On the Optimality of Napoleon Triangles

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Abstract. An elementary geometric construction known as Napoleon's theorem produces an equilateral triangle built on the sides of any initial triangle: the centroids of each equilateral triangle meeting the original sides, all outward or all inward, comprise the vertices of the new equilateral triangle. In this note we observe that two Napoleon iterations yield triangles with useful optimality properties. Two inner transformations result in a (degenerate) triangle whose vertices coincide at the original centroid. Two outer transformations yield an equilateral triangle whose vertices are closest to the original in the sense of minimizing the sum of the three squared distances.

1. INTRODUCTION. In elementary geometry, one way of constructing an equilateral triangle from any given triangle is as follows: in a plane the centroids of equilateral triangles erected, either all externally or all internally, on the sides of the given triangle form an equilateral triangle, illustrated in Figure 1 [1]. This result is generally referred to as *Napoleon's theorem*, notwithstanding its dubious origins — see [2] for a detailed history of the theorem. We will refer to these constructions as *the outer and inner Napoleon triangles* of the original triangle, respectively. Conversely, given its outer and inner Napoleon triangles in position (i.e. they are oppositely oriented and have the same centroid), the original triangle is uniquely determined [3]. A fascinating application of Napoleon triangles is the planar tessellation used by Escher: a plane can be tiled using congruent copies of the hexagon defined by the vertices of any triangle and its outer Napoleon triangle, known as *Escher's theorem* [4].



Figure 1. An illustration of (left) the Fermat point *F*, outer Torricelli configuration $\triangle_{A^TB^TC^T}$ and outer Napoleon triangle $\triangle_{A^NB^NC^N}$, and (right) inner Torricelli configuration $\triangle_{A_TB_TC_T}$ and inner Napoleon triangle $\triangle_{A_NB_NC_N}$ of a triangle \triangle_{ABC} . Note that centroids of Torricelli configurations, Napoleon triangles and the original triangle all coincide, i.e. $c(\triangle_{ABC}) = c(\triangle_{A^TB^TC^T}) = c(\triangle_{A^NB_NC_N}) = c(\triangle_{A_TB_TC_T}) = c(\triangle_{A_NB_NC_N}).$

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Equilaterals built on the sides of a triangle make a variety of appearances in the classical literature. Torricelli uses this construction to locate Fermat's point minimizing the sum of distances to the vertices of a given triangle, now called *the Fermat-Torricelli problem* [5]. The unique solution of this problem is known as *the Fermat-Torricelli point* of the given triangle, located as follows [6]. If an internal angle of the triangle is greater than 120°, then the Fermat point is at that obtuse vertex. Otherwise, the three lines joining opposite vertices of the original triangle and externally erected triangles are concurrent, and they intersect at the Fermat point, see Figure 1. The triangle defined by the new vertices of the erected equilateral triangles is referred to as *the Torricelli configuration* [7, 8].

In this paper we demonstrate some remarkable, but not immediately obvious, optimality properties of twice iterated Napoleon triangles. First, two composed inner Napoleon transformations of a triangle collapse the original to a point located at its centroid which, by definition, minimizes the sum of squared distances to the vertices of the given triangle. Surprisingly, two composed outer Napoleon transformations yield an equilateral triangle optimally aligned with the original by virtue of minimizing the sum of squared distances between the paired vertices. More precisely, for any triangle \triangle_{ABC} with the vertices ${}^1 A, B, C \in \mathbb{R}^d$, we will say that the triangle $\triangle_{A'B'C'}$ is an *optimally aligned* equilateral triangle of \triangle_{ABC} if it solves the following constrained optimization problem:

minimize
$$||A - A'||^2 + ||B - B'||^2 + ||C - C'||^2$$

subject to $||A' - B'||^2 = ||A' - C'||^2 = ||B' - C'||^2$ (1)

where $A', B', C' \in \mathbb{R}^d$ and $\|.\|$ denotes the standard Euclidean norm on \mathbb{R}^d . As we show below, this optimization problem has a unique solution so long as A, B, C are not collinear.

2. TORRICELLI AND NAPOLEON TRANSFORMATIONS. For any ordered triple $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]^T \in \mathbb{R}^{3d}$, let $\mathbf{R}_{\mathbf{x}}$ denote the rotation matrix corresponding to a counter-clockwise rotation by $\pi/2$ in the plane, defined by orthonormal vectors n and t, in which the triangle $\Delta_{\mathbf{x}}$ formed by \mathbf{x} is positively oriented (i.e. its vertices in counter-clockwise order follow the sequence $\ldots \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow \ldots$), ²

$$\mathbf{R}_{\mathbf{x}} := \begin{bmatrix} \mathbf{n}, \mathbf{t} \end{bmatrix} \begin{bmatrix} 0 & -1\\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{n}, \mathbf{t} \end{bmatrix}^{\mathrm{T}},$$
(2)

where ³

$$n := \begin{cases} \frac{x_2 - x_1}{\|x_2 - x_1\|_2}, \text{ if } x_1 \neq x_2, \\ \frac{x_3 - x_2}{\|x_3 - x_2\|_2}, \text{ otherwise,} \end{cases} \quad t := \begin{cases} \in \{z \in \mathbb{S}^{d-1} | n^T z = 0\}, \text{ if } x \text{ is collinear,} \\ \mathbf{P}(n) \frac{x_3 - x_1}{\|x_3 - x_1\|_2} \end{cases}, \text{ otherwise,} \end{cases}$$
(3)

where $\mathbf{P}(n) := \mathbf{I}_d - nn^T$ is the projection onto $T_n \mathbb{S}^{d-1}$ (the tangent space of \mathbb{S}^{d-1} at point $n \in \mathbb{S}^{d-1}$), and \mathbf{I}_d is the $d \times d$ identity matrix. Note that $\Delta_{\mathbf{x}}$ is both positively and negatively oriented if \mathbf{x} is collinear. Consequently, to define a plane containing

¹Here, \mathbb{R} denotes the set of real numbers, and \mathbb{R}^d is the *d*-dimensional Euclidean space.

 $^{^{2}\}mathbf{A}^{\mathrm{T}}$ denotes the transpose of matrix \mathbf{A} .

³For any trivial triangle $\Delta_{\mathbf{x}}$ all of whose vertices are located at the same point we fix $\mathbf{R}_{\mathbf{x}} = \mathbf{0}$ by setting $\frac{\mathbf{x}}{\|\mathbf{x}\|} = 0$ whenever $\mathbf{x} = \mathbf{0}$.

such **x** we select an arbitrary vector t perpendicular to n in (3). It is also convenient to have $c(\mathbf{x})$ denote the centroid of $\triangle_{\mathbf{x}}$, i.e. $c(\mathbf{x}) := \frac{1}{3} \sum_{i=1}^{3} x_i$.

In general, the Torricelli and Napoleon transformations of three points in Euclidean d-space can be defined based on their original planar definitions in a 2-dimensional subspace of \mathbb{R}^d containing \mathbf{x} . That is to say, for any $\mathbf{x} \in \mathbb{R}^{3d}$, select a 2-dimensional subspace of \mathbb{R}^d containing \mathbf{x} , and then construct the erected triangles on the side of $\triangle_{\mathbf{x}}$ in this subspace to obtain the Torricelli and Napoleon transformations of \mathbf{x} . Accordingly, let $T_{\pm} : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$ and $N_{\pm} : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$ denote the Torricelli and Napoleon transformations where the sign, + and -, determines the type of the transformation, inner and outer, respectively. One can write closed-form expressions of the Torricelli and Napoleon transformations as:

Lemma 1. The Torricelli and Napoleon transformations of any triple $\mathbf{x} \in \mathbb{R}^{3d}$ on a plane containing \mathbf{x} are, respectively, given by ⁴

$$\Gamma_{\pm}(\mathbf{x}) = \left(\frac{1}{2}\mathbf{K} \pm \frac{\sqrt{3}}{2} (\mathbf{I}_3 \otimes \mathbf{R}_{\mathbf{x}}) \mathbf{L}\right) \mathbf{x},\tag{4}$$

$$N_{\pm}(\mathbf{x}) = \frac{1}{3} \left(\mathbf{K} \mathbf{x} + T_{\pm}(\mathbf{x}) \right), \tag{5}$$

where

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \otimes \mathbf{I}_{d} \quad and \quad \mathbf{L} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \otimes \mathbf{I}_{d}.$$
(6)

Proof. One can locate the new vertex of an equilateral triangle, inwardly or outwardly, constructed on one side of \triangle_x in the plane containing x using different geometric properties of equilateral triangles. We find it convenient to use the perpendicular bisector of the corresponding side of \triangle_x , the line passing through its midpoint and being perpendicular to it, such that the new vertex is on this bisector and a proper distance away from the side of \triangle_x .

For instance, let $\mathbf{y} = [y_1, y_2, y_3]^T = T_+(\mathbf{x})$. Consider the side of $\Delta_{\mathbf{x}}$ joining x_1 and x_2 , using the bisector, $b_{12} := \frac{1}{2}(x_1 + x_2)$, to locate the new vertex, y_3 , of inwardly erected triangle on this side as

$$y_3 = b_{12} + \frac{\sqrt{3}}{2} \mathbf{R}_{\mathbf{x}} (x_2 - x_1),$$
 (7)

where $\mathbf{R}_{\mathbf{x}}$ (2) defines a counter-clockwise rotation by $\frac{\pi}{2}$ in the plane where \mathbf{x} is positively oriented. Note that the height of an equilateral triangle from any side is $\frac{\sqrt{3}}{2}$ times its side length. Hence, by symmetry, one can conclude (4).

Given a Torricelli configuration $\mathbf{y} = [y_1, y_2, y_3]^T = T_{\pm}(\mathbf{x})$, by definition, the vertices of associated Napoleon triangle $\mathbf{z} = [z_1, z_2, z_3]^T = N_{\pm}(\mathbf{x})$ are given by

$$z_1 = \frac{1}{3}(y_1 + x_2 + x_3), \ z_2 = \frac{1}{3}(x_1 + y_2 + x_3) \text{ and } z_3 = \frac{1}{3}(x_1 + x_2 + y_3), \ (8)$$

which is equal to (5), and so the result follows.

⁴Here, \otimes denotes the Kronecker product [9].

Note that the Torricelli and Napoleon transformations of \mathbf{x} are unique if and only if $\mathbf{x} \in \mathbb{R}^{3d}$ is non-collinear. If, contrarily, \mathbf{x} is collinear, then $\triangle_{\mathbf{x}}$ is both positively and negatively oriented and for $d \ge 3$ there is more than one 2-dimensional subspace of \mathbb{R}^d containing \mathbf{x} .

Remark 1 ([3]). For any $\mathbf{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^{3d}$, the centroid of the Torricelli configuration $\mathbf{y} = [y_1, y_2, y_3]^T = T_{\pm}(\mathbf{x})$, the Napoleon configuration $\mathbf{z} = N_{\pm}(\mathbf{x})$ and the original triple \mathbf{x} all coincide,

$$c(\mathbf{x}) = c(\mathbf{y}) = c(\mathbf{z}), \qquad (9)$$

and distances between the associated elements of x and y are all the same, i.e. for any $i \neq j \in \{1, 2, 3\}$

$$\|\mathbf{y}_{i} - \mathbf{x}_{i}\|_{2} = \|\mathbf{y}_{j} - \mathbf{x}_{j}\|_{2}.$$
 (10)

An observation key to all further results is that Napoleon transformations of equilateral triangles are very simple.

Lemma 2. The inner Napoleon transformation N_+ of any triple $\mathbf{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^{3d}$ comprising the vertices of an equilateral triangle $\triangle_{\mathbf{x}}$ collapses it to the trivial triangle all of whose vertices are located at its centroid $c(\mathbf{x})$,

$$N_{+}(\mathbf{x}) = \mathbf{1}_{3} \otimes c(\mathbf{x}), \qquad (11)$$

whereas the outer Napoleon transformation N_ reflects the vertices of \triangle_x with respect to its centroid c(x), ⁵

$$N_{-}(\mathbf{x}) = 2 \cdot \mathbf{1}_{3} \otimes c(\mathbf{x}) - \mathbf{x}.$$
(12)

Proof. Observe that the inwardly erected triangle on any side of an equilateral triangle is equal to the equilateral triangle itself, i.e. $T_{+}(\mathbf{x}) = \mathbf{x}$, and so, by definition, one has (11). Alternatively, using (5), one can obtain

$$N_{+}(\mathbf{x}) = \frac{1}{3} (\mathbf{K}\mathbf{x} + T_{+}(\mathbf{x})) = \frac{1}{3} (\mathbf{K}\mathbf{x} + \mathbf{x}) = \mathbf{1}_{3} \otimes c(\mathbf{x}), \qquad (13)$$

where \mathbf{K} is defined as in (6).

Now consider outwardly erected equilateral triangles on the sides of an equilateral triangle, and let $\mathbf{y} = [y_1, y_2, y_3]^T = T_{-}(\mathbf{x})$. Note that each erected triangle has a common side with the original triangle. Since $\triangle_{\mathbf{x}}$ is equilateral, observe that the midpoint of the unshared vertices of an erected triangle and the original triangle is equal to the midpoint of their common sides, i.e. $\frac{1}{2}(y_1 + x_1) = \frac{1}{2}(x_2 + x_3)$ and so on. Hence, we have $T_{-}(\mathbf{x}) = \mathbf{K}\mathbf{x} - \mathbf{x}$. Thus, one can verify the result using (5) as

$$N_{-}(\mathbf{x}) = \frac{1}{3} (\mathbf{K}\mathbf{x} + T_{-}(\mathbf{x})) = \frac{1}{3} (\mathbf{K}\mathbf{x} + \mathbf{K}\mathbf{x} - \mathbf{x}) = 2 \cdot \mathbf{1}_{3} \otimes c(\mathbf{x}) - \mathbf{x}.$$
 (14)

⁵Here, $\mathbf{1}_3$ is the \mathbb{R}^3 column vector of all ones, and \cdot denotes the standard array product.

Since the Napoleon transformation of any triangle results in an equilateral triangle, motivated from Lemma 2, we now consider the iterations of the Napoleon transformation. For any $k \ge 0$ let $N_{\pm}^k : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$ denote the *k*-th Napoleon transformation defined to be

$$\mathbf{N}_{\pm}^{k+1} := \mathbf{N}_{\pm} \circ \mathbf{N}_{\pm}^k,\tag{15}$$

where we set $N^0_{\pm} := id$, and $id : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$ is the identity map on \mathbb{R}^{3d} .

It is evident from Lemma 2 that:

Lemma 3. For any $\mathbf{x} \in \mathbb{R}^{3d}$ and $k \geq 1$,

$$N_{+}^{k+1}(\mathbf{x}) = \mathbf{1}_{3} \otimes c(\mathbf{x}), \quad and \quad N_{-}^{k+2}(\mathbf{x}) = N_{-}^{k}(\mathbf{x}).$$
(16)

As a result, the basis of iterations of the Napoleon transformations consists of N_{\pm} and N_{\pm}^2 , whose explicit forms, except N_{-}^2 , are given above. Using (5) and (12), the closed-form expression of the double outer Napolean transformation N_{-}^2 can be obtained as:

Lemma 4. An arbitrary triple, $\mathbf{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^{3d}$ gives rise to the double outer Napoleon triangle, $N_-^2 : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$, according to the formula

$$N_{-}^{2}(\mathbf{x}) = \frac{2}{3}\mathbf{x} + \frac{1}{3}T_{+}(\mathbf{x}).$$
(17)

Proof. By Napoleon's theorem, $N_{-}(x)$ is an equilateral triangle. Using (5) and Lemma 2, one can obtain the result as follows:

$$N_{-}^{2}(\mathbf{x}) = N_{-}(N_{-}(\mathbf{x})) = 2 \cdot \mathbf{1}_{3} \otimes c(x) - N_{-}(\mathbf{x}) = 2 \cdot \mathbf{1}_{3} \otimes c(x) - \frac{1}{3} (\mathbf{K}\mathbf{x} + \mathbf{T}_{-}(\mathbf{x})), (18)$$
$$= \frac{2}{3} (\mathbf{K}\mathbf{x} + \mathbf{x}) - \frac{1}{3} (\mathbf{K}\mathbf{x} + \mathbf{T}_{-}(\mathbf{x})) = \frac{2}{3}\mathbf{x} + \frac{1}{3} (\mathbf{K}\mathbf{x} - \mathbf{T}_{-}(\mathbf{x})) = \frac{2}{3}\mathbf{x} + \frac{1}{3}\mathbf{T}_{+}(\mathbf{x}), (19)$$

where \mathbf{K} is defined as in (6).

Notice that $N_{-}^{2}(\mathbf{x})$ is a convex combination of \mathbf{x} and $T_{+}(\mathbf{x})$, see Figure 2.

3. OPTIMALITY OF NAPOLEON TRANSFORMATIONS. To best of our knowledge, the Napoleon transformation N_{\pm} is mostly recognized as being a function into the space of equilateral triangles. In addition to this inherited property, N_{\pm}^2 has an optimality property that is not immediately obvious. Although the double inner Napoleon transformation N_{\pm}^2 is not really that interesting to work with, it gives a hint about the optimality of N_{\pm}^2 : for any given triangle N_{\pm}^2 yields a trivial triangle all of whose vertices are located at the centroid of the given triangle which, by definition, minimizes the sum of squared distances to the vertices of the original triangle. Surprisingly, one has a similar optimality property for N_{\pm}^2 :

Theorem 1. The double outer Napoleon transformation $N_{-}^{2}(\mathbf{x})$ (17) yields the equilateral triangle most closely aligned with $\triangle_{\mathbf{x}}$ in the sense that it minimizes the total sum of squared distances between corresponding vertices. That is to say, for any



Figure 2. (left) Outer, $\triangle_{A^N B^N C^N}$, and double outer, $\triangle_{A^D B^D C^D}$, Napoleon transformations of a triangle \triangle_{ABC} . (right) The double outer Napoleon triangle $\triangle_{A^D B^D C^D}$ is a convex combination of the original triangle \triangle_{ABC} and its inner Torricelli configuration $\triangle_{A_T B_T C_T}$.

 $\mathbf{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^{3d}$, $N^2_{-}(\mathbf{x})$ is an optimal solution of the following problem

minimize
$$\sum_{i=1}^{3} \|\mathbf{x}_{i} - \mathbf{y}_{i}\|^{2}$$
 (20)
subject to $\|\mathbf{y}_{1} - \mathbf{y}_{2}\|^{2} = \|\mathbf{y}_{1} - \mathbf{y}_{3}\|^{2} = \|\mathbf{y}_{2} - \mathbf{y}_{3}\|^{2}$

where $\mathbf{y} = [y_1, y_2, y_3]^T \in \mathbb{R}^{3d}$. Further, if \mathbf{x} is non-collinear, then (20) has a unique solution.

Proof. Using the method of Lagrange multipliers [10], we first show that an optimal solution of (20) lies in the plane containing the triangle \triangle_x . Then, to show the result, we solve (20) using a proper parametrization of equilateral triangles in \mathbb{R}^2 .

The Lagrangian formulation of (20) minimizes

$$L(\mathbf{y}, \lambda_1, \lambda_2) = \sum_{i=1}^3 \|\mathbf{x}_i - \mathbf{y}_i\|_2^2 + \lambda_1 \Big(\|\mathbf{y}_1 - \mathbf{y}_2\|_2^2 - \|\mathbf{y}_1 - \mathbf{y}_3\|_2^2 \Big) \\ + \lambda_2 \Big(\|\mathbf{y}_1 - \mathbf{y}_2\|_2^2 - \|\mathbf{y}_2 - \mathbf{y}_3\|_2^2 \Big), \quad (21)$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are Lagrange multipliers. Necessary conditions for the locally optimal solutions of (20) is ⁶

$$\nabla_{\mathbf{y}} L(\mathbf{y}, \lambda_1, \lambda_2) = 2 \begin{bmatrix} (y_1 - x_1) + \lambda_1 (y_3 - y_2) + \lambda_2 (y_1 - y_2) \\ (y_2 - x_2) + \lambda_1 (y_2 - y_1) + \lambda_2 (y_3 - y_1) \\ (y_3 - x_3) - \lambda_1 (y_3 - y_1) - \lambda_2 (y_3 - y_2) \end{bmatrix} = 0, \quad (22)$$

from which one can conclude that an optimal solution of (20) lies in the plane containing \triangle_x . Accordingly, without loss of generality, suppose that \triangle_x is a positively oriented triangle in \mathbb{R}^2 , i.e. its vertices are in counter-clockwise order in \mathbb{R}^2 .

In general, an equilateral triangle $\triangle_{\mathbf{y}}$ in \mathbb{R}^2 with vertices $\mathbf{y} = [y_1, y_2, y_3]^T \in \mathbb{R}^6$ can be uniquely parametrized using two of its vertices, say y_1 and y_2 , and a binary

⁶Here, $\nabla_{\mathbf{y}}$ denotes the gradient taken with respect to the coordinate \mathbf{y} .

variable $k \in \{-1, +1\}$ specifying the orientation of \triangle_y ; for instance, k = +1 if \triangle_y is positively oriented, and so on. Consequently, the remaining vertex, y_3 , can be located as

$$\mathbf{y}_{3} = \frac{1}{2}(\mathbf{y}_{1} + \mathbf{y}_{2}) + k\frac{\sqrt{3}}{2}\mathbf{R}_{\pi/2}(\mathbf{y}_{2} - \mathbf{y}_{1}),$$
(23)

where $\mathbf{R}_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the rotation matrix defining a rotation by $\pi/2$.

Hence, one can rewrite the optimization problem (20) in term of new parameters as an unconstrained optimization problem as: for $y_1, y_2 \in \mathbb{R}^2$ and $k \in \{-1, 1\}$,

minimize
$$\|\mathbf{x}_1 - \mathbf{y}_1\|_2^2 + \|\mathbf{x}_2 - \mathbf{y}_2\|_2^2 + \|\mathbf{x}_3 - \mathbf{M}\mathbf{y}_1 - \mathbf{M}^{\mathrm{T}}\mathbf{y}_2\|_2^2$$
 (24)

where $\mathbf{M} := \frac{1}{2}\mathbf{I} - k\frac{\sqrt{3}}{2}\mathbf{R}_{\pi/2}$, and \mathbf{I} is the 2 × 2 identity matrix. Note that $\mathbf{M} + \mathbf{M}^{\mathrm{T}} = \mathbf{I}, \mathbf{M}^{\mathrm{T}}\mathbf{M} = \mathbf{M}\mathbf{M}^{\mathrm{T}} = \mathbf{I}$ and $\mathbf{M}^{2} = -\mathbf{M}^{\mathrm{T}}$.

For a fixed $k \in \{-1, 1\}$, (24) is a convex optimization problem of y_1 and y_2 because every norm on \mathbb{R}^n is convex and compositions of convex functions with affine transformations preserve convexity [11]. Hence, a global optimal solution of (24) occurs where the gradient of the objective function is zero at,

$$\begin{bmatrix} (\mathbf{I} + \mathbf{M}^{\mathrm{T}}\mathbf{M}) & (\mathbf{M}^{2})^{\mathrm{T}} \\ \mathbf{M}^{2} & (\mathbf{I} + \mathbf{M}\mathbf{M}^{\mathrm{T}}) \end{bmatrix} \begin{bmatrix} y_{1} \\ y_{2} \end{bmatrix} = \begin{bmatrix} x_{1} + \mathbf{M}^{\mathrm{T}}x_{3} \\ x_{2} + \mathbf{M}x_{3} \end{bmatrix},$$
(25)

which simplifies to

$$\begin{bmatrix} 2\mathbf{I} & -\mathbf{M} \\ -\mathbf{M}^{\mathrm{T}} & 2\mathbf{I} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + \mathbf{M}^{\mathrm{T}} x_3 \\ x_2 + \mathbf{M} x_3 \end{bmatrix}.$$
 (26)

Note that the objective function, $f(\mathbf{y})$, is strongly convex since its Hessian, $\nabla^2 f(\mathbf{y})$, satisfies

$$\nabla^2 f(\mathbf{y}) = \begin{bmatrix} 2\mathbf{I} & -\mathbf{M} \\ -\mathbf{M}^{\mathrm{T}} & 2\mathbf{I} \end{bmatrix} \succeq \mathbf{I},$$
(27)

which means that for a fixed $k \in \{-1, +1\}$ the optimal solution of (24) is unique.

Now observe that

$$\frac{1}{3} \begin{bmatrix} 2\mathbf{I} & \mathbf{M} \\ \mathbf{M}^{\mathrm{T}} & 2\mathbf{I} \end{bmatrix} \begin{bmatrix} 2\mathbf{I} & -\mathbf{M} \\ -\mathbf{M}^{\mathrm{T}} & 2\mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$
(28)

hence the solution of linear equation (26) is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2\mathbf{I} & \mathbf{M} \\ \mathbf{M}^{\mathrm{T}} & 2\mathbf{I} \end{bmatrix} \begin{bmatrix} x_1 + \mathbf{M}^{\mathrm{T}} x_3 \\ x_2 + \mathbf{M} x_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2x_1 + 2\mathbf{M}^{\mathrm{T}} x_3 + \mathbf{M} x_2 + \mathbf{M}^2 x_3 \\ \mathbf{M}^{\mathrm{T}} x_1 + (\mathbf{M}^2)^{\mathrm{T}} x_3 + 2x_2 + 2\mathbf{M} x_3 \end{bmatrix}, \quad (29)$$

$$= \frac{1}{3} \begin{bmatrix} 2\mathbf{x}_1 + \mathbf{M}^{\mathrm{T}} \mathbf{x}_3 + \mathbf{M} \mathbf{x}_2 \\ 2\mathbf{x}_2 + \mathbf{M}^{\mathrm{T}} \mathbf{x}_1 + \mathbf{M} \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \left(\mathbf{x}_1 + \frac{\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3}{3} \right) + k \frac{1}{2\sqrt{3}} \mathbf{R}_{\pi/2} \left(\mathbf{x}_3 - \mathbf{x}_2 \right) \\ \frac{1}{2} \left(\mathbf{x}_2 + \frac{\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3}{3} \right) + k \frac{1}{2\sqrt{3}} \mathbf{R}_{\pi/2} \left(\mathbf{x}_1 - \mathbf{x}_3 \right) \end{bmatrix}.$$
(30)

Here, substituting y_1 and y_2 back into (23) yields

$$y_3 = \frac{1}{2} \left(x_3 + \frac{x_1 + x_2 + x_3}{3} \right) + k \frac{1}{2\sqrt{3}} \mathbf{R}_{\pi/2} (x_2 - x_1) \,. \tag{31}$$

Thus, overall, we have

$$\mathbf{y} = \frac{2}{3}\mathbf{x} + \frac{1}{3}\left(\frac{1}{2}\mathbf{K}\mathbf{x} + k\frac{\sqrt{3}}{2}(\mathbf{I}_3 \otimes \mathbf{R}_{\pi/2})\mathbf{L}\mathbf{x}\right) = \begin{cases} \frac{2}{3}\mathbf{x} + \frac{1}{3}\mathbf{T}_+(\mathbf{x}), & \text{if } k = +1, \\ \frac{2}{3}\mathbf{x} + \frac{1}{3}\mathbf{T}_-(\mathbf{x}), & \text{if } k = -1. \end{cases}$$
(32)

where **K** and **L** are defined as in (6). Recall that $\triangle_{\mathbf{x}}$ is assumed to be positively oriented, i.e. $\mathbf{R}_{\mathbf{x}} = \mathbf{R}_{\pi/2}$, and so it is convenient to have the results in terms of Torricelli transformations T_{\pm} (4). As a result, the difference of **y** and **x** is simply given by

$$\mathbf{y} - \mathbf{x} = \begin{cases} \frac{1}{3} (\mathbf{T}_{+}(\mathbf{x}) - \mathbf{x}), & \text{if } k = +1, \\ \frac{1}{3} (\mathbf{T}_{-}(\mathbf{x}) - \mathbf{x}), & \text{if } k = -1. \end{cases}$$
(33)

Finally, one can easily verify that the optimum value of k is equal to +1 since the distance of x to its inner Torricelli configuration $T_{+}(x)$ is always less than or equal to its distance to the outer Torricelli configuration $T_{-}(x)$. Here, the equality only holds if x is collinear. Thus, an optimal solution of (20) coincides with the double outer Napoleon transformation, $N_{-}^{2}(x)$ (17), and it is the unique solution of (20) if x is non-collinear.

As a final remark we would like to note that our particular interest in the optimality of Napoleon triangles comes from our research on coordinated robot navigation, where a group of robots require to interchange their (structural) adjacencies through a minimum cost configuration determined by the double outer Napoleon transformation [12].

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