On Semi-discrete Monge Kantorovich and generalized partitions

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Abstract

Let X a probability measure space and $\psi_1...,\psi_N$ measurable, real valued functions on X. Consider all possible partitions of X into N disjoint subdomains X_i on which $\int_{X_i} \psi_i$ are prescribed. We address the question of characterizing the set $(m_1, ..., m_N) \in \mathbb{R}^N$ for which there exists a partition $X_1, ..., X_N$ of X satisfying $\int_{X_i} \psi_i = m_i$ and discuss some optimization problems on this set of partitions. The relation of this problem to semi-discrete version of optimal mass transportation is discussed as well.

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1 Introduction

1.1 Semi-discrete Monge problem

Optimal Transportation, known also as *Monge-Kantorovich theory*, became very popular in last decades. The first publication by Monge [8] goes back to 1781. Excellent modern reviews are the books of C. Villani [10, 11].

The object of optimal transportation is to find an optimal map transporting a given, prescribed probability measure into another. In general setting, it deals with a pair of probability measure spaces $(X, \mathcal{B}_X, \mu), (Y, \mathcal{B}_Y, \nu)$ and a $\mu \oplus \nu$ measurable cost function $c : X \times Y \to \mathbb{R}$. The Monge problem is to maximize¹ the functional

$$T \to \int_X c(x, T(x)) d\mu \in \mathbb{R}$$
 (1)

among all measurable maps $T : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ which transport the measure μ to ν , i.e. $T_{\#}\mu = \nu$, that is

$$\mu(T^{-1}(B)) = \nu(B)$$
 (2)

for any $B \in \mathcal{B}_Y$.

In the special case where Y is a *finite* space, i.e $Y := \{y_1, \ldots, y_N\}$ and $\mathcal{B}_Y = 2^Y$, ν is characterized by a vector

$$\vec{m} := (m_1, \dots, m_N) \in S_I := \left\{ \vec{m} \in \mathbb{R}^N , \sum_{i \in I} m_i = 1 , m_i \ge 0 \right\}$$
 (3)

via $\nu(\{y_i\}) := m_i$. Here and thereafter, $I := \{1 \dots N\}$.

In this case, any mapping $T_{\#}\mu = \nu$ induces a *partition* of X into a finite number of components $X_i := T^{-1}(\{y_i\}) \in \mathcal{B}_X$ where $\mu(X_i) = m_i$.

¹Traditionally, the MK problem deals with minimization of the cost. In the current setting it is more natural to talk about maximization. The two options are, of course, equivalent under a sign change of the cost c.

The optimal transport plan T is then reduced to an *optimal partition*² of Xwithin the class

$$\mathcal{P}_{\vec{m}} := \{ \overrightarrow{X} := (X_1, \dots, X_N) ; \quad X_i \in \mathcal{B}_X \\ \cup_1^N X_i = X , \quad \mu(X_i \cap X_j) = 0 \text{ if } i \neq j, \quad \mu(X_i) = m_i \} .$$
(4)

In the above case we can replace $c : X \times Y \to \mathbb{R}$ by N measurable functions $\phi_i : X \to \mathbb{R}$ via $\phi_i(x) := c(x, y_i)$. The semi-discrete (or *optimal* partition) Monge problem of maximizing (1, 2) takes the form

$$\Xi_{\phi}^{*}(\vec{m}) := \sup_{\vec{X}} \left\{ \sum_{1}^{N} \int_{X_{i}} \phi_{i}(x) d\mu \; ; \quad \vec{X} \in \mathcal{P}_{\vec{m}} \right\}, \tag{5}$$

where, again, $\vec{m} \in S_I$.

This paper generalizes the concept of optimal partition in three directions to be described below.

1.1.1 Individual prices

Let $\vec{\psi} := (\psi_1 \dots \psi_N)$ where $\psi_i : X \to \mathbb{R}$ are measurable functions on (X, \mathcal{B}_X) . Let

$$\mathcal{P}_{\vec{m}}^{\vec{\psi}} := \{ \overrightarrow{X} := (X_1, \dots, X_N) ; \quad X_i \in \mathcal{B}_X \\ \cup_1^N X_i = X , \quad \mu(X_i \cap X_j) = 0 \text{ if } i \neq j, \quad \int_{X_i} \psi_i d\mu = m_i \} \quad (6)$$

and set

$$S_{I}^{\psi} := \left\{ \vec{m} \in \mathbb{R}^{N} \; ; \; \mathcal{P}_{\vec{m}}^{\vec{\psi}} \neq \emptyset \right\} \; . \tag{7}$$

The generalized optimal partition problem (5) takes the form of

$$\Xi_{\phi}^{*}(\vec{m}) := \sup_{\vec{X}} \left\{ \sum_{1}^{N} \int_{X_{i}} \phi_{i}(x) d\mu \; ; \; \vec{X} \in \mathcal{P}_{\vec{m}}^{\vec{\psi}} \right\}$$
(8)

where $\vec{m} \in S_I^{\psi}$. ² See [9].

1.1.2 Subpartition

The definition of $\mathcal{P}_{\vec{m}}^{\vec{\psi}}$ requires the partition to exhaust the space $X = \bigcup_{1}^{N} X_{i}$. We extend the set of partitions $\mathcal{P}_{\vec{m}}^{\vec{\psi}}$ to sub partitions where $\bigcup_{1}^{N} X_{i} \subseteq X$:

$$\underline{\mathcal{P}}_{\vec{m}}^{\vec{\psi}} := \{ \overrightarrow{X} := (X_1, \dots, X_N) ; \quad X_i \in \mathcal{B}_X \\ \cup_1^N X_i \subseteq X , \quad \mu(X_i \cap X_j) = 0 \text{ if } i \neq j, \quad \int_{X_i} \psi_i d\mu = m_i \}$$
(9)

and, respectively,

$$\underline{S}_{I}^{\psi} := \left\{ \vec{m} \in \mathbb{R}^{N} \; ; \; \underline{\mathcal{P}}_{\vec{m}}^{\vec{\psi}} \neq \emptyset \right\} \; , \tag{10}$$

$$\Xi_{\phi}^{+*}(\vec{m}) := \sup_{\vec{X}} \left\{ \sum_{1}^{N} \int_{X_{i}} \phi_{i}(x) d\mu \; ; \quad \vec{X} \in \underline{\mathcal{P}}_{\vec{m}}^{\vec{\psi}} \right\} \quad , \quad \vec{m} \in \underline{S}_{I}^{\psi} \tag{11}$$

1.1.3 Optimal selections

To motivate the above we consider the following cooperative game:

Let $\{X, \mathcal{B}, \mu\}$ be a probability measure space (the "cake").

For each agent $i \in \{1 \dots N\}$ and $x \in X$ we associate the price $\psi_i(x) \in \mathbb{R}$ of purchase of x by the the agent i.

Let $C_i \geq 0$ be the *capital* of agent *i*, we set $\overrightarrow{C} = \{C_1 \dots C_N\} \in \mathbb{R}^N$. An *affordable share* for *i* is a part of the cake $X_i \in \mathcal{B}$ such that $\int_{X_i} \psi_i d\mu \leq C_i$. An *admissible partition* of *X* is defined as a partition of *X* into *N* essentially disjoint affordable shares of the agents $\overrightarrow{X} := (X_1, \dots, X_N)$, that is

$$\mu(X_i \cap X_i) = 0 \quad \text{if} \quad i \neq j \quad ; \quad \int_{X_i} \psi_i d\mu \le C_i \quad , \ \cup_1^N X_i \subseteq X \; .$$

More generally, let $K \subset \mathbb{R}^N$ be a closed set. The set of subpartitions $\underline{\mathcal{P}}_K^{\psi}$ is defined by

$$\underline{\mathcal{P}}_{K}^{\psi} := \bigcup_{\vec{m} \in K} \underline{\mathcal{P}}_{\vec{m}}^{\vec{\psi}} .$$

$$(12)$$

For each agent i and $x \in X$ we associate the *profit* $\phi_i(x)$ of x for this agent. Again $\phi_i : X \to \mathbb{R}$ are measurable functions. The profit of agent i under a given partition is

$$F_i(X_i) := \int_{X_i} \phi_i(x) d\mu$$
.

The total profit of all agents is

$$\mathcal{F}_N(\overrightarrow{X}) := \sum_1^N F_i(X_i) \; .$$

The object of the game is to maximize the total profit, that is,

$$\max_{\overrightarrow{X}} \mathcal{F}_N(\overrightarrow{X}) \tag{13}$$

over all admissible partitions subjected in $\underline{\mathcal{P}}_{K}^{\psi}$.

The paradigm for the selection problem is as follows:

1. Maximize the function

$$\vec{m} \mapsto \Xi_{\phi}^{+*}(\vec{m}) \tag{14}$$

where Ξ_{ϕ}^{+*} given by (11), on $\underline{S}_{I}^{\psi} \cap K$.

2. For a maximizer \vec{m} of (14), evaluate the optimal subpartitions \vec{X} realizing the maximum (13) within $\underline{\mathcal{P}}_{m}^{\psi}$.

1.2 Description of main results

Obviously, if all prices ψ_i are identical (say $\psi_i \equiv 1$) then the set S_I^{ψ} is just the simplex S_I (3). In that case (8) is reduced into the semi discrete Monge problem (5).

Since the semi-discrete Monge problem is a special case of the Monge problem, a lot is known on its solvability and uniqueness. The essential condition for solvability and uniqueness of the classical Monge problem is the *twist condition* which, in the present case (and for a smooth ϕ_i on a smooth manifold X) takes the form

$$\phi_i - \phi_j$$
 has no critical point $\forall i \neq j$. (15)

see [4, 7, 6, 9].

The twist condition for non-smooth ϕ_i an abstract topological measure space X takes the form

$$\mu (x \in X ; \phi_i(x) - \phi_j(x) = r) = 0$$
(16)

for any $i \neq j \in I$ and for any $r \in \mathbb{R}$ (Section 4.3, Theorem 4.3).

The generalization for this in the case of individual price takes the form

$$\mu\left(x \in X ; \phi_i(x) - \phi_j(x) = \alpha \psi_i(x) - \beta \psi_j(x)\right) = 0 \tag{17}$$

for any $i \neq j \in I$ and any $\alpha, \beta \in \mathbb{R}$ (Theorem 4.1, Section 4.2). Indeed, (17) is reduced to (16) where $\vec{\psi}$ is a constant.

In the case of subpartitions we need an additional assumption to guarantee the unique solvability, namely

$$\mu \left(x \in X ; \phi_i(x) = \alpha \psi_i(x) \right) = 0 \tag{18}$$

for any $\alpha \in \mathbb{R}$ and any $i \in I$. (Theorem 4.1-(ii), Section 4.2). In particular, we need the condition

$$\mu (x \in X ; \phi_i(x) = r) = 0 \tag{19}$$

for any $r \in \mathbb{R}$ and any $i \in I$, in addition to (16) to obtain the unique solvability of the subpartition version of the Monge problem. (Corollary 4.2, Section 4.3).

In contrast, (17, 18) are not enough, in general, for the unique solvability in the general case. The additional condition

$$\mu \left(x \in X ; \alpha \psi_i(x) - \beta \psi_j(x) = 0 \right) = 0 \tag{20}$$

for any $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 > 0$ and $i \neq j \in I$, together with (17, 18), are enough to guarantee the unique solvability of the problems introduced above (sec. 1.1.1-1.1.3).

1.3 Structure of the paper

In Section 2 we relax the notion of (sub)partition to that of a weak (sub)partition. In Theorem 2.1, section 2.2, we prove that the weak (sub)partition and strong (sub)partition sets are the same. In section 2.3 we characterize these sets using a dual formalism.

Section 3 deals with optimal weak (sub)partitions. In section 3.1 we set up the condition for the existence of optimal weak (sub)partitions and prove the existence of such subpartition for the selection problem (Theorem 3.1). In sections 3.2 and 3.3 we use the dual formulation to characterize the optimal weak sub(partition) (Theorem 3.2).

In Section 4 we discuss strong (sub)partitions. Section 4.1 sets up the assumption (20) for the existence of unique strong partition for any \vec{m} in the boundary of the (sub)partition set S_I^{ψ} , (\underline{S}_I^{ψ}) , in Proposition 4.1. In section 4.2 we prove the main result for uniqueness of optimal strong (sub)partition-Theorem 4.1, and for the optimal selection (14) where K is a convex set - Theorem 4.2. Finally, in section 4.3 we discuss the Monge selection problem in light of the above results and prove the uniqueness of an optimal subpartition for the Monge problem under conditions (16, 19), in Corollary 4.2.

1.4 Notations and conventions

- i) Unless otherwise stated explicitly, any assumption cited below is valid form its citation point to the rest of the text.
- ii) $I := \{1 \dots N\}$. $\mathbb{R}^I := \mathbb{R}^N$.
- iii) For $J \subseteq I$, $\mathbb{R}^J = \{ \vec{p} = (p_1, \dots p_N) \in \mathbb{R}^N : p_i = 0 \text{ if } i \notin J \}.$
- iv) The partial order relation $\vec{p} > \vec{q} \ (\vec{p} \ge \vec{q})$ on \mathbb{R}^J means $p_i > q_i \ (p_i \ge q_i)$ for any $i \in J$.
- $\mathbf{v})\ \mathbb{R}^J_+:=\{\vec{p}\in\mathbb{R}^J\ ; \vec{p}\geq\vec{0}\}.$
- vi) (X, \mathcal{B}, μ) is a compact Polish space \mathcal{B} is the Borel- σ algebra and μ is a Borel non-atomic measure.
- vii) $\vec{\psi} := (\psi_1 \dots \psi_N) \in C(X; \mathbb{R}^N).$
- viii) $\vec{\mu} := (\mu_1, \dots, \mu_N)$ where μ_i are non-negative Borel measures on $\mathcal{B}(X)$.

$$\begin{aligned} &\text{ix}) \ \mathcal{P}_{\vec{m}}^{\vec{\psi},w} := \left\{ \vec{\mu} \ ; \quad \int_{\Omega} \psi_i d\mu_i = m_i \ , \ \sum_1^N \mu_i = \mu \right\} \\ &\text{x}) \ \underline{\mathcal{P}}_{\vec{m}}^{\vec{\psi},w} := \left\{ \vec{\mu} \ ; \quad \int_{\Omega} \psi_i d\mu_i = m_i \ , \ \sum_1^N \mu_i \le \mu \right\} \\ &\text{xi}) \ S_I^{\vec{\psi},\vec{w}} := \left\{ \vec{m} \in \mathbb{R}^I \ ; \quad \mathcal{P}_{\vec{m}}^{w,\vec{\psi}} \neq \emptyset \right\} \\ &\text{xii}) \ \underline{S}_I^{\vec{\psi},\vec{w}} := \left\{ \vec{m} \in \mathbb{R}^I \ ; \ \underline{\mathcal{P}}_{\vec{m}}^{w,\vec{\psi}} \neq \emptyset \right\} . \end{aligned}$$

2 Weak (sub)partitions

2.1 Back to Kantorovich

The Monge problem (1, 2) is relaxed into the *Kantorovich problem* as follows: maximize of the linear functional

$$\int_{X \times Y} c(x, y) d\pi(x, y) : \Pi(\mu, \nu) \to \mathbb{R}$$

where $\Pi(\mu, \eta)$ is the convex set of measures on $X \times Y$ whose marginals are μ, ν , that is

$$\pi(A \times Y) = \mu(A) \quad ; \quad \pi(X \times B) = \nu(B)$$

for all measurable sets $A \in \mathcal{B}_X(B \in \mathcal{B}_Y)$.

Again, in the special case where Y is a discrete space $Y = \{y_1, \dots, y_N\}$ and $\nu(\{y_i\}) := m_i \ge 0$, the set $\Pi(\mu, \nu)$ is reduced into the set of decompositions of the measure μ into n non-negative measures

$$\mathcal{P}_{\vec{m}}^{w} := \left\{ \vec{\mu} := (\mu_{1}, \dots \mu_{N}) ; \quad \int_{X} d\mu_{i} = m_{i} , \sum_{1}^{N} \mu_{i} = \mu \right\} ,$$

Indeed, $\pi \in \Pi(\mu, \nu)$ iff $\pi = \sum_{1}^{N} \mu_i \delta_{\{y_i\}}$ where $\vec{\mu} \in \mathcal{P}_{\vec{m}}^w$. Note that the set of partitions $\mathcal{P}_{\vec{m}}$ can be embedded in $\mathcal{P}_{\vec{m}}^w$ by identifying a set $X_i \in \mathcal{B}$ with the measure μ restricted to X_i , that is, $\mu_i := \mu \lfloor X_i$, whence $\int_{X_i} d\mu = \int_X d\mu_i.$ In the same way we consider the set of *relaxed* (weak) partitions corre-

sponding to ψ . Here (6) is generalized into

$$\mathcal{P}_{\vec{m}}^{\vec{\psi},w} := \left\{ \vec{\mu} := (\mu_1, \dots \mu_N) ; \quad \int_{\Omega} \psi_i d\mu_i = m_i \ , \ \sum_{1}^{N} \mu_i = \mu \right\} .$$
(21)

where, again, $\vec{m} \in \mathbb{R}^{I}$. Let also S_{I}^{ψ} (7) generalized into

$$S_{I}^{\vec{\psi,w}} := \left\{ \vec{m} \in \mathbb{R}^{I} \; ; \; \mathcal{P}_{\vec{m}}^{w,\vec{\psi}} \neq \emptyset \right\} \; . \tag{22}$$

Naturally, (8) is generalized into

$$\Xi_{\phi,w}^*(\vec{m}) := \sup_{\vec{\mu}} \left\{ \sum_{1}^N \int_X \phi_i(x) d\mu_i \quad ; \quad \vec{\mu} \in \mathcal{P}_{\vec{m}}^{w,\vec{\psi}} \right\}$$
(23)

and $\Xi_{\phi,w}^*(\vec{m}) = -\infty$ iff $\vec{m} \notin S_I^{\psi,w}$. In analogy to (9-10) we also define the *weak subpartition*

$$\underline{\mathcal{P}}_{\vec{m}}^{\vec{\psi},w} := \left\{ \vec{\mu} := (\mu_1, \dots \mu_N) ; \quad \int_{\Omega} \psi_i d\mu_i = m_i \quad , \quad \sum_1^N \mu_i \le \mu \right\}$$
(24)

and

$$\underline{S}_{I}^{\vec{\psi,w}} := \left\{ \vec{m} \in \mathbb{R}^{I} \; ; \; \underline{\mathcal{P}}_{\vec{m}}^{\vec{\psi,w}} \neq \emptyset \right\} \; , \tag{25}$$

$$\Xi_{\phi,w}^+(\vec{m}) := \sup_{\vec{\mu}} \left\{ \sum_{1}^N \int_X \phi_i(x) d\mu_i \; ; \; \vec{\mu} \in \underline{\mathcal{P}}_{\vec{m}}^{w,\vec{\psi}} \right\}$$
$$= -\infty \text{ iff } \vec{m} \notin S_{\vec{v},w}^{\vec{\psi},w}.$$

 $\Xi_{\phi,w}^+(\vec{m}) = -\infty \text{ iff } \vec{m} \notin \underline{S}_N^{\vec{\psi}}$

Since, as remarked above, any (sub)partition $\overrightarrow{X} \in \mathcal{P}_{\vec{m}}^{\vec{\psi}}$ ($\overrightarrow{X} \in \underline{\mathcal{P}}_{\vec{m}}^{\vec{\psi}}$) induces a weak (sub)partition $\vec{\mu} \in \mathcal{P}_{\vec{m}}^{\vec{\psi},w}$ ($\vec{\mu} \in \underline{\mathcal{P}}_{\vec{m}}^{\vec{\psi},w}$) via $\mu_i := \mu \lfloor X_i$ it follows

$$S_I^{\psi} \subseteq S_I^{\psi, \overline{w}} , \quad \underline{S}_I^{\psi} \subseteq \underline{S}_I^{\overline{\psi}, w} .$$
 (26)

2.2 Properties of the partition set

Lemma 2.1. The sets S_I^{ψ} , $\underline{S}_I^{\psi,w} \subset \mathbb{R}^I$ are compact and convex.

Proof. Since $|m_i| = |\int_X \psi_i d\mu_i| \le ||\psi_i||_{\infty} \int_X d\mu = ||\psi_i||_{\infty}$, so $S_I^{\psi, w}$ is bounded. Compactness follows from the weak- C^* compactness of the set of probability measures on a compact set. Convexity follows directly from the definition.

Recalling the definition of the strong (sub)partition sets (7, 10) we now prove

Theorem 2.1.

$$S_I^{\psi} = S_I^{\psi, w}$$
 and $\underline{S}_I^{\psi} = \underline{S}_I^{\psi, w}$

Proof. We have to prove the opposite inclusion of (26). If $\vec{m} \in S_I^{\psi,w}$, consider the set of weak partitions $\mathcal{P}_{\vec{m}}^{\vec{\psi},w}$. By Radom-Nikodym Theorem, any $\vec{\mu} = (\mu_1, \dots, \mu_N) \in \mathcal{P}_{\vec{m}}^{\vec{\psi},w}$ is characterized by $\vec{h} = (h_1, \dots, h_N)$ where h_i are μ -measurable functions, $\mu_i = h_i \mu$, satisfying $0 \leq h_i \leq 1$ on X. Moreover we have $\sum_{1}^{N} h_i = 1 \mu$ -a.e on X. Now $\mathcal{P}_{\vec{m}}^{\vec{\psi},w}$ is convex and compact in the weak topology. By Krein-Milman Theorem there exists an exposed point of $\mathcal{P}_{\vec{m}}^{\vec{\psi},w}$. We show that for an exposed point, $h_i \in \{0,1\}$ μ -a.e on X, for all $i \in I$.

Assume a set $D \subset X$ on which both $h_1 > \epsilon$ and $h_i > \epsilon$ for some $i \neq 1$. Since $h_1 + h_i \in [0, 1]$ it follows also that h_1 , h_i are smaller than $1 - \epsilon$ on D as well. Using Lyapunov partition theorem [5] we can find a subset $C \subset D$ such that $\int_C \psi_1 d\mu_1 = \int_D \psi_1 d\mu_1/2$ and $\int_C \psi_i d\mu_i = \int_D \psi_i d\mu_i/2$. Set $w := \mathbf{1}_D - 2\mathbf{1}_C$ where $\mathbf{1}_A$ stands for the indicator function of a measurable set $A \subset X$. It follows that w is supported on D, $||w||_{\infty,D} = 1$ and $\int_X w\psi_1 d\mu_1 = \int_X w\psi_i d\mu_i = 0$. By assumption, $h_1(x) \pm \epsilon w(x) \in [0, 1]$ and $h_i(x) \pm \epsilon w(x) \in [0, 1]$ for any $x \in D$. Set

$$\vec{\mu}_1 := (\mu_1 + \epsilon w \mu, \mu_2, \dots, \mu_i - \epsilon w \mu, \dots, \mu_N)$$
 and

 $\vec{\mu}_2 := (\mu_1 - \epsilon w \mu, \mu_2, \dots, \mu_i + \epsilon w \mu, \dots, \mu_N)$. Then both $\vec{\mu}_1, \vec{\mu}_2$ are in $\mathcal{P}_{\vec{m}}^{\vec{\psi}, w}$ and $\vec{\mu} = \frac{1}{2}\vec{\mu}_1 + \frac{1}{2}\vec{\mu}_2$. This is in contradiction to the assumption that μ is an exposed point. It follows that either $h_i = 0$ or $h_1 = 0$ μ -a.e. Since i is arbitrary and $\sum_I h_j = 1$ μ -a.e. it follows that $h_1 \in \{0,1\}$ μ -a.e. hence $h_j \in \{0,1\}$ for any $j \in I$ μ -a.e. The proof $\underline{S}_I^{\vec{\psi}} = \underline{S}_I^{\vec{\psi},w}$ follows identically. \Box

2.3 Dual representation of weak (sub)partitions

Let now, for $\vec{p} = (p_1, \dots, p_N) \in \mathbb{R}^I$

$$\xi_0(x, \vec{p}) := \max_{i \in I} p_i \psi_i(x) : X \times \mathbb{R}^I \to \mathbb{R}$$
(27)

$$\xi_0^+(x,\vec{p}) := \max(\xi_0(x,\vec{p}), 0) \tag{28}$$

$$\Xi_0(\vec{p}) := \int_X \xi_0(x, \vec{p}) d\mu : \mathbb{R}^I \to \mathbb{R} \quad .$$
⁽²⁹⁾

$$\Xi_0^+(\vec{p}) := \int_X \xi_0^+(x, \vec{p}) d\mu(x) \ . \tag{30}$$

Theorem 2.2. $\vec{m} \in S_I^{\psi}$ (res. $\vec{m} \in \underline{S}_I^{\psi}$) if and only if

a)
$$\Xi_0(\vec{p}) - \vec{m} \cdot \vec{p} \ge 0$$
 ; b) res. $\Xi_0^+(\vec{p}) - \vec{m} \cdot \vec{p} \ge 0$ (31)

for any $\vec{p} \in \mathbb{R}^I$. Here $\vec{m} \cdot \vec{p} := \sum_1^N p_i m_i$

Corollary 2.1. \vec{m} is an inner point of S_I^{ψ} (res. \underline{S}_I^{ψ}) iff $\vec{p} = 0$ is a strict minimizer of (31-a) (res (31-b)).

Proof of Corollary 2.1: Since $\Xi_0(\vec{p}) - \vec{m} \cdot \vec{p}$ is an homogeneous function and $\Xi_0(\vec{0}) = 0$, it follows that $\vec{0}$ is a minimizer of (31) for any $\vec{m} \in S_I^{\psi}$. If it is a strict minimizer then $\Xi_0(\vec{p}) - \vec{m} \cdot \vec{p} > 0$ for any $\vec{p} \neq 0$ hence there exists a neighborhood of \vec{m} for which $\Xi_0(\vec{p}) - \vec{m'} \cdot \vec{p} > 0$ for any $\vec{m'}$ in this neighborhood, so $\vec{m'} \in S_I^{\psi}$ by Theorem 2.2. Otherwise, there exists $\vec{p}_0 \neq 0$ for which $\Xi_0(\vec{p}_0) - \vec{m} \cdot \vec{p}_0 < 0$ for any $\vec{m'}$ for which $\Xi_0(\vec{p}_0) - \vec{m} \cdot \vec{p}_0 < 0$ for any $\vec{m'}$ for which $(\vec{m} - \vec{m'}) \cdot \vec{p}_0 < 0$. By Theorem 2.2 it follows that $\vec{m'} \notin S_I^{\psi}$ so \vec{m} is not an inner point of S_I^{ψ} . The second case is proved similarly. \Box

The set S_I^{ψ} may contain inner points. As an example, consider the case where N = 2, ψ_1 and ψ_2 are continuous, positive functions and there exists pair of point $x, y \in X$ such that $\psi_2(x) - \psi_1(x) = \psi_1(y) - \psi_2(y) > 0$. If x is in the support of μ_1 and y in the support of μ_2 then S_I^{ψ} contains an interior point. Indeed, we can move a neighborhood of x from 1 to 2, and a neighborhood of y from 2 to 1. This way we increased both m_1 and m_2 to obtain $(m'_1, m'_2) \in S_I^{\psi}$ satisfying $m'_1 > m_1$ and $m'_2 > m_2$. On the other hand we can evidently increase one of them (say m_1) while decreasing m_2 by transferring a mass from 2 to 1. Then we obtain $(m_1'', m_2'') \in S_I^{\psi}$ satisfying $m_1'' > m_1$ and $m_2'' < m_2$. By convexity, S_I^{ψ} contains the triangle whose vertices are $(m_1, m_2), (m_1', m_2'), (m_1'', m_2'')$, and in particular an interior point.

To prove Theorem 2.2 we need some auxiliary lemmas:

Lemma 2.2. Ξ_0 and Ξ_0^+ are convex functions on \mathbb{R}^I .

Proof. By definition, ξ_0 and ξ_0^+ are convex function in \vec{p} for any $x \in X$. Hence Ξ_0, Ξ_0^+ are convex as well from definition (29, 30).

Lemma 2.3. If $\vec{m} \in S_I^{\psi}$ then

$$\Xi_0(\vec{p}) - \vec{m} \cdot \vec{p} \ge 0$$

for any $\vec{p} \in \mathbb{R}^{I}$. Likewise, if $\vec{m} \in \underline{S}_{I}^{\psi}$ then

$$\Xi_0^+(\vec{p}) - \vec{m} \cdot \vec{p} \ge 0$$

for any $\vec{p} \in \mathbb{R}^I$.

Proof. Assume $\vec{m} \in S^{\psi}$. Since $S_I^{\psi} = S_I^{\psi, \vec{w}}$ by Theorem 2.1 then, by definition, there exists $\vec{\mu} \in \mathcal{P}_m^{\psi, w}$ such that $\int_X \psi_i d\mu_i = m_i$. Also, from (21) $(\sum_{i=1}^{N} \mu_i = \mu)$ and (29)

$$\Xi_0(\vec{p}) = \int_X \xi_0(x, \vec{p}) d\mu = \sum_1^N \int_X \xi_0(x, \vec{p}) d\mu_i$$

while from (28) $\xi_0(x, \vec{p}) \ge p_i \psi_i(x)$ so

$$\Xi_0(\vec{p}) \ge \sum_1^N p_i \int_X \psi_i d\mu_i = \vec{p} \cdot \vec{m} \; .$$

The case for Ξ_0^+ is proved similarly.

In order to prove the second direction of Theorem 2.2 we need the following definition of *regularized maximizer*:

Definition 2.1. Let $\vec{a} \in \mathbb{R}^{I}$. Then, for $\epsilon > 0$,

$$max_{\epsilon}(\vec{a}) := \epsilon \ln\left(\sum_{i \in I} e^{a_i/\epsilon}\right)$$

Lemma 2.4. For any $\epsilon > 0$ max_{ϵ}(·) is a smooth convex function on \mathbb{R}^{I} . In addition $\max_{\epsilon_{1}}(\vec{a}) \geq \max_{\epsilon_{2}}(\vec{a}) \geq \max_{i \in I}(a_{i})$ for any $\vec{a} \in \mathbb{R}^{I}$, $\epsilon_{1} > \epsilon_{2} > 0$ and

$$\lim_{\epsilon \searrow 0} \max_{\epsilon}(\vec{a}) = \max_{i \in I} a_i .$$
(32)

Proof. Follows from

$$max_{\epsilon}(\vec{a}) = \max_{\vec{\beta}} \left\{ -\epsilon \sum_{1}^{N} \beta_i \ln \beta_i + \vec{\beta} \cdot \vec{a} \right\}$$
(33)

where the maximum is taken on the simplex $0 \leq \vec{\beta}, \ \vec{\beta} \cdot \vec{1} = 1$. Note that the maximizer is

$$\beta_i^0 = \frac{e^{a_i/\epsilon}}{\sum_j e^{a_j/\epsilon}} < 1$$

for $i \in I$. Since $\sum_{1}^{N} \beta_i \ln \beta_i \leq 0$, the term in brackets in (33) is monotone non-decreasing in $\epsilon > 0$. Finally, (32) follows from the Jensen's inequality via $-\sum_{1}^{N} \beta_i \ln \beta_i \leq \ln N$.

Definition 2.2.

$$\xi_{\epsilon}(x,\vec{p}) := max_{\epsilon} \left(p_1\psi_1(x), \dots p_N\psi_N(x) \right) : X \times \mathbb{R}^I \to \mathbb{R}$$
(34)

$$\Xi_{\epsilon}(\vec{p}) := \int_{X} \xi_{\epsilon}(x, \vec{p}) d\mu : \mathbb{R}^{I} \to \mathbb{R} \quad .$$
(35)

Also, for each $\vec{p} \in \mathbb{R}^{I}$ and $i \in I$ set

$$\mu_i^{(\vec{p})}(dx) := \frac{e^{p_i \psi_i(x)/\epsilon}}{\sum_{j \in I} e^{p_j \psi_j(x)/\epsilon}} \mu(dx)$$
(36)

Likewise

$$\xi_{\epsilon}^{+}(x,\vec{p}) := \max_{\epsilon} \left(p_1 \psi_1(x), \dots p_N \psi_N(x), 0 \right) : X \times \mathbb{R}^I \to \mathbb{R}$$
(37)

$$\Xi_{\epsilon}^{+}(\vec{p}) := \int_{X} \xi_{\epsilon}^{+}(x, \vec{p}) d\mu : \mathbb{R}^{I} \to \mathbb{R} \quad .$$
(38)

and

$$\mu_i^{(\vec{p},+)}(dx) := \frac{e^{p_i \psi_i(x)/\epsilon}}{1 + \sum_{j \in 1} e^{p_j \psi_j(x)/\epsilon}} \mu(dx)$$
(39)

Since \max_{ϵ} is smooth and convex due to lemma 2.4 it follows from the above definition via an explicit differentiation.

Lemma 2.5. For each $\epsilon > 0$, Ξ_{ϵ} (res. Ξ_{ϵ}^+) is a convex and C^{∞} on \mathbb{R}^I . In addition

$$\frac{\partial \Xi_{\epsilon}(\vec{p})}{\partial p_{i}} = \int_{X} \psi_{i}(x) d\mu_{i}^{(\vec{p})} \quad res. \quad \frac{\partial \Xi_{\epsilon}^{+}(\vec{p})}{\partial p_{i}} = \int_{X} \psi_{i}(x) d\mu_{i}^{(\vec{p},+)}$$

The proof of Theorem 2.2 follows from the following Lemma

Lemma 2.6. For any $\epsilon, \delta > 0$ and $\vec{m} \in \mathbb{R}^{I}$

$$\vec{p} \to \Xi_{\epsilon}(\vec{p}) + \frac{\delta}{2} |\vec{p}|^2 - \vec{m} \cdot \vec{p}$$
 (40)

is a strictly convex function on \mathbb{R}^{I} . In addition

$$\Xi_{\epsilon}(\vec{p}) + \frac{\delta}{2} |\vec{p}|^2 - \vec{m} \cdot \vec{p} \ge \Xi_0(\vec{p}) - \vec{m} \cdot \vec{p} + \frac{\delta}{2} |\vec{p}|^2 \tag{41}$$

so, if (31) is satisfied, then $\vec{p} \to \Xi_{\epsilon}(\vec{p}) + \frac{\delta}{2} |\vec{p}|^2 - \vec{m} \cdot \vec{p}$ is a coercive function as well. The same statement holds for Ξ_{ϵ}^+ as well.

Proof of Theorem 2.2

From Lemma 2.6 we obtain at once the existence of a minimizer $\vec{p}_{\epsilon,\delta} \in \mathbb{R}^I$ of (40) for any $\epsilon, \delta > 0$, provided (31) holds. Moreover, from Lemma 2.5 we also get for that minimizer $\vec{p}^{\epsilon,\delta}$ satisfies

$$m_i = \frac{\partial \Xi_{\epsilon}}{\partial p_i^{\epsilon,\delta}} + \delta p_i^{\epsilon,\delta} = \int_X \psi_i d\mu_i^{(p^{\epsilon,\delta})} + \delta p_i^{\epsilon,\delta} \tag{42}$$

By convexity of Ξ_{ϵ} :

$$\nabla \Xi_{\epsilon}(\vec{p}) \cdot \vec{p} \ge \Xi_{\epsilon}(\vec{p}) - \Xi_{\epsilon}(\vec{0})$$

Multiply (42) by $\vec{p}^{\epsilon,\delta}$ to obtain

$$\vec{p}^{\epsilon,\delta} \cdot \nabla \Xi(\vec{p}^{\epsilon,\delta}) + \delta \left| \vec{p}^{\epsilon,\delta} \right|^2 - \vec{m} \cdot \vec{p}^{\epsilon,\delta} = 0 \ge \Xi_{\epsilon}(\vec{p}^{\epsilon,\delta}) - \Xi_{\epsilon}(\vec{0}) + \delta \left| \vec{p}^{\epsilon,\delta} \right|^2 - \vec{m} \cdot \vec{p}^{\epsilon,\delta}$$
(43)

It follows from (31, 41, 43) that

$$-\Xi_{\epsilon}(\vec{0}) + \delta \left| \vec{p}^{\epsilon,\delta} \right|^2 \le 0$$

hence

$$\delta \left| \vec{p}^{\epsilon,\delta} \right| \leq \sqrt{\delta} \sqrt{\Xi_{\epsilon}(\vec{0})} \; .$$

Hence (42) implies

$$\lim_{\delta \to 0} \int_X \psi_i d\mu_i^{(p^{\epsilon,\delta})} = m_i$$

By compactness of $C^*(X)$ and since $\sum_1^N \mu_i^{(p^{\epsilon,\delta})} = \mu$ via (36) we can choose a subsequence $\delta \to 0$ along which the limits

$$\lim_{\delta \to 0} \mu_i^{(p^{\epsilon,\delta})} := \mu_i^{\epsilon}$$

holds. It follows that

$$\sum_{1}^{N} \mu_i^{(p^{\epsilon})} = \mu \quad ; \quad \int_X \psi_i d\mu_i^{\epsilon} = m_i \; .$$

Again, the proof for $\vec{m} \in \underline{S}_I^{\psi}$ is analogous. \Box

3 Weak optimal (sub)partitions

3.1 Existence and characterization of weak (sub)partitions

Let $K \subset \mathbb{R}^I$ be a closed set. Recall

$$\underline{\mathcal{P}}_{K}^{\psi} := \bigcup_{\vec{m} \in K} \underline{\mathcal{P}}_{\vec{m}}^{\psi} . \tag{44}$$

Assumption 3.1. The components of the function $\vec{\phi} = (\phi_1, \dots, \phi_N) : X \to \mathbb{R}^I$ are upper sami continuous (usc) and bounded on X.

Theorem 3.1. There exists a weak subpartition $\vec{\mu}$ which maximize the total profit $\int_X \vec{\phi} \cdot d\vec{\mu}$ in $\underline{\mathcal{P}}_K^{\psi}$.

The proof of Theorem 3.1 is almost immediate. Since ψ_i are continuous by standing assumption, the set $\underline{\mathcal{P}}_K^{\psi}$ is weakly closed. Since ϕ_i are u.s.c by Assumption 3.1, the limit of a maximizing sequence is a maximizer.

3.2 Dual representation

Our next object is to characterize the set of optimal (sub)partitions. For this we turn back to the dual formulation. Define the function $\xi_{\phi}: X \times \mathbb{R}^{I} \to \mathbb{R}$ as

$$\xi_{\phi}(x,\vec{p}) := \max \left\{ \phi_1(x) + p_1 \psi_1(x), \dots, \phi_N(x) + p_N \psi_N(x) \right\} .$$
(45)

Likewise

$$\xi_{\phi}^{+}(x,\vec{p}) := \max\left\{\phi_{1}(x) + p_{1}\psi_{1}(x), \dots, \phi_{N}(x) + p_{N}\psi_{N}(x), 0\right\}$$
(46)

Set

$$\Xi_{\phi}(\vec{p}) := \int_{X} \xi_{\phi}(x, \vec{p}) d\mu(x) : \mathbb{R}^{I} \to \mathbb{R}$$
(47)

$$\Xi_{\phi}^{+}(\vec{p}) := \int_{X} \xi_{\phi}^{+}(x, \vec{p}) d\mu(x) : \mathbb{R}^{I} \to \mathbb{R}$$

$$\tag{48}$$

and $\Xi_{\phi}^*, \, (\Xi_{\phi}^{+\,*}) : \mathbb{R}^I \to \mathbb{R} \cup \{-\infty\}$ as

$$\Xi_{\phi}^{*}(\vec{m}) = \inf_{\vec{p} \in \mathbb{R}^{I}} \left[\Xi_{\phi}(\vec{p}) - \vec{m} \cdot \vec{p} \right] \quad ; \quad \Xi_{\phi}^{+*}(\vec{m}) = \inf_{\vec{p} \in \mathbb{R}^{I}} \left[\Xi_{\phi}^{+}(\vec{p}) - \vec{m} \cdot \vec{p} \right] \tag{49}$$

for $\vec{m} \in \mathbb{R}^{I}$.

Recall that the essential domain of the concave function $F : \mathbb{R}^I \to \mathbb{R} \cup \{-\infty\}$ is the set $\{\vec{m}; F(\vec{m}) > -\infty\}$.

Lemma 3.1. Ξ_{ϕ}^* (res. Ξ_{ϕ}^{+*}) is a concave function on \mathbb{R}^I . The essential domain of Ξ_{ϕ}^* (res. Ξ_{ϕ}^{+*}) is S_I^{ψ} (res. \underline{S}_I^{ψ}).

Proof. Comparing the definitions of Ξ_{ϕ} to that of Ξ_{0} we obtain $\Xi_{\phi}(x,\vec{p}) - \|\vec{\phi}\|_{\infty} \leq \Xi_{0}(x,\vec{p}) \leq \Xi_{\phi}(x,\vec{p}) + \|\vec{\phi}\|_{\infty}$ for any $x \in X$ and any $\vec{p} \in \mathbb{R}^{I}$. It follows $\Xi_{0}(\vec{p}) - \|\vec{\phi}\|_{\infty} \leq \Xi_{\phi}(\vec{p}) \leq \Xi_{0}(\vec{p}) + \|\vec{\phi}\|_{\infty}$ for any $\vec{p} \in \mathbb{R}^{I}$ as well. It follows that $\Xi_{\phi}(\vec{p}) - \vec{p} \cdot \vec{m}$ is bounded from below iff $\Xi_{0}(\vec{p}) - \vec{p} \cdot \vec{m}$ is bounded from below. Note that $\Xi_{0}(\vec{p}) - \vec{p} \cdot \vec{m}$ is bounded from below on \mathbb{R}^{I} iff $\Xi_{0}(\vec{p}) - \vec{p} \cdot \vec{m} \geq 0$ on \mathbb{R}^{I} . Theorem 2.2 then implies that $\vec{m} \in S_{I}^{\psi}$ iff \vec{m} is in the essential domain of Ξ_{ϕ}^{*} . Same proof for Ξ_{ϕ}^{+*} .

3.3 From duality to optimal partition

We now investigate the sub-gradient of Ξ_{ϕ} and Ξ_{ϕ}^+ . Recall that $\vec{m} \in \partial_{\vec{p}} F$ iff

$$F(\vec{q}) - F(\vec{p}) \ge \vec{m} \cdot (\vec{q} - \vec{p})$$

for any $\vec{q} \in \mathbb{R}^I$.

Let us consider the positive simplex of measures

$$\underline{\mathcal{P}} := \left\{ \vec{\mu} = (\mu_1, \dots, \mu_N), \quad \mu_i \ge 0, \quad \sum_1^N \mu_i \le \mu \right\}$$
$$\mathcal{P} := \left\{ \vec{\mu} = (\mu_1, \dots, \mu_N), \quad \mu_i \ge 0, \quad \sum_1^N \mu_i = \mu \right\}$$

For each $\vec{\mu} \in \underline{\mathcal{P}}$ we consider the vector

$$\vec{m}(\vec{\mu}) := \left(\int \psi_1 d\mu_1, \dots \int \psi_N d\mu_N\right) \in \mathbb{R}^I .$$
(50)

Lemma 3.2. For any $\vec{p} \in \mathbb{R}^I$ there exists $\mathcal{P}_{\vec{p}} \subset \mathcal{P}, \ \mathcal{P}_{\vec{p}} \neq \emptyset$, (res. $\underline{\mathcal{P}}_{\vec{p}} \subset \underline{\mathcal{P}}, \ \underline{\mathcal{P}}_{\vec{p}} \neq \emptyset$,) such that

$$i) \ \vec{m} \in \partial_{\vec{p}} \Xi_{\phi} \ (res. \ \vec{m} \in \partial_{\vec{p}} \Xi_{\phi}^+) \ iff \ \vec{m} = \vec{m}(\vec{\mu}) \ for \ some \ \vec{\mu} \in \mathcal{P}_{\vec{p}} \ (res. \ \vec{\mu} \in \underline{\mathcal{P}}_{\vec{p}}).$$

ii) For any $\vec{\mu} \in \mathcal{P}_{\vec{p}}$ (res. $\vec{\mu} \in \underline{\mathcal{P}}_{\vec{p}}$), $\Xi_{\phi}(\vec{p}) = \vec{m}(\vec{\mu}) \cdot \vec{p} + \int_{X} \vec{\phi} \cdot d\vec{\mu}$ (res. $\Xi_{\phi}^{+}(\vec{p}) = \vec{m}(\vec{\mu}) \cdot \vec{p} + \int_{X} \vec{\phi} \cdot d\vec{\mu}$).

Proof. We present the proof for Ξ_{ϕ} . The proof for Ξ_{ϕ}^+ is analogous.

i) Let

$$\xi_{\phi}^{\epsilon}(x,\vec{p}) := max_{\epsilon} \left(p_{1}\psi_{1}(x) + \phi_{1}(x), \dots p_{N}\psi_{N}(x) + \phi_{N}(x) \right) : X \times \mathbb{R}^{I} \to \mathbb{R}$$

$$(51)$$

$$\Xi_{\phi}^{\epsilon}(\vec{p}) := \int_{X} \xi_{\phi}^{\epsilon}(x,\vec{p}) d\mu : \mathbb{R}^{I} \to \mathbb{R} \quad .$$

$$(52)$$

and

$$\mu_{\epsilon,i}^{\vec{\phi},\vec{p}}(dx) := \frac{\exp\left(\frac{p_i\psi_i(x) + \phi_i(x)}{\epsilon}\right)}{\sum_{j=1}^N \exp\left(\frac{p_j\psi_j(x) + \phi_j(x))}{\epsilon}\right)} \mu(dx) \quad , \quad i \in \{1\dots N\} \quad (53)$$

As in Lemma 2.5 we obtain that Ξ_{ϕ}^{ϵ} is a smooth, convex function and the sequence Ξ_{ϕ}^{ϵ} satisfies $\lim_{\epsilon \to 0} \Xi_{\phi}^{\epsilon} = \Xi_{\phi}$ pointwise. In addition, Lemma 2.4 also implies that this sequence is monotone decreasing. This implies, in particular, that $\Xi_{\phi}^{\epsilon} \to \Xi_{\phi}$ in the *Mosco- sense* (c.f. [1]). In addition

$$\frac{\partial \Xi_{\phi}^{\epsilon}(\vec{p})}{\partial p_{i}} = \int_{X} \psi_{i}(x) d\mu_{\epsilon,i}^{\vec{\phi},\vec{p}} .$$
(54)

By Theorem 3.66 in [1] it follows that $\partial \Xi_{\phi}^{\epsilon} \to \partial \Xi_{\phi}$ in the sense of G-convergence, that is:

$$\forall (\vec{p}, \vec{\zeta}) \in \partial \Xi_{\phi}, \ \exists (\vec{p}_{\epsilon}, \zeta_{\epsilon}) \in \partial \Xi_{\phi}^{\epsilon} \ , \ \vec{p}_{\epsilon} \to \vec{p} \text{ and } \vec{\zeta}_{\epsilon} \to \vec{\zeta} \text{ for } \epsilon \searrow 0.$$

Since $\partial_{\vec{p}}\Xi_{\phi}^{\epsilon} = \{\nabla_{\vec{p}}\Xi_{\phi}^{\epsilon}\}$ we obtain that $\vec{\zeta} \in \partial_{\vec{p}}\Xi_{\phi}$ iff there exists a sequence $\vec{p}_{\epsilon} \to \vec{p}$ and $\nabla_{\vec{p}_{\epsilon}}\Xi_{\phi}^{\epsilon} \to \vec{\zeta}$ as $\epsilon \to 0$. By (54)

$$\nabla_{\vec{p}_{\epsilon}} \Xi_{\phi}^{\epsilon} = \vec{m} \left(\overrightarrow{\mu_{\epsilon}^{\vec{\phi}, \vec{p}_{\epsilon}}} \right)$$

where $\overrightarrow{\mu_{\epsilon}^{\vec{\phi},\vec{p}_{\epsilon}}} = \left(\mu_{\epsilon,1}^{\vec{\phi},\vec{p}_{\epsilon}}, \dots, \mu_{\epsilon,N}^{\vec{\phi},\vec{p}_{\epsilon}}\right)$. Let $\mathcal{P}_{\vec{p}}$ be the sets of limits (in $C^*(X)$) of all sequences

$$\mu_{\epsilon}^{\vec{\phi},\vec{p}_{\epsilon}'}, \quad \epsilon \to 0$$

Since X is compact, $\mathcal{P}_{\vec{p}}$ is non-empty for any $\vec{p} \in \mathbb{R}^{I}$. In addition we obtain $\vec{m} \in \partial_{\vec{p}} \Xi_{\phi}$ iff there exists $\vec{\mu} \in \mathcal{P}_{\vec{p}}$ for which $\vec{m} = \vec{m}(\vec{\mu})$.

ii) By Lemma 2.4 with $a_i := p_{\epsilon,i} \vec{\psi}_i + \phi_i$ we obtain, after integration of $\max_{\epsilon}(\vec{a})$ over X with respect to μ :

Note that $\frac{d\mu_{\epsilon,i}^{\phi,\vec{p}_{\epsilon}}}{d\mu} \leq 1$ from (53). Taking the limit $\epsilon \to 0, \vec{p}_{\epsilon} \to \vec{p}$ we get

$$\Xi_{\phi}^{\epsilon}(\vec{p}_{\epsilon}) \to \Xi_{\phi}(\vec{p}) = \sum_{i \in I} p_i \int \psi_i d\mu_i + \int_X \vec{\phi} \cdot d\vec{\mu}$$
(55)

where

$$\vec{\mu} \in \{\lim_{\epsilon \to 0} \overrightarrow{\mu_{\epsilon}^{\vec{\phi}, \vec{p}_{\epsilon}}}\} \in \mathcal{P}_{\vec{p}}$$

and $\lim_{\epsilon \to 0} \vec{m} \left(\overrightarrow{\mu^{\phi, \vec{p}_{\epsilon}}} \right) = \vec{m}(\vec{\mu}) \in \partial_{\vec{p}} \Xi_{\phi}$. The limit (55) then takes the form

$$\Xi_{\phi}(\vec{p}) = \vec{m}(\vec{\mu}) \cdot \vec{p} + \int_X \vec{\phi} \cdot d\vec{\mu} \quad .$$

Again, the alternative case holds similarly.

Let $\widehat{\mathcal{P}}$ (res. $\underline{\widehat{\mathcal{P}}}$) be the weak (C^*) closure of the union of all $\mathcal{P}_{\vec{p}}$ (res. $\underline{\mathcal{P}}_{\vec{p}}$) for $\vec{p} \in \mathbb{R}^I$:

$$\widehat{\mathcal{P}} := \overline{\bigcup_{\vec{p} \in \mathbb{R}^I} \mathcal{P}_{\vec{p}}}^{C^*} \quad , \quad res. \ \underline{\widehat{\mathcal{P}}} := \overline{\bigcup_{\vec{p} \in \mathbb{R}^I} \underline{\mathcal{P}}_{\vec{p}}}^{C^*} \quad .$$
(56)

Lemma 3.3. For any $\vec{m} \in S_I^{\psi}$ (res. $\vec{m} \in \underline{S}_I^{\psi}$) there exists $\vec{\mu} \in \widehat{\mathcal{P}}$ (res. $\vec{\mu} \in \underline{\widehat{\mathcal{P}}}$) for which $\vec{m} = \vec{m}(\vec{\mu})$. In particular, this $\vec{\mu}$ is a maximizer of $\int_X \vec{\phi} \cdot d\vec{\mu}$ in $\mathcal{P}_m^{\psi,w}$ (res. $\underline{\mathcal{P}}_m^{\psi,w}$) and satisfies

$$\int_X \vec{\phi} \cdot d\vec{\mu} = \Xi_{\phi}^*(\vec{m}) \qquad res. \quad \int_X \vec{\phi} \cdot d\vec{\mu} = \Xi_{\phi}^{+*}(\vec{m}) \ .$$

Proof. Following the argument of Lemma 2.6, set

$$\vec{p} \to \Xi_{\phi}(\vec{p}) + \frac{\delta}{2} |\vec{p}|^2 - \vec{m} \cdot \vec{p}$$
(57)

for some $\delta > 0$. By Lemma 3.1, $\vec{m} \in S_I^{\psi}$ iff

$$\Xi_{\phi}(\vec{p}) + \frac{\delta}{2}|\vec{p}|^2 - \vec{m} \cdot \vec{p} \ge \Xi_{\phi}^*(\vec{p}) + \frac{\delta}{2}|\vec{p}|^2$$

so $\vec{p} \to \Xi_{\phi}(\vec{p}) + \frac{\delta}{2} |\vec{p}|^2 - \vec{m} \cdot \vec{p}$ is a convex coercive function. Hence there exists $\vec{p}_{\delta} \in \mathbb{R}^I$ which minimize (57),

$$\Xi_{\phi}(\vec{p}_{\delta}) + \frac{\delta}{2} |\vec{p}_{\delta}|^2 - \vec{m} \cdot \vec{p}_{\delta} = \min_{\vec{p} \in \mathbb{R}^I} \left[\Xi_{\phi}(\vec{p}) + \frac{\delta}{2} |\vec{p}|^2 - \vec{m} \cdot \vec{p} \right]$$
(58)

and

$$\vec{m} \in \partial_{\vec{p}_{\delta}} \Xi_{\phi} + \delta \vec{p}_{\delta} .$$
⁽⁵⁹⁾

By Lemma 3.2-(i) it follows that there exists $\vec{\mu}_{\delta} \in \mathcal{P}_{\vec{p}_{\delta}}$ for which $\vec{m} = \vec{m}(\vec{\mu}_{\delta}) + \delta \vec{p}_{\delta}$. We now proceed as in the proof of Theorem 2.2. By the definition of $\partial_p \Xi_{\phi}$:

$$\partial_{\vec{p}} \Xi_{\phi}(\vec{p}) \cdot \vec{p} \ge \Xi_{\phi}(\vec{p}) - \Xi_{\phi}(\vec{0})$$

Multiply (59) by \vec{p}_{δ} to obtain

$$\partial_{\vec{p}_{\delta}}\Xi_{\phi}\cdot\vec{p}_{\delta}+\delta\left|\vec{p}_{\delta}\right|^{2}-\vec{m}\cdot\vec{p}_{\delta}=0\geq\Xi_{\phi}(\vec{p}_{\delta})-\Xi_{\phi}(\vec{0})+\delta\left|\vec{p}_{\delta}\right|^{2}-\vec{m}\cdot\vec{p}_{\delta}\quad(60)$$

It follows from (58,60) that $\delta |\vec{p}^{\epsilon,\delta}|^2$ is bounded uniformly in $\delta > 0$, so $\delta |\vec{p}_{\delta}| \leq C\sqrt{\delta}$ for some C > 0 independent of δ . Hence (59) implies $\partial_{\vec{p}_{\delta}} \Xi_{\phi} \to \vec{m}$ as $\delta \to 0$. Hence $\vec{m}(\vec{\mu}_{\delta}) \to \vec{m}$. By compactness of $C^*(X)$ we can choose a subsequence $\delta \to 0$ along which $\vec{\mu}_{\delta}$ converges to some $\vec{\mu} \in \hat{\mathcal{P}}$ for which $\vec{m} = \vec{m}(\vec{\mu})$.

Theorem 3.2. There exists a maximizer of $\int_X \vec{\phi} \cdot d\vec{\mu}$ in $\mathcal{P}_m^{\psi,w}$ and any such maximizer is in $\widehat{\mathcal{P}}$. Likewise, there exists a maximizer of $\int_X \vec{\phi} \cdot d\vec{\mu}$ in $\underline{\mathcal{P}}_m^{\psi}$ and any such maximizer is in $\widehat{\underline{\mathcal{P}}}$.

Proof. First, any $\mu \in \mathcal{P}_m^{\psi,w}$ satisfies

$$\int_X \vec{\phi} \cdot d\vec{\mu} \le \Xi_\phi(\vec{p}) - \vec{p} \cdot \vec{m}$$

for any $\vec{p} \in \mathbb{R}^{I}$. Indeed, since

$$\phi_i(x) \le \xi(x, p) - p_i \psi_i(x) \quad i \in I$$

we get

$$\int_{X} \vec{\phi} \cdot d\vec{\mu} \leq \sum_{1}^{N} \int_{X} \xi(x, \vec{p}) d\mu_{i} - \sum_{1}^{N} p_{i} \int_{X} \psi_{i} d\mu_{i}$$
$$\leq \int_{X} \xi(x, \vec{p}) (\sum_{1}^{N} d\mu_{i}) - \vec{p} \cdot \vec{m} \leq \Xi_{\phi}(\vec{p}) - \vec{p} \cdot \vec{m} \leq \Xi_{\phi}^{*}(\vec{m}) . \quad (61)$$

Let now $\vec{m} \in S_I^{\psi}$ By Lemma 3.3 there exists $\vec{\mu} \in \widehat{\mathcal{P}}$ such that $\vec{m}(\vec{\mu}) = \vec{m}$. By definition (56) there exists a sequence $\vec{p}_n \in \mathbb{R}^I$ such that $\vec{\mu} = \lim_{n \to \infty} \vec{\mu}^n$ where $\vec{\mu}^n \in \mathcal{P}_{\vec{p}_n}$.

In particular, $\vec{m}_n := \vec{m}(\vec{\mu}^n) \to \vec{m}$. By lemma 3.2-(ii) we obtain that $\Xi_{\phi}(\vec{p}_n) = \vec{m}(\vec{\mu}^n) \cdot \vec{p}_n + \int_X \vec{\phi} \cdot d\vec{\mu}^n$. From Lemma 3.2-(i)

$$\Xi_{\phi}^*(\vec{m}_n) = \Xi_{\phi}(\vec{p}_n) - \vec{m}_n \cdot \vec{p}_n = \int_X \vec{\phi} d\vec{\mu}^n$$

Taking the limit $n \to \infty$ and the lower-semi-continuity of Ξ_ϕ^* we get

$$\Xi_\phi^*(\vec{m}) \leq \int_X \vec{\phi} d\vec{\mu}$$

This, with (61), implies that $\vec{\mu}$ is the maximizer.

4 Strong (sub)partitions

4.1 Structure of the strong partition sets

Assumption 4.1. For any $i \in I$ and $x \in X$ $\psi_i > 0$ and is positive. In addition, for any $i \neq j \in I$

$$\mu \left[x \in X; \alpha \psi_i(x) + \beta \psi_j(x) = 0 \right] = 0$$

for any $\alpha, \beta \in \mathbb{R}, \ \alpha^2 + \beta^2 > 0$,

Lemma 4.1. Under Assumption 4.1, Ξ_0 is differentiable at any point $\vec{p} := (p_1, \ldots, p_N)$ for which $\Pi_1^N p_i \neq 0$. In particular

$$\frac{\partial \Xi_0}{\partial p_i}(\vec{p}) = \int_{X_i(\vec{p})} \psi_i d\mu \tag{62}$$

is continuous, where $X_i(\vec{p}) := \{x \in X ; p_i\psi_i(x) = \xi_0(x, \vec{p})\}$ is a strong partition.

If, in addition, $\vec{p} > \vec{0}$ then Ξ_0^+ is differentiable at \vec{p} as well and

$$\frac{\partial \Xi_0^+}{\partial p_i}(\vec{p}) = \int_{X_i(\vec{p})} \psi_i d\mu \ . \tag{63}$$

Proof. (63) follows from (62) by definition (compare (27) to (28), using the assumption $\psi_i > 0$). Assumption 4.1 yields the existence of a strong partition $\overrightarrow{X}(\vec{p}) := (X_1(\vec{p}), \dots, X_N(\vec{p}))$ in $\mathcal{P}_{\vec{m}}^{\vec{\psi}}$ associated with each \vec{p} :

$$X_i(\vec{p}) := \{ x \in X \ ; \ p_i \psi_i(x) = \xi_0(x, \vec{p}) \} \ .$$
(64)

where ξ_0 as defined in (27). In particular $\mu(X_i(\vec{p}) \cap X_j(\vec{p})) = 0$ for $i \neq j$. Note that $\xi_0(x, \vec{p})$ is differentiable a.e and

$$\frac{\partial \xi_0(x,\vec{p})}{\partial p_i} = \begin{cases} \psi_i(x) & \text{if } x \in X_i(\vec{p}) \text{ a.e} \\ 0 & \text{if } x \notin X_i(\vec{p}) \text{ a.e} \end{cases}$$

A direct integration of the above over X yields (62). Under Assumption 4.1, the sets $X_i(\vec{p})$ are continuous with \vec{p} in the Hausdorff metric at $\vec{p} \neq 0$, hence it yields that the right side of (62) is, indeed, continuous, hence Ξ_0 is differentiable at any \vec{p} satisfying the assumption of the Lemma. The same proof holds for Ξ_0^+ where this time

$$\underline{X}_i(\vec{p}) := \{ x \in X ; \ p_i \psi_i(x) = \xi_0^+(x, \vec{p}) \}$$

is a partition.

Proposition 4.1. Assume $\vec{m} > \vec{0}$. Under assumption 4.1, if $\vec{m} \in \partial S_I^{\psi}$ then

- i) \vec{m} is an exposed point in S_I^{ψ} . That is, \vec{m} is not an interior point of any segment contained in S_I^{ψ} .
- ii) There exists a unique partition in $\mathcal{P}_m^{\psi,w}$. Moreover, this partition is a strong one.

Lemma 4.2. Under assumption 4.1, if $\vec{m} \in \partial \underline{S}_{I}^{\psi}$, $\vec{m} > \vec{0}$, then $\vec{m} \in \partial S_{I}^{\psi}$. Moreover, $\underline{\mathcal{P}}_{m}^{\psi,w} = \mathcal{P}_{m}^{\psi}$.

Corollary 4.1. If \vec{m} is supported on $J \subset I$ so $\vec{m}_J > \vec{0}$ (see section 1.4-(iii) and either $\vec{m} \in \partial S_J^{\psi}$ or $\vec{m} \in \partial \underline{S}_J^{\psi}$ then the conclusion of Proposition 4.1 hold.

Proof. of Proposition 4.1: Using Corollary 2.1 we obtain that if $\vec{m} \in \partial S_I^{\psi}$ there exists $\vec{p}^0 \neq \vec{0}$ for which $\Xi_0(\vec{p}^0) - \vec{m} \cdot \vec{p}^0 = 0 \leq \Xi_0(\vec{p}) - \vec{m} \cdot \vec{p}$ for any $\vec{p} \in \mathbb{R}^I$. We claim that if $\vec{m} > \vec{0}$ then \vec{p}^0 can be chosen to satisfy the assumption of Lemma 4.1. In particular, we prove that either $\vec{p}^0 > \vec{0}$ or $\vec{p}^0 < \vec{0}$.

Assume that, say, $p_1^0 > 0$. Since $\psi_1 > c$ on X for some c > 0 by assumption, then $\xi_0(\vec{p}^0, x) \ge p_1^0\psi_1(x) > p_1c$ on X. If $p_j^0 \le 0$ for some $j \ne 1$, let $\epsilon > 0$ for which $(p_j^0 + \epsilon)\psi_j < p_1^0\psi_1$ on X. Then by definition $\xi_0(\vec{p}^0, x) = \xi^0(\vec{p}^0 + \epsilon \vec{e_j}, x)$ on X. Here $\vec{e_j}$ is the unit coordinate vector pointing in the positive j direction. hence $\Xi_0(\vec{p}^0) = \Xi_0(\vec{p}^0 + \epsilon \vec{e_j})$ so

$$\Xi_0(\vec{p}^0 + \epsilon \vec{e}_j) - \vec{m} \cdot (\vec{p}^0 + \epsilon \vec{e}_j) = \Xi_0(\vec{p}^0) - \vec{m} \cdot \vec{p}_0 - \epsilon m_j = -\epsilon m_j \; .$$

Since $m_j > 0$ by assumption it follows that we get a contradiction to $\bar{p}^0 \in S_I^{\psi}$ by Theorem 2.2.

Alternatively, if $p_1^0 < 0$ and $p_j^0 \ge 0$ for some $j \ne 1$, then $\Xi_0(\vec{p}^0 + \epsilon \vec{e}_1) = \Xi_0(\vec{p}^0)$ for any $0 < \epsilon < -p_1^0$ so

$$\Xi_0(\vec{p}^0 + \epsilon \vec{e}_1) - \vec{m} \cdot (\vec{p}^0 + \epsilon \vec{e}_1) = \Xi_0(\vec{p}^0) - \vec{m} \cdot \vec{p}_0 - \epsilon m_1 < 0$$

as well. Hence either $p^0 > \vec{0}$ or $p^0 < \vec{0}$ and, in particular, the condition of Lemma 4.1 is satisfied.

Proof of (i):

Suppose now that ∂S_I^{ψ} contains an interval centered at $\vec{m} > \vec{0}$. In particular there exists $\vec{m}_1, \vec{m}_2 \in \partial S_I^{\psi}, \vec{m}_1 \neq \vec{m}_2$ such that $\vec{m} = (\vec{m}_1 + \vec{m}_2)/2$. Let \vec{p}^0 corresponding to \vec{m} as above:

$$\Xi(\vec{p}^0) - \frac{\vec{m}_1 + \vec{m}_2}{2} \cdot \vec{p}^0 = 0 .$$
 (65)

Since $\vec{m}_1, \vec{m}_2 \in S_I^{\psi}$ we get by Theorem 2.2

$$\Xi(\vec{p}^0) - \vec{m}_1 \cdot \vec{p}^0 \ge 0 \quad ; \quad \Xi(\vec{p}^0) - \vec{m}_2 \cdot \vec{p}^0 \ge 0 \; . \tag{66}$$

Averaging these two inequalities we get

$$\Xi(\vec{p}^0) - \frac{\vec{m}_1 + \vec{m}_2}{2} \cdot \vec{p}^0 \ge 0$$

and, from (65) we get that the two inequalities in (66) are, in fact, equalities:

$$\Xi(\vec{p}^0) - \vec{m}_1 \cdot \vec{p}^0 = 0 \quad ; \quad \Xi(\vec{p}^0) - \vec{m}_2 \cdot \vec{p}^0 = 0$$

which implies that $\vec{m}_1, \vec{m}_2 \in \partial_{\vec{p}^0} \Xi_0$. In particular Ξ_0 is not differentiable at \vec{p}^0 , which is a contradiction to Lemma 4.1. Hence $\vec{m}_1 = \vec{m}_2$.

Proof of (ii):

From Lemma 4.1 we also get that

$$X_i^0(\bar{p}^0) := \{ x \in X ; p_i^0 \psi_i(x) = \xi_0(x, \bar{p}^0) \}$$

is a strong partition. If $\vec{m} \in \partial \underline{S}^{\psi}$ and $\vec{m} > \vec{0}$ then, necessarily, $\vec{p}^0 > \vec{0}$.

We now show that any weak partition in $\mathcal{P}_m^{\psi,w}$ is the strong partition given by \vec{X}^0 . Indeed, if $\vec{\mu} \in \mathcal{P}_m^{\psi,w}$, then

$$\Xi_0(\vec{p}^0) = \int_X \xi_0(x, \vec{p}^0) d\mu(x) = \sum_1^N \int_X \xi_0(x, \vec{p}^0) d\mu_i(x) \ge \sum_1^N p_i^0 \int_X \psi_i d\mu_i = \vec{p}^0 \cdot \vec{m}$$

Since $\Xi_0(\vec{p}^0) = \vec{p}^0 \cdot \vec{m}$, it follows that

$$\sum_{1}^{N} \int_{X} \left(\xi_0(x, \bar{p}^0) - p_i^0 \psi_i(x) \right) d\mu_i(x) = 0 \; .$$

Note that $\xi_0(\vec{p}, x) \geq p_i \psi_i(x)$ for any $i \in I$ and a.e $x \in X$ with strong inequality only for $x \in X_j^0(\vec{p}^0)$, $j \neq i$, by definition of ξ_0 . Hence $\mu_i = h_i \mu$ where h = 0 on $X - X_i^0(\vec{p}^0)$ μ -a.e. Since $\sum_{1}^{N} h_i = 1$ μ -a.e., it follows that, necessarily, h_i is the indicator function of $X_i^0(\vec{p}^0)$. In particular, $\vec{\mu}$ is a strong partition, and is a singleton in $\mathcal{P}_{\vec{m}}^{w,\vec{\psi}}$.

Proof. of Lemma 4.2:

Following the proof of Proposition 4.1 we get the existence of $\vec{p}^0 > 0$ for which $\Xi_0^+(\vec{p}^0) - \vec{m} \cdot \vec{p}^0 = 0$. If $\vec{\mu} \in \underline{\mathcal{P}}_m^{\psi,w}$ is a weak subpartition, then as in the above proof we get

$$\Xi_0^+(\vec{p}^0) = \int_X \xi_0^+(x, \vec{p}^0) d\mu(x) \ge \sum_1^N \int_X \xi_0^+(x, \vec{p}^0) d\mu_i(x) \ge \sum_1^N p_i^0 \int_X \psi_i d\mu_i = \vec{p}^0 \cdot \vec{m}$$

In particular

$$\int_X \xi_0^+(x, \bar{p}^0) d\mu(x) = \sum_1^N \int_X \xi_0^+(x, \bar{p}^0) d\mu_i(x)$$

Since $\xi_0^+(x, \vec{p}^0)$ is positive and continuous on X and $\sum_1^N \mu_i \leq \mu$ it follows that $\vec{\mu}$ is, in fact, a weak partition.

4.2 Uniqueness of optimal strong (sub)partitions

Assumption 4.2. $\phi_i \in C(X)$ for all $i \in I$.

- i) For any $i, j \in I$ and any $\alpha, \beta \in \mathbb{R}$, $\mu(x \in X; \alpha \psi_i(x) - \beta \psi_j(x) + \phi_i(x) - \phi_j(x) = 0) = 0$.
- ii) For any $i \in I$ and any $\alpha \in \mathbb{R}$, $\mu (x \in X ; \phi_i(x) = \alpha \psi_i(x)) = 0$.

Recall (45,46). For each $\vec{p} \in \mathbb{R}^I$ let

$$X_i(\vec{p}) := \{ x \in X \ ; \ p_i \psi_i(x) + \phi_i(x) = \xi_\phi(x, \vec{p}) \} \ , \tag{67}$$

$$\underline{X}_{i}(\vec{p}) := \{ x \in X \; ; \; p_{i}\psi_{i}(x) + \phi_{i}(x) = \xi_{\phi}^{+}(x,\vec{p}) \} \; , \tag{68}$$

By Assumption 4.2-(i) it follows that $\overrightarrow{X}(\vec{p})$ is, indeed, a strong partition for any $\vec{p} \in \mathbb{R}^{I}$. Likewise, Assumption 4.2-(i,ii) implies that $\underline{\overrightarrow{X}}(\vec{p})$ is a strong subpartition. In particular, $\mu(X_{i}(\vec{p}) \cap X_{j}(\vec{p})) = \mu(\underline{X}_{i}(\vec{p}) \cap \underline{X}_{j}(\vec{p})) = 0$ for $i \neq j$. Moreover, (62, 63) are generalized into

$$\frac{\partial \Xi_{\phi}}{\partial p_i}(\vec{p}) = \int_{X_i(\vec{p})} \psi_i d\mu \quad , \quad \frac{\partial \Xi_{\phi}^+}{\partial p_i}(\vec{p}) = \int_{\underline{X}_i(\vec{p})} \psi_i d\mu \tag{69}$$

where the right sides of (69) are continuous in \vec{p} . It follows

Lemma 4.3. Under Assumption 4.2(i), Ξ_{ϕ} is differentiable on \mathbb{R}^{I} . If, in addition, Assumption 4.2(ii) is granted, then Ξ_{ϕ}^{+} is differentiable as well.

Theorem 4.1.

- i) Let \vec{m} be an interior point of S_I^{ψ} . Under Assumption 4.2(i), there exists a unique partition in $\mathcal{P}_m^{\psi,w}$ which maximize $\int_X \vec{\phi} \cdot d\vec{\mu}$, and this partition is a strong one.
- ii) If \vec{m} be an interior point of \underline{S}_{I}^{ψ} and, in addition, Assumption 4.2(ii) is granted, then for there exists a unique subpartition in $\underline{\mathcal{P}}_{m}^{\psi,w}$ which maximize $\int_{X} \vec{\phi} \cdot d\vec{\mu}$, and this subpartition is a strong one.
- iii) If, in addition, Assumption 4.1 is granted that both (i, ii) hold for any $\vec{m} \in S_I^{\psi}$ ($\vec{m} \in \underline{S}_I^{\psi}$).

Proof. We may assume that $\vec{m} > \vec{0}$ for otherwise, if $m_i > 0$ for $i \in J \subset I$ and $m_i = 0$ for $i \notin J$, we can restrict our discussion from I to J (c.f Corollary 4.1).

Same argument holds if $\vec{m} \in \partial \underline{S}_J^{\psi}$. Hence we assume that \vec{m} is an interior point of S_J^{ψ} (res. \underline{S}_J^{ψ}).

(*i*): Recall that, for any $\vec{\mu} \in \mathcal{P}_m^{\psi, w}$,

$$\int_X \vec{\phi} \cdot d\vec{\mu} \le \Xi_{\phi}^*(\vec{m}) := \inf_{\vec{p} \in \mathbb{R}^J} \left[\Xi_{\phi}(\vec{p}) - \vec{m} \cdot \vec{p} \right]$$

and S_J^{ψ} is the essential domain of Ξ_{ϕ}^* . By Lemma 3.3 any maximizer satisfies the equality above $\Xi_{\phi}^*(\vec{m}) = \int_X \vec{\phi} \cdot d\vec{\mu}$.

If \vec{m} is an interior point then (see [2]) there exists $\vec{p} \in \mathbb{R}^J$ for which the equality

$$\Xi_{\phi}(\vec{p}) - \vec{m} \cdot \vec{p} = \Xi_{\phi}^*(\vec{m})$$

holds. For any $\vec{\mu}$ (in particular, for the maximizer) we get from the definition of Ξ_ϕ

$$\Xi_{\phi}(\vec{p}) - \vec{m} \cdot \vec{p} = \sum_{1}^{N} \int_{X} \left(\xi_{\phi}(x, \vec{p}) - p_{i}\psi_{i}(x)\right) d\mu_{i}$$
(70)

so, by Lemma 3.3 any maximizer satisfies

$$\sum_{1}^{N} \int_{X} \left(\xi_{\phi}(x, \vec{p}) - p_i \psi_i(x) - \phi_i(x) \right) d\mu_i(x) = 0 .$$
 (71)

By Assumption 4.2-(i) and (67), the *i* integrand above is positive on $X - X_i(\vec{p})$ and a.e zero on $X_i(\vec{p})$, so μ_i is supported on $X_i(\vec{p})$. Since

$$\int_{X_i(\vec{p})} \psi_i d\mu = \frac{\partial \Xi_\phi}{\partial p_i}(\vec{p}) = m_i = \int_X \psi_i d\mu_i \tag{72}$$

it follows that $\mu_i = \mu \lfloor X_i(\vec{p})$, that is, $\vec{\mu}$ is a strong partition.

(ii) In the case $\vec{\mu} \in \underline{\mathcal{P}}_{m}^{\psi,w}$ (70) turns into an inequality

$$\Xi_{\phi}^{+}(\vec{p}) - \vec{m} \cdot \vec{p} \ge \sum_{1}^{N} \int_{X} \left(\xi_{\phi}^{+}(x, \vec{p}) - p_i \psi_i(x) \right) d\mu_i$$

but $\xi_{\phi}^{+} - p_i \psi_i - \phi_i \ge 0$ on X by definition (46) so

$$\Xi_{\phi}^{+}(\vec{p}) - \vec{m} \cdot \vec{p} \ge \sum_{1}^{N} \int_{X} \left(\xi_{\phi}^{+}(x, \vec{p}) - p_{i}\psi_{i}(x) - \phi_{i}(x) \right) d\mu_{i} + \int_{X} \vec{\phi} \cdot d\vec{\mu} \ge \Xi_{\phi}^{+*}(\vec{m})$$

By Lemma 3.3 again we have $\Xi_{\phi}^{++}(\vec{m}) = \Xi_{\phi}^{+}(\vec{p}) - \vec{m} \cdot \vec{p}$ so we have equality in (72) and the rest of the proof as above.

(iii): If $\vec{m} \in \partial S_J^{\psi}$ ($\vec{m} \in \partial \underline{S}_I^{\psi}$) then by Proposition 4.1-(ii) the set $\mathcal{P}_m^{\psi,w}$ ($\underline{\mathcal{P}}_m^{\psi,w}$) is composed a unique strong (sub)partition, so the Theorem follows trivially.

We turn now to the case of optimal selection.

Theorem 4.2. Given a closed convex set $K \subset \mathbb{R}^I$. There exists a unique subpartition which optimize (13), and this subpartition is a strong one.

Proof. By Theorem 4.1 we only have to prove the uniqueness of the maximizer of Ξ_{ϕ}^{+*} on $\underline{S}_{I}^{\psi} \cap K$ (14).

To show the uniqueness of this maximizer we use Corollary 18.12(ii) on page 268 of [2]. It implies that a function Ξ_{ϕ}^{+*} is strictly convex in the interior of its domain \underline{S}_{I}^{ψ} if it is the convex dual of a differentiable convex function. In our case $-\Xi_{\phi}^{+*}$ is the convex dual of Ξ_{ϕ}^{+} which is differentiable by Corollary 4.3. Hence, if the a maximizer \vec{m} of Ξ_{ϕ}^{+*} in the convex set $K \cap \underline{S}_{I}^{\psi}$ is an interior point of \underline{S}_{I}^{ψ} , then it is unique by its strong concavity. If, on the other hand, $\vec{m} \in \partial \underline{S}_{I}^{\psi} \cap K$ is a maximizer and $\vec{m} > \vec{0}$ then Proposition 4.1-(i) implies that \vec{m} is an exposed point of \underline{S}_{I}^{ψ} . This implies that, again, this maximizer is unique. If the components of \vec{m} are not all positive then we reduce the problem to the subset $J \subset I$ which support the maximizer \vec{m} and apply Corollary 4.1.

4.3 Back to Monge

It is interesting to compare Assumption 4.2-(i) with the *twist condition* (15). Recall that the Monge problem corresponds to the case where all ψ_i are equal, say $\psi_i \equiv 1$ for any $i \in I$. In that case Assumption 4.2-(i) takes the form

$$\mu (x \in X ; \phi_i(x) - \phi_j(x) = r) = 0$$
(73)

for any $i \neq j$ and $r \in \mathbb{R}$. This seems to be a weaker version of (15). Theorem 4.1-(i) yields the uniqueness of of optimal partition for any \vec{m} in the interior of the set S_I^{ψ} . Embarrassingly, S_I^{ψ} is the simplex S_I (3), and does not contain any interior point! Part (iii) of Theorem 4.1 is of no help either, since Assumption 4.1 is never satisfied in that case.....

On the other hand, if we add Assumption 4.2-(ii) which, in the above case, takes the form

$$\mu(x \in X ; \phi_i(x) = r) = 0$$
 (74)

for any $i \in I$ and any $r \in \mathbb{R}$, then Theorem 4.1-(ii) yields

Corollary 4.2. Under conditions (73, 74) there is a unique, strong subpartition for the Monge partition problem for any $\vec{m} \in \underline{S}_I^0 :=$

$$\left\{ \vec{m} \in \mathbb{R}^I ; \quad 0 \le m_i , \quad \sum_{i \in I} m_i < 1 \right\} .$$

However, it turns out that condition (73) alone is also sufficient for the uniqueness of strong partition in the Monge case:

Theorem 4.3. Suppose $\psi_1 = \ldots = \psi_N \equiv 1$ and the components of $\vec{\phi}$ are continuous on X. If (73) is satisfied then there is a unique optimal partition for any $\vec{m} \in S_I$ (3), and this unique partition is a strong one.

Proof. In the case under consideration, (45, 64) takes the form

$$\xi_1(x,\vec{p}) := \max \left\{ \phi_1(x) + p_1, \dots, \phi_N(x) + p_N \right\}$$

$$\Xi_1(\vec{p}) := \int_X \xi_1(x,\vec{p}) d\mu(x) : \mathbb{R}^I \to \mathbb{R} ,$$

Note that Ξ_1 is additively invariant under shift

$$\Xi_1(\vec{p} + \alpha \vec{1}) = \Xi_1(\vec{p}) + \alpha \quad \forall \vec{p} \in \mathbb{R}^I, \ \alpha \in \mathbb{R} .$$

So, $\Xi_1(\vec{p}) - \vec{m} \cdot \vec{p}$ is invariant under such shift for any $\vec{m} \in S_I$. Thus, we may set to zero the first coordinate p_1 of \vec{p} and obtain for $\Xi_1^0(p_2, \ldots p_N) := \Xi_1(0, p_2, \ldots p_N)$

$$\Xi_1^0(p_2, \dots p_N) - \sum_2^N m_i p_i = \Xi_1(\vec{p}) - \vec{m} \cdot \vec{p} .$$

Now, (73) implies that $\Xi_1^0 \in C^1(\mathbb{R}^{N-1})$ and the range of $\nabla \Xi_1^0$ is the whole N-1 simplex $\sum_2^N m_i \leq 1$. Thus for any point $\vec{m} \in S_I$ for which $m_1 > 0$, we get $(m_2, \ldots m_N)$ as an interior point in the range of $\nabla \Xi_1^0$. This yields the proof of uniqueness as in Theorem 4.1-(i).

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