$\begin{aligned} -\Delta\lambda - k^2\lambda &= 0 && \text{in } \Omega, \\ \frac{\jmath}{\varrho_\circ\omega}\frac{\partial\lambda}{\partial n} &= 0 && \text{on } \Gamma_\mathrm{N}, \\ \frac{\jmath}{\varrho_\circ\omega}\frac{\partial\lambda}{\partial n} &= -\overline{A}\lambda && \text{on } \Gamma_\mathrm{R}. \end{aligned} \tag{2.6}$$

3) *Finally,* (2.4a) *has a weak solution if and only if*

$$\int_\Omega q\overline{\lambda}\,\mathrm{d}\mathbf{x} = 0 \quad \text{for all } \lambda \in \mathcal{N}^*.$$

Remark 2.3. (1) From the theory of compact operators and the Fredholm alternative it follows that case (2.5) appears only for a countable set.
(2) If (2.5) is true, we infer that its solutions ψ satisfies

$$\sum_{i=1}^{d} \left\| \frac{\partial\psi}{\partial \mathbf{x_i}} \right\|_H^2 - k^2 \|\psi\|_H^2 + \jmath\varrho_\circ\omega A \|\psi\|^2_{L^2_\mathbb{C}(\Gamma_\mathrm{R})} = 0.$$

Hence, $\varrho_\circ\omega\Re e(A)\|\psi\|^2_{L^2_\mathbb{C}(\Gamma_\mathrm{R})} = 0$, i.e., $\psi = 0$ on Γ_R provided $\Re e(A) \neq 0$. $\Diamond$

If a weak solution p to (2.4a) exists the following regularity result.

Corollary 2.4. *Let $A \in \mathbb{C}$ be given. Suppose that there exists a weak solution $p = p(A) \in V$ to* (2.4a)*. Then, p lies in the space $C_\mathbb{C}(\overline{\Omega})$ of all continuous functions defined on Ω with values in $\mathbb{C}$. Moreover, there exists a constant $C > 0$ independent of f, q, A, and p such that*

$$\|p\|_V + \|p\|_{C_\mathbb{C}(\overline{\Omega})} \leq C \left(\|q\|_H + \|p\|_H + |A|_\mathbb{C} \, \|p\|_{L^s_\mathbb{C}(\Gamma_\mathrm{R})} \right)$$

for $s \in (2, 4]$. Furthermore, $p \in H^{3/2}_\mathbb{C}(\Omega)$.

Proof. The solution p solves

$$-\Delta p + a_0 p = h \text{ in } \Omega, \quad \frac{\partial p}{\partial n} = g \text{ on } \Gamma$$

weakly with $a_0 > 0$, $h = q + (a_0 + k^2)p$ in Ω, $g = 0$ on Γ_N and $g = -\jmath \varrho_\circ \omega A p$ on Γ_R. From $q \in H$ and $p \in V$ we infer that $h \in H$ holds. Moreover, $p|_{\Gamma_\mathrm{R}} \in H^{1/2}_\mathbb{C}(\Gamma_\mathrm{R})$. From the Sobolev embedding theorem [1, p. 144] we infer that $p|_{\Gamma_\mathrm{R}} \in L^s_\mathbb{C}(\Gamma_\mathrm{R})$ holds for $s \in (2, 4]$. Thus, $g \in L^s_\mathbb{C}(\Gamma)$ and $s > d - 1$. Therefore, the claim follows from [3] and regularity results for elliptic equations. ☐

Remark 2.5. For $p \in V$ and $A \in \mathbb{C}$ the function

$$g = \begin{cases} 0 & \text{on } \Gamma_\mathrm{N}, \\ -\jmath \varrho_\circ \omega A p & \text{on } \Gamma_\mathrm{R} \end{cases}$$

may not belong to $H^{1/2}_\mathbb{C}(\Gamma)$. Therefore, we can not guarantee H^2-regularity for a solution p to (2.4a). ♢

2.3. **The Helmholtz equation on a frequency band.** Let f be a given frequency and δf a positive scalar with $f_a = f - \delta f > 0$. Then, we set $f_b = f + \delta f$ and define the frequency band $\mathcal{I} = [f_a, f_b] \subset \mathbb{R}_+$ and introduce a function space taking into account the varying frequency in (2.1). By $\mathcal{H} = L^2(\mathcal{I}; H)$ we denote the space of all functions $\varphi : \mathcal{I} \to H$, $f \mapsto \varphi(\cdot\,; f)$, which are square integrable, i.e.,

$$\|\varphi\|_\mathcal{H} = \sqrt{\int_\mathcal{I} \|\varphi(\cdot\,; f)\|_H^2 \, \mathrm{d}f} < \infty \quad \text{for every } \varphi \in \mathcal{H}.$$

Analogously, the Hilbert space $\mathcal{V} = L^2(\mathcal{I}; V)$ is defined. Throughout we denote by $\varphi(f)$ the complex-valued function defined on Ω when the frequency $f \in \mathcal{I}$ is fixed.

The admissible admittance values belong to the bounded, closed and convex set

$$\mathcal{A}_\mathrm{ad} = \left\{ A \in \mathbb{C} \,\middle|\, \Re e(A) \in [\underline{A}_\Re, \overline{A}_\Re] \text{ and } \Im m(A) \in [\underline{A}_\Im, \overline{A}_\Im] \right\}$$

with constants $\underline{A}_\Re \leq \overline{A}_\Re$ and $\underline{A}_\Im \leq \overline{A}_\Im$. For given source term $q \in C(\mathcal{I}; V)$ and for admittance $A \in \mathcal{A}_\mathrm{ad}$ we consider the Helmholtz equation over the band $\mathcal{I}$:

$$-\Delta p(f) - k^2 p(f) = q(f) \qquad \text{in } \Omega \text{ and f.a.a. } f \in \mathcal{I}, \tag{2.7a}$$

$$\frac{\jmath}{\varrho_\circ \omega} \frac{\partial p}{\partial n}(f) = 0 \qquad \text{on } \Gamma_\mathrm{N} \text{ and f.a.a. } f \in \mathcal{I}, \tag{2.7b}$$

$$\frac{\jmath}{\varrho_\circ \omega} \frac{\partial p}{\partial n}(f) = A p(f) \qquad \text{on } \Gamma_\mathrm{R} \text{ and f.a.a. } f \in \mathcal{I}, \tag{2.7c}$$

where ‘f.a.a.’ stands for ‘for almost all’.

Let us introduce the Banach spaces

$$\mathcal{P} = \mathcal{V} \cap L^2(\mathcal{I}; C_\mathbb{C}(\overline{\Omega})) \quad \text{and} \quad X = \mathcal{P} \times \mathbb{C},$$

where $\mathcal{P}$ is endowed with the norm

$$\|p\|_{\mathcal{P}} = \|p\|_{\mathcal{V}} + \|p\|_{L^2(\mathcal{I};C_{\mathbb{C}}(\overline{\Omega}))} \quad \text{for } p \in \mathcal{P}.$$

We identify the dual space $\mathcal{V}'$ of $\mathcal{V}$ with $L^2(\mathcal{I};V')$. Moreover, let us define the nonlinear operator $e : X \to \mathcal{V}'$ by

$$\langle e(p,A),\varphi\rangle_{\mathcal{V}',\mathcal{V}} = \int_{\mathcal{I}} B(p(f),\varphi(f);A)\,\mathrm{d}f \quad \text{for } (p,A) \in X \text{ and } \varphi \in \mathcal{V},$$

where the parameter dependent bilinear form B has been introduced in (2.3). Then, $e(p,A) = 0$ in Y' holds if and only if $(p,A) \in X$ satisfies (2.7). Finally, we set

$$X_{\mathrm{ad}} = \mathcal{P} \times \mathcal{A}_{\mathrm{ad}}.$$

Assumption 1. *Let $q \in C^1_{\mathbb{C}}(\mathcal{I} \times \overline{\Omega}))$ be given and $A \in \mathcal{A}_{ad}$ arbitrarily chosen.*

1) Existence: *There exists at least one solution $p \in \mathcal{P}$ to $e(p,A) = 0$ in $\mathcal{V}'$.*
2) Estimate: *If (p,A) satisfies $e(p,A) = 0$ in $\mathcal{V}'$, then we have*

$$\|p\|_{\mathcal{P}} \le C\big(1 + |A|_{\mathbb{C}}\big)$$

for a constant $C > 0$ independent of A and $f \in \mathcal{I}$.

Remark 2.6.
1) Notice that the existence of a solution p to $e(p,A) = 0$ depends on the chosen source term q, the values of k^2, and the boundary conditions. In our numerical experiments we observe unique solvability of the discretized Helmholtz equation for the used frequency grid.
2) Due to Theorem 2.2, part 3), there exists a solution to $e(p,A) = 0$ in $\mathcal{V}'$ for every $A \in \mathcal{A}_{\mathrm{ad}}$ if and only if

$$\int_\Omega q(f)\overline{\lambda}(f)\,\mathrm{d}\mathbf{x} = 0 \quad \text{for all } \lambda \in \mathcal{V} \text{ and f.a.a. } f \in \mathcal{I},$$

where $\lambda \in \mathcal{V}$ solves

$$\int_{\mathcal{I}}\bigg(\int_\Omega \nabla\lambda\cdot\nabla\overline{\varphi} - k^2\lambda\overline{\varphi}\,\mathrm{d}\mathbf{x} - \jmath\varrho_\circ\omega\overline{A}\int_{\Gamma_{\mathrm{R}}}\lambda\overline{\varphi}\,\mathrm{d}\mathbf{s}\bigg)\mathrm{d}f = 0$$

for all $\varphi \in \mathcal{V}$. ◊

2.4. **The optimization problem.** The goal of the parameter identification problem is to identify the complex admittance A from given real-valued quantities p_i^{m} for the amplitude of the sound pressure at the given the microphon positions $\mathbf{x}_i \in \Omega \cup \Gamma_{\mathrm{N}}$, for $1 \le i \le n_{\mathrm{m}}$. For $1 \le i \le n_{\mathrm{m}}$ we choose $0 < \gamma_i \ll 1$ and introduce the bounded operators $\mathcal{C}_i : \mathcal{P} \to \mathbb{R}$ by

$$\mathcal{C}_i(p) = \eta \ln\bigg(\gamma_i + \int_{\mathcal{I}} \big|p(\mathbf{x}_i;f)\big|^2_{\mathbb{C}}\,\mathrm{d}f\bigg) \quad \text{for } p \in \mathcal{P}. \tag{2.8}$$

From $\gamma_i > 0$ it follows that $\mathcal{C}_i$ is well-defined for $i \in \{1,\dots,n_{\mathrm{m}}\}$.

Now we introduce the cost functional $J : X \to \mathbb{R}$ by

$$J(p,A) = \frac{1}{n_{\mathrm{m}}}\sum_{i=1}^{n_{\mathrm{m}}}\big|\mathcal{C}_i(p) - p_i^{\mathrm{m}}\big|^2 + \frac{\sigma}{2}\big|A - \hat{A}\big|^2_{\mathbb{C}} \quad \text{for } (p,A) \in X.$$

The first term of the cost measures the distance between the sound pressure and the corresponding measurement at each microphon position $\mathbf{x}_i$ using the averaging over the intervall $\mathcal{I}$ as well as the logarithmic scaling. The last term is needed to regularize the ill-posed inverse problem. For that purpose we assume that σ is a positive regularization parameter ensuring that A is not too far from a nominal value $\hat{A} \in \mathbb{C}$ for the admittance.

Remark 2.7. For every $p \in \mathcal{P}$ the function $\mathcal{G}_i(\cdot\,; p) : \overline{\Omega} \to \mathbb{R}$, $1 \le i \le n_\mathrm{m}$, defined by

$$\mathcal{G}_i(\mathbf{x}; p) = \eta \ln\bigg(\gamma_i + \int_{\mathcal{I}} \big|p(\mathbf{x}; f)\big|_{\mathbb{C}}^2 \,\mathrm{d}f\bigg), \quad \mathbf{x} \in \overline{\Omega},$$

is continuous, i.e., $\mathcal{G}_i(\cdot\,; p) \in C(\overline{\Omega})$. Let $\delta_{\mathbf{x}_i}$ denote the Dirac delta function that is continuous on $C(\overline{\Omega})$. Thus, using the point measure $\mu_i = \delta_{\mathbf{x}_i}$ we can express the cost functional in integral form as follows:

$$J(p, A) = \frac{1}{n_\mathrm{m}} \sum_{i=1}^{n_\mathrm{m}} \int_\Omega \big|\mathcal{G}_i(\cdot\,; p) - p_i^\mathrm{m}\big|^2 \,\mathrm{d}\mu_i + \frac{\sigma}{2} \big|A - \hat{A}\big|_{\mathbb{C}}^2 \quad \text{for } (p, A) \in X.$$

In particular, we have

$$\int_\Omega \big|\mathcal{G}_i(\cdot\,; p) - p_i^\mathrm{m}\big|^2 \,\mathrm{d}\mu_i = \big|\mathcal{C}_i(p) - p_i^\mathrm{m}\big|^2 \quad \text{for } i \in \{1, \ldots, n_\mathrm{m}\},$$

where $p \in L^2(\mathcal{I}; C_{\mathbb{C}}(\overline{\Omega}))$. ◊

Now we formulate our parameter estimation problem in terms of an infinite dimensional optimization problem:

$$\textbf{(P)} \qquad \min J(x) \quad \text{subject to (s.t.)} \quad x \in \mathcal{F}(\mathbf{P}) = \big\{\tilde{x} \in X_\mathrm{ad} \,\big|\, e(\tilde{x}) = 0 \text{ in } \mathcal{V}'\big\}.$$

To prove existence of optimal solutions the next assumption is required.

Assumption 2. *If $(p, A) \in X_\mathrm{ad}$ satisfies $e(p, A) = 0$ in $\mathcal{V}'$, then*

$$\|p(\mathbf{x}_i; \cdot)\|_{H_{\mathbb{C}}^s(\mathcal{I})} \le C\big(1 + |A|_{\mathbb{C}}\big) \quad \textit{for } 1 \le i \le n_\mathrm{m}$$

with some $s > 0$ and with a constant $C > 0$.

Remark 2.8. 1) By the Sobolev embedding theorem [1, p. 144] the Hilbert space $H_{\mathbb{C}}^s(\mathcal{I})$ is compactly embedded into $L_{\mathbb{C}}^2(\mathcal{I})$ for all $s > 0$.
2) One possibility to ensure more regularity for $p(\mathbf{x}_i; \cdot)$ is to add the quadratic term $\|p(\mathbf{x}_i; \cdot)\|_{H_{\mathbb{C}}^1(\mathcal{I})}^2$ to the cost functional. In our numerical experiments it turns out that even $\|p(\mathbf{x}_i; \cdot)\|_{W_{\mathbb{C}}^{1,\infty}(\mathcal{I})}$ is bounded on $\mathcal{I}$. ◊

Theorem 2.9. *Let Assumptions 1 and 2 hold. Then,* (**P**) *admits at least one local optimal solution denoted by $x^* = (p^*, A^*)$.*

Theorem 2.9 is proved in the Appendix.

2.5. **The reduced problem.** Problem (**P**) is an infinite-dimensional optimization problem with equality and inequality constraints. In this section we introduce the so-called reduced problem, where the equality constraint $e(\tilde{x}) = 0$ in $\mathcal{V}'$ is eliminated by considering a reduced cost functional that is defined on the admittance parameter only. For that purpose we assume instead of Assumption 1 the following stronger condition.

Assumption 3. *Let $q \in C_{\mathbb{C}}^1(\mathcal{I} \times \overline{\Omega})$ and $A \in \mathcal{A}_{ad}$ be given. Then there exists a unique solution $p \in \mathcal{P}$ to $e(p, A) = 0$ in $\mathcal{V}'$ satisfying*

$$\|p\|_{\mathcal{P}} \le C\big(1 + |A|_{\mathbb{C}}\big)$$

for a constant $C > 0$ independent of A and $f \in \mathcal{I}$.

Proposition 2.10. *Let Assumption* 3 *hold. Then, the non-linear solution operator* $\mathcal{S} : \mathcal{A}_{\mathrm{ad}} \to \mathcal{P}$ *with* $e(\mathcal{S}(A), A) = 0$ *in* $\mathcal{V}'$ *is well-defined, bounded, and continuous.*

Proof. By Assumption 3 the operator $\mathcal{S}$ is well-defined. Using Assumption 1 and $A \in \mathcal{A}_{\mathrm{ad}}$ we have

$$\|\mathcal{S}(A)\|_{\mathcal{P}} \leq C\big(1 + |A|_{\mathbb{C}}\big) \leq C\left(1 + \max\left\{|\underline{A}|_{\mathbb{C}}, |\overline{A}|_{\mathbb{C}}\right\}\right),$$

i.e., $\mathcal{S}$ is bounded. To prove the continuity of $\mathcal{S}$ let $\{A^n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_{\mathrm{ad}}$ and $A^* \in \mathcal{A}_{\mathrm{ad}}$ satisfying (A.9d). Then, by Assumption 1 the sequence $\{p^n\}_{n\in\mathbb{N}}$ defined by $p^n = \mathcal{S}(A^n)$ is bounded in $\mathcal{P}$. Thus, the continuity follows by analogous arguments as in the proof of Theorem 2.9. □

Utilizing the operator $\mathcal{S}$ we define the reduced cost functional

$$\hat{J} : \mathcal{A}_{\mathrm{ad}} \to \mathbb{R}, \quad \hat{J}(A) = J(\mathcal{S}(A), A) \text{ for } A \in \mathcal{A}_{\mathrm{ad}}.$$

Then, we consider the reduced problem

$$(\hat{\mathbf{P}}) \qquad \min \hat{J}(A) \quad \text{s.t.} \quad A \in \mathcal{A}_{\mathrm{ad}}.$$

Clearly, A^* is a local solution to $(\hat{\mathbf{P}})$ if and only if $x^* = (\mathcal{S}(A^*), A^*)$ solves $(\mathbf{P})$.

Remark 2.11. Note the $A \mapsto \hat{J}(A)$ is a real-valued function of a complex number $A = A_{\Re} + \jmath A_{\Im} \mapsto \hat{J}(A)$. In our numerical realization we consider $\hat{J}$ as a mapping from $\mathbb{R}^2 \to \mathbb{R}$ (for brevity, we use the same notation for the mapping), i.e., $(A_{\Re}, A_{\Im}) \mapsto \hat{J}(A_{\Re}, A_{\Im})$. ◊

3. Optimality conditions

To solve $(\mathbf{P})$ or $(\hat{\mathbf{P}})$ we make use of the associated optimality conditions. This section is devoted to derive these first- and second-order necessary optimality conditions. For that purpose we have to ensure differentiablity properties of the mappings e and J. The second-order conditions are important for the numerical solution procedure.

For the derivation of the optimlaity conditions let us introduce the Lagrange functional associated to $(\mathbf{P})$ by

$$\begin{aligned} L(x,\lambda) &= J(x) + \Re e\big(\langle e(x), \lambda\rangle_{\mathcal{V}',\mathcal{V}}\big) \\ &= \int_\Omega \frac{1}{n_{\mathrm{m}}} \sum_{i=1}^{n_{\mathrm{m}}} \int_\Omega \big|\mathcal{G}_i(\,\cdot\,; p) - p_i^{\mathrm{m}}\big|^2 \,\mathrm{d}\mu_i + \frac{\sigma}{2}\,\big|A - \hat{A}\big|_{\mathbb{C}}^2 \\ &\quad + \int_{\mathcal{I}} \Re e\big(B(p(f), \varphi(f); A)\big)\,\mathrm{d}f \quad \text{for } (x,\lambda) \in X \times \mathcal{V}, \end{aligned}$$

where the mappings $\mathcal{G}_i$, $1 \leq i \leq n_{\mathrm{m}}$, and the parameter-dependent bilinear form B have been introduced in (2.3) and Remark 2.7, respectively.

3.1. Differentiability properties of constraints and the cost functional.

In Section 3.2 we will study optimality conditions for $(\mathbf{P})$. For that purpose we investigate differentiability properties of the functions J, e, and $\hat{J}$. The next proposition is proved in the Appendix.

Proposition 3.1. *The operator* $e : X \to \mathcal{V}'$ *and the cost functional are twice continuously Fréchet-differentiable. Moreover, their second Fréchet-derivatives are Lipschitz-continuous on* X.

Let Assumption 3 be valid. In Section 2.5 we have introduced the reduced problem. From Proposition 3.1 we infer that e is Fréchet-differentiable. Using $e(\mathcal{S}(A), A) = 0$ for all $A \in \mathcal{A}_{\mathrm{ad}}$ we obtain

$$0 = \nabla_A\big(e(\mathcal{S}(A), A)\big)A_\delta = \nabla_p e(\mathcal{S}(A), A)\mathcal{S}'(A)A_\delta + \nabla_A e(\mathcal{S}(A), A)A_\delta$$

for any direction $A_\delta \in \mathbb{C}$.

The following assumption ensures a standard constraint qualification for $(\hat{\mathbf{P}})$.

Assumption 4. *For every $A \in \mathcal{A}_{\mathrm{ad}}$ and $A_\delta \in \mathbb{C}$ there exists a unique weak solution $p_\delta \in \mathcal{P}$ such that*

$$\begin{aligned} -\Delta p_\delta(f) - k^2 p_\delta(f) &= 0 && \textit{in } \Omega,\\ \frac{\jmath}{\varrho_\circ\omega}\,\frac{\partial p_\delta}{\partial n}(f) &= 0 && \textit{on } \Gamma_{\mathrm{N}},\\ \frac{\jmath}{\varrho_\circ\omega}\,\frac{\partial p_\delta}{\partial n}(f) &= Ap_\delta(f) + A_\delta p(f) && \textit{on } \Gamma_{\mathrm{R}} \end{aligned} \tag{3.1}$$

f.a.a. $f \in \mathfrak{I}$, where $p = \mathcal{S}(A)$. Moreover, p_δ satisfies

$$\|p_\delta\|_{\mathcal{P}} \le C\big(1 + |A_\delta|_{\mathbb{C}}\big)$$

with a constant $C > 0$ independent of A_δ.

Remark 3.2. 1) Note that Assumption 4 is closely related to Assumption 3.
2) Let Assumptions 1, 3-4 hold, then $\mathcal{S}$ is Fréchet-differentiable and its Fréchet derivative is $p_\delta = \mathcal{S}'(A)A_\delta$, where p_δ solves (3.1) in the weak sense. In particular, $p_\delta = -\big(\nabla_p e(\mathcal{S}(A), A)\big)^{-1}\nabla_A e(\mathcal{S}(A), A)A_\delta$. ◊

3.2. **First-order necessary optimality conditons.** Problem $(\hat{\mathbf{P}})$ is a non-convex programming problem so that different local minima might occur. A numerical method will produce a local minimum close to its starting value. Hence, we do not restrict our investigations to global solutions of $(\hat{\mathbf{P}})$. We will assume that a fixed reference solution is given satisfying certain first- and second-order optimality conditions (ensuring local optimality of the solution).

In our non-convex optimization problem the cost functional is defined pointwise in Ω. This leads to an adjoint problem with measures on the right-hand side. For this reason suppose that $A^* \in \mathcal{A}_{\mathrm{ad}}$ is a local solution to $(\hat{\mathbf{P}})$ and $p^* = \mathcal{S}(A^*)$. We define $\lambda^* \in \mathcal{W} = L^2(\mathfrak{I}; W^{1,s}_{\mathbb{C}}(\Omega))$, $s \in [1, 3/2)$, to be a weak solution to

$$-\Delta\lambda^*(f) - k^2\lambda^*(f) = \frac{4\eta}{n_{\mathrm{m}}}\sum_{i=1}^{n_{\mathrm{m}}}\frac{\big(p_i^{\mathrm{m}} - \mathcal{C}_i(p^*)\big)p^*\delta_{\mathbf{x}_i}}{\gamma_i + \|p(\mathbf{x}_i;\cdot)\|^2_{L^2_{\mathbb{C}}(\mathfrak{I})}} \quad \text{in } \Omega, \tag{3.2a}$$

$$\frac{\jmath}{\varrho_\circ\omega}\,\frac{\partial\lambda^*}{\partial n}(f) = 0 \quad \text{on } \Gamma_{\mathrm{N}}, \tag{3.2b}$$

$$\frac{\jmath}{\varrho_\circ\omega}\,\frac{\partial\lambda^*}{\partial n}(f) + \overline{A^*}\lambda^*(f) = 0 \quad \text{on } \Gamma_{\mathrm{R}} \tag{3.2c}$$

for almost all $f \in \mathfrak{I}$, where $\delta_{\mathbf{x}_i}$ denotes the Dirac delta distribution, i.e.,

$$\delta_{\mathbf{x}_i}(\varphi) = \varphi(\mathbf{x}_i) \quad \text{for } \varphi \in C_{\mathbb{C}}(\overline{\Omega}) \text{ and } 1 \le i \le n_{\mathrm{m}}.$$

Remark 3.3. Let $r = s/(s-1)$ for $s \in (1, 3/2)$. Then, $3 < r < \infty$ and $s^{-1} + r^{-1} = 1$. The function λ^* is a weak solution to (3.2) provided

$$\begin{aligned}(3.3)\qquad &\int_{\mathcal{J}} \int_\Omega \nabla\lambda^* \cdot \nabla\overline{\varphi} - k^2\lambda^*\overline{\varphi}\,\mathrm{d}\mathbf{x} - \jmath\varrho_\circ\omega\overline{A^*}\int_{\Gamma_\mathrm{R}} \lambda^*\overline{\varphi}\,\mathrm{d}\mathbf{s}\mathrm{d}f \\ &= \frac{4\eta}{n_\mathrm{m}} \sum_{i=1}^{n_\mathrm{m}} \frac{\big(p_i^\mathrm{m} - \mathcal{C}_i(p^*)\big)}{\gamma_i + \|p(\mathbf{x}_i;\cdot)\|^2_{L^2_\mathbb{C}(\mathcal{J})}} \int_{\mathcal{J}} p^*(\mathbf{x}_i,\cdot)\overline{\varphi(\mathbf{x}_i,\cdot)}\,\mathrm{d}f\end{aligned}$$

for all $\varphi \in L^2(\mathcal{J}; W^{1,r}_\mathbb{C}(\Omega))$. Notice that $W^{1,r}_\mathbb{C}(\Omega)$ is continuously embedded into the space $C_\mathbb{C}(\overline{\Omega})$; see [1, p. 144]. Thus, $\varphi \in L^2(\mathcal{J}; C_\mathbb{C}(\overline{\Omega}))$ in (3.3). Moreover, from $\lambda^*(f) \in W^{1,s}_\mathbb{C}(\Omega)$ f.a.a. $f \in \mathcal{J}$ we infer that $\lambda^*(f)|_{\Gamma_\mathrm{R}} \in L^\nu_\mathbb{C}(\Gamma_\mathrm{R})$ for $1 \le \nu \le 6s/(6-s)$. Therefore, in (3.3) all integrals are well-defined. ♢

The necessary optimality conditions are stated in the next theorem which is proved in the Appendix.

Theorem 3.4 (KKT conditions)**.** *Let Assumptions* 1-4 *hold. Suppose that* $A^* \in \mathcal{A}_\mathrm{ad}$ *is an optimal solution to* $(\hat{\mathbf{P}})$ *and* $p^* = \mathcal{S}(A^*)$ *holds. Suppose that there exists a weak solution* $\lambda^* \in \mathcal{W}$ *to* (3.3)*. Then, the variational inequality*

$$(3.4)\qquad \Re e\bigg(\Big(\sigma\big(A^* - \hat{A}\big) - \jmath\varrho_\circ \int_{\mathcal{I}} \Big(\omega\int_{\Gamma_\mathrm{R}} \lambda^*\overline{p^*}\,\mathrm{d}\mathbf{s}\Big)\mathrm{d}f\Big)\big(\overline{A^*} - \overline{A_\delta}\big)\bigg) \ge 0$$

holds for all $A_\delta \in \mathcal{A}_\mathrm{ad}$.

Summarizing, the first-order necessary optimality conditions consist in the state equation (2.1) with (p, A) replaced by (p^*, A^*), the dual equation (3.2) and the optimality condition (3.4).

Remark 3.5. It follows from Theorem 3.4 that we obtain the gradient $(A_\Re, A_\Im) \mapsto \hat{J}'(A_\Re, A_\Im)$ of the reduced cost is given by

$$(3.5)\qquad \hat{J}'(A_\Re, A_\Im) = \begin{pmatrix} \sigma\big(A_\Re - \hat{A}_\Re\big) + \varrho_\circ\Im m\Big(\int_{\mathcal{I}}\big(\omega\int_{\Gamma_\mathrm{R}}\lambda\overline{p}\,\mathrm{d}\mathbf{s}\big)\mathrm{d}f\Big) \\ \sigma\big(A_\Im - \hat{A}_\Im\big) - \varrho_\circ\Re e\Big(\int_{\mathcal{I}}\big(\omega\int_{\Gamma_\mathrm{R}}\lambda\overline{p}\,\mathrm{d}\mathbf{s}\big)\mathrm{d}f\Big)\end{pmatrix},$$

where $A = A_\Re + \jmath A_\Im$, $\hat{A} = \hat{A}_\Re + \jmath A_\Im$, $p \in \mathcal{P}$ solves (2.7) and $\lambda \in L^2(\mathcal{J}; W^{1,s}(\Omega;\mathbb{C}))$, $s \in [1, 3/2)$, is the solution to

$$\begin{aligned}(3.6)\qquad -\Delta\lambda(f) - k^2\lambda(f) &= \frac{4\eta}{n_\mathrm{m}}\sum_{i=1}^{n_\mathrm{m}} \frac{\big(p_i^\mathrm{m} - \mathcal{C}_i(p^*)\big)p^*\delta_{\mathbf{x}_i}}{\gamma_i + \|p(\mathbf{x}_i;\cdot)\|^2_{L^2(\mathcal{J};\mathbb{C})}} && \text{in } \Omega, \\ \frac{\jmath}{\varrho_\circ\omega}\frac{\partial\lambda}{\partial n}(f) &= 0 && \text{on } \Gamma_\mathrm{N}, \\ \frac{\jmath}{\varrho_\circ\omega}\frac{\partial\lambda}{\partial n}(f) &= -\overline{A}\lambda(f) && \text{on } \Gamma_\mathrm{R}\end{aligned}$$

f.a.a. $f \in \mathcal{J}$; compare Remark 2.11. ♢

3.3. **Second-order conditions.** We infer from Proposition 3.1 that the second Fréchet derivative of the Lagrange functional with respect to $x = (p, A) \in X$ at a point $(x^\circ,\lambda^\circ)\in X\times\mathcal{W}$, $x^\circ=(p^\circ,A^\circ)$, is given by

$$\begin{aligned}\nabla_{xx}L(x^\circ,\lambda^\circ)(x_\delta,x_\delta)&=\frac{2}{n_\mathrm{m}}\sum_{i=1}^{n_\mathrm{m}}\big|\mathcal{C}_i'(p^\circ)\tilde p_\delta\big|^2+\Re e\bigg(2\jmath\varrho_\circ A_\delta\int_\mathcal{I}\omega\int_{\Gamma_\mathrm{R}}p_\delta\overline{\lambda^\circ}\,\mathrm{d}\mathbf{s}\mathrm{d}f\bigg)\\&\quad+\frac{2}{n_\mathrm{m}}\sum_{i=1}^{n_\mathrm{m}}\big(\mathcal{C}_i(p^\circ)-p_i^\mathrm{m}\big)\mathcal{C}_i''(p^\circ)(p_\delta,p_\delta)+\sigma\,\big|A_\delta\big|_\mathbb{C}^2\end{aligned}$$

and, by (A.14b), we have

$$\mathcal{C}_i''(p^\circ)(p_\delta,p_\delta)=\frac{2\eta\,\Re e\big(\int_\mathcal{I}\int_\Omega|\tilde p_\delta|_\mathbb{C}^2\,\mathrm{d}\mu_i\mathrm{d}f\big)}{\gamma_i+\|p^\circ(\mathbf{x}_i;\cdot)\|^2_{L^2(\mathcal{I};\mathbb{C})}}-\frac{4\eta\,\Re e\big(\int_\mathcal{I}\int_\Omega p^\circ\overline{p}_\delta\,\mathrm{d}\mu_i\mathrm{d}f\big)^2}{\big(\gamma_i+\|p^\circ(\mathbf{x}_i;\cdot)\|^2_{L^2(\mathcal{I};\mathbb{C})}\big)^2}.$$

Thus, we have the following result.

Theorem 3.6. *Assume that Assumptions* 1-4 *are satisfied. Let $x^*=(p^*,A^*)\in\mathcal{F}(\mathbf{P})$ be a feasible point for* ($\mathbf{P}$) *satisfying together with $\lambda^*\in\mathcal{W}$ the dual system* (3.2)*. Suppose that $x_\delta=(p_\delta,A_\delta)\in\ker\nabla e(x^*)$, i.e., $\nabla e(x^*)x_\delta=0$ in $\mathcal{V}'$. If*

$$\|p_\delta\|_\mathcal{P}^2\le C_\mathrm{ker}\,\big|A_\delta\big|_\mathbb{C}^2, \tag{3.7a}$$

$$\bigg|\frac{2}{n_\mathrm{m}}\sum_{i=1}^{n_\mathrm{m}}\big(\mathcal{C}_i(p^*)-p_i^\mathrm{m}\big)\mathcal{C}_i''(p^*)(p_\delta,p_\delta)\bigg|\le\frac{\sigma}{4}\,\big|A_\delta\big|_\mathbb{C}^2, \tag{3.7b}$$

$$\bigg|\Re e\bigg(2\jmath\varrho_\circ A_\delta\int_\mathcal{I}\omega\int_{\Gamma_\mathrm{R}}p_\delta\overline{\lambda^*}\,\mathrm{d}\mathbf{s}\mathrm{d}f\bigg)\bigg|\le\frac{\sigma}{4}\,\big|A_\delta\big|_\mathbb{C}^2 \tag{3.7c}$$

hold, the point (x^*,λ^*) satisfies the second-order sufficient optimality conditions, i.e., there exists a $\kappa>0$ such that

$$\nabla_{xx}L(x^*,\lambda^*)(x_\delta,x_\delta)\ge\kappa\,\|x_\delta\|_X^2\quad\textit{for all } x_\delta\in\ker\nabla e(x^*). \tag{3.8}$$

Proof. From (3.7) we infer that

$$\nabla_{xx}L(x^*,\lambda^*)(x_\delta,x_\delta)\ge\frac{\sigma}{4C_\mathrm{ker}}\,\|p_\delta\|_\mathcal{P}^2+\frac{\sigma}{4}\,\big|A_\delta\big|_\mathbb{C}^2\ge\kappa\,\|x_\delta\|_X^2$$

with $\kappa=\sigma/\max(4C_\mathrm{ker},4)>0$. This gives (3.8). □

Remark 3.7. 1) From $x_\delta\in\ker\nabla e(x^*)$ we infer that $x_\delta=(p_\delta,A_\delta)$ satisfies

$$\int_\mathcal{I}B(p_\delta(f),\varphi(f);A)\,\mathrm{d}f=-\jmath\varrho_\circ A_\delta\int_\mathcal{I}\omega\int_{\Gamma_\mathrm{R}}p^*\overline{\varphi}\,\mathrm{d}\mathbf{s}\mathrm{d}f\quad\text{for all }\varphi\in\mathcal{V}.$$

Hence, f.a.a. $f\in\mathcal{I}$ the function $p_\delta(f)$ is a weak solution to

$$\begin{aligned}-\Delta p_\delta(f)-k^2p_\delta(f)&=0&&\text{in }\Omega,\\\frac{\jmath}{\varrho_\circ\omega}\frac{\partial p_\delta}{\partial n}(f)&=0&&\text{on }\Gamma_\mathrm{N},\\\frac{\jmath}{\varrho_\circ\omega}\frac{\partial p_\delta}{\partial n}(f)-Ap_\delta(f)&=A_\delta p^*(f)&&\text{on }\Gamma_\mathrm{R}.\end{aligned}\tag{3.9}$$

For $A_\delta=0$ we observe that $p_\delta(f)=0$ solves (3.9) f.a.a. $f\in\mathcal{I}$. To ensure (3.7a) the norm $\|p_\delta\|_\mathcal{P}$ has to be small provided $|A_\delta|$ is small. Of course, the constant C_ker can depend on p^*.

2) If for $0 < \tau \ll 1$ it follows that

$$\big|\mathcal{C}_i(p^*) - p_i^{\mathrm{m}}\big| \le \frac{\sigma\,\big|A_\delta\big|_{\mathbb{C}}^2}{8\big(\tau + |\mathcal{C}_i''(p^*)(p_\delta,p_\delta)|\big)}, \quad 1 \le i \le n_{\mathrm{m}}.$$

Then, we have

$$\begin{aligned}\left|\frac{2}{n_{\mathrm{m}}}\sum_{i=1}^{n_{\mathrm{m}}}\big(\mathcal{C}_i(p^*) - p_i^{\mathrm{m}}\big)\mathcal{C}_i''(p^*)(p_\delta,p_\delta)\right| &\le \frac{\sigma\,\big|A_\delta\big|_{\mathbb{C}}^2}{4n_{\mathrm{m}}}\sum_{i=1}^{n_{\mathrm{m}}}\frac{\big|\mathcal{C}_i''(p^*)(p_\delta,p_\delta)\big|}{\tau + |\mathcal{C}_i''(p^*)(p_\delta,p_\delta)|}\\ &\le \frac{\sigma}{4}\,\big|A_\delta\big|_{\mathbb{C}}^2.\end{aligned}$$

Thus, (3.7b) can be guaranteed if $|\mathcal{C}_i(p^*) - p_i^{\mathrm{m}}|$ is small.

3) Using (3.7a), Lemma A.1 with $\varepsilon = 1$ and $0 < \omega \le 2\pi f_b$ we estimate

$$\left|\Re e\left(2\jmath\varrho_\circ A_\delta\int_{\mathcal{I}}\omega\int_{\Gamma_{\mathrm{R}}}p_\delta\overline{\lambda^*}\,\mathrm{d}s\mathrm{d}f\right)\right| \le C\,\|\lambda^*\|_{L^2(\mathcal{I};L^1(\Gamma_{\mathrm{R}};\mathbb{C}))}\big|A_\delta\big|_{\mathbb{C}}^2,$$

where $C = 4\pi\varrho_\circ f_b C_\Gamma C_{\mathrm{ker}}$. Thus, (3.7c) holds provided

$$\|\lambda^*\|_{L^2(\mathcal{I};L^1(\Gamma_{\mathrm{R}};\mathbb{C}))} \le \frac{\sigma}{4C}.$$

Recall that λ^* is a weak solution to (3.2). In particular, if $|\mathcal{C}_i(p^*) - p_i^{\mathrm{m}}| = 0$, $1 \le i \le n_{\mathrm{m}}$, holds, we have $\lambda^* = 0$. Thus, if $\|\lambda^*\|_{L^2(\mathcal{I};L^1(\Gamma_{\mathrm{R}};\mathbb{C}))}$ is small for small residuals $|\mathcal{C}_i(p^*) - p_i^{\mathrm{m}}|$, condition (3.7c) can be ensured; compare part 2). $\Diamond$

4. POD reduced-order modelling

In this section we recall briefly the POD method and explain the reduced-order model for the Helmholtz equation as well as for the identification problem. Moreover, we present two numerical examples.

4.1. **The POD method.** Let $w_1, \ldots, w_n \in \mathbb{R}^m$ be given vectors and $\bar{w} = \sum_{i=1}^n w_i$ the corresponding mean value. We set $y_j = w_j - \bar{w}$, $j = 1, \ldots, n$, and $\mathcal{V} = \operatorname{span}\{y_j\}_{j=1}^n$ with $d = \dim\mathcal{V} \le m$ and $Y = [y_1, \ldots, y_n] \in \mathbb{R}^{m\times n}$. On $\mathbb{R}^m$ we use the inner product

$$\langle u, v\rangle_W = u^T W v \quad \text{for } u, v \in \mathbb{R}^m \tag{4.1}$$

with a symmetric, positive definite weighting matrix $W \in \mathbb{R}^{m\times m}$ and its induced norm $\|u\|_W = (u^T W u)^{1/2}$. Since W is a symmetric and positive definite matrix, $W^{1/2}$ is also defined via the eigenvalue decomposition of W. Then, for an arbitrary $\ell \le d$ we consider the minimization problem

$$\min_{\psi_1,\ldots,\psi_\ell}\sum_{j=1}^n \alpha_j\Big\|y_j - \sum_{i=1}^\ell \langle y_j, \psi_i\rangle_W\,\psi_i\Big\|_W^2 \quad \text{subject to } \langle\psi_i,\psi_j\rangle_W = \delta_{ij} \text{ for } 1 \le i, j \le \ell, \tag{4.2}$$

where $\{\alpha_j\}_{j=1}^n$ are nonnegative weights, δ_{ij} stands for the Kronecker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $j \ne i$. A solution to (4.2) is called a *POD basis of rank ℓ*.

We have $D^{1/2} = \operatorname{diag}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})$. Let us define the $m \times n$ matrix $\hat{Y} = W^{1/2} Y D^{1/2}$. A solution (4.2) is characterized by the *first-order necessary optimality conditions*: Solve the $n \times n$ symmetric eigenvalue problem

$$\hat{Y}^T\hat{Y}\hat{v}_i = \lambda_i\hat{v}_i, \quad 1 \le i \le \ell, \tag{4.3}$$

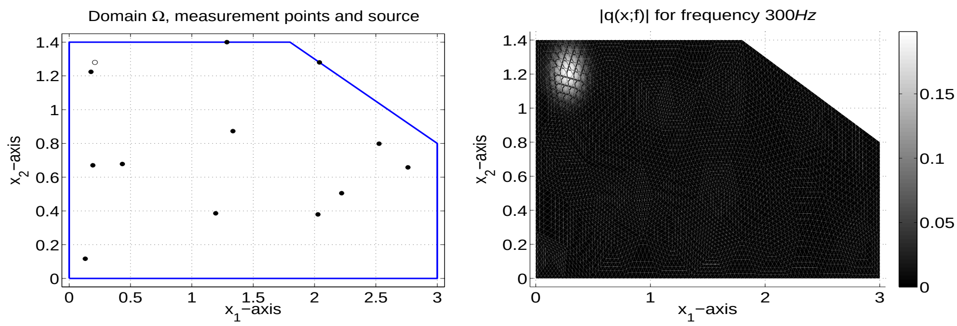

FIGURE 4.1. Acoustic domain Ω and $n_{\mathrm{m}} = 12$ measurement points ($*$) for the sound pressure and loud speker ($\circ$) (left plot); absolut value of the source function for frequency $f = 300$ (right plot).

where $\lambda_1 \geq \ldots \geq \lambda_\ell > 0$, and set $\psi_i = YD^{1/2}v_i/\sqrt{\lambda_i}$; see, e.g., [14]. Note that $\hat{Y}^T\hat{Y} = D^{1/2}Y^TWYD^{1/2}$. It can be shown that $\{\psi_i\}_{i=1}^{\ell}$ is already a solution to (4.2).

For the application of POD to concrete problems the choice of ℓ is certainly of central importance for applying POD. It appears that no general a-priori rules are available. Rather the choice of ℓ is based on heuristic considerations combined with observing the ratio of the modeled to the total energy contained in the system Y, which is expressed by

$$\mathcal{E}(\ell) = \frac{\sum_{i=1}^{\ell} \lambda_i}{\sum_{i=1}^{d} \lambda_i} = \frac{\sum_{i=1}^{\ell} \lambda_i}{\operatorname{trace}(\hat{Y}^T\hat{Y})}.$$

4.2. **Computation of the POD basis and reduced-order modelling.** The acoustic domain is presented in the left plot of Figure 4.1. The impedance boundary is $\Gamma_{\mathrm{R}} = \{(x,0)\,|\,0.5 \leq x \leq 2.5\}$ and the loud speaker is located in $\mathbf{x}_q = (0.21, 1.28) \in \Omega$. Therefore, we use the complex-valued source term

$$q(\mathbf{x}; f) = \frac{1}{5}\exp\left(\frac{\jmath\pi(f-200)}{50}\right)\exp\big(-50|\mathbf{x}-\mathbf{x}_q|^2\big) \text{ for } \mathbf{x} \in \Omega \text{ and } f \in [200, 500],$$

see Figure 4.1 (right plot). We apply a standard piecewise linear finite element (FE) discretization with $m = 4957$ degrees of freedom. Let $\{\varphi_i\}_{i=1}^m$ denote the piecewise linear finite element ansatz functions. Then, a finite element function is described by a coefficient vector in $\mathbb{R}^m$ containing the values of the finite element function at each grid points. We introduce the symmetric and positive definite stiffness matrix $S \in \mathbb{R}^{m\times m}$ with the elements

$$S_{ij} = \int_\Omega \varphi_j(\mathbf{x})\varphi_i(\mathbf{x}) + \nabla\varphi_j(\mathbf{x}) \cdot \nabla\varphi_i(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad 1 \leq i, j \leq m.$$

Then the H^1-inner product of the two FE functions is given by the the weihgthed inner product (4.1) of their coefficient vectors with $W = S$.

The POD basis is computed from FE solutions $p^h = p^h_{\Re} + \jmath p^h_{\Im}$ to (2.1) for the frequencies $f = 200, 201, \ldots, 500\,Hz$, where for every f we vary the admittance

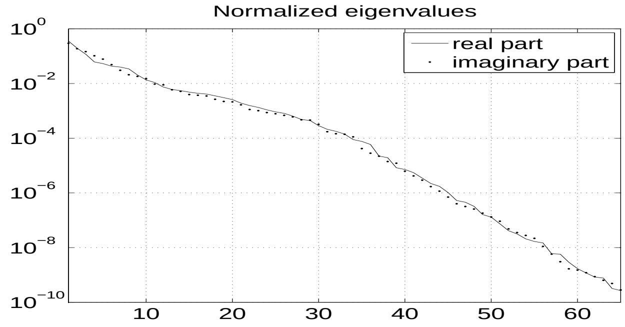

FIGURE 4.2. Decay of the first $\ell = 65$ eigenvalues of $\hat{Y}^T\hat{Y}$.

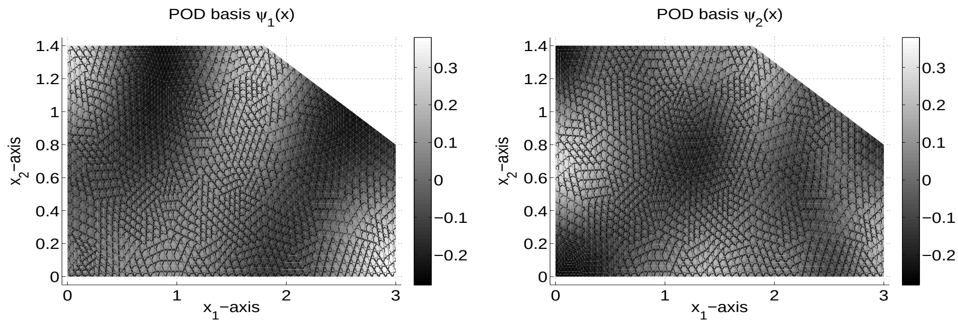

FIGURE 4.3. The first two POD basis functions for the imaginary part.

$A = A_{\Re} + \jmath A_{\Im}$ as follows

$$A_{\Re} = 2 \cdot 10^{-4}, 4 \cdot 10^{-4}, 6 \cdot 10^{-4}, \qquad (\text{in } \tfrac{m/s}{Pa}),$$
$$A_{\Im} = 6 \cdot 10^{-4}, 10^{-3}, 1.6 \cdot 10^{-3}, \qquad (\text{in } \tfrac{m/s}{Pa}),$$

i.e., we compute $n = 301 * 9 = 2709$ FE solutions p^h. In particular, we obtain for the real part coefficient vectors $a_j \in \mathbb{R}^m$, $j = 1, \ldots, n$. Note that that $2 \cdot 10^{-4}\,\frac{m/s}{Pa} \le A_{\Re}^{id} \le 6 \cdot 10^{-4}\,\frac{m/s}{Pa}$ and $6 \cdot 10^{-4}\,\frac{m/s}{Pa} \le A_{\Im}^{id} \le 1.6 \cdot 10^{-3}\,\frac{m/s}{Pa}$ hold; see Figure 2.1. First we compute a POD basis for the real part. The CPU time for the snapshot computation was 359 seconds. In the context of Section 4.1 we choose $W = S$ for the weighting matrix and the vectors $w_j = a_j$, $j = 1, \ldots, n$. In (**P**) we take the weights $\alpha_j = 1$ for all j. Then, we compute the POD basis $\{\psi_i\}_{i=1}^{\ell}$ of rank $\ell = 65$ for the approximation of the real part of the sound pressure. For the imaginary part we proceed analogously and determine a POD basis $\{\phi_i\}_{i=1}^{\ell}$. In Figure 4.2 the decay of the eigenvalues of the correlation matrix $\hat{Y}^T\hat{Y}$ are shown. The first two POD eigenfunctions are presented in Figures 4.3 and 4.4. The CPU time for the computation of the POD basis was 36 seconds. The values for ratio $\mathcal{E}(\ell)$ for different ℓ's are presented in Table 4.4.

Next we utilize the computed POD basis functions to derive a POD Galerkin scheme for (2.1). For that purpose let $\chi_{ik} = \psi_i + \jmath\phi_k : \Omega \to \mathbb{C}$ for $1 \le j, k \le \ell$.

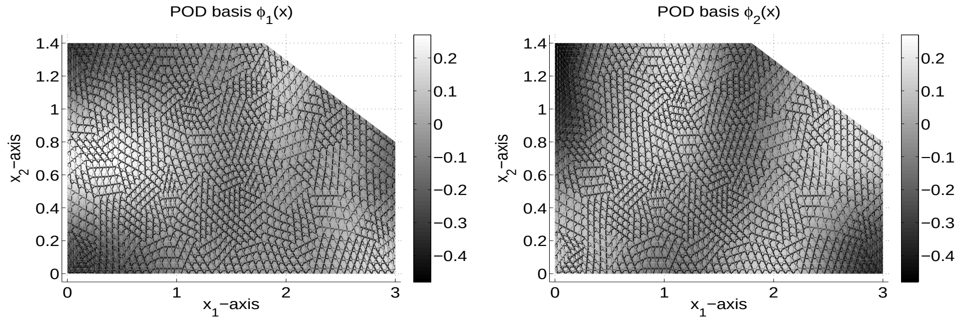

FIGURE 4.4. The first two POD basis functions for the imaginary part.

	$\ell = 5$	$\ell = 10$	$\ell = 15$	$\ell = 20$	$\ell = 30$	$\ell = 35$
$\mathcal{E}(\ell)$ for real part	78.59%	93.73%	97.20%	98.97%	99.92%	99.99%
$\mathcal{E}(\ell)$ for imaginary part	80.99%	94.35%	97.71%	99.13%	99.93%	99.99%

TABLE 4.4. Values for ratio $\mathcal{E}$.

Then, we make the ansatz

$$p^\ell(\mathbf{x}) = \sum_{i=1}^{\ell} a^i \psi_i(\mathbf{x}) + \jmath b^i \phi_i(\mathbf{x}), \quad a^i,\, b^i \in \mathbb{R},$$

multiply (2.1a) by the test functions $\psi_i + \jmath\phi_j$, $i, j = 1, \ldots, \ell$ and integrate over Ω. Integration by part and the boundary conditions (2.1b)-(2.1c) we end up with a linear system in the 2ℓ real coefficients a^i, b^i, $1 \le i \le \ell$, whereas in the FE case we have a linear system of the size $2m = 9914 \gg 2\ell = 130$. In an analogous way we derive a POD Galerkin scheme for the adjoint equation.

5. Numerical experiments

This section is devoted to present numerical examples for the identification problem. We apply a projected gradient method for the solution of $(\hat{\mathbf{P}})$. For the line search we apply the Armijo step size rule. The existence of second-order derivatives justifies the use of a second-order information. Therefore, we utilize a BFGS-quasi approximation for the second derivative to scale our gradient direction. If the BFGS approximation is not positive definite, we use the BFGS approximation of the previous iteration. In particular, if no inequality constraints are active, we obtain a variant of the quasi Newton method. For more details we refer, e.g., to [19].

In Figure 4.1 the $n_{\mathrm{m}} = 12$ measurement points $\mathbf{x}_i$ for the sound pressure are plotted.

Run 5.1. In the first test we want to identify the reference admittance A_{id} for Melamin $50mm$ (see Figure 2.1) from exact simulation data. Let us devide the frequency interval from 200 to 499 Hz into $n_{\mathcal{I}} = 100$ disjunct intervals $\mathcal{I}_k = [200 + 3(k-1), 199 + 3k]$, $1 \le k \le n_{\mathcal{I}}$, and solve sequentially $(\mathbf{P})$ for all k with $\mathcal{I} = \mathcal{I}_k$. The computed optimal admittance values are denoted by A^k_{opt}, $k = 1, \ldots, n_{\mathcal{I}}$. We

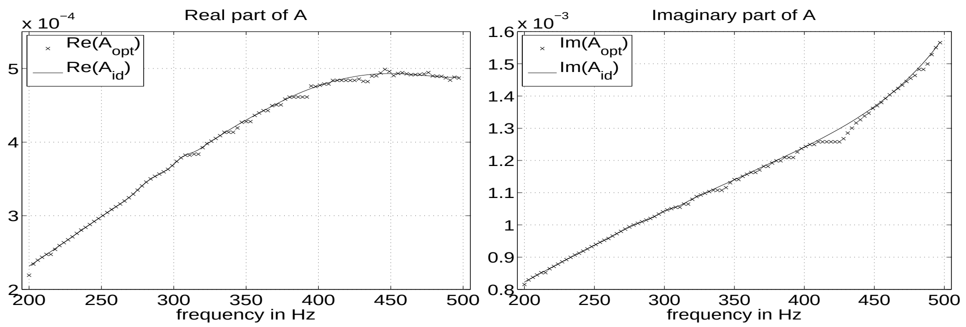

FIGURE 5.5. Run 5.1: Real part (left plot) and imaginary part (right plot) of the optimal and ideal admittance.

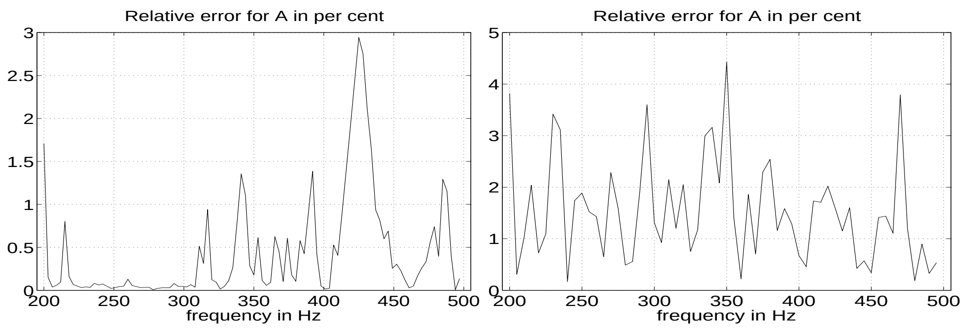

FIGURE 5.6. Error in the admittance: Run 5.1 with interval length $\delta f = 3$ (left plot) and Run 5.2 with $\delta f = 5$ (right plot).

use the corresponding FE solution p^h to get the p_i^{m}, $i = 1, \ldots, n_{\mathrm{m}}$, by

$$p_i^{\mathrm{m}} = \eta \ln \left(0.01 + \int_{\mathcal{I}_k} |p^h(\mathbf{x}_i; f)|^2 \, \mathrm{d}f \right), \quad k = 1, \ldots, n_{\mathcal{I}}. \tag{5.4}$$

For the cost functional we take $\gamma_i = 0.01$, $\eta = 5000$, and $\sigma = 10^6$. The nominal admittance $\hat{A}$ is chosen as follows: For $k = 1$ we set the nominal admittance to be $\hat{A} = 0.0002 + 0.0006\jmath$ and solve $(\mathbf{P})$ to get an optimal solution $A_{opt}^{(1)}$. The relative error for the starting admittance is about 21%. To improve the result we solve $(\mathbf{P})$ again on $\mathcal{I}_1$ with $\hat{A} = A_{opt}^{(1)}$ to get $A_{opt}^1 = A_{opt}^{(2)}$. Then we go to $k = 2$ and solve $(\mathbf{P})$ on $\mathcal{I}_2$ with $\hat{A} = A_{opt}^1$. We compute the optimal solution A_{opt}^2 and continue analogously for $i = 3, \ldots, n_{\mathcal{I}}$. The numerical method stops after 254 seconds, i.e., each optimization solve requires about 2.5 seconds on average. In Figure 5.5 the real and imaginary part of the optimal admittances as well as the corresponding values for A_{id} are plotted. The relative error can be seen in Figure 5.6. It turns out that by our initialization strategy the error for $k = 1$ can be reduced from 15,1% to 5,3% for the real part. for a summary of the CPU times we refer to Table 5.6.$\Diamond$

Run 5.2. Now we repeat Run 5.1, but we enlarge the intervals $\mathcal{I}_k$. We set $\mathcal{I}_k = [200 + 5(k-1), 199 + 5k]$, $1 \leq k \leq n_{\mathcal{I}}$ with $n_{\mathcal{I}} = 60$. The error in A is presented in the right plot of Figure 5.6. We observe that – compared to the previous run – the

Compute snapshot	359
Compute POD basis of rank $\ell = 65$	36
Run 5.1	254
Run 5.2	301
Run 5.3	261

TABLE 5.6. CPU-times in seconds.

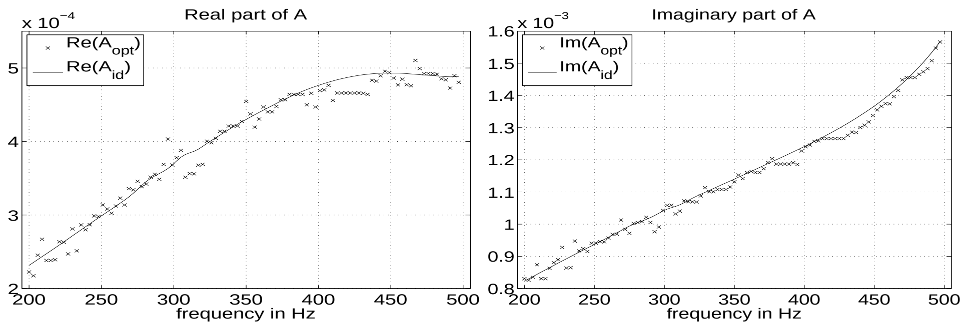

FIGURE 5.7. Run 5.3: Real part (left plot) and imaginary part (right plot) of the optimal and ideal admittance.

error is slightly enlarged but still smaller than 5%. Note the the CPU time is larger (although the number of frequency bands is reduced from 100 to 60. Therefore, on average each optimization solve requires more iterations. $\Diamond$

Run 5.3. In the third test run we want to identify the same reference admittance as in Run 5.1, but now from perturbed measurement data. Instead of (5.4) we utilize

$$p_i^{\mathrm{m}} = \eta \ln \Big(0.01 + \int_{\mathfrak{I}_k} |p^h(\mathbf{x}_i; f) + \varepsilon_i(f)|^2 \, \mathrm{d}f \Big), \quad k = 1, \ldots, n_{\mathfrak{I}},$$

where $\varepsilon_i : \mathfrak{I} \to \mathbb{C}$, $1 \le i \le n_{\mathrm{m}}$, is a perturbation satisfying

$$|\varepsilon_i(f)|/|p^h(\mathbf{x}_i; f)| \le 5\% \quad \text{for } f \in \mathfrak{I}_k \text{ and } i = 1, \ldots, n_{\mathrm{m}}.$$

For the cost functional we take the same parameters as in Run 5.1. We also choose the same POD basis as in the previous run. The numerical method stops after 261 seconds. In Figure 5.7 the real and imaginary part of the optimal admittances as well as the corresponding values for A_{id} are plotted. The relative error can be seen in the right plot of Figure 5.8. Again, our initialization strategy reduces significantly the error for $k = 1$. In particular, from 25% to 9% for the real part. $\Diamond$

APPENDIX

A.1. **Proof of Theorem 2.2.** To prove the claim of the theorem we make use of the following two lemmas.

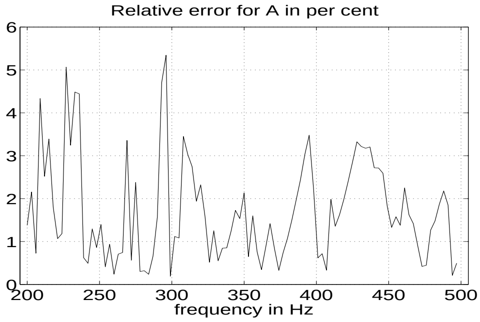

FIGURE 5.8. Error in the admittance: Run 5.3.

Lemma A.1. *There exists a constant* $C_\Gamma > 0$ *such that, for every* $\varphi \in V$ *and* $\varepsilon > 0$*, we have*

$$\|\varphi\|^2_{L^2_{\mathbb{C}}(\Gamma)} \le \frac{C_\Gamma}{\varepsilon}\,\|\varphi\|^2_H + C_\Gamma\varepsilon\sum_{i=1}^d \Big\|\frac{\partial\varphi}{\partial\mathbf{x}_i}\Big\|^2_H.$$

Proof. The lemma is a complex variant of Lemma 3.3 in [20]. □

For given $\mu \ge 0$ we introduce the parameter-dependent bilinear form $B_\mu : V \times V \to \mathbb{C}$ as

$$B_\mu(p,\varphi) = \int_\Omega \nabla p\cdot\overline{\nabla\varphi} + \big(\mu - k^2\big)p\overline{\varphi}\,\mathrm{d}\mathbf{x} + \jmath\varrho_\circ\omega A\int_{\Gamma_\mathrm{R}} p\overline{\varphi}\,\mathrm{ds}\quad \text{for } p,\,\varphi\in V.$$

Lemma A.2. *Suppose that* $\mu > k^2 + 2C_\Gamma^2\varrho_\circ^2\omega^2|A|^2$ *hold, where* C_Γ *has been introduced in Lemma* A.1*. Then, for every* $g \in H$ *there exists a unique solution* $p \in V$ *solving*

$$B_\mu(p,\varphi) = \int_\Omega g\overline{\varphi}\,\mathrm{d}\mathbf{x}\quad \textit{for all } \varphi\in V. \tag{A.1}$$

If Ω *is bounded and convex,* $p \in H^2_{\mathbb{C}}(\Omega)$ *holds.*

Proof. First, we prove that B_μ is bounded. Therefore, we estimate

$$\begin{aligned}\big|B_\mu(p,\varphi)\big| &\le \sum_{i=1}^d\Big\|\frac{\partial p}{\partial\mathbf{x}_i}\Big\|_H\Big\|\frac{\partial\varphi}{\partial\mathbf{x}_i}\Big\|_H + \big|\mu-k^2\big|\,\|p\|_H\|\varphi\|_H\\ &\quad+\big|\varrho_\circ\omega(f)A(f)\big|\,\|p\|_{L^2_{\mathbb{C}}(\Gamma)}\|\varphi\|_{L^2_{\mathbb{C}}(\Gamma)}\end{aligned}$$

for all $p,\,\varphi\in V$. From Lemma A.1 with $\varepsilon = 1$ we infer

$$\big|B_\mu(p,\varphi)\big| \le \big(1+\big|\mu-k^2\big|+2C_\Gamma\varrho_\circ\omega|A|\big)\,\|p\|_V\|\varphi\|_V\quad\text{for all } p,\,\varphi\in V,$$

i.e., B_μ is a bounded bilinear form. Applying Lemma A.1 with $\varepsilon = 1/(2C_\Gamma \varrho_\circ\omega|A|)$ and using $\eta := \mu - k^2 - 2C_\Gamma^2\varrho_\circ^2\omega^2|A|^2 > 0$ we find

$$\begin{aligned} B_\mu(p,p) &\geq \sum_{i=1}^d \Big\|\frac{\partial p}{\partial \mathbf{x}_i}\Big\|_H^2 + \big(\mu - k^2\big)\,\|p\|_H^2 - \varrho_\circ\omega|A|\,\|p\|^2_{L^2(\Gamma;\mathbb{C})} \\ &\geq \frac{1}{2}\sum_{i=1}^d \Big\|\frac{\partial p}{\partial \mathbf{x}_i}\Big\|_H^2 + \big(\mu - k^2 - 2C_\Gamma^2\varrho_\circ^2\omega^2|A|^2\big)\,\|p\|_H^2 \\ &\geq \min\left(\frac{1}{2},\eta\right)\|p\|_V^2 \quad \text{for all } p\in V, \end{aligned} \tag{A.2}$$

which implies that B_μ is coercive. Thus, the existence of a unique weak solution $p \in V$ to (2.1) follows from a complex variant of the Lax-Milgram theorem [5, p. 297]. For the regularity results we refer to [8]. $\square$

Remark A.3. If $p \in V$ satisfies (A.1) then p is a weak solution to the elliptic boundary value problem

$$\begin{aligned} -\Delta p + \big(\mu - k^2\big)p &= g && \text{in } \Omega,\\ \frac{\jmath}{\varrho_\circ\omega}\frac{\partial p}{\partial n} &= 0 && \text{on } \Gamma_{\mathrm{N}},\\ \frac{\jmath}{\varrho_\circ\omega}\frac{\partial p}{\partial n} &= Ap && \text{on } \Gamma_{\mathrm{R}} \end{aligned}$$

for given $g \in H$. $\Diamond$

Observe that p solves (2.4a) weakly if and only if

$$B_\mu(p,\varphi) = \int_\Omega \big(\mu p + q\big)\overline{\varphi}\,\mathrm{d}\mathbf{x} \quad \text{for all } \varphi \in V \tag{A.3}$$

holds. Introducing the linear solution operator $S_\mu : H \to H$ as follows: $v = S_\mu g$ is the unique solution to

$$B_\mu(v,\varphi) = \int_\Omega g\overline{\varphi}\,\mathrm{d}\mathbf{x} \quad \text{for all } \varphi \in V.$$

By Lemma A.2 the operator S_μ is well-defined. Moreover, we infer from $\mu > k^2 + 2C_\Gamma^2\varrho_\circ^2\omega^2|A|^2$ and (A.2) that for given $g \in H$ and for $p = S_\mu(g)$ we have

$$\frac{1}{2}\min\big(1,2\eta\big)\,\|p\|_V^2 \leq B_\mu(p,p) = \int_\Omega g\overline{p}\;\mathrm{d}\mathbf{x} \leq \|g\|_H\|p\|_H \leq \|g\|_H\|p\|_V$$

i.e.,

$$\|S_\mu(g)\|_H \leq \|S_\mu(g)\|_V \leq \frac{2}{\min(1,2\eta)}\,\|g\|_H.$$

Therefore, S_μ is also bounded. Since $p = S_\mu(g)$ belongs to V and V is compactly embedded into H, we conclude that S_μ is also a compact operator.

The function p solves (2.4a) weakly if and only if

$$p = S_\mu(\mu p + q) = \mu S_\mu(p) + S_\mu(q) \quad \text{in } H. \tag{A.4}$$

or, equivalently

$$p - \mu S_\mu(p) = S_\mu(q) \quad \text{in } H. \tag{A.5}$$

We set $h = S_\mu(q) \in V \subset H$. Since S_μ is compact, we can apply the Fredholm alternative [5, p. 641]: either

$$\text{for each } h \in V \text{ the equation } p - \mu S_\mu(p) = h \text{ has a unique solution } p \in H \tag{A.6}$$

or else

$$\text{the equation } p - \mu S_\mu(p) = 0 \text{ has nonzero solutions in } H. \tag{A.7}$$

If (A.6) hold, then it follows from (A.3), (A.4) and (A.5) that there exists a unique weak solution to (2.4-a). On the other hand, if (A.7) is satisfied, then $\mu \neq 0$ holds and the dimension of the space $\mathcal{N}$ of the solutions to (A.7) is finite [5, p. 641] and equals the dimension of the space $\mathcal{N}^*$ of solutions to

$$\lambda - \mu S_\mu(\lambda) = 0 \quad \text{in } H, \tag{A.8}$$

where $S_\mu^* : H \to H$ denotes the adjoint operator to S_μ. Note that p solves (A.7) if and only if p is a weak solution to (2.5-a). Furthermore, λ satisfies (A.8) if and only if λ is a weak solution to (2.6).

Finally, it follows from the Fredholm alternative that (A.6) has a solution if and only if

$$\langle h, \lambda \rangle_H = \langle S_\mu(q), \lambda \rangle_H = \langle q, S_\mu^*(\lambda) \rangle_H = \frac{1}{\mu} \langle q, \lambda \rangle_H.$$

Hence, (2.4a) has a solution provided $\langle q, \lambda \rangle_H = 0$ for all weak solutions to (2.6).

A.2. **Proof of Theorem 2.9.** By Assumptions 1 and 2 the set X_{ad} is non-empty. Hence, there exists a minimizing sequence $\{x^n\}_{n\in\mathbb{N}}$ in X_{ad} with $x^n = (p^n, A^n)$ so that

$$0 \leq \inf_{x \in \mathcal{F}(\mathbf{P})} J(x) = \lim_{n\to\infty} J(x^n) < \infty.$$

Since $A^n \in \mathcal{A}_{\mathrm{ad}}$ holds for all n, the sequence $\{A^n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{C}$. By Assumption 1, the sequence $\{p^n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{P}$. Furthermore, we infer from Assumption 2 and from the boundedness of $\{A^n\}_{n\in\mathbb{N}}$ that the sequence $\{p^n(\mathbf{x}_i;\cdot)\}_{n\in\mathbb{N}}$ is bounded in $H^s_{\mathbb{C}}(\mathcal{I})$. Using Remark 2.8, part 1), it follows that there exist a subsequence $\{x^{n_k}\}_{k\in\mathbb{N}}$ with $x^{n_k} = (p^{n_k}, A^{n_k})$ and an element $x^* = (p^*, A^*) \in X$ such that

$$\lim_{k\to\infty} \int_{\mathcal{I}} \langle p^{n_k}(f) - p^*(f), \varphi(f) \rangle_V \, \mathrm{d}f = 0 \qquad \text{for all } \varphi \in \mathcal{V}, \tag{A.9a}$$

$$\lim_{k\to\infty} \|p^{n_k}(\mathbf{x}_i;\cdot) - p^*(\mathbf{x}_i;\cdot)\|_{L^2_{\mathbb{C}}(\mathcal{I})} = 0 \qquad \text{for } 1 \leq i \leq n_{\mathrm{m}}, \tag{A.9b}$$

$$\lim_{k\to\infty} \int_{\mathcal{I}} \langle p^{n_k}(f) - p^*(f), \varphi(f) \rangle_{L^2_{\mathbb{C}}(\Gamma_{\mathrm{R}})} \, \mathrm{d}f = 0 \qquad \text{for all } \varphi \in \mathcal{V}, \tag{A.9c}$$

$$\lim_{k\to\infty} \left| A^{n_k} - A^* \right|_{\mathbb{C}} = 0, \tag{A.9d}$$

where we have used Lemma A.1, $p|_\Gamma \in L^2(0,T; H^{1/2}_{\mathbb{C}}(\Gamma))$ and the Sobolev embedding theorem [1, p. 144]. Since $\mathcal{A}_{\mathrm{ad}}$ is closed in $\mathbb{C}$, we have $A^* \in \mathcal{A}_{\mathrm{ad}}$. Moreover, we infer from (A.9a) that

$$\lim_{k\to\infty} \int_{\mathcal{I}} \Big(\int_\Omega \nabla p^{n_k} \cdot \nabla \overline{\varphi} - k^2 p^{n_k} \overline{\varphi} \, \mathrm{d}\mathbf{x} \Big) \mathrm{d}f = 0 \quad \text{for all } \varphi \in \mathcal{V}. \tag{A.10}$$

Recall that $\omega \le 2\pi f_b$ for $f \in \mathcal{I}$. Utilizing (A.9c), (A.9d), and Lemma A.1 imply the existence of a constant $C > 0$ satisfying

$$\begin{aligned}&\Big|\jmath\varrho_\circ\big(A^{n_k}-A^*\big)\int_{\mathcal{I}}\Big(\omega\int_\Omega p^{n_k}\overline{\varphi}\,\mathrm{d}\mathbf{s}\Big)\mathrm{d}f\Big|\\&\quad\le\varrho_\circ 2\pi f_b\big|A^{n_k}-A^*\big|_{\mathbb{C}}\|p^{n_k}\|_{L^2(\mathcal{I};L^2_{\mathbb{C}}(\Gamma_\mathrm{R}))}\|\varphi\|_{L^2(\mathcal{I};L^2_{\mathbb{C}}(\Gamma_\mathrm{R}))}\\&\quad\le\varrho_\circ 2\pi f_b C\big|A^{n_k}-A^*\big|_{\mathbb{C}}\|p^{n_k}\|_{\mathcal{V}}\|\varphi\|_{\mathcal{V}}\to 0\text{ as }k\to\infty\end{aligned}\tag{A.11}$$

for all $\varphi \in \mathcal{V}$. Using (A.9c) we conclude

$$\lim_{k\to\infty}\Big|\jmath\varrho_\circ A^*\int_{\mathcal{I}}\Big(\omega\int_{\Gamma_\mathrm{R}}\big(p^{n_k}-p^*\big)\overline{\varphi}\,\mathrm{d}\mathbf{s}\Big)\mathrm{d}f\Big|=0\quad\text{for all }\varphi\in\mathcal{V}.\tag{A.12}$$

Combining (A.10)–(A.12) it follows that

$$\lim_{k\to\infty}\langle e(x^{n_k})-e(x^*),\varphi\rangle_{\mathcal{V}',\mathcal{V}}=0\quad\text{for all }\varphi\in\mathcal{V}.$$

Due to $e(x^{n_k}) = 0$ in $\mathcal{V}'$ for all k we have $e(x^*) = 0$ in $\mathcal{V}'$. Thus, $x^* \in \mathcal{F}(\mathbf{P})$. By (A.9b) we conclude that

$$\lim_{k\to\infty}\mathcal{C}_i(p^{n_k})=\mathcal{C}_i(p^*)\quad\text{for }1\le i\le n_\mathrm{m}.\tag{A.13}$$

Hence, it follows from (A.9d) and (A.13) that

$$\inf_{x\in\mathcal{F}(\mathbf{P})}J(x)=\lim_{k\to\infty}J(x^{n_k})=J(x^*),$$

i.e., $x^* = (p^*, A^*)$ is a (global) solution to (**P**). □

A.3. **Proof of Proposition 3.1.** Recall that the operators $\mathcal{C}_i$, $1 \le i \le n_\mathrm{m}$, have been introduced in (2.8). To prove differentiability properties of the cost functional we make use of the following lemma.

Lemma A.4. *Then, for every $i \in \{1, \dots, n_\mathrm{m}\}$ the operator $\mathcal{C}_i : \mathcal{P} \to \mathbb{R}$ is twice Fréchet-differentiable. Its first and second Fréchet derivatives are given by*

$$\mathcal{C}_i'(p)p_\delta=\frac{2\eta\,\Re e\big(\int_{\mathcal{I}}\int_\Omega p\overline{p}_\delta\,\mathrm{d}\mu_i\mathrm{d}f\big)}{\gamma_i+\|p(\mathbf{x}_i;\cdot)\|^2_{L^2_{\mathbb{C}}(\mathcal{I})}},\tag{A.14a}$$

$$\begin{aligned}\mathcal{C}_i''(p)(\tilde{p}_\delta,p_\delta)&=\frac{2\eta\,\Re e\big(\int_{\mathcal{I}}\int_\Omega \tilde{p}_\delta\overline{p}_\delta\,\mathrm{d}\mu_i\mathrm{d}f\big)}{\gamma_i+\|p(\mathbf{x}_i;\cdot)\|^2_{L^2_{\mathbb{C}}(\mathcal{I})}}\\&\quad-\frac{4\eta\,\Re e\big(\int_{\mathcal{I}}\int_\Omega p\overline{\tilde{p}}_\delta\,\mathrm{d}\mu_i\mathrm{d}f\big)\,\Re e\big(\int_{\mathcal{I}}p\overline{p}_\delta\,\mathrm{d}\mu_i\mathrm{d}f\big)}{\big(\gamma_i+\|p(\mathbf{x}_i;\cdot)\|^2_{L^2_{\mathbb{C}}(\mathcal{I})}\big)^2}\end{aligned}\tag{A.14b}$$

for any directions p_δ, $\tilde{p}_\delta \in \mathcal{P}$. Moreover, the second Fréchet-derivative of $\mathcal{C}_i$ is locally Lipschitz-continuous on $\mathcal{P}$.

Proof. Let $i \in \{1, \dots, n_\mathrm{m}\}$. We introduce the operator $\mathcal{D}_i : \mathcal{P} \to \mathbb{R}$ by

$$\mathcal{D}_i(p)=\gamma_i+\|p(\mathbf{x}_i;\cdot)\|^2_{L^2_{\mathbb{C}}(\mathcal{I})}\quad\text{for }p\in\mathcal{P}.$$

It follows by standard arguments that $\mathcal{D}_i$ is twice Fréchet-differentiable. Its first and second Fréchet derivatives at a point $p \in \mathcal{P}$ are given as

$$\begin{aligned}\mathcal{D}_i'(p)p_\delta &= 2\Re e\bigg(\int_\mathcal{I}\int_\Omega p(\mathbf{x};f)p_\delta(\mathbf{x};f)\,\mathrm{d}\mu_i\mathrm{d}f\bigg),\\ \mathcal{D}_i''(p)(\tilde{p}_\delta,p_\delta) &= 2\Re e\bigg(\int_\mathcal{I}\int_\Omega \tilde{p}_\delta(\mathbf{x};f)p_\delta(\mathbf{x};f)\,\mathrm{d}\mu_i\mathrm{d}f\bigg)\end{aligned} \tag{A.15}$$

for any directions p_δ, $\tilde{p}_\delta \in \mathcal{P}$. Moreover, $\mathcal{D}_i''$ is independent of p. Thus, in particular, $\mathcal{D}_i''$ is Lipschitz-continuous on $\mathcal{P}$. Notice that $\mathcal{D}_i(p) > 0$ and $\mathcal{C}_i = 2(\ln\circ\mathcal{D}_i)$. Since $s \mapsto \ln(s)$ is three times continuously differentiable on $(0,\infty)$, the claim follows from (A.15) and the chain rule for Fréchet derivatives; see, e.g., [16, p. 176]. $\square$

Next we prove Proposition 3.1. Let $x = (p, A)$ be chosen. Utilizing $\omega \le 2\pi f_b$ for all $f \in \mathcal{I}$ and Lemma A.1 with $\varepsilon = 1$ we find

$$\begin{aligned}\|e(x+x_\delta)-e(x)-e'(x)x_\delta\|_{\mathcal{V}'} &= \sup_{\|\varphi\|_\mathcal{V}=1}\bigg|\varrho_\circ A_\delta\int_\mathcal{I}\omega\int_{\Gamma_\mathrm{R}} p_\delta\overline{\varphi}\,\mathrm{d}\mathbf{s}\mathrm{d}f\bigg|\\ &\le 2C_1\,|A_\delta|_\mathbb{C}\|p_\delta\|_\mathcal{V} \le C_1\,\Big(|A_\delta|_\mathbb{C}^2+\|p_\delta\|_\mathcal{V}^2\Big)\\ &\le C_1\,\|x_\delta\|_X^2\end{aligned}$$

for any direction $x_\delta = (p_\delta, A_\delta) \in X$ with the constant $C_1 = \pi\varrho_\circ f_b C_\Gamma > 0$. Hence,

$$\lim_{\|x_\delta\|_X\searrow 0}\frac{\|e(x+x_\delta)-e(x)-e'(x)x_\delta\|_{\mathcal{V}'}}{\|x_\delta\|_X} = 0,$$

i.e., the operator e is Fréchet-differentiable and its first Fréchet derivative is given by (A.16). Moreover, we infer from

$$\begin{aligned}&\|e'(x+\tilde{x}_\delta)-e'(x)-e''(x)\tilde{x}_\delta\|_{L(X,\mathcal{V}')}\\ &= \sup_{\|x_\delta\|_X=1}\sup_{\|\varphi\|_\mathcal{V}=1}\Big|\langle e'(x+\tilde{x}_\delta)x_\delta - e'(x)x_\delta - e''(x)(\tilde{x}_\delta,x_\delta),\varphi\rangle_{\mathcal{V}',\mathcal{V}}\Big| = 0\end{aligned}$$

that the operator e is also twice Fréchet-differentiable and its second Fréchet derivative is given by

$$\begin{aligned}\langle e'(x)x_\delta,\varphi\rangle_{\mathcal{V}',\mathcal{V}} &= \int_\mathcal{I}\int_\Omega \nabla p_\delta\cdot\nabla\overline{\varphi} - k^2 p_\delta\overline{\varphi}\,\mathrm{d}\mathbf{x}\mathrm{d}f\\ &\quad + \jmath\varrho_\circ\int_\mathcal{I}\omega\int_{\Gamma_\mathrm{R}}\big(A_\delta p + A p_\delta\big)\overline{\varphi}\,\mathrm{d}\mathbf{s}\mathrm{d}f,\end{aligned} \tag{A.16}$$

for any directions $x_\delta = (p_\delta, A_\delta)$, $\tilde{x}_\delta = (\tilde{p}_\delta, \tilde{A}_\delta) \in X$ and for $\varphi \in \mathcal{V}$, where the Fréchet derivatives of the $\mathcal{C}_i$'s are stated in Lemma A.4. Since $e''(x)$ is independent of x, the second Fréchet-derivative of e is Lipschitz-continuous on X. Here, $L(X, \mathcal{V}')$ denotes the Banach space of all bounded linear operators from X to $\mathcal{V}'$. It follows thatIts first and second Fréchet derivatives at a point $x = (p, A) \in X$ are given by

$$\langle e''(x)(\tilde{x}_\delta,x_\delta),\varphi\rangle_{\mathcal{V}',\mathcal{V}} = \jmath\varrho_\circ\int_\mathcal{I}\omega\int_{\Gamma_\mathrm{R}}\big(A_\delta\tilde{p}_\delta + \tilde{A}_\delta p_\delta\big)\overline{\varphi}\,\mathrm{d}\mathbf{s}\mathrm{d}f$$

for any directions $x_\delta = (p_\delta, A_\delta)$, $\tilde{x}_\delta = (\tilde{p}_\delta, \tilde{A}_\delta) \in X$ and for $\varphi \in \mathcal{V}$. The proof for the cost functional follows directly from the chain rule for Fréchet derivatives [16,

p. 176] and Lemma A.4. In particular, we find

$$J'(x)x_\delta = \frac{2}{n_\mathrm{m}} \sum_{i=1}^{n_\mathrm{m}} \big(\mathcal{C}_i(p) - p_i^\mathrm{m}\big)\mathcal{C}_i'(p)p_\delta + \sigma \Re e\big((A-\hat{A})\overline{A}_\delta\big),$$

$$\begin{aligned} J''(x)(\tilde{x}, x_\delta) &= \frac{2}{n_\mathrm{m}} \sum_{i=1}^{n_\mathrm{m}} \big(\mathcal{C}_i'(p)\tilde{p}_\delta\big)\big(\mathcal{C}_i'(p)p_\delta\big) \\ &\quad + \frac{2}{n_\mathrm{m}} \sum_{i=1}^{n_\mathrm{m}} \big(\mathcal{C}_i(p) - p_i^\mathrm{m}\big)\mathcal{C}_i''(p)(\tilde{p}_\delta, p_\delta) + \sigma \Re e\big(\tilde{A}_\delta \overline{A}_\delta\big) \end{aligned}$$

for any directions $x_\delta = (p_\delta, A_\delta)$, $\tilde{x}_\delta = (\tilde{p}_\delta, \tilde{A}_\delta) \in X$. □

A.4. **Proof of Theorem 3.4.** By Proposition 3.1 and Remark 3.2-2) the reduced cost functional $\hat{J}$ is Fréchet-differentiable. It follows from [15] that first-order necessary optimality conditions for $(\hat{\mathbf{P}})$ are given by

$$\Re e\Big(\langle \hat{J}'(A^*), A^* - A_\delta\rangle_\mathbb{C}\Big) \geq 0 \quad \text{for all } A_\delta \in \mathcal{A}_\mathrm{ad}. \tag{A.17}$$

Using Proposition 3.1, (3.1), and (3.3) we find for the gradient $\hat{J}'$ at $A^* \in \mathcal{A}_\mathrm{ad}$

$$\begin{aligned} &\langle \hat{J}'(A^*), A^* - A_\delta\rangle_\mathbb{C} = \langle \nabla_A J(x^*), A^* - A_\delta\rangle_\mathbb{C} + \langle \nabla_p J(x^*), p_\delta\rangle_{\mathcal{P}',\mathcal{P}} \\ &= \sigma\langle A^* - \hat{A}, A^* - A_\delta\rangle_\mathbb{C} + \frac{4\eta}{n_\mathrm{m}} \sum_{i=1}^{n_\mathrm{m}} \frac{\big(p_i^\mathrm{m} - \mathcal{C}_i(p^*)\big)}{\gamma_i + \|p(\mathbf{x}_i;\cdot)\|^2_{L^2(\mathcal{I};\mathbb{C})}} \int_\mathcal{I} p^*(\mathbf{x}_i,\cdot)\overline{p_\delta(\mathbf{x}_i,\cdot)}\,\mathrm{d}f \\ &= \sigma\langle A^* - \hat{A}, A^* - A_\delta\rangle_\mathbb{C} + \int_\mathcal{I}\int_\Omega \nabla\lambda^*\cdot\nabla\overline{p_\delta} - k^2\lambda^*\overline{p_\delta}\,\mathrm{d}\mathbf{x}\mathrm{d}f \\ &\quad - \jmath\varrho_\circ \int_\mathcal{I} \omega\overline{A^*}\int_{\Gamma_\mathrm{R}} \lambda^*\overline{p}_\delta\,\mathrm{d}\mathbf{s}\mathrm{d}f \\ &= \sigma\langle A^* - \hat{A}, A^* - A_\delta\rangle_\mathbb{C} + \overline{\int_\mathcal{I}\int_\Omega \nabla p_\delta\cdot\nabla\overline{\lambda^*} - k^2 p_\delta\overline{\lambda^*}\,\mathrm{d}\mathbf{x}\mathrm{d}f} \\ &\quad - \jmath\varrho_\circ \int_\mathcal{I} \omega\overline{A^*}\int_{\Gamma_\mathrm{R}} \lambda^*\overline{p_\delta}\,\mathrm{d}\mathbf{s}\mathrm{d}f \\ &= \sigma\langle A^* - \hat{A}, A^* - A_\delta\rangle_\mathbb{C} - \overline{\jmath\varrho_\circ \int_\mathcal{I} \omega\int_{\Gamma_\mathrm{R}} A_\delta p^*\overline{\lambda^*}\,\mathrm{d}\mathbf{s}\mathrm{d}f} \\ &= \sigma\Big\langle A^* - \hat{A} + \jmath\varrho_\circ \int_\mathcal{I}\omega\int_{\Gamma_\mathrm{R}} \lambda^*\overline{p^*}\,\mathrm{d}\mathbf{s}\mathrm{d}f, A^* - A_\delta\Big\rangle_\mathbb{C}. \end{aligned}$$

Inserting this expression for the gradient into (A.17) we obtain (3.4). □

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