QUASIDIFFERENTIALS IN KANTOROVICH SPACES

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Abstract. This is an overview of the quasidifferential calculus for the mappings that arrive at Kantorovich spaces. The necessary optimality conditions are also derived for multiple criteria optimization problems with quasidifferentiable data.

Introduction

A mapping is *quasidifferentiable* at an interior point of the domain of definition provided that there is a directional derivative at the point which can be presented as difference of two sublinear operators. In this event the quasidifferential is introduces by a natural extension of the Minkowski duality. This leads to a rather broad class of the mappings admitting linearization which contains convex and concave operators.

The central problem of the calculus consists in providing some formulas for the quasidifferential of a composite mapping. This problem splits into the three stages: (1)—search for the explicit representation of the directional derivative of the mapping through the directional derivatives of the terms of the composition; (2)—representation of the so-obtained directional derivative as difference of sublinear operators; and (3)—calculation of the quasidifferential of the composite through the quasidifferentials of the terms of the composition. The first stage consists in calculating the relevant limits and uses the tools of the classical analysis with due technical modifications. The second stage is either obvious or involves some artificial tricks. The third stage bases on the Minkowski duality in the version extended to the broader class of quasilinear operators, i.e., differences of sublinear operators.

Using this approach we can derive all the main formulas of quasidifferential calculus, i.e., the quasidifferentials of a sum, a product, a fraction, a composition, a supremum, and an infimum (cp. § 3).

Quasidifferential calculus enables us to derive the necessary optimality conditions with quasidifferentiable constraints given as along the lines of subdifferential calculus; cp. §§ 4, 5. More details are collected in [16, Chapter 5].)

1. Kantorovich Spaces

In this section we briefly outline the main definitions and facts of the theory of Kantorovich spaces which we will use later. More details are collected in [4], [12], [13], and [23].

Key words and phrases. Kantorovich space, sublinear operator, quasidifferential, nonsmooth extremal problem.

1.1. In what follows, $\mathbb{R}$ is the field of the reals. An *ordered vector space* (over $\mathbb{R}$) is some pair $(E, \leq)$, with E a real vector space and $\leq$ an order on E, satisfying the conditions:

(1) if $x \leq y$ and $u \leq v$, then $x + u \leq y + v$ for all $x, y, u, v \in E$;

(2) if $x \leq y$ then $\lambda x \leq \lambda y$ for all $x, y \in E$ and $0 \leq \lambda \in \mathbb{R}$.

Introduction such a relation on a vector space amounts to distinguishing some set $E_+ \subset E$, called the *positive cone* in E and enjoying the properties::

$$E_+ + E_+ \subset E_+, \qquad \lambda E_+ \subset E_+ \quad (0 \leq \lambda \in \mathbb{R}).$$

In this event the quasiorder $\leq$ and the positive cone E_+ are connected as follows: $x \leq y \iff y - x \in E_+$ $(x, y \in E)$. The elements of E_+ are referred to as *positive.* The quasiorder $\leq$ is an order if and only if E_+ is an *ordering* or *salient* cone, i.e., $E_+ \cap -E_+ = \{0\}$.

1.2. An order vector space whose order makes it into a lattice is called a *vector lattice.* To every two elements x and y of a vector lattice E there correspond their *join* or *supremum* $x \vee y := \sup\{x, y\}$ and their *meet* or *infimum* $x \wedge y := \inf\{x, y\}$. In particular, each x in a vector lattice E has the *positive part* $x^+ := x \vee 0$, the *negative part* $x^- := (-x)^+ := -x \wedge 0$, and the *modulus* $|x| := x \vee (-x)$.

A vector lattice E is *Dedekind complete* whenever each nonempty order bounded subset of E has a supremum and an infimum. A Dedekind complete vector lattice is called a *Kantorovich space* or, briefly, a *K-space* in memory of Leonid Kantorovich who was the first to distinguish this class of space in [11]. Below E is an arbitrary K-space unless specified otherwise.

1.3. Elements $x, y \in E$ are called *disjoint*, which is written as $x \perp y$, provided that $|x| \wedge |y| = 0$. The set

$$M^\perp := \{x \in E : (\forall y \in M)\, x \perp y\},$$

with $M \subset E$, is the *disjoint complement* of M.

A *band* of E is a set of the form $M^\perp$, with $M \subset E$ and $M \neq \varnothing$. The inclusion ordered set of all bands of E is the complete Boolean algebra $\mathfrak{B}(E)$ with the Boolean operations $\vee$, the meet; $\wedge$, the join; and $(\cdot)^*$, the Boolean complement acting as follows:

$$L \wedge K = L \cap K, \quad L \vee K = (L \cup K)^{\perp\perp}, \quad L^* = L^\perp \quad (L, K \in \mathfrak{B}(E)).$$

The algebra $\mathfrak{B}(E)$ is called the *base* of E.

1.4. A K-space E is called *extended* or *universally complete* provided that each nonempty set of pairwise disjoint elements of E has a supremum. Universally complete K-spaces are isomorphic if and only if so are their bases. An arbitrary K-space admits an embedding in a universally complete K-space with the same base. This completion is unique to within isomorphism. We now list the main examples of universally complete K-spaces:

(1) The space $L^0(\Omega, \Sigma, \mu)$ of cosets of almost finite measurable functions on Ω, where (Ω, Σ, μ) is a measure space and μ is σ-finite (or, which is more general, has the direct sum property; cp. [12], [17]. The base of $L^0(\Omega, \Sigma, \mu)$ is isomorphic with $\Sigma/\mu^{-1}(0)$, the Boolean factor algebra of measurable sets over negligible sets.

(2) The space $C_\infty(Q)$ of continuous functions on an extremally disconnected compact space Q ranging in the extended real axis and each taking the value $\pm\infty$ only on a nowhere dense subset; cp. [4] and [23]. The base of this K-space is isomorphic to the Boolean algebra of clopen subsets of Q.

1.5. The order of a vector lattice provides a few types of convergence of nets and sequences. A net $(x_\alpha)_{\alpha\in\mathrm{A}}$ in E is *order convergent* or *o-converges* to $x \in E$ provided that there is a decreasing net $(e_\beta)_{\beta\in\mathrm{B}}$ in E such that $\inf\{e_\beta : \beta \in \mathrm{B}\} = 0$ and to each $\beta \in \mathrm{B}$ there is $\alpha(\beta) \in \mathrm{A}$ satisfying $|x_\alpha - x| \le e_\beta$ for all $\alpha(\beta) \le \alpha \in \mathrm{A}$. In this event x is the *order limit* or *o-limit* of (x_α) and we write $x = o\text{-}\lim x_\alpha$ or $x_\alpha \xrightarrow{(o)} x$.

If we replace (e_β) by a sequence $(\lambda_n e)_{n\in\mathbb{N}}$, where $0 \le e \in E_+$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a numeric sequence satisfying $\lim_{n\to\infty} \lambda_n = 0$, then we say that $(x_\alpha)_{\alpha\in\mathrm{A}}$ (relatively uniformly) converges to $x \in E$ with *regulator* e. The element x is the *r-limit* of (x_α). We use the notations $x = r\text{-}\lim_{\alpha\in\mathrm{A}} x_\alpha$ and $x_\alpha \xrightarrow{(r)} x$.

1.6. Let E and F be vector lattices. By $L(E, F)$ we will denote the space of linear operators from E to F. We call a linear operator $T : E \to F$ *positive* in case $T(E_+) \subset F_+$; *regular*, in case T can be presented as difference of positive operators; *order bounded* or, briefly, *o-bounded* in case T maps each order bounded subset of E to an order bounded subset of F.

Every positive operator is obviously order bounded; hence, so is the difference of order bounded operators. Therefore, every regular operator is order bounded. The converse, failing in general, is valid provided that F is Dedekind complete, as follows from the definitive Riesz–Kantorovich Theorem.

1.7. **Riesz–Kantorovich Theorem.** *Let E be a vector lattice and let F be a K-space. The set of all order bounded operators $L^\sim(E, F)$, ordered by the positive cone $L^\sim(E, F)_+$, is a K-space.*

The basics of the theory of regular operators in K-spaces were laid by L. V. Kantorovich in [11] where the above theorem had appeared firstly. F. Riesz formulated a similar theorem for the space of continuous linear functionals on the vector lattice $C([a, b])$ in his celebrated talk [28] at the International Mathematical Congress in Bologna in 1928.

1.8. A linear operator $T : E \to F$ is called *order continuous* (respectively, *sequentially o-continuous* or *order σ-continuous*), provided that $Tx_\alpha \xrightarrow{(o)} 0$ in F for every net (x_α) o-converging to the zero of E (respectively, $Tx_n \xrightarrow{(o)} 0$ in F for every sequence (x_n) that o-converges to the zero of E). The sets of all order continuous and order σ-continuous operators from E to F are denoted by $L^\sim_n(E, F)$ and $L^\sim_\sigma(E, F)$.

1.9. **Theorem.** *Let E and F be vector lattices, with F Dedekind complete. Then $L^\sim_n(E, F)$ and $L^\sim_\sigma(E, F)$ are bands of $L^\sim(E, F)$.*

1.10. Consider some K-space E. Let $L^\sim(E) := L^\sim(E, E)$ be the space of regular endomorphisms of E. Denote by $\mathrm{Orth}(E)$ the least band of $L^\sim(E)$ which contains the identity operator I_E; i.e., $\mathrm{Orth}(E) := \{I_E\}^{\perp\perp}$. The elements of $\mathrm{Orth}(E)$ are called *orthomorphisms*. The orthomorphisms enjoy a few remarkable properties:

(1) $T \in L^\sim(E)$ is an orthomorphism if and only if T commutes with order projections; i.e., $\pi T = T\pi$ for all $\pi \in \mathfrak{P}(E)$.

(2) Every orthomorphism is order continuous.

(3) Every two orthomorphisms commute.

(4) If $\pi, \rho \in \mathrm{Orth}(E)$, then $\rho \circ \pi \in \mathrm{Orth}(E)_+$.

(5) The K-space $\mathrm{Orth}(E)$ with composition as multiplication is an *f-algebra*; i.e., $\mathrm{Orth}(E)$ is a vector lattice and a commutative algebra such that

$$\alpha, \pi, \rho \in \mathrm{Orth}(E)_+, \ \ \pi \perp \rho \Longrightarrow (\alpha\pi) \perp \rho.$$

In what follows $A := \mathrm{Orth}(E)$ is the orthomorphism algebra of E.

1.11. Let X be a vector space, while E is a K-space. An operator $p : X \to E$ is *sublinear* provided that $p(x + y) \le p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for all $x, y \in X$ and $0 \le \lambda \in \mathbb{R}$. Denote the set of all sublinear operators from X to E by $\mathrm{Sbl}(X, E)$. Define addition and multiplication on $\mathrm{Sbl}(X, E)$ pointwise, i.e., $(p + q)x := p(x) + q(x)$ and $p \le q \iff (\forall x \in X)\, p(x) \le q(x)\ (x \in X)$. Clearly, the addition and order agree with one another as usual: $p_1 \le p_2 \implies p_1 + q \le p_2 + q$.

To each $p \in \mathrm{Sbl}(X, E)$ we can uniquely assign its *support set*

$$\partial p := \{T \in L(X, E) :\ Tx \le p(x)\ (x \in X)\}.$$

Put $\mathrm{CS}_c(X, E) := \{\partial p :\ p \in \mathrm{Sbl}(X, E)\}$.

1.12. Denote by $\mathrm{QL}(X, E)$ the set of all mappings from X to E that are representable as differences of sublinear operators. Under the natural algebraic operations, $\mathrm{QL}(X, E)$ becomes an A-module. Indeed, if $f = p - q$ for some $p, q \in \mathrm{Sbl}(X, E)$ and $\alpha \in \mathrm{Orth}(E)$ then

$$\alpha f := \alpha \circ f = (\alpha^+ p + \alpha^- q) - (\alpha^- p + \alpha^+ q),$$

and so $\alpha f \in \mathrm{QL}(X, E)$. The elements of $\mathrm{QL}(X, E)$ are called *quasilinear operators.*

In other words, $\mathrm{QL}(X, E) := \mathrm{Sbl}(X, E) - \mathrm{Sbl}(X, E)$ and the structure of an ordered A-module is induced from E^X, i.e., this is done pointwise. In particular, the order on $\mathrm{QL}(X, E)$ is determined from the positive cone $\{f \in \mathrm{QL}(X, E) :\ f(x) \geqslant 0\ (x \in X)\}$.

To each pair of sublinear operators $p, q \in \mathrm{Sbl}(X, E)$ we put into correspondence the quasilinear operator $\phi(p, q) : x \mapsto p(x) - q(x)\ (x \in X)$. Observe that the pairs (p, q) and (p', q') yield the same quasilinear operator, i.e. $\phi(p, g) = \phi(p', g')$, provided that $p - q = p' - q'$. Wi will call these pairs *equivalent.* The coset of (p, q) will be denoted by $[p, q]$.

1.13. To each pair $p, q \in \mathrm{Sbl}(X, E)$ we can assign the pair of their support sets ∂p and ∂q. Now, to each quasilinear operator $l = p - q$ there corresponds the coset $[p, q]$ as well as the coset $[\partial p, \partial q]$.

Finally, if $l = p - q$ then we can put $\mathcal{D}l := [\partial p, \partial q]$ and observe that $\mathcal{D}l$ does not depend on the particular representation of l as difference of sublinear operators. The element $\mathcal{D}l$ of the A-module $\mathrm{CS}_c(X, E)$ is called the *quasidifferential* of l (at the zero). Moreover, for the support sets ∂p and ∂q we use the following terms and denotations: $\underline{\partial} l := \partial p$ is the *subdifferential* of l (at the zero) and $\overline{\partial} l := \partial q$ is the *superdifferential* of l (at the zero).

1.13. Let us inspect what happens to various operations under the mapping $\mathcal{D} : l \mapsto \mathcal{D}$. If $\alpha \in \mathrm{Orth}(E)$ and $l, l_1, \dots, l_n \in \mathrm{QL}(X, E)$ then

$$\begin{aligned}
\mathcal{D}(\alpha l) &= \alpha \mathcal{D} l = [\alpha^+ \underline{\partial} l + \alpha^- \overline{\partial} l,\ \alpha^- \underline{\partial} l + \alpha^+ \overline{\partial} l], \\
\mathcal{D}(l_1 + \ldots + l_n) &= \mathcal{D} l_1 + \ldots + \mathcal{D} l_n = \\
&= [\,\underline{\partial} l_1 + \ldots + \underline{\partial} l_n,\ \overline{\partial} l_1 + \ldots + \overline{\partial} l_n], \\
\mathcal{D}\big(l_1 \vee \ldots \vee l_n\big) &= \left[\mathrm{op} \bigcup_{i=1}^{n} \Big(\underline{\partial} l_i + \sum_{j=1, j \ne i}^{n} \overline{\partial} l_j\Big),\ \sum_{j=1}^{n} \overline{\partial} l_j\right], \\
\mathcal{D}\big(l_1 \wedge \ldots \wedge l_n\big) &= \left[\sum_{j=1}^{n} \underline{\partial} l_j,\ \mathrm{op} \bigcup_{i=1}^{n} \Big(\overline{\partial} l_i + \sum_{j=1, j \ne i}^{n} \underline{\partial} l_j\Big)\right].
\end{aligned}$$

For more details see [16, Chapter 5]):

2. Quasidifferentiable Mappings

Here we will recall the concept of quasidifferential and consider the simple properties of directional derivatives.

2.1. Let X be a vector space, and let E be a K-space. Put $E^\bullet := E \cup \{+\infty\}$. Consider $f : X \to E^\bullet$ and $x_0 \in \operatorname{core}(\operatorname{dom}(f))$. The record $x \in \operatorname{core}(C)$ means that $C - x$ is an absorbing set. If, given $h \in X$, there exists

$$f'(x_0)h := o\text{-}\lim_{\alpha \downarrow 0} \frac{f(x_0 + \alpha h) - f(x_0)}{\alpha}$$
$$= \inf_{\varepsilon > 0} \sup_{0<\alpha<\varepsilon} \frac{f(x_0 + \alpha h) - f(x_0)}{\alpha} = \sup_{\varepsilon > 0} \inf_{0<\alpha<\varepsilon} \frac{f(x_0 + \alpha h) - f(x_0)}{\alpha},$$

then we call $f'(x_0)h$ the *one-sided derivative* or, rarely, the *Dini derivative* of f at x_0 in direction h. Assume that, given x_0, the element $f'(x_0)h$ exists for all $h \in X$. Then the mapping $f'(x_0) : X \to E$ appears which is also called the *one-sided directional derivative* or *Dini derivative* at x_0. In this event we also say that f is *directionally differentiable* at x_0.

2.2. A mapping f is *quasidifferentiable* at x_0 provided that

(1) f is directionally differentiable at x_0;

(2) $f'(x_0) : X \to E$ is a quasilinear mapping.

If f is quasidifferentiable at x_0, then the Minkowski duality assign to the quasilinear operator $f'(x_0) \in \mathrm{QL}(X, E)$ the element $\mathcal{D}\big(f'(x_0)\big) \in [\mathrm{CS}_c(X, E)]$ which is called the *quasidifferential* of f at x_0 and denoted by $\mathcal{D}f(x_0)$.

If $f'(x_0)$ can be presented as difference of sublinear operators $p, q \in \mathrm{Sbl}(X, E)$ so that $\mathcal{D}f(x_0) = [\partial p, \partial q]$, then

$$f'(x_0)h = \sup_{S \in \partial p} S(h) - \sup_{T \in \partial q} T(h) = p(h) - q(h) \quad (h \in X).$$

In this event ∂p and ∂q are respectively called the *subdifferential* and *superdifferential* of f at x_0 and denoted by $\underline{\partial} f(x_0)$ and $\overline{\partial} f(x_0)$. In other words,

$$\mathcal{D}f(x_0) := [\partial p, \partial q] := [\underline{\partial} f(x_0), \overline{\partial} f(x_0)].$$

Assume that a quasidifferentiable mapping f has the quasidifferential at x_0 of the form $\mathcal{D}f(x_0) = [\underline{\partial} f(x_0), \{0\}]$ or $\mathcal{D}f(x_0) = [\{0\}, \overline{\partial} f(x_0)]$). Then we say that f is *subdifferentiable* or, respectively, *superdifferentiable* at x_0. If f has the directional derivative $T := f'(x_0)$ at some point $x_0 \in \operatorname{core}(\operatorname{dom}(f))$ which is a linear operator, then f is subdifferentiable and superdifferentiable simultaneously and, moreover, $\mathcal{D}f(x_0) = [\{T\}, \{0\}] = [\{0\}, \{-T\}]$.

Each convex operator f is subdifferentiable at every point $x_0 \in \operatorname{core}(\operatorname{dom}(f))$, since we have the sublinear directional derivative $f'(x_0)$. In this event $\underline{\partial} f(x_0) = \partial f(x)$. An operator f is called *concave* provided that $-f$ is a convex operator. A concave operator f is superdifferentiable at every point $x_0 \in \operatorname{core}(\operatorname{dom}(-f))$ and, moreover, $\overline{\partial} f(x_0) = -\partial(-f)(x_0)$. In this event the directional derivative $f'(x_0)$ exists too and presents a *superlinear operator*; i.e., $-f'(x_0)$ is a sublinear operator.

The differences of convex operators or, which is the same, the sums of convex and concave operators comprise a much broader class of quasidifferentiable mappings.

2.3. Let E and F be K-spaces. Consider a mapping $g : E \to F^\bullet$ directionally differentiable at $e_0 \in \operatorname{core}(\operatorname{dom}(g))$. Take $u \in E$ and $d \in F$. Suppose that for every sequence $(e_n) \subset E$, $e_n \downarrow 0$, we have

$$\inf_{m\in\mathbb{N}} \sup_{\substack{0<\alpha<1/m \\ |u'-u|\leqslant e_m}} \left| \frac{g(e_0+\alpha u') - g(e_0)}{\alpha} - d \right| = 0.$$

Then d is called the *Hadamard derivative* of g at e_0 in direction u, and we put $g'(e_0)u := d$. This denotation is justified by the obvious reason that if the Hadamard derivative exists then so does the Dini derivative at the same point and in the same direction and the two derivatives coincide. Therefore, we can define the Hadamard derivative of g at e_0 in direction u by the formulas

$$g'(e_0)u := g'_{e_0}(u) := \inf_{m\in\mathbb{N}} \sup_{\substack{0<\alpha<1/m \\ |u'-u|\leqslant e_m}} \frac{g(e_0+\alpha u') - g(e_0)}{\alpha}$$

$$= \sup_{m\in\mathbb{N}} \inf_{\substack{0<\alpha<1/m \\ |u'-u|\leqslant e_m}} \frac{g(e_0+\alpha u') - g(e_0)}{\alpha}.$$

If the Hadamard derivative $g'(e_0)u$ exists at e_0 in every direction $u \in E$ then we say that g is *Hadamard differentiable* at e_0.

The definition of Hadamard derivative is simplified if F is a regular K-space. Recall that a K-space F is called *regular* provided that for every nested sequence of subsets $F \supset A_1 \supset \ldots \supset A_n \supset \ldots$ satisfying $a = \inf_n \sup(A_n)$ there are finite subsets $A'_n \subset A_n$ enjoying the property $o\text{-}\lim_{n\to\infty} \sup(A'_n) = a$.

Let F be a regular K-space. An element $d \in F$ is the Hadamard derivative of $g : E \to F^\bullet$ at $e_0 \in \operatorname{core}(\operatorname{dom}(g))$ in direction $u \in E$ if and only if for all sequences $(\alpha_n) \subset \mathbb{R}$ and $(u_n) \subset E$ such that $\alpha_n \downarrow 0$ and $u_n \xrightarrow{(o)} u$ we have

$$d = o\text{-}\lim_{n\to\infty} \frac{g(e_0+\alpha_n u_n) - g(e_0)}{\alpha_n}$$

$$= \inf_{m\in\mathbb{N}} \sup_{n\geqslant m} \frac{g(e_0+\alpha_n u_n) - g(e_0)}{\alpha_n} = \sup_{m\in\mathbb{N}} \inf_{n\geqslant m} \frac{g(e_0+\alpha_n u_n) - g(e_0)}{\alpha_n}.$$

2.4. In the situation under study the Hadamard differentiability of g does not guarantee the continuity of the directional derivative $g'(e_0)(\cdot)$ as this happens in case $E = \mathbb{R}^n$ and $F = \mathbb{R}$; cp. [10]. Let us consider the two cases in which the Hadamard differentiable mapping has the directional derivative continuous. We will understand continuity of the directional derivative as follows:

A mapping $\varphi : E \to F$ is called *mo-continuous* at $u_0 \in E$ provided that for every sequence $(e_n) \subset E$, $e_n \downarrow 0$, we have

$$\inf_{n\in\mathbb{N}} \sup_{|u-u_0|\leqslant e_n} |\varphi(u) - \varphi(u_0)| = 0.$$

If F is a regular K-space, then *mo*-continuity means sequential *o*-continuity. Recall that $U \subset E$ is called a *normal* subset whenever $u_1 \leqslant e \leqslant u_2$ implies that $e \in U$ for all $u_1, u_2 \in U$ and $e \in E$.

(1) *Assume that $g : E \to F^\bullet$ is Dini differentiable at $e_0 \in \operatorname{core}(\operatorname{dom}(g))$. Assume further that there are a normal subset $U \subset E$ and an mo-continuous sublinear operator $p : E \to F$ such that $e_0 \in \operatorname{core}(U)$ and*

$$|g(u_1) - g(u_2)| \leqslant p(u_1 - u_2) \quad (u_1, u_2 \in U).$$

Then g is Hadamard differentiable at e_0 and the directional derivative $g'(e_0)(\cdot)$ is mo-continuous.

(2) *Assume that F is a regular K-space. If $g : E \to F^\bullet$ is Hadamard differentiable at $e_0 \in \operatorname{core}(\operatorname{dom}(g))$ then the directional derivative $g'(e_0)(\cdot)$ is sequentially o-continuous.*

2.5. In [21] V. F. Demyanov and A. M. Rubinov gave the definition of the quasidifferential of a mapping from a Banach space to a Banach K-space. We provide a somewhat more general definition of quasidifferentiability.

Let X be a topological vector space, and let E be a topological K-space. The latter means that E is simultaneously a topological vector space and a K-space having a'filter base of neighborhoods of the zero which consists of normal subsets of E. If we replace the order limit in the definition of directional derivative in 2.1 by the topological limit an require in the definition of quasidifferentiability in 2.2 that the directional derivative may be presented as difference of two continuous sublinear operators then we arrive at the definition of topological quasidifferential we will use later in 5.2.

We thus encounter the problem of how the two definitions of quasidifferential in 2.1 and 2.5 are related. Note that if E enjoys condition (A), i.e., order convergence implies topological convergence; then the two definitions yield the same.

3. Quasidifferential Calculus

Let us consider the main formulas for calculating the quasidifferentials of mapping with range in a K-space. Quasidifferential calculus of scalar functions on $\mathbb{R}^n$ was developed by V. F. Demyanov, L. N. Polyakova, and A. M. Rubinov in [8] and [9]; also see [7] and [10]. In [10] there was showed how to apply the methods of quasidifferential calculus to the mappings with range in a Banach K-space. The quasidifferentials of the operators with range an arbitrary K-space were investigated in [1] and [3]; also see [16].

3.1. Let $f_1, \dots, f_n : X \to E^\bullet$ be quasidifferentiable at $x_0 \in \bigcap_{i=1}^n \operatorname{core}(\operatorname{dom}(f_i))$. Then the sum $f := f_1 + \dots + f_n$, supremum $g := f_1 \vee \dots \vee f_n$, and infimum $h := f_1 \wedge \dots \wedge f_n$ of $f_1, \dots, f_n$ are quasidifferentiable at x_0 as well and we have the formulas:

$$\begin{aligned}\mathcal{D}f(x_0) &= \mathcal{D}f_1(x_0) + \dots + \mathcal{D}f_n(x_0)\\ &= [\underline{\partial} f_1(x_0) + \dots + \underline{\partial} f_n(x_0),\, \overline{\partial} f_1(x_0) + \dots + \overline{\partial} f_n(x_0)].\end{aligned}$$

$$\underline{\partial} g(x_0) = \bigcup_{(\alpha_1,\dots,\alpha_n) \in \Gamma_n(x_0)} \sum_{k=1}^n \alpha_k \Big(\underline{\partial} f_k(x_0) + \sum_{l \neq k} \overline{\partial} f_l(x_0)\Big),$$

$$\overline{\partial} g(x_0) = \sum_{k=1}^n \overline{\partial} f_k(x_0), \quad \underline{\partial} h(x_0) = \sum_{k=1}^n \underline{\partial} f_k(x_0),$$

$$\overline{\partial} h(x_0) = \bigcup_{(\alpha_1,\dots,\alpha_n) \in \Delta_n(x_0)} \sum_{k=1}^n \alpha_k \Big(\overline{\partial} f_k(x_0) + \sum_{l \neq k} \underline{\partial} f_l(x_0)\Big),$$

where

$$\Gamma_n(x_0) := \Gamma_n(x_0; f_1, \dots, f_n) := \Big\{ (\alpha_1, \dots, \alpha_n) : \ \alpha_k \in \operatorname{Orth}_+(E), \\ \sum_{k=1}^{n} \alpha_k = I_E, \ \sum_{k=1}^{n} \alpha_k f_k(x_0) = f(x_0) \Big\}.$$

$$\Delta_n(x_0) := \Delta_n(x_0; f_1, \dots, f_n) := \Big\{ (\alpha_1, \dots, \alpha_n) : \ \alpha_k \in \operatorname{Orth}_+(E), \\ \sum_{k=1}^{n} \alpha_k = I_E, \ \sum_{k=1}^{n} \alpha_k f_k(x_0) = g(x_0) \Big\}.$$

3.2. Let $f : X \to E^\bullet$ and $g : X \to \operatorname{Orth}(E)^\bullet$ be quasidifferentiable at $x_0 \in \operatorname{core}(\operatorname{dom}(f)) \cap \operatorname{core}(\operatorname{dom}(g))$. Then the mapping $gf = g \cdot f : X \to E^\bullet$, acting by the rule $gf : x \mapsto g(x)f(x)$, is quasidifferentiable at x_0 too and we have the formulas

$$\mathcal{D}(g \cdot f)(x_0) = g(x_0)\mathcal{D}f(x_0) + \mathcal{D}g(x_0)f(x_0),$$

with

$$\begin{aligned} \underline{\partial}(gf)(x) &= g^+(x_0)\underline{\partial}f(x_0) + g^-(x_0)\overline{\partial}f(x_0) \\ &\quad + \underline{\partial}g(x_0)f^+(x_0) + \overline{\partial}g(x_0)f^-(x_0), \\ \overline{\partial}(gf)(x) &= g^+(x_0)\overline{\partial}f(x_0) + g^-(x_0)\underline{\partial}f(x_0) \\ &\quad + \overline{\partial}g(x_0)f^+(x_0) + \underline{\partial}g(x_0)f^-(x_0). \end{aligned}$$

3.3. We now list some conditions for the composite of quasidifferentiable mappings be quasidifferentiable as well.

Theorem. *Let X be a vector space, while E and F are K-spaces. Assume that $f : X \to E^\bullet$ is quasidifferentiable at $x_0 \in \operatorname{core}(\operatorname{dom}(f))$, and $g : E \to F^\bullet$ is quasidifferentiable and Hadamard differentiable at $e_0 := f(x_0) \in \operatorname{core}(\operatorname{dom}(g))$ with mo-continuous derivative $g'(e_0)(\cdot)$. Assume further that the quasidifferential $\mathcal{D}g(e_0)$ is defined by the pair of order bounded support sets $\underline{\partial}g(e_0)$ and $\overline{\partial}g(e_0)$ in $L^\sim(E, F)$. Then $g \circ f$ is quasidifferentiable at x_0. If $\underline{\partial}g(e_0) \cup \overline{\partial}g(e_0) \subset [\Lambda_1, \Lambda_2]$ for some $\Lambda_1, \Lambda_2 \in L^\sim(E, F)$, then*

$$\mathcal{D}(g \circ f)(x_0) = \left[\bigcup_{C \in \underline{\partial}g(e_0)} \partial(P_C), \ \bigcup_{C \in \overline{\partial}g(e_0)} \partial(P_C) \right],$$

where

$$P_C(x_0) := (C - \Lambda_1) \sup_{S \in \underline{\partial}f(x_0)} S(x_0) + (\Lambda_2 - C) \sup_{T \in \overline{\partial}f(x_0)} T(x_0).$$

This formula was derived under somewhat different assumptions by A. M. Rubinov; cp. [21].

3.4. Let X and Y be Banach spaces, while E is an arbitrary K-space. A mapping $f : X \times Y \to E$ is called *convex-concave* on $X \times Y$ provided that $x \mapsto f(x, y)$ is convex at every fixed $y \in Y$, and $y \mapsto f(x, y)$ is concave at every fixed $x \in X$. The *partial subdifferentials* $\partial_x(x_0, y_0)$ and $\partial_y(x_0, y_0)$ are defined as

$$\begin{aligned}\partial_x(x_0, y_0) &:= \underline{\partial} f_1(x_0)\\ &= \{T \in L(X, E) : \ f(x, y_0) - f(x_0, y_0) \geq T(x - x_0) \ \ (x \in X)\};\\ \partial_y(x_0, y_0) &:= \overline{\partial} f_2(y_0)\\ &= \{S \in L(Y, E) : \ f(x_0, y) - f(x_0, y_0) \leq S(y - y_0) \ \ (y \in Y)\},\end{aligned}$$

where $f_1(x) := f(x, y_0)$ and $f_2(y) := f(x_0, y)$.

The problem of quasidifferentiability of a convex-concave function was studied by V. F. Demyanov and L. V. Vasilieva in [7]. In more detail, assume that $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and $E = \mathbb{R}$. Then a convex-concave function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^\bullet$ is quasidifferentiable at an interior point (x_0, y_0) of its domain of definition. Moreover,

$$\mathcal{D}f(x_0, y_0) = [\partial_x(x_0, y_0) \times \mathbb{0}_m, \ \mathbb{0}_n \times \partial_y(x_0, y_0)].$$

In this connection it is of interest to find some conditions for a convex-concave operator to be quasidifferentiable.

3.5. V. V. Gorokhovik introduced in [5] and [6] the concept of ε-quasidifferential for a real function on a finite dimensional space. He also studies the properties of ε-quasidifferential; cp. [10]. The article [6] contains the main formulas for calculating ε-quasidifferentials and demonstrates then the necessary and sufficient conditions of a local extremum can be formulated for a broad class of functions in terms of upper and lower local approximations.

This raises the problem of deriving the formulas for calculating the ε-quasidifferentials of the mapping from an arbitrary vector space to an arbitrary K-space and studying the Gorokhovik quasidifferentiability of these mappings.

4. Necessary Optimality Conditions

In this section we will present the necessary conditions of an extremum for quasidifferentiable mappings with range in a K-space, staying in the algebraic framework. Furthermore, we will keep the same terminology and notations as in [16, § 5.1]. The main results of this section were obtained in [2]; also see [16, Chapter 6].

4.1. Assume that X is a vector space and E is an arbitrary K-space. Consider some program (C, f), i.e., a multiple criteria extremal problem $x \in C, \ f(x) \to \inf$, where $C \subset X$ is some set and $f : X \to E^\bullet$ is a mapping that is assumed quasidifferentiable at the appropriate point of $\operatorname{core}(\operatorname{dom}(f))$. A point $x_0 \in C$ is an *ideal local infimum (supremum)* of the program $x \in C, \ f(x) \to \inf$ (or $x \in C, \ f(x) \to \sup$) provided that there is $U \subset X$ such that $0 \in \operatorname{core} U$ and $f(x_0) = \inf\{f(x) : \ x \in C \cap (x_0 + U)\}$ (respectively, $f(x_0) = \sup\{f(x) : \ x \in C \cap (x_0 + U)\}$). We will understand a local extremum by analogy, unless specified otherwise.

4.2. Let us formulate the necessary optimality conditions in the unconstrained program, i.e., in case $C = X$.

(1) Theorem. *Let $f : X \to E^\bullet$ be quasidifferentiable at $x_0 \in \operatorname{core}(\operatorname{dom}(f))$. If $x_0$is an ideal local optimum in the unconstrained vector program $f(x) \to \inf$, then $\overline{\partial} f(x_0) \subset \underline{\partial} f(x_0)$ or, which is the same, $\mathcal{D}f(x_0) \geq 0$.*

The necessary optimality conditions admit the equivalent reformulation:

$$\overline{\partial} f(x_0) \subset \underline{\partial} f(x_0) \iff 0 \in \bigcap_{v \in \overline{\partial} f(x_0)} \big(\underline{\partial} f(x_0) - v\big).$$

4.3. Consider some vector program of the form (C, f), with $C := \{x \in X : g(x) \leqslant 0\}$, on assuming that f and g are quasidifferentiable at the appropriate point. We will denote this program by (g, f). Let us introduce the quasiregularity condition we will use below.

Let X be an arbitrary vector space, while E and F are some K-spaces. Recall that a mapping T from F to E is a *Maharam operator* provided that T is order continuous and preserves order intervals; cp. [16, Chapter 4]. Consider $f : X \to E^{\bullet}$ and $g : X \to F^{\bullet}$. The vector program (g, f) is called *quasiregular* at $x_0 \in \operatorname{core}(\operatorname{dom}(g))$ provided that the following hold:

(a) there are a sublinear Maharam operator $r : F \to E$ and an absorbing $U \subset X$ such that $\pi_x f(x_0) \leq \pi_x f(x)$ for all $x \in x_0 + U$, where $\pi_x := [(r \circ g(x))^{-}]$ is the projection on the bang generated by $(r \circ g(x))^{-}$;

(b) $\pi T \circ \overline{\partial} g(x_0) \cap \pi T \circ \underline{\partial} g(x_0) = \varnothing$ for all $T \in \partial r(g(x_0))$ and every nonzero projection $\pi \in \mathfrak{P}(E)$.

Condition (a) is satisfied for instance in the case that there is a sublinear Maharam operator $r : F \to E$ such that $g(x) \nleq 0$ implies $r \circ g(x) \geq 0$ for all $x \in X$.

4.4. **Theorem.** *Assume the quasiregularity condition 4.3. If a feasible point x_0 is an ideal local optimum of the quasiregular quasidifferentiable problem (g, f), then to given $s \in \overline{\partial} f(x_0)$ and $S \in \overline{\partial} g(x_0)$ there are a positive orthomorphism $\alpha \in \operatorname{Orth}_{+}(E)$ and a Maharam operator $\gamma \in L_{+}(F, E)$ such that the following system of conditions is compatible:*

$$\ker \alpha = \{0\}, \quad \gamma \circ g(x_0) = 0,$$
$$0 \in \alpha(\underline{\partial} f(x_0) - s) + \gamma \circ (\underline{\partial} g(x_0) - S).$$

4.5. Observe a few corollaries of Theorem 4.4:

(1) *Assume that the quasidifferentiable vector program (g, f) enjoys the quasiregularity condition 4.3 at a feasible point $x_0 \in X$ and, moreover, $\pi r \circ g(x_0) < 0$ for every nonzero projection $\pi \in \mathfrak{P}(E)$. If x_0 is an ideal optimum of (g, f), then $\overline{\partial} f(x_0) \subset \underline{\partial} f(x_0)$.*

(2) In the case of convex programs, the quasiregularity in the sense of 4.3 agrees with the quasiregularity in the sense of [16, 5.2.1]. Indeed, if f and $\tilde{g} := r \circ g$ are convex operators, then

$$\underline{\partial} f(x_0) = \partial f(x_0), \quad \overline{\partial} f(x_0) = \{0\}, \quad \underline{\partial} \tilde{g}(x_0) = \partial \tilde{g}(x_0), \quad \overline{\partial} \tilde{g}(x_0) = \{0\}.$$

Furthermore, (b) of the quasiregularity condition 4.3, i.e., the equality $\pi T \circ \overline{\partial} g(x_0) \cap \pi T \circ \underline{\partial} g(x_0) = \varnothing$ valid for all $T \in \partial r$ means in this event that $0 \notin \pi \partial \tilde{g}(x_0)$ which amounts to the existence of $h_0 \in X$ satisfying $\pi \tilde{g}'(x_0) h_0 < 0$. But

$$\pi \tilde{g}'(x_0) h_0 = \inf_{t > 0} \frac{\pi \tilde{g}(x_0 + t h_0) - \pi \tilde{g}(x_0)}{t};$$

hence, there are a projection $0 \neq \pi' \leq \pi$ and a real $t_0 > 0$ such that $\pi' g(x_0 + t_0 h_0) < \pi' g(x_0) \leq 0$. Therefore, the conditions of 4.3 are as follows: For some sublinear Maharam operator $r : F \to E$ and an absorbing set U, we have firstly that $\pi_x f(x_0) \leq \pi_x f(x)$ for all $x \in x_0 + U$ where $\pi_x := [g(x)^{-}]$ and, secondly, for each nonzero projection $\pi \in \mathfrak{P}(E)$ there are a projection $0 \neq \pi' \leq \pi$ and $x' \in X$ with $x' := x_0 + t_0 h_0$ such that $(r \circ g)(x') < 0$.

(3) *A feasible point* $x_0 \in \operatorname{core}(\operatorname{dom}(f)) \cap \operatorname{core}(\operatorname{dom}(g))$ *is an ideal optimum of a quasiregular convex program* (g, f) *if and only if there are a positive orthomorphism* $\alpha \in \operatorname{Orth}_+(E)$ *and a Maharam operator* $\gamma \in L_+(F, E)$ *such that the following system of conditions is compatible:*

$$\ker \alpha = \{0\}, \quad \gamma \circ g(x_0) = 0, \quad 0 \in \alpha \partial f(x_0) + \gamma \circ \partial g(x_0).$$

(4) Let $f, \varphi : X \to E^\bullet$ and $g, \psi : X \to F^\bullet$ be quasidifferentiable at an appropriate point. Reduce each of the extremal problems

$$\psi(x) \geq 0, \quad f(x) \to \inf;$$
$$g(x) \leq 0, \quad \varphi(x) \to \sup;$$
$$\psi(x) \geq 0, \quad \varphi(x) \to \sup$$

to the above problem (g, f) by letting $g := -\psi$ and $f := -\varphi$. In this event we encounter some obvious modifications of the quasiregularity condition. Indeed, the quasiregularity condition for the program $\psi(x) \geq 0,\ f(x) \to \inf$ means the existence of a sublinear Maharam operator $r : F \to E$ such that, firstly,, $\pi_x f(x_0) \leq \pi_x f(x)$ for all $x \in X$ with $\pi_x := [(r \circ \psi(x))^+]$ and, secondly, $\pi T \circ \underline{\partial}\psi(x_0) \cap \pi T \circ \overline{\partial}\psi(x_0) = \varnothing$ for every $T \in \partial r(\psi(x_0))$ and every nonzero projection $\pi \in \mathfrak{P}(E)$.

If a feasible point x_0 *is an ideal local optimum of a general quasiregular quasidifferentiable program* $\psi(x) \geq 0,\ f(x) \to \inf$, *then to all* $s \in \overline{\partial} f(x_0)$ *and* $S \in \underline{\partial}\psi(x_0)$ *there are a positive orthomorphism* $\alpha \in \operatorname{Orth}_+(E)$ *and a Maharam operator* $\gamma \in L_+(F, E)$ *such that the following system of conditions is compatible:*

$$\ker \alpha = \{0\}, \quad \gamma \circ \psi(x_0) = 0,$$
$$0 \in \alpha(\underline{\partial} f(x_0) - s) + \gamma \circ (\overline{\partial}\psi(x_0) - S).$$

(5) Theorem 4.4, the main result of 2.4, was obtained by E. K. Basaeva in [2]; also see [16]. In the scalar case $E = F = \mathbb{R}$ and $X = \mathbb{R}^n$ Theorem 4.4 is well known; for instance, see [16, Theorem V.3.2]. In this event the quasiregularity condition may be slackened, but this will lead to slackening the necessary optimality conditions. In more detail, if the closure of $\{h \in X : g'(x_0)h < 0\}$ is $\{h \in X : g'(x_0)h \leqslant 0\}$ (which is regularity), then

$$0 \in (\underline{\partial} f(x_0) - s) + \operatorname{cl}\operatorname{cone}(\underline{\partial} g(x_0) - S),$$

for all $s \in \overline{\partial} f(x_0)$ and $S \in \overline{\partial} g(x_0)$, where $\operatorname{cl}\operatorname{cone}(\mathcal{U})$ stands for the closed cone hull of $\mathcal{U}$.

4.6. The quasiregularity condition 4.3 allows us to write down the necessary optimality conditions for $S \in \overline{\partial} g(x_0)$ provided that $\pi T \circ \overline{\partial} g(x_0) \cap \pi T \circ \underline{\partial} g(x_0) = \varnothing$ for every $T \in \partial r(g(x_0))$ and every nonzero $\pi \in \mathfrak{P}(E)$. If the last condition is not fulfilled then there is a maximal projection ρ such that the condition is valid for all projections $\pi \leq \rho$, and part of the necessary conditions may be written as in 4.4 but with a constraint on ρ. Namely,

$$0 \in \rho\alpha(\underline{\partial} f(x_0) - s) + \rho\gamma(\underline{\partial} g(x_0) - S).$$

The necessary optimality conditions at ρ^d will principally different: To all $s \in \overline{\partial} f(x_0)$ and $S \in \overline{\partial} g(x_0)$ there is a positive orthomorphism $\alpha \in \operatorname{Orth}_+(\rho^d E)$, $\ker \alpha =$

$\{0\}$ such that

$$0 \in \rho^d \alpha(\underline{\partial} f(x_0) - s) + \rho^d \operatorname{Cop}(\partial r(g(x_0)) \circ \underline{\partial} g(x_0) - S),$$

were $\operatorname{Cop}(\mathcal{U})$ stands for $\operatorname{cl}(\operatorname{mix}(\operatorname{co}\mathcal{U}))$, while $\operatorname{co}(A)$ is the convex hull of A, while $\operatorname{mix}(A)$ is the collection of all mixings of the elements of A by all partitions of unity in $\mathfrak{P}(E)$ (cp. [16, Appendix 4]), and $\operatorname{cl}(A)$ is the closure of A with respect to pointwise o-convergence. This fact can be established on using the Vector Bipolar Theorem [16, p. 190]; also see [15].

5. Accounting for the Containment Constraints

Let us consider the necessary optimality conditions in the case that the program under study involves the constraint that the solution must belong to a given set. The regularity condition on the set is convenient to formulate in terms of the topology of the ambient vector space. In other words, we have to define topological quasidifferentials. To this end it suffices to change the scope of the concept of quasilinear mapping so as to imply now that a quasilinear operator must admit representation as difference of *continuous* sublinear operators.

5.1. Let X be a topological vector space, while E is a topological K-space an A^c is the algebra of continuous orthomorphisms of E. Assume that the positive cone of a topological K-space is normal by default. In this event the Minkowski duality ∂ defines a bijection between the sets of (total = everywhere defined) continuous sublinear operators and the collection of equicontinuous support sets; cp. [16, 3.2.2 (1)].

Let $\mathrm{QL}^c(X, E)$ stand for the part of $\mathrm{QL}(X, E)$ consisting of the quasilinear operators that are presentable as differences of continuous sublinear operators. Clearly, $\mathrm{QL}^c(X, E)$ is a lattice ordered A^c-module. The module and lattice operations as well as order, are induced from $\mathrm{QL}(X, E)$. We call the members of $\mathrm{QL}^c(X, E)$ *continuous quasilinear operators*.

By analogy, the collection of equicontinuous support sets $\mathrm{CS}_c^c(X, E)$ is defined as the part of $\mathrm{CS}_c(X, E)$ consisting of the support sets of sublinear operators; cp. [16, .2.2 (1)]. The relevant restriction of the isomorphism $\mathcal{D}$ of 1.13 will be denoted by $\mathcal{D}^c$. Clearly, $\mathcal{D}^c$ is an isomorphism between the A^c-modules $[\mathrm{CS}_c^c(X, E)]$ and $\mathrm{QL}^c(X, E)$.

5.2. Therefore, to preserve the formulas of quasidifferential calculus of § 4 in the topological setting it suffices ti require that the directional derivative in the definition of 2.2 may be presented as difference of continuous sublinear operators.

Let X be a topological vector space and let E be a topological K-space. Consider $f : X \to E^\bullet$ and $x_0 \in \operatorname{core}(\operatorname{dom}(f))$. We say that f is *topologically quasidifferentiable* at x_0 provided that the directional derivative $f'(x_0)$ exists at x_0 and presents a continuous quasilinear operator.

Therefore, if f is topologically quasidifferentiable at x_0, then the Minkowski duality put in correspondence to the quasilinear operator $f'(x_0) \in \mathrm{QL}^c(X, E)$ the element $\mathcal{D}(f'(x_0)) \in [\mathrm{CS}_c^c(X, E)]$ called the *topological quasidifferential* of f at x_0 and denoted by $\mathcal{D}^c f(x_0) := [\underline{\partial}^c f(x_0), \overline{\partial}^c f(x_0)]$. Here $\underline{\partial}^c f(x_0)$ and $\overline{\partial}^c f(x_0)$ stand respectively for the *topological subdifferential* and *topological superdifferential* of f at x_0.

The formulas comprising the calculus of topological quasidifferentials coincide with their algebraic analogs in 2.2 if we replace $\mathcal{D}$ with $\mathcal{D}^c$.

5.3. Let X be a topological vector space, while $C \subset X$ and $x_0 \in C$. The *feasible direction cone* $\mathrm{Fd}(C, x_0)$ of C at x_0 is introduced as follows:

$$\mathrm{Fd}(C, x_0) := \{h \in X : \ (\exists \varepsilon > 0)\, x_0 + [0, \varepsilon)h \subset C\}.$$

We say that C is *K-regular* at x_0 whenever K is a convex cone and $K \subset \mathrm{cl}\,\big(\mathrm{Fd}(C, x_0)\big)$. Given some set C that is K-regular at x_0, we introduce a *normal cone* of C at x_0 as follows: $N_E(C, x_0) := \pi_E(K) := \{T : Tk \le 0,\, k \in K\}$. Obviously, this definition yields a nonunique cone.

If $C \subset X$ is K-regular at $x_0 \in C$ and $f : X \to E^\bullet$ is quasidifferential at x_0, where $x_0 \in \mathrm{core}(\mathrm{dom}(f))$. For x_0 to be an ideal local optimum of (C, f) it is necessary that

$$\overline{\partial}^c f(x_0) \subset \underline{\partial}^c f(x_0) + N_E(C, x_0).$$

5.4. Consider a vector program (g, f) with the extra constraint $x \in C$ for some $C \subset X$. We denote this program by (C, g, f). Assume that $x_0 \in C \cap \mathrm{core}(\mathrm{dom}(f)) \cap \mathrm{core}(\mathrm{dom}(g))$ and suppose that $f : X \to E^\bullet$ and $g : X \to F^\bullet$ are topologically quasidifferentiable at x_0. The vector program (C, g, f) is called *quasiregular* at x_0 provided that

(a) there are a continuous Maharam operator $r : F \to E$ and a neighborhood U of x_0 such that $\pi_x f(x_0) \le \pi_x f(x)$ for all $x \in C \cap U$, where $\pi_x := [(r \circ g(x))^-]$ is the projection to the band generated by $(r \circ g(x))^-$;

(b) C is K-regular at x_0;

(c) for every $T \in \partial r(g(x_0))$ and every nonzero $\pi \in \mathfrak{P}(E)$ we have

$$\pi T \circ \overline{\partial}^c g(x_0) \cap \big(\pi T \underline{\partial}^c g(x_0) + \pi N_E(C, x_0)\big) = \varnothing.$$

5.5. **Theorem.** *Let f and g be quasidifferentiable at $x_0 \in C \cap \mathrm{core}(\mathrm{dom}(f)) \cap \mathrm{core}(\mathrm{dom}(g))$, and let the vector program (C, g, f) be quasiregular at x_0. If x_0 is an ideal local optimum of (C, g, f), then to all $s \in \overline{\partial}^c f(x_0)$ and $S \in \overline{\partial}^c g(x_0)$ there are a continuous orthomorphism $\alpha \in \mathrm{Orth}(E)$, a continuous Maharam operator $\gamma \in L_+(F, E)$, and a continuous linear operator $\lambda \in L(X, E)$ such that the following system of conditions is compatible:*

$$0 \le \alpha \le I_E, \quad \ker \alpha = \{0\}, \quad \lambda \in N_E(C, x_0), \quad \gamma \circ g(x_0) = 0,$$
$$-\lambda \in \alpha(\underline{\partial}^c f(x_0) - s) + \gamma \circ (\underline{\partial}^c g(x_0) - S).$$

5.6. We call $\{x_1^0, \dots, x_n^0\} \subset C$ a *generalized local optimum* of a program (C, f) provided that there is a neighborhood U of the zero such that $f(x_1^0) \wedge \dots \wedge f(x_n^0) \le f(x_1) \wedge \dots \wedge f(x_n)$ for all $x_i \in (x_i^0 + U) \cap C$ and $i := 1, \dots, n$.

Let $f : X \to E^\bullet$ be quasidifferentiable at each of the feasible points $x_1^0, \dots, x_n^0 \in \mathrm{core}(\mathrm{dom}(f))$. If $\{x_1^0, \dots, x_n^0\}$ is a generalized local optimum of the unconstrained program $f(x) \to \inf$, then for all $\alpha_1, \dots, \alpha_n \in \mathrm{Orth}_+(E)$ such that

$$\alpha_1 + \dots + \alpha_n = I_E, \quad \sum_{i=1}^{n} \alpha_i f(x_i^0) = f(x_1^0) \wedge \dots \wedge f(x_n^0),$$

we have the inclusions

$$\alpha_k \overline{\partial}^c f(x_k) \subset \alpha_k \underline{\partial}^c f(x_k) \quad (k := 1, \dots, n).$$

5.7. **Theorem.** *Let $f : X \to E^{\bullet}$ be quasidifferentiable at $x_1^0, \dots, x_n^0 \in \operatorname{core}(\operatorname{dom}(f))$ and let $C \subset X$ be K_l-regular at $x_l^0 \in C$ for all $l := 1, \dots, n$, where $K_1, \dots, K_n$ are convex cones. If $\{x_1^0, \dots, x_n^0\}$ is a generalized local optimum of the program (C, f), the for all $\alpha_1, \dots, \alpha_n \in \operatorname{Orth}_+(E)$ such that*

$$\alpha_1 + \ldots + \alpha_n = I_E, \quad \sum_{k=1}^{n} \alpha_k f(x_k^0) = f(x_1^0) \wedge \ldots \wedge f(x_n^0),$$

we have the inclusions

$$\alpha_k \overline{\partial}^c f(x_k^0) \subset \alpha_k \underline{\partial}^c f(x_k^0) + N_E(C, x_k^0) \quad (k := 1, \dots, n).$$

5.8. Consider some vector program (C, g, f). Assume that

$$x_i^0 \in C \cap \operatorname{core}(\operatorname{dom}(f)) \cap \operatorname{core}(\operatorname{dom}(g)),$$

while $f : X \to E^{\bullet}$ and $g : X \to F^{\bullet}$ are topologically quasidifferentiable at x_i^0 for all $i := 1, \dots, n$. The vector program (C, g, f) is called *quasiregular* at $\{x_1^0, \dots, x_n^0\}$ provided that the following conditions are fulfilled:

(a) there are a continuous Maharam operator $r : F \to E$ and neighborhoods U_i of x_i^0 such that $\pi_x e \le \pi_x f(x)$ for all $x \in C \cap U_i$, where $e := f(x_1^0) \wedge \ldots \wedge f(x_n^0)$ and $\pi_x := [(r \circ g(x))^-]$ is the projection on the band generated by $(r \circ g(x))^-$;

(b) C is K_i-regular at x_i^0;

(c) $\pi T \circ \overline{\partial}^c g(x_i^0) \cap \big(\pi T \circ \underline{\partial}^c g(x_i^0) + \pi N_E(C, x_i^0)\big) = \varnothing$ for all $T \in \partial r(g(x_i^0))$, $i := 1, \dots, n$, and $\pi \in \mathfrak{P}(E)$.

5.9. **Theorem.** *Assume that $g : X \to F^{\bullet}$ and $f : X \to E^{\bullet}$ are quasidifferentiable at $x_1^0, \dots, x_n^0 \in C \cap \operatorname{core}(\operatorname{dom}(f)) \cap \operatorname{core}(\operatorname{dom}(g))$. Assume further that the vector program (C, g, f) is quasiregular in the sense of* 5.8 *at $\{x_1^0, \dots, x_n^0\}$. If $\{x_1^0, \dots, x_n^0\}$ is a generalized local optimum of (C, g, f), then to all $s_i \in \overline{\partial} f(x_i^0)$ and $S_i \in \overline{\partial} g(x_i^0)$ there are orthomorphisms $\alpha_1, \dots, \alpha_n \in \operatorname{Orth}(E)$, continuous Maharam operators $\gamma_1, \dots, \gamma_n \in L_+(F, E)$, and continuous linear operators $\lambda_i \in L(X, E)$ such that*

$$0 \leqslant \alpha_i \leqslant I_E, \quad \ker(\alpha_1) \cap \ldots \cap \ker(\alpha_n) = \{0\},$$

$$\gamma_i \circ g(x_0) = 0, \quad \lambda_i \in N_E(K_{\xi_i}),$$

$$-\lambda_i \in \alpha_i (\underline{\partial}^c f(x_i^0) - s_i) + \gamma_i \circ (\underline{\partial}^c g(x_i^0) - S_i) \quad (i := 1, \dots, n).$$

5.10. It seems worthwhile to study necessary optimality conditions for multiple criteria extremal problems of the type (C, g, f) with g and f integral quasidifferentiable operators

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