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Dynamic Optimization of Large-Population System with Partial Information

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Abstract We consider the dynamic optimization of large-population system with partial information. The associated mean-field game is formulated and its consistency condition is equivalent to the wellposedness of some Riccati equation system. The limiting state average is represented by a mean-field stochastic differential equation driven by the common Brownian motion. The decentralized strategies with partial information are obtained and the approximate Nash equilibrium is verified.

Keywords Dynamic optimization · Forward-backward stochastic differential equation · Large-population system · Mean-field game · Partial information

Mathematics Subject Classification 65K10 · 91A25 · 93E20 · 93C41

1 Introduction

The starting point of our work is the large-population (LP) systems which are strongly grounded in various fields. One efficient methodology to study LP system, is the mean-field game which enables us to obtain the decentralized optimal control. The interested readers may refer the pioneering work [1] for the motivation and methodology of mean-field games. Based on [1], considerable research attention has been drawn along this research line. Some recent literature include [2–4] for linear-quadratic-Gaussian (LQG) mean-field games of large-population system, [5] for large population systems with major and

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minor players. In addition, the stochastic control problems with a mean-field term in dynamics and (or) cost functional can be found in [6–8] etc.

This paper focuses on the dynamic optimization of large-population system in its linear-quadratic (LQ) case by taking into account the partial information structure. One systematic introduction of stochastic LQ control can be found in [9] and the references therein. An extensive review of LQ control with partial information is provided in [10] and other related works include [11, 12] etc. The backward or recursive LQ control with partial information can be found in [13]. Herein, we turn to study the partial information structures of linear large-population systems. Here, the individual agents can only access the filtration generated by one observable component of underlying Brownian motion. The unobservable Brownian motion component may be interpreted as the effect of a passive version of major player of [5], or be framed into a partial observation problem (see [10]). We remark that a class of mean-field LQG games with noisy observations is also addressed in [14] but defined on an infinite-time horizon so the algebra Riccati equations are involved there. Moreover, the limiting state-average in [14] is deterministic as there is no common noise. The random limiting state-average in our problem makes our analysis different to that in [14].

The rest of this paper is organized as follows. Section 2 introduces the mean-field LQG games with partial filtration structure. Section 2 also discusses the related filtering equation and consistency conditions. Section 3 is devoted to the asymptotic analysis of the related ϵ-Nash equilibrium. Section 4 concludes our work.

2 Mean-Field Games with Partial Filtration

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which a standard $(1+N)$-dimensional Brownian motion $\{W(t), W_i(t),\ 1 \leq i \leq N\}_{0\leq t\leq T}$ is defined. The information structure of large-population system is as follows. We denote by $\{\mathcal{F}_t^{w_i}\}_{0\leq t\leq T}$ the filtration generated by the component W_i; $\{\mathcal{F}_t^{w}\}_{0\leq t\leq T}$ the filtration generated by the component W. Here, $\{\mathcal{F}_t^{w_i}\}_{0\leq t\leq T}$ stands for the individual information owning by the i^{th} agent; $\{\mathcal{F}_t^{w}\}_{0\leq t\leq T}$ the common information taking effects on all agents. $\mathcal{F}_t^i := \sigma(\mathcal{F}_t^{w_i} \cup \mathcal{F}_t^{w})$ represents the full information of i^{th} agent, $\mathcal{F}_t := \sigma(\cup_{i=1}^N \mathcal{F}_t^i)$ denotes the complete information of system. For simplicity, we set $\mathcal{F} = \mathcal{F}_T$. In decentralized setup, it is infeasible for the i^{th} agent to access the information of other agents, i.e., $\{\mathcal{F}_t^{w_j}\}_{0\leq t\leq T}$ for $j \neq i$. This is reasonable due to the asymmetric information (for example, the individual firm's own operation information will not be released to the public or its peer firms). $\{\mathcal{F}_t^{w}\}_{0\leq t\leq T}$ can represent the information of some macro process imposing on all agents (firms) due to the common external economic factors.

We consider a large-population system with N individual agents $\{\mathcal{A}_i\}_{1\le i\le N}$. The state x_i for $\mathcal{A}_i$ satisfies the following controlled linear stochastic system:

$$\begin{cases} dx_i(t) =[A(t)x_i(t)+B(t)u_i(t)+\alpha x^{(N)}(t)+m(t)]dt+\sigma(t)dW_i(t)+\tilde{\sigma}(t)dW(t), \\ x_i(0)=x \end{cases} \tag{1}$$

where $x^{(N)}(t)=\frac{1}{N}\sum_{i=1}^N x_i(t)$ is the state-average, $\alpha\in\mathbb{R}$ denotes the coupling constant which maybe positive or negative. In (1), W_i denotes the individual random noise while W denotes the common random noise. Other work discussing the large-population system with common noise W includes [15]. Thus, the admissible control $u_i\in\mathcal{U}_i$ where the admissible control set $\mathcal{U}_i$ is defined by

$$\mathcal{U}_i:=\{u_i(\cdot)|u_i(\cdot)\in L^2_{\mathcal{F}^{w_i}_t}(0,T;\mathbb{R})\},\ 1\le i\le N.$$

Denote $u_{-i}=(u_1,\cdots,u_{i-1},\ u_{i+1},\cdots u_N)$ the strategies of all agents except $\mathcal{A}_i$. The cost functional of $\mathcal{A}_i$ is

$$\mathcal{J}_i(u_i(\cdot),u_{-i}(\cdot))=\mathbb{E}\left[\int_0^T\Big(Q(t)(x_i(t)-x^{(N)}(t))^2+R(t)u_i^2(t)\Big)\,dt+Gx_i^2(T)\right]. \tag{2}$$

Moreover, we have the following assumption:

$$\text{(H)}\qquad \begin{aligned} &A(\cdot),B(\cdot),m(\cdot),\sigma(\cdot),\tilde{\sigma}(\cdot),Q(\cdot),R(\cdot)\in L^\infty(0,T;\mathbb{R}),\\ &\alpha\in\mathbb{R},Q(\cdot)\ge 0,R(\cdot)>0,G\ge 0.\end{aligned}$$

Here, $L^\infty(0,T;\mathbb{R})$ denotes the space of uniformly bounded functions. Now, we formulate the large population LQG games with partial filtration (**PF**).

Problem (PF). Find a control strategies set $\bar{u}=(\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_N)$ which satisfies

$$\mathcal{J}_i(\bar{u}_i(\cdot),\bar{u}_{-i}(\cdot))=\inf_{u_i(\cdot)\in\mathcal{U}_i}\mathcal{J}_i(u_i(\cdot),\bar{u}_{-i}(\cdot))$$

where $\bar{u}_{-i}$ represents $(\bar{u}_1,\cdots,\bar{u}_{i-1},\bar{u}_{i+1},\cdots,\bar{u}_N)$.

To study (**PF**), one efficient protocol is the mean-field LQG games which bridges the "centralized" LQG problems via the limiting state-average, as the number of agents tends to infinity. Due to partial filtration structure, it is natural to set the following feedback control on filters

$$u_i(t)=-a(t)\mathbb{E}(x_i(t)|\mathcal{F}^{w_i}_t)+\sum_{j=1,j\neq i}^{N}\tilde{a}(t)\mathbb{E}(x_j(t)|\mathcal{F}^{w_i}_t)+b(t) \tag{3}$$

where the coefficients $a(\cdot), \tilde{a}(\cdot)$ and $b(\cdot)$ are deterministic functions and $\tilde{a}(\cdot) = O(\frac{1}{N})$. Inserting (3) into state equation (1), we get the following realized state dynamics

$$\begin{aligned} dx_i(t) =& [A(t)x_i(t) - B(t)a(t)\mathbb{E}(x_i(t)|\mathcal{F}_t^{w_i}) + B(t)\tilde{a}(t)\sum_{j=1,j\neq i}^{N}\mathbb{E}(x_j(t)|\mathcal{F}_t^{w_i}) + B(t)b(t) \\ &+ \alpha x^{(N)}(t) + m(t)]dt + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t),\ 1 \le i \le N. \end{aligned} \tag{4}$$

Take summation of the above N equations and divide by N,

$$\begin{aligned} d\Big(\frac{1}{N}\sum_{i=1}^{N}x_i(t)\Big) =& \Big[A(t)\frac{1}{N}\sum_{i=1}^{N}x_i(t) - B(t)a(t)\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(x_i(t)|\mathcal{F}_t^{w_i}) + B(t)b(t) + \alpha x^{(N)}(t) + m(t) \\ &+ B(t)\tilde{a}(t)\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\mathbb{E}(x_j(t)|\mathcal{F}_t^{w_i})\Big]dt + \sigma(t)\frac{1}{N}\sum_{i=1}^{N}dW_i(t) + \tilde{\sigma}(t)dW(t). \end{aligned}$$

Letting $N \to \infty$, we obtain the following limiting process which is a mean-field stochastic differential equation (SDE):

$$\begin{cases} dx_0(t) = [(A(t)+\alpha)x_0(t) - \tilde{\alpha}(t)\mathbb{E}x_0(t) + \tilde{b}(t)]dt + \tilde{\sigma}(t)dW(t), \\ x_0(0) = x \end{cases} \tag{5}$$

where the functions $\tilde{\alpha}(\cdot), \tilde{b}(\cdot)$ are to be determined. Now, we introduce an auxiliary state:

$$\begin{cases} dx_i(t) = [A(t)x_i(t) + B(t)u_i(t) + \alpha x_0(t) + m(t)]dt + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t), \\ x_i(0) = x \end{cases} \tag{6}$$

with the auxiliary cost functional

$$J_i(u_i(\cdot)) = \mathbb{E}\left[\int_0^T \big(Q(t)(x_i(t) - x_0(t))^2 + R(t)u_i^2(t)\big)\,dt + Gx_i^2(T)\right] \tag{7}$$

where $x_0(\cdot)$ is given by (5). Note that (6) and (7) are obtained from (1) and (2) with $x^{(N)}(\cdot)$ replaced by $x_0(\cdot)$. Thus, we formulate the following limiting partial filtration (**LPF**) LQG game.

Problem (LPF). For the i^{th} agent, $i = 1, 2, \cdots, N$, find $\bar{u}_i(\cdot) \in \mathcal{U}_i$ satisfying

$$J_i(\bar{u}_i(\cdot)) = \inf_{u_i(\cdot)\in\mathcal{U}_i} J_i(u_i(\cdot)).$$

Then $\bar{u}_i(\cdot)$ is called an optimal control for Problem (**LPF**).

Remark 2.1 The state of (**LPF**) in (6) differs from the state of (**PF**) in (1). Specifically, the latter is affected by the state-average $x^{(N)}$. Here, we still write them in the same notation to ease the presentation.

Applying the variational method, we have the following result to the optimal control of (**LPF**).

Theorem 2.1 *Let* (H) *hold. Suppose there exists an optimal control* $\bar{u}_i(\cdot)$ *of Problem* (**LPF**) *and* $\bar{x}_i(\cdot)$ *is the corresponding optimal state, then there exists an adjoint process* $p_i(\cdot) \in L^2_{\mathcal{F}^i_t}(0,T;\mathbb{R})$ *satisfying the following backward stochastic differential equation (BSDE) for some* $\beta(\cdot)$ *and* $\tilde{\beta}(\cdot)$*:*

$$\begin{cases} dp_i(t) =[-A(t)p_i(t) - Q(t)(\bar{x}_i(t) - x_0(t))]dt + \beta(t)dW_i(t) + \tilde{\beta}(t)dW(t), \\ p_i(T) =G\bar{x}_i(T), \ i = 1, 2, \cdots, N \end{cases} \tag{8}$$

such that

$$\bar{u}_i(t) = -R^{-1}(t)B(t)\mathbb{E}(p_i(t)|\mathcal{F}^{w_i}_t)$$

where the conditional expectation is defined in its optional projection version.

Consequently, we get the following Hamiltonian system for $\mathcal{A}_i$:

$$\begin{cases} dx_0(t) =[(A(t)+\alpha)x_0(t) - \tilde{\alpha}(t)\mathbb{E}x_0(t) + \tilde{b}(t)]dt + \tilde{\sigma}(t)dW(t), \\ d\bar{x}_i(t) =[A(t)\bar{x}_i(t) - B^2(t)R^{-1}(t)\mathbb{E}(p_i(t)|\mathcal{F}^{w_i}_t) + \alpha x_0(t) + m(t)]dt \\ \qquad\qquad + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t), \\ dp_i(t) =[-A(t)p_i(t) - Q(t)(\bar{x}_i(t) - x_0(t))]dt + \beta(t)dW_i(t) + \tilde{\beta}(t)dW(t), \\ x_0(0) =\bar{x}_i(0) = x, \ p_i(T) = G\bar{x}_i(T), \ i = 1, 2, \cdots, N. \end{cases} \tag{9}$$

After obtaining $\tilde{\alpha}(\cdot), \tilde{b}(\cdot)$ in Theorem 2.2 (see below), by the monotonic conditions of forward-backward stochastic differential equation (FBSDE) (see [16]), it is easy to see that (9) admits a unique solution $(x_0(\cdot), \bar{x}_i(\cdot), p_i(\cdot)) \in L^2_{\mathcal{F}^w_t}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}^i_t}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}^i_t}(0,T;\mathbb{R})$. Note that in system (9), the forward optimal state $\bar{x}_i(\cdot)$ depends on the backward adjoint process $p_i(\cdot)$ through its filtering state $\mathbb{E}(p_i(t)|\mathcal{F}^{w_i}_t)$. In this sense, (9) becomes a filtered FBSDE system and its decoupling should be proceeded through some FBSDE that involves the filtering state only. To this end, we introduce the following filter notations

$$\hat{\bar{x}}_i(t) = \mathbb{E}[\bar{x}_i(t)|\mathcal{F}^{w_i}_t], \quad \hat{p}_i(t) = \mathbb{E}[p_i(t)|\mathcal{F}^{w_i}_t]$$

where the conditional expectations to the partial filtration $\mathcal{F}_t^{w_i}$ should be understood in the version of optional projection. Then we reach a FBSDE system involving the state filters only:

$$\begin{cases} d\hat{\hat{x}}_i(t) =[A(t)\hat{\hat{x}}_i(t) - B^2(t)R^{-1}(t)\hat{p}_i(t) + \alpha\mathbb{E}x_0(t) + m(t)]dt + \sigma(t)dW_i(t), \\ \hat{\hat{x}}_i(0) =x, \\ d\hat{p}_i(t) =[-A(t)\hat{p}_i(t) - Q(t)(\hat{\hat{x}}_i(t) - \mathbb{E}x_0(t))]dt + \beta(t)dW_i(t), \\ \hat{p}_i(T) =G\hat{\hat{x}}_i(T),\ i = 1, 2, \cdots, N. \end{cases} \tag{10}$$

Note that system (10) is driven by W_i only so it becomes observable to agent $\mathcal{A}_i$. It can be viewed a filtering system of (9) that is unobservable as driven by W_i and W both. Taking expectation on (5),

$$\begin{cases} d\mathbb{E}x_0(t) =[(A(t) + \alpha - \tilde{\alpha}(t))\mathbb{E}x_0(t) + \tilde{b}(t)]dt, \\ \mathbb{E}x_0(0) =x \end{cases} \tag{11}$$

where $\tilde{\alpha}(\cdot),\ \tilde{b}(\cdot)$ are functions to be determined. One key step in mean-field game is to analyze the related consistency condition (which is also called Nash certainty equivalence (NCE) principle, see [14], [3], etc).

Remark 2.2 To intuitively explain the consistency condition, we give some remarks.

(1) Unlike most literature on mean-field games, there is no fixed-point argument involved here (e.g., some contraction mapping based on the datum of our problem) to characterize the consistency condition. Instead, our consistency condition is transformed into the wellposedness of Riccati equation system (12) (see below). Actually, $(\hat{P}(\cdot), \Phi(\cdot))$ depend on $(\tilde{\alpha}(\cdot), \tilde{b}(\cdot))$, thus (17) (see below) can be rewritten by

$$\begin{cases} \tilde{\alpha} = \mathcal{T}_1(\tilde{\alpha}) := B^2R^{-1}(P + \hat{P}(\tilde{\alpha})), \\ \tilde{b} = \mathcal{T}_2(\tilde{b}) := -B^2R^{-1}\Phi(\tilde{\alpha}, \tilde{b}) + m. \end{cases}$$

In this sense, (12) can be understood as the consistency condition of (**LPF**).

(2) The advantages of handling the consistency condition of $(\tilde{\alpha}(\cdot), \tilde{b}(\cdot))$ are as follows. The consistency condition imposed on $(\tilde{\alpha}(\cdot), \tilde{b}(\cdot))$ is equivalent to the wellposedness of Riccati equation (12) (see below) which can be ensured in an arbitrary time interval. On the other hand, as addressed in [17], the fixed-point analysis on x will preferably lead to the consistency condition only on a small time interval.

Now we first state the following result.

Theorem 2.2 *Suppose* (H) *hold true and the following Riccati equation system*

$$\begin{cases} \dot{\Pi}(t) + (2A(t) + \alpha)\Pi(t) - B^2(t)R^{-1}(t)\Pi^2(t) = 0, \\ \dot{\Phi}(t) + [A(t) - B^2(t)R^{-1}(t)\Pi(t)]\Phi(t) + m(t)\Pi(t) = 0, \\ \Pi(T) = G,\ \Phi(T) = 0 \end{cases} \tag{12}$$

admits unique solution $(\Pi(\cdot), \Phi(\cdot))$, *then* $(\tilde{\alpha}(\cdot),\ \tilde{b}(\cdot))$ *can be uniquely determined by*

$$\begin{cases} \tilde{\alpha}(t) = B^2(t)R^{-1}(t)\Pi(t), \\ \tilde{b}(t) = -B^2(t)R^{-1}(t)\Phi(t) + m(t). \end{cases} \tag{13}$$

Proof By the terminal condition of (9) or (10), we suppose

$$\hat{p}_i(t) = P(t)\hat{\hat{x}}_i(t) + \hat{P}(t)\mathbb{E}x_0(t) + \Phi(t) \tag{14}$$

for some $P(\cdot), \hat{P}(\cdot) \in L^\infty(0, T; \mathbb{R})$ and $\Phi(t) \in L^\infty(0, T; \mathbb{R})$ with terminal conditions

$$P(T) = G,\ \hat{P}(T) = \Phi(T) = 0.$$

Applying Itô's formula to (14) and noting (9), we have

$$\begin{aligned} d\hat{p}_i(t) = &\Big(\dot{P}(t) + P(t)A(t) - B^2(t)R^{-1}(t)P^2(t)\Big)\hat{\hat{x}}_i(t)dt \\ &+\Big(\dot{\hat{P}}(t) + \hat{P}(t)(A(t) + \alpha - \tilde{\alpha}(t)) - P(t)B^2(t)R^{-1}(t)\hat{P}(t) + \alpha P(t)\Big)\mathbb{E}x_0(t)dt \\ &+\Big(\dot{\Phi}(t) - P(t)B^2(t)R^{-1}(t)\Phi(t) + P(t)m(t) + \hat{P}(t)\tilde{b}(t)\Big)dt + P(t)\sigma(t)dW_i(t) \\ = &\Big[\,(-Q(t) - A(t)P(t))\,\hat{\hat{x}}_i(t) + (Q(t) - A(t)\hat{P}(t))\mathbb{E}x_0(t) - A(t)\Phi(t)\Big]dt + \beta(t)dW_i(t). \end{aligned}$$

Comparing coefficients, we obtain

$$\begin{cases} \dot{P}(t) + P(t)A(t) - B^2(t)R^{-1}(t)P^2(t) = -Q(t) - A(t)P(t), \\ \dot{\hat{P}}(t) + \hat{P}(t)(A(t) + \alpha - \tilde{\alpha}(t)) - P(t)B^2(t)R^{-1}(t)\hat{P}(t) + \alpha P(t) = Q(t) - A(t)\hat{P}(t), \\ \dot{\Phi}(t) - P(t)B^2(t)R^{-1}(t)\Phi(t) + P(t)m(t) + \hat{P}(t)\tilde{b}(t) = -A(t)\Phi(t), \\ \beta(t) = P(t)\sigma(t). \end{cases} \tag{15}$$

Note that the above Riccati equations are parameterized by the undetermined functions $(\tilde{\alpha}(t), \tilde{b}(t))$ which are to be specified below. To this end, note that the optimal state $\bar{x}_i(t)$ can be represented by

$$\begin{aligned}d\bar{x}_i(t) =&[A(t)\bar{x}_i(t) - B^2(t)R^{-1}(t)(P(t)\hat{\bar{x}}_i(t) + \hat{P}(t)\mathbb{E}x_0(t) + \varPhi(t)) + \alpha x_0(t) + m(t)]dt\\ &+ \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t).\end{aligned}$$

Therefore the state-average satisfies:

$$\begin{aligned}d\bar{x}^{(N)}(t) =&\Big[A(t)\bar{x}^{(N)}(t) - B^2(t)R^{-1}(t)(P(t)\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(\bar{x}_i(t)|\mathcal{F}_t^{w_i}) + \hat{P}(t)\\ &\cdot \mathbb{E}x_0(t) + \varPhi(t)) + \alpha x_0(t) + m(t)\Big]dt + \sigma(t)\frac{1}{N}\sum_{i=1}^{N}dW_i(t) + \tilde{\sigma}(t)dW(t).\end{aligned}$$

Let $N \to +\infty$, the limiting process x_0 is given by

$$\begin{aligned}dx_0(t) =&\Big[(A(t) + \alpha)x_0(t) - B^2(t)R^{-1}(t)(P(t) + \hat{P}(t))\mathbb{E}x_0(t)\\ &- B^2(t)R^{-1}(t)\varPhi(t) + m(t)\Big]dt + \tilde{\sigma}(t)dW(t).\end{aligned} \tag{16}$$

Comparing the coefficients with (9), we have

$$\begin{cases}\tilde{\alpha}(t) = B^2(t)R^{-1}(t)(P(t) + \hat{P}(t)),\\ \tilde{b}(t) = -B^2(t)R^{-1}(t)\varPhi(t) + m(t).\end{cases} \tag{17}$$

Thus we rewrite (15) as

$$\begin{cases}\dot{P}(t) + 2A(t)P(t) - B^2(t)R^{-1}(t)P^2(t) + Q(t) = 0,\\ \dot{\hat{P}}(t) + \hat{P}(t)[2A(t) + \alpha - B^2(t)R^{-1}(t)(P(t) + \hat{P}(t)) - B^2(t)R^{-1}(t)P(t)] + \alpha P(t) - Q(t) = 0,\\ \dot{\varPhi}(t) + [A(t) - (P(t) + \hat{P}(t))B^2(t)R^{-1}(t)]\varPhi(t) + (P(t) + \hat{P}(t))m(t) = 0,\\ P(T) = G,\ \hat{P}(T) = \varPhi(T) = 0.\end{cases}$$

Letting $\varPi(t) = P(t) + \hat{P}(t)$, we get

$$\begin{cases}\dot{\varPi}(t) + (2A(t) + \alpha)\varPi(t) - B^2(t)R^{-1}(t)\varPi^2(t) = 0,\\ \varPi(T) = G.\end{cases} \tag{18}$$

This completes the proof. $\square$

Moreover, the filtering system (10) can be decoupled as

$$\begin{cases} d\hat{\hat{x}}_i(t) = \Big[\Big(A(t) - B^2(t)R^{-1}(t)P(t)\Big)\hat{\hat{x}}_i(t) + \Big(\alpha - B^2(t)R^{-1}(t)(\Pi(t) - P(t))\Big)\mathbb{E}x_0(t) \\ \qquad\qquad - B^2(t)R^{-1}(t)\Phi(t) + m(t)\Big]dt + \sigma(t)dW_i(t), \\ \hat{p}_i(t) = P(t)\hat{\hat{x}}_i(t) + (\Pi(t) - P(t))\mathbb{E}x_0(t) + \Phi(t), \\ \hat{\hat{x}}_i(0) = x_i(0),\ \hat{p}_i(T) = G\hat{\hat{x}}_i(T). \end{cases} \tag{19}$$

Taking average of all and sending $N \to +\infty$, we regenerate

$$\begin{cases} d\mathbb{E}x_0(t) = \Big[\big(A(t) + \alpha - B^2(t)R^{-1}(t)\Pi(t)\big)\mathbb{E}x_0(t) - B^2(t)R^{-1}(t)\Phi(t) + m(t)\Big]dt, \\ \mathbb{E}x_0(0) = x. \end{cases} \tag{20}$$

Remark 2.3 To conclude this section, we give some remarks concerning Theorem 2.2.

(1) By [8], [9], it follows $P(\cdot)$ is determined uniquely as a nonnegative constant. One sufficient condition for the existence and uniqueness of $\Pi(\cdot)$ can be found in [18] hence the solvability of $\hat{P}(\cdot)$ follows directly by noting $\Pi(t) = P(t) + \hat{P}(t)$. In addition, the solvability of $\Phi(\cdot)$ follows from that of $\Pi(\cdot)$.

(2) As referred in Remark 2.2, in [17] the fixed-point analysis on x preferably leads to the consistency condition defined only on a small time interval. This finding also corresponds to the standard result in forward-backward SDE theory: as discussed in [19], the usual contraction mapping on forward-backward system will always lead to its existence and uniqueness in a very small time interval.

3 ϵ-Nash Equilibrium for Problem (PF)

Now we show that $(\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_N)$ satisfies the ϵ-Nash equilibrium for (**PF**).

Definition 3.1 A set of controls $u_k(\cdot) \in \mathcal{U}_k,\ 1 \le k \le N$, for N agents is called an ϵ-Nash equilibrium with respect to the costs $\mathcal{J}_k,\ 1 \le k \le N$, iff there exists $\epsilon \ge 0$ such that for any fixed $1 \le i \le N$, we have

$$\mathcal{J}_i(u_i, u_{-i}) \le \mathcal{J}_i(u_i', u_{-i}) + \epsilon \tag{21}$$

when any alternative control $u_i'(\cdot) \in \mathcal{U}_i$ is applied by $\mathcal{A}_i$.

Theorem 3.1 *Let* (H) *hold and* (12) *admit a solution* (Π, Φ), *then* $(\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_N)$ *satisfies the* ϵ*-Nash equilibrium of Problem* (**PF**). *Here, for* $1 \le i \le N$, $\bar{u}_i$ *is given by*

$$\bar{u}_i(t) = -R^{-1}(t)B(t)\Big[P(t)\hat{\hat{x}}_i(t) + (\Pi(t) - P(t))\mathbb{E}x_0(t) + \Phi(t)\Big] \tag{22}$$

where $\hat{\bar{x}}_i$ *and* $\mathbb{E}x_0$ *satisfy* (19) *and* (20) *respectively.*

As preliminaries of proving the theorem, several lemmas are presented to produce some estimates on the state and cost difference between Problem (**PF**) and (**LPF**) and the proofs are available upon request. Recall that

$$\begin{cases} d\bar{x}_i(t) = & \Big[A(t)\bar{x}_i(t) - B^2(t)R^{-1}(t)\Big(P(t)\hat{\bar{x}}_i(t) + \hat{P}(t)\mathbb{E}x_0(t) + \Phi(t)\Big) + \alpha x_0(t) + m(t)\Big]dt \\ & + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t), \\ d\hat{\bar{x}}_i(t) = & \Big[\Big(A(t) - B^2(t)R^{-1}(t)P(t)\Big)\hat{\bar{x}}_i(t) + \Big(\alpha - B^2(t)R^{-1}(t)\hat{P}(t)\Big)\mathbb{E}x_0(t) \\ & - B^2(t)R^{-1}(t)\Phi(t) + m(t)\Big]dt + \sigma(t)dW_i(t), \\ \bar{x}_i(0) = & \hat{\bar{x}}_i(t) = x_i(0), \end{cases} \tag{23}$$

and denote

$$\bar{x}^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N}\bar{x}_i(t), \qquad \hat{\bar{x}}^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N}\hat{\bar{x}}_i(t).$$

Here, $\bar{x}^{(N)}(t)$ denotes the average of state (in (**LPF**)) while $\hat{\bar{x}}^{(N)}$ denotes the average of filtered states. Note that $\hat{\bar{x}}_i(t)$ is driven by W_i only thus it is observable to the individual agent $\mathcal{A}_i$. It enters the state dynamics (23) as an input process when applying the optimal strategy. Some estimates are as follows.

Lemma 3.1

$$\sup_{0\le t\le T}\mathbb{E}\Big|\hat{\bar{x}}^{(N)}(t) - \mathbb{E}x_0(t)\Big|^2 = O\Big(\frac{1}{N}\Big),$$
$$\sup_{0\le t\le T}\mathbb{E}\Big|\bar{x}^{(N)}(t) - x_0(t)\Big|^2 = O\Big(\frac{1}{N}\Big).$$

Denote $y_i, 1\le i\le N$, the state of $\mathcal{A}_i$ to the control $\bar{u}_i, 1\le i\le N$ in Problem (**PF**), namely,

$$\begin{cases} dy_i(t) = & \Big[A(t)y_i(t) - B^2(t)R^{-1}(t)\Big(P(t)\hat{\bar{x}}_i(t) + \hat{P}(t)\mathbb{E}x_0(t) + \Phi(t)\Big) \\ & + \alpha y^{(N)}(t) + m(t)\Big]dt + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t), \\ y_i(0) = & x_i(0) \end{cases}$$

where $y^{(N)}(t) = \frac{1}{N}\sum_{j=1}^{N} y_j(t)$. By the difference of states related to $\bar{u}_i$ in (**PF**) and (**LPF**), we have the following estimates:

Lemma 3.2

$$\sup_{0\le t\le T}\mathbb{E}\Big|y^{(N)}(t)-x_0(t)\Big|^2=O\Big(\frac{1}{N}\Big),$$
$$\sup_{1\le i\le N}\left[\sup_{0\le t\le T}\mathbb{E}\Big|y_i(t)-\bar{x}_i(t)\Big|^2\right]=O\Big(\frac{1}{N}\Big),$$
$$\sup_{1\le i\le N}\left[\sup_{0\le t\le T}\mathbb{E}\Big||y_i(t)|^2-|\bar{x}_i(t)|^2\Big|\right]=O\Big(\frac{1}{\sqrt{N}}\Big).$$

This lemma can be proved by applying the same method in Lemma 3.1 and Cauchy-Schwarz inequality.

As to the difference of cost functionals, it holds

Lemma 3.3 *For* $\forall\ 1\le i\le N$,

$$\Big|\mathcal{J}_i(\bar{u}_i,\bar{u}_{-i})-J_i(\bar{u}_i)\Big|=O\Big(\frac{1}{\sqrt{N}}\Big).$$

After addressing the above estimates of states and costs corresponding to control $\bar{u}_i, 1\le i\le N$, given by (22), our goal is to prove that the control strategies set $(\bar{u}_1,\cdots,\bar{u}_N)$ is an ϵ-Nash equilibrium for Problem (**PF**). For any fixed i, $1\le i\le N$, consider an admissible control $u_i\in\mathcal{U}_i$ for $\mathcal{A}_i$ and denote z_i the corresponding state process in Problem (**PF**), that is

$$\begin{cases} dz_i(t)=\Big[A(t)z_i(t)+B(t)u_i(t)+\alpha z^{(N)}(t)+m(t)\Big]dt+\sigma(t)dW_i(t)+\tilde{\sigma}(t)dW(t),\\ z_i(0)=x_i(0)\end{cases}\tag{24}$$

whereas other agents keep the control $\bar{u}_j, 1\le j\le N, j\ne i$, i.e.,

$$\begin{cases} dz_j(t)=\Big[A(t)z_j(t)-B^2(t)R^{-1}(t)\Big(P(t)\hat{\bar{x}}_j(t)+\hat{P}(t)\mathbb{E}x_0(t)+\Phi(t)\Big)\\ \qquad\qquad +\alpha z^{(N)}(t)+m(t)\Big]dt+\sigma(t)dW_j(t)+\tilde{\sigma}(t)dW(t),\\ z_j(0)=x_j(0)\end{cases}\tag{25}$$

where $z^{(N)}(t)=\frac{1}{N}\sum_{j=1}^N z_j(t)$ and $\hat{\bar{x}}_j(t)$ is given by (23). If $\bar{u}_i,\ 1\le i\le N$ is an ϵ-Nash equilibrium with respect to cost $\mathcal{J}_i$, it holds that

$$\mathcal{J}_i(\bar{u}_i,\bar{u}_{-i})\ge\inf_{u_i\in\mathcal{U}_i}\mathcal{J}_i(u_i,\bar{u}_{-i})\ge\mathcal{J}_i(\bar{u}_i,\bar{u}_{-i})-\epsilon.$$

Then, when making the perturbation, we just need to consider $u_i\in\mathcal{U}_i$ such that $\mathcal{J}_i(u_i,\bar{u}_{-i})\le\mathcal{J}_i(\bar{u}_i,\bar{u}_{-i})$, which implies

$$\mathbb{E}\int_0^T R(t)u_i^2(t)dt\le\mathcal{J}_i(u_i,\bar{u}_{-i})\le\mathcal{J}_i(\bar{u}_i,\bar{u}_{-i})=J_i(\bar{u}_i)+O\Big(\frac{1}{\sqrt{N}}\Big).$$

In the limiting cost functional, by the optimality of $(\bar{x}_i, \bar{u}_i)$, we get that $(\bar{x}_i, \bar{u}_i)$ is L^2-bounded. Then we obtain the boundedness of $J_i(\bar{u}_i)$, i.e.,

$$\mathbb{E}\int_0^T R(t)u_i^2(t)dt \leq C_0$$

where C_0 is a positive constant, independent of N. Thus we have

Proposition 3.1 *For any fixed $i, 1 \leq i \leq N$, $\sup_{0\leq t\leq T} \mathbb{E}|z_i(t)|^2$ is bounded.*

Correspondingly, the state process $\bar{x}_i^0$ for agent $\mathcal{A}_i$ under control u_i in Problem (**LPF**) satisfies

$$\begin{cases} d\bar{x}_i^0(t) = \Big[A(t)\bar{x}_i^0(t) + B(t)u_i(t) + \alpha x_0(t) + m(t)\Big]dt + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t), \\ \bar{x}_i^0(0) = x_i(0) \end{cases}$$

and for agent $\mathcal{A}_j$, $j \neq i$,

$$\begin{cases} d\bar{x}_j(t) = \Big[A(t)\bar{x}_j(t) - B^2(t)R^{-1}(t)\Big(P(t)\hat{\bar{x}}_j(t) + \hat{P}(t)\mathbb{E}x_0(t) + \Phi(t)\Big) \\ \qquad\qquad + \alpha x_0(t) + m(t)\Big]dt + \sigma(t)dW_j(t) + \tilde{\sigma}(t)dW(t), \\ \bar{x}_j(0) = x_j(0) \end{cases}$$

where $\hat{\bar{x}}_j$ and x_0 are given in (19).

In order to give necessary estimates of perturbed states and costs in Problem (**PF**) and (**LPF**), we introduce some intermediate states and present some of their properties. Denote

$$z^{(N-1)}(t) = \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} z_j(t), \quad \hat{\bar{x}}^{(N-1)}(t) = \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} \hat{\bar{x}}_j(t).$$

Then by (25), we have

$$\begin{cases} dz^{(N-1)}(t) = \Big[(A(t) + \frac{N-1}{N}\alpha)z^{(N-1)}(t) - B^2(t)R^{-1}(t)\Big(P(t)\hat{\bar{x}}^{(N-1)}(t) + \hat{P}(t)\mathbb{E}x_0(t) \\ \qquad\qquad + \Phi(t)\Big) + \frac{\alpha}{N}z_i(t) + m(t)\Big]dt + \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} \sigma(t)dW_j(t) + \tilde{\sigma}(t)dW(t), \\ z^{(N-1)}(0) = x^{(N-1)}(0) \end{cases}$$

where $x^{(N-1)}(0)=\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}x_j(0)$. Besides, we introduce

$$\begin{cases} d\check{z}_i(t)=&\Big[A(t)\check{z}_i(t)+B(t)u_i(t)+\dfrac{N-1}{N}\alpha\check{z}^{(N-1)}(t)+m(t)\Big]dt\\ &+\sigma(t)dW_i(t)+\tilde{\sigma}(t)dW(t),\\ \check{z}_i(0)=&x_i(0)\end{cases}$$

and for $j\neq i$,

$$\begin{cases} d\check{z}_j(t)=&\Big[A(t)\check{z}_j(t)-B^2(t)R^{-1}(t)\Big(P(t)\hat{\bar{x}}_j(t)+\hat{P}(t)\mathbb{E}x_0(t)+\Phi(t)\Big)\\ &+\dfrac{N-1}{N}\alpha\check{z}^{(N-1)}(t)+m(t)\Big]dt+\sigma(t)dW_j(t)+\tilde{\sigma}(t)dW(t),\\ \check{z}_j(0)=&x_j(0)\end{cases}$$

where $\check{z}^{(N-1)}(t)=\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\check{z}_j(t)$.

We have the following estimates on these states.

Proposition 3.2

$$\begin{aligned}
&\sup_{0\le t\le T}\mathbb{E}\Big|\hat{\bar{x}}^{(N-1)}(t)-\mathbb{E}x_0(t)\Big|^2=O\Big(\frac{1}{N}\Big),\\
&\sup_{0\le t\le T}\mathbb{E}\Big|z^{(N)}(t)-z^{(N-1)}(t)\Big|^2=O\Big(\frac{1}{N}\Big),\\
&\sup_{0\le t\le T}\mathbb{E}\Big|\check{z}^{(N-1)}(t)-z^{(N-1)}(t)\Big|^2=O\Big(\frac{1}{N^2}\Big),\\
&\sup_{0\le t\le T}\mathbb{E}\Big|\check{z}^{(N-1)}(t)-x_0(t)\Big|^2=O\Big(\frac{1}{N}\Big),\\
&\sup_{0\le t\le T}\mathbb{E}\Big|z_i(t)-\check{z}_i(t)\Big|^2=O\Big(\frac{1}{N^2}\Big),\\
&\sup_{0\le t\le T}\mathbb{E}\Big|\check{z}_i(t)-\bar{x}_i^0(t)\Big|^2=O\Big(\frac{1}{N}\Big).
\end{aligned}$$

Further, more direct estimates about states and costs of Problem (**PF**) and (**LPF**) under perturbed controls can be obtained, which enable us to prove Theorem 3.1.

Lemma 3.4

$$\begin{aligned}
&\sup_{0\le t\le T} \mathbb{E}\Big|z_i(t) - \bar{x}_i^0(t)\Big|^2 = O\Big(\frac{1}{N}\Big),\\
&\sup_{0\le t\le T} \mathbb{E}\Big|z^{(N)}(t) - x_0(t)\Big|^2 = O\Big(\frac{1}{N}\Big),\\
&\sup_{0\le t\le T} \mathbb{E}\Big||z_i(t)|^2 - |\bar{x}_i^0(t)|^2\Big| = O\Big(\frac{1}{\sqrt{N}}\Big),\\
&\Big|\mathcal{J}_i(u_i, \bar{u}_{-i}) - J_i(u_i)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big).
\end{aligned}$$

It is worth pointing out that we mainly apply Gronwall's inequality, theory of SDE and the fact

$$\mathbb{E}\left|\int_0^t \frac{1}{N}\sigma(s)\sum_{i=1}^N dW_i(s)\right|^2 = O\Big(\frac{1}{N}\Big)$$

to obtain the proofs of Lemma 3.1, Proposition 3.1 and 3.2. As to the proofs of Lemma 3.3 and 3.4, the similar technique as Lemma 3.2 and theory of SDE are used to derive them. However, due to the paper length, the corresponding proofs of these lemmas and propositions are omitted. In the following, we are going to state the proof of the main theorem in this paper.

Proof of Theorem 3.1: Consider the ϵ-Nash equilibrium for $\mathcal{A}_i$. Combining Lemma 3.3 and 3.4, we have

$$\begin{aligned}
\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) &= J_i(\bar{u}_i) + O\Big(\frac{1}{\sqrt{N}}\Big)\\
&\le J_i(u_i) + O\Big(\frac{1}{\sqrt{N}}\Big)\\
&= \mathcal{J}_i(u_i, \bar{u}_{-i}) + O\Big(\frac{1}{\sqrt{N}}\Big).
\end{aligned}$$

Thus, Theorem 3.1 follows by taking $\epsilon = O\Big(\frac{1}{\sqrt{N}}\Big)$. □

4 Conclusions

Here, we focus on a class of linear-quadratic-Gaussian game of large-population system with partial information. The decentralized strategies and ϵ-Nash equilibrium property are derived by investigating the associated mean-field game. The future works include backward and forward-backward optimization problems of large-population system with full or partial information.

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