Martínez Legaz, Juan Enrique; Hadjisavvas, Nicolas; Lara, Felipe. «A quasiconvex asymptotic function with applications in optimization». Journal of Optimization Theory and Applications, Vol. 180, Núm. 1 (January 2019), p. 170-186. DOI 10.1007/s10957-018-1317-2

A Quasiconvex Asymptotic Function with Applications in Optimization

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December 15, 2017

Abstract

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Keywords: Asymptotic cones, Asymptotic functions, Quasiconvexity, Nonconvex optimization.

1 Introduction

2 Preliminaries and Basic Definitions

In this paper, we denote the duality pairing between two elements of $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$ and the norm by $\|\cdot\|$. Let $K \subseteq \mathbb{R}^n$, its closure is denoted by $\mathrm{cl}\ K$, its boundary by $\mathrm{bd}\ K$, its topological interior by $\mathrm{int}\ K$, its relative interior by $\mathrm{ri}\ K$ and its convex hull by $\mathrm{co}\ K$. By K^* we denote the positive polar cone of K. The indicator function of K is defined by $\delta_K(x) = 0$ if $x \in K$ and $\delta_K(x) = +\infty$ elsewhere, and its support function is defined by $\sigma_K(y) := \sup_{x \in K} \langle x, y \rangle$. By $B(x, \delta)$ we mean the open ball with center at $x \in \mathbb{R}^n$ and radius $\delta > 0$.

Given any function $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the effective domain of f is defined by $\mathrm{dom}\ f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. We say that f is a proper function if $f(x) > -\infty$ for every $x \in \mathbb{R}^n$ and $\mathrm{dom}\ f$ is nonempty. For a function f, we adopt the usual convention $\inf_{\emptyset} f = +\infty$ and $\sup_{\emptyset} f = -\infty$.

We denote by $\mathrm{epi}\, f := \{(x, t) \in \mathrm{dom}\ f \times \mathbb{R} : f(x) \leq t\}$ its epigraph and for a given $\lambda \in \mathbb{R}$ by $S_\lambda(f) := \{x \in \mathbb{R}^n : f(x) \leq \lambda\}$ its level set at height λ. By definition, a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex if $\mathrm{epi}\, f$ is convex. As usual, $\mathrm{argmin}_K f := \{x \in K : \forall y \in K, f(y) \geq f(x)\}$.

A proper function f with convex domain is said to be:

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(a) semistrictly quasiconvex if for every $x, y \in \mathrm{dom}\, f$ with $f(x) \neq f(y)$,

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}, \ \forall \, \lambda \in]0,1[.$$

(b) quasiconvex if for every $x, y \in \mathrm{dom}\, f$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \ \forall \, \lambda \in [0,1].$$

Every convex function is quasiconvex, and every semistrictly quasiconvex and lower semicontinuous (lsc from now on) function is quasiconvex. The continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \min\{|x|, 1\}$, is quasiconvex without being semistrictly quasiconvex.

Recall that,

$$\begin{aligned} f \text{ is convex} &\Longleftrightarrow \text{epi } f \text{ is a convex set.} \\ f \text{ is quasiconvex} &\Longleftrightarrow S_\lambda(f) \text{ is a convex set, for all } \lambda \in \mathbb{R}. \end{aligned}$$

For a further study on generalized convexity, we refer to [3, 4, 7].

As explained in [2], the notions of asymptotic cone and the associated asymptotic function have been employed in optimization theory in order to handle unbounded and/or nonsmooth situations. In particular, when standard compactness hypotheses are absent. We recall some basic definitions and properties of asymptotic cones and functions, which can be found in [2].

For a nonempty set K from $\mathbb{R}^n$ its asymptotic cone is defined by

$$K^\infty := \left\{ u \in \mathbb{R}^n : \ \exists \, t_k \to +\infty, \ \exists \, x_k \in K, \ \frac{x_k}{t_k} \to u \right\}.$$

In case K is a closed convex set, it is known that the asymptotic cone is equal to

$$K^\infty = \left\{ u \in \mathbb{R}^n : \ x_0 + \lambda u \in K, \ \forall \, \lambda \geq 0 \right\} \text{ for any } x_0 \in K. \tag{2.1}$$

The basic properties of the asymptotic cone are listed below.

Proposition 2.1 *Let $\emptyset \neq K \subseteq \mathbb{R}^n$, then*

(a) *If $K_0 \subseteq K$, then $(K_0)^\infty \subseteq K^\infty$.*

(b) *$(K + x_0)^\infty = K^\infty$ for all $x_0 \in \mathbb{R}^n$.*

(c) *$K^\infty = (\overline{K})^\infty$.*

(d) *$K^\infty = \{0\}$ iff K is bounded.*

(e) *Let $\{K_i\}_{i \in I}$ be a family of sets from $\mathbb{R}^n$, then $\bigcup_{i \in I} (K_i)^\infty \subseteq (\bigcup_{i \in I} K_i)^\infty$. The equality holds when $|I| < +\infty$.*

(*f*) *Let* $\{K_i\}_{i\in I}$ *be a family of sets from* $\mathbb{R}^n$ *satisfying* $\bigcap_{i\in I} K_i \neq \emptyset$, *then*

$$\left(\bigcap_{i\in I} K_i\right)^{\infty} \subseteq \bigcap_{i\in I}(K_i)^{\infty}.$$

The equality holds when every K_i *is closed and convex.*

The asymptotic function $f^{\infty} : \mathbb{R}^n \to \mathbb{R}\cup\{\pm\infty\}$ of a proper function f as before, is the function for which

$$\text{epi } f^{\infty} := (\text{epi } f)^{\infty}. \tag{2.2}$$

From this, one may show that

$$f^{\infty}(u) = \inf\left\{\liminf_{k\to+\infty} \frac{f(t_k u_k)}{t_k} : \ t_k \to +\infty,\ u_k \to u\right\}. \tag{2.3}$$

Moreover, when f is lsc and convex, for all $x_0 \in \text{dom } f$ we have

$$f^{\infty}(u) = \sup_{t>0} \frac{f(x_0+tu) - f(x_0)}{t} = \lim_{t\to+\infty} \frac{f(x_0+tu) - f(x_0)}{t}. \tag{2.4}$$

A function f is called coercive if $f(x) \to +\infty$ as $\|x\| \to +\infty$. If $f^{\infty}(u) > 0$ for all $u \neq 0$, then f is coercive. In addition, if f is convex and lsc, then

$$f \text{ is coercive} \Longleftrightarrow f^{\infty}(u) > 0,\ \forall\ u \neq 0 \Longleftrightarrow \operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset \text{ and compact.} \tag{2.5}$$

The problem to find an adequate definition of an asymptotic function has been studied in the last years, since the usual asymptotic function is not well suited for the description of the behavior of a nonconvex function at infinity. Several attempts to deal with the quasiconvex case has been made in [1, 5, 6, 10] while applications to optimization can be found in [6, 9].

The following two asymptotic functions to deal with quasiconvexity were introduced in [5]. Recall that, given a proper function $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, the q-asymptotic function is defined by

$$f_q^{\infty}(u) := \sup_{x\in\text{dom } f} \sup_{t>0} \frac{f(x+tu) - f(x)}{t}. \tag{2.6}$$

Given $\lambda \in \mathbb{R}$ with $S_\lambda(f) \neq \emptyset$, the λ-asymptotic function is defined by

$$f^{\infty}(u;\lambda) := \sup_{x\in S_\lambda(f)} \sup_{t>0} \frac{f(x+tu) - \lambda}{t}. \tag{2.7}$$

If f is lsc and quasiconvex, by [5, Theorem 4.7] we have

$$f_q^{\infty}(u) > 0,\ \forall\ u \neq 0 \iff \operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset \text{ and compact,} \tag{2.8}$$

and by [5, Proposition 5.3]

$$f^\infty(u;\lambda) > 0,\ \forall\ u \neq 0 \iff S_\lambda(f) \neq \emptyset \text{ and compact,} \tag{2.9}$$

If f is quasiconvex (resp. lsc), then $f^q(\cdot)$ and $f^\infty(\cdot;\lambda)$ are quasiconvex (resp. lsc). Furthermore, the following relations hold for any $\lambda \in \mathbb{R}$ with $S_\lambda(f) \neq \emptyset$,

$$f^\infty \leq f^\infty(\cdot;\lambda) \leq f^\infty_q. \tag{2.10}$$

Both inequalities could be strict even for quasiconvex functions, as was proved in [5, Example 5.6].

Finally, it is important to point out that $f^\infty_q(u) > 0$ for all $u \neq 0$ does not imply that f is coercive as the function $f(x) = \frac{|x|}{1+|x|}$ shows. Hence, the characterization (2.8) goes beyond coercivity.

3 A Quasiconvex Asymptotic Function

In this section we introduce a new definition of an asymptotic function to deal with quasiconvex functions. Properties, calculus rules, geometric interpretation and comparison with previous attempts presented in the literature are established.

Definition 3.1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. We define the r-asymptotic function of f denoted by $f^r : \mathbb{R}^n \to \overline{\mathbb{R}}$ as the function for which*

$$f^r(u) := \inf\left\{\lambda :\ u \in (S_\lambda(f))^\infty\right\}. \tag{3.1}$$

We adopt the convention that $(\emptyset)^\infty = \emptyset$.

The function is natural for quasiconvex functions, since it is defined through the asymptotic cone of the level sets of the original function.

Since f is lsc and quasiconvex, $S_\lambda(f)$ is a closed convex set. For any λ such that $S_\lambda(f) \neq \emptyset$, by Proposition 2.1($f$) we have

$$S_\lambda(f^r) = \bigcap_{\mu>\lambda} (S_\mu(f))^\infty = \left(\bigcap_{\mu>\lambda} S_\mu(f)\right)^\infty = (S_\lambda(f))^\infty. \tag{3.2}$$

Next remark follows immediately from the previous equation.

Remark 3.1

(i) *Since $S_\lambda(f)$ is a closed convex set for any $\lambda > \inf_{\mathbb{R}^n} f$, by the previous equation we have that $S_\lambda(f^r)$ is a closed convex set, hence f^r is lsc and quasiconvex. Furthermore, by definition of f^r, all its level sets are closed convex cones, i.e., it is positively homogeneous of degree 0.*

(ii) The r-asymptotic function is monotone in the sense that $f_1 \le f_2$ implies that $(f_1)^r \le (f_2)^r$. In fact, take $\lambda \in \mathbb{R}$ such that $S_\lambda(f_2) \neq \emptyset$, then

$$S_\lambda(f_2) \subseteq S_\lambda(f_1) \implies (S_\lambda(f_2))^\infty \subseteq (S_\lambda(f_1))^\infty \iff S_\lambda(f_2)^r \subseteq S_\lambda(f_1)^r,$$

which means that $(f_1)^r \le (f_2)^r$.

The previous monotonicity property does not hold for f_q^∞, as the quasiconvex functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ given by $f_1(x) = \frac{|x|}{1+|x|}$ and $f_2(x) \equiv 1$ show.

An analytic formula for the r-asymptotic function in the lsc and quasiconvex case is given below.

Proposition 3.1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc and quasiconvex function, then for any $u \in \mathbb{R}^n$ we have*

$$f^r(u) = \inf_{x \in \mathbb{R}^n} \sup_{t \ge 0} f(x + tu). \tag{3.3}$$

Proof. For every $\lambda > f^r(u)$ we have $u \in (S_\lambda(f))^\infty$. Then we can find $x \in S_\lambda(f)$, such that for all $t \ge 0$ we have $x + tu \in S_\lambda(f)$. Thus,

$$\exists\, x \in \mathbb{R}^n : \sup_{t \ge 0} f(x + tu) \le \lambda \implies \inf_{x \in \mathbb{R}^n} \sup_{t \ge 0} f(x + tu) \le \lambda.$$

Since this is true for all $\lambda > f^r(u)$, we deduce that $\inf_{x \in \mathbb{R}^n} \sup_{t \ge 0} f(x+tu) \le f^r(u)$.

Conversely, if $\inf_{x \in \mathbb{R}^n} \sup_{t \ge 0} f(x + tu) < \lambda$, then there exists $x \in \mathbb{R}^n$ such that for all $t \ge 0$, $x + tu \in S_\lambda(f)$. Hence $u \in (S_\lambda(f))^\infty$, so $f^r(u) \le \lambda$.

This shows that $f^r(u) \le \inf_{x \in \mathbb{R}^n} \sup_{t \ge 0} f(x + tu)$ and proves equality (3.3). ■

Remark 3.2 *Let $C \subseteq \mathbb{R}^n$ be a closed convex set. Then $(\delta_C)^r = \delta_{C^\infty}$. For the usual asymptotic function, a similar result is [2, Corollary 2.5.1].*

Another analytic formula for f^r is given below.

Proposition 3.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc and quasiconvex function. Then for each $u \in \mathbb{R}^n$,*

$$f^r(u) = \inf_{x \in \mathbb{R}^n} \lim_{t \to +\infty} f(x + tu). \tag{3.4}$$

Proof. We know by [4] that for a quasiconvex function defined on an interval I in $\mathbb{R}$, there exist two consecutive disjoint intervals I_1, I_2 (one of them might be empty) with $I = I_1 \cup I_2$, such that the function is nonincreasing on I_1 and nondecreasing on I_2. Thus,

$$\sup_{t \ge 0} f(x + tu) = \max\left\{ f(x), \lim_{t \to +\infty} f(x + tu) \right\},$$

and,

$$f^r(u) = \inf_{x \in \mathbb{R}^n} \max \left\{ f(x), \lim_{t \to +\infty} f(x+tu) \right\}.$$

Since obviously

$$\inf_{x \in \mathbb{R}^n} \max \left\{ f(x), \lim_{t \to +\infty} f(x+tu) \right\} \geq \inf_{x \in \mathbb{R}^n} \lim_{t \to +\infty} f(x+tu), \tag{3.5}$$

to show (3.4) it is enough to show that strict inequality in (3.5) is not possible. Assume that strict inequality holds. Then there exists $x_0 \in \mathbb{R}^n$ such that $\lim_{t\to+\infty} f(x_0 + tu) < f^r(u)$. Take t_0 large enough so that $f(x_0+t_0u) < f^r(u)$. Set $x_1 = x_0 + t_0u$. Then obviously $\lim_{t\to+\infty} f(x_1 + tu) = \lim_{t\to+\infty} f(x_0 + tu)$. Thus,

$$\max \left\{ f(x_1), \lim_{t \to +\infty} f(x_1+tu) \right\} < f^r(u) = \inf_{x \in \mathbb{R}^n} \max \left\{ f(x), \lim_{t \to +\infty} f(x+tu) \right\},$$

which is a contradiction. ■

Remark 3.3 *Take $t > 0$ and $u \in \mathbb{R}^n$, then $\sup_{t\geq 0} f(x+tu) \geq f(x) \geq \inf_{x\in\mathbb{R}^n} f(x)$, thus*

$$\inf_{x \in \mathbb{R}^n} f(x) = f^r(0) \geq \inf_{u \in \mathbb{R}^n} f^r(u) \geq \inf_{x \in \mathbb{R}^n} f(x).$$

This implies

$$\inf_{x \in \mathbb{R}^n} f(x) = \inf_{u \in \mathbb{R}^n} f^r(u) = f^r(0). \tag{3.6}$$

Hence, f and f^r has the same optimal value, and the r-asymptotic function obtain the optimal value at $u = 0$.

From the geometric point of view, the r-asymptotic function provides the behavior of the value of the original quasiconvex function at the infinity, rather than the behavior of the slope, as does f^∞. The next example ilustrates our interpretation.

Example 3.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be the continuous quasiconvex function given by*

$$f(x) = \begin{cases} x^2, & x \leq 0, \\ \frac{|x|}{1+|x|}, & x > 0. \end{cases}$$

An easy calculus shows that

$$f^r(u) = \begin{cases} +\infty, & u < 0, \\ 0, & u = 0, \\ 1, & x > 0. \end{cases}$$

Next proposition provides calculus rules for the r-asymptotic function. We recall that the composition of a nondecreasing function h with a quasiconvex function g is also quasiconvex.

Proposition 3.3 *The following assertion holds,*

(a) *Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. If $h : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ is a nondecreasing continuous function, then $(h \circ g)^r = h(g^r)$.*

(b) *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a family of proper, lsc and quasiconvex functions with I an arbitrary index set. Then*

$$\left(\sup_{i\in I} f_i\right)^r \geq \sup_{i\in I}(f_i)^r. \tag{3.7}$$

Proof. (a) : Take $u \in \mathbb{R}^n$, thus

$$\begin{aligned}(h \circ g)^r(u) &= \inf_{x\in\mathbb{R}^n} \lim_{t\to+\infty} (h \circ g)(x + tu) = \inf_{x\in\mathbb{R}^n} \lim_{t\to+\infty} h(g(x + tu)) \\ &= \inf_{x\in\mathbb{R}^n} h\left(\lim_{t\to+\infty} g(x + tu)\right) = h\left(\inf_{x\in\mathbb{R}^n} \lim_{t\to+\infty} g(x + tu)\right) \\ &= h(g^r(u)).\end{aligned}$$

(b) Set $f := \sup_{i\in I} f_i$. Then

$$\begin{aligned}f^r(u) &= \inf_{x\in\mathbb{R}^n} \sup_{t\geq0} \sup_{i\in I} f_i(x + tu) = \inf_{x\in\mathbb{R}^n} \sup_{i\in I} \sup_{t\geq0} f_i(x + tu) \\ &\geq \sup_{i\in I} \inf_{x\in\mathbb{R}^n} \sup_{t\geq0} f_i(x + tu) = \sup_{i\in I}(f_i)^r(u).\end{aligned}$$

Hence (3.7) holds. ∎

Note that equality does not hold in general in (3.7).

Example 3.2 *Define on $\mathbb{R}^2$ the convex functions given by $f_1(x_1, x_2) = |x_1 - 1|$ and $f_2(x_1, x_2) = |x_1 + 1|$, and $f = \max\{f_1, f_2\} = 1 + |x_1|$. Take $u = (0, 1)$. Then*

$$(f_1)^r(u) = \inf_{(x_1,x_2)\in\mathbb{R}^2} \sup_{t\geq0} |x_1 - 1| = 0, \quad (f_2)^r(u) = \inf_{(x_1,x_2)\in\mathbb{R}^2} \sup_{t\geq0} |x_1 + 1| = 0,$$

$$f^r(u) = \inf_{(x_1,x_2)\in\mathbb{R}^2} \sup_{t\geq0} (1 + |x_1|) = 1.$$

Thus, $f^r(u) > \max\{(f_1)^r, (f_2)^r\}$.

Another formula for computing the r-asymptotic function is given below.

Proposition 3.4 *Let $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function, let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with $A(\mathbb{R}^n) \cap \operatorname{dom} g \neq \emptyset$, and let $f(x) := g(Ax)$. Then f is lsc, quasiconvex and*

$$f^r(u) \geq g^r(Au), \ \forall\, u \in \mathbb{R}^n. \tag{3.8}$$

Whenever A is onto, equality holds in (3.8).

Proof. It is clear that f is lsc and quasiconvex. Now, take any $u \in \mathbb{R}^n$, then

$$\begin{aligned} f^r(u) &= \inf_{x\in\mathbb{R}^n} \sup_{t\geq 0} f(x+tu) = \inf_{x\in\mathbb{R}^n} \sup_{t\geq 0} g(Ax+t(Au)) \\ &\geq \inf_{z\in\mathbb{R}^m} \sup_{t\geq 0} g(z+t(Au)) = g^r(Au). \end{aligned}$$

If A is onto, then Ax takes on all values $z \in \mathbb{R}^m$ so equality holds. ■

3.1 Comparison with other Asymptotic Functions

Let us compare the three asymptotic functions f^∞, f_q^∞ and $f^\infty(\cdot;\lambda)$ which are known from the literature, with the function f^r introduced in the previous section.

When f is convex, the three functions f^∞, f_q^∞ and $f^\infty(\cdot;\lambda)$ are equal, see also [5, Proposition 5.4]:

Proposition 3.5 *Let f be convex and λ be such that $S_\lambda(f) \neq \emptyset$. Then $f^\infty = f_q^\infty = f^\infty(\cdot;\lambda)$.*

Proof. Only $f^\infty = f^\infty(\cdot;\lambda)$ needs a proof. We note that for $x \in S_\lambda(f)$, the functions $t \to \frac{f(x+tu)-f(x)}{t}$ and $t \to \frac{f(x)-\lambda}{t}$ are nondecreasing, thus $t \to \frac{f(x+tu)-\lambda}{t}$ is nondecreasing too, and

$$\sup_{t>0} \frac{f(x+tu)-\lambda}{t} = \lim_{t\to+\infty} \frac{f(x+tu)-\lambda}{t} = \lim_{t\to+\infty} \frac{f(x+tu)-f(x)}{t} = f^\infty(u).$$

It follows that

$$f^\infty(u;\lambda) = \sup_{x\in S_\lambda(f)} \sup_{t>0} \frac{f(x+tu)-\lambda}{t} = f^\infty(u).$$

■

In contrast to the above, when f is convex, f^r is in general not equal to f^∞. For example, consider the constant function $f(x) = \alpha$. Here, $f^\infty \equiv 0$ and $f^r \equiv \alpha$. Hence, for $\alpha > 0$ we have $f^\infty < f^r$, while for $\alpha < 0$ we have $f^r < f^\infty$. The same example shows that there is no connection between f^r and f_q^∞ or $f^\infty(\cdot;\lambda)$. This difference is not surprising, since f^∞ is related to the slope of the function f at infinity, whereas f^r is related to the value of f at infinity.

The r-asymptotic function f^r is also convex whenever f is convex. In fact, it is constant in its domain:

Proposition 3.6 *Let f be proper, convex and lsc. Then $f^r = \inf f + \delta_C$, where $C := \{u \in \mathbb{R}^n : f^\infty(u) \leq 0\}$.*

Proof. By [2, Proposition 2.5.3], for each $\alpha \in \mathbb{R}$ such that $S_\alpha(f) \neq \emptyset$, one has the equality: $(S_\alpha(f))^\infty = S_0(f^\infty)$. Thus, f^r has just one level set, and its value on this level set is $\inf f^r = \inf f$. ■

The asymptotic functions f^r, f^∞ and $f^\infty(\cdot;\lambda)$ are not convex in general if f is not convex. In contrast, f_q^∞ is always convex, for any proper function f. To see this, we first recall the notion of generalized recession cone [14].

Definition 3.2 *Let K be any nonempty subset of $\mathbb{R}^n$. Its generalized recession cone is the set*

$$\operatorname{rec} K := \{u \in \mathbb{R}^n : \ x + tu \in K, \ \forall \ x \in K, \ \forall \ t > 0\}.$$

Note that K is not required to be closed or convex. If K is closed and convex, then $\operatorname{rec} K = K^\infty$, the usual asymptotic cone of K.

It was proved in [14, Lemma 2.1] that for any nonempty set K from $\mathbb{R}^n$, the set rec K is always a convex cone. This result will be important in our further analysis.

A natural definition is the following.

Definition 3.3 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. We define the (generalized) recession function of f as the function $f^{rec} : \mathbb{R}^n \to \overline{\mathbb{R}}$ for which*

$$\text{epi } f^{rec} := \text{rec}(\text{epi } f). \tag{3.9}$$

The recession function is well-defined, as shown by the following proposition, which is useful to understand the nature of the q-asymptotic function.

Proposition 3.7 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then for every $u \in \mathbb{R}^n$*

$$f^{rec}(u) = \sup_{x \in \text{dom } f} \sup_{t > 0} \frac{f(x + tu) - f(x)}{t} = f_q^\infty(u). \tag{3.10}$$

Proof. Observe that

$$\begin{aligned}
(u, \alpha) \in \text{rec (epi } f) &\iff (x, \lambda) + t(u, \alpha) \in \text{epi } f, \ \forall \ (x, \lambda) \in \text{epi } f, \ \forall \ t > 0 \\
&\iff (x + tu, f(x) + t\alpha) \in \text{epi } f, \ \forall \ x \in \text{dom } f, \ \forall \ t > 0 \\
&\iff f(x + tu) \le f(x) + t\alpha, \ \forall \ x \in \text{dom } f, \ \forall \ t > 0 \\
&\iff \frac{f(x + tu) - f(x)}{t} \le \alpha, \ \forall \ x \in \text{dom } f, \ \forall \ t > 0 \\
&\iff \sup_{x \in \text{dom } f} \sup_{t > 0} \frac{f(x + tu) - f(x)}{t} \le \alpha \\
&\iff (u, \alpha) \in \text{epi } f_q^\infty.
\end{aligned}$$

This shows that rec (epi f) = epi f_q^∞, so f^{rec} is well defined and is equal to f_q^∞. ∎

As a result, we have:

Proposition 3.8 *For any proper function f, its q-asymptotic function f_q^∞ is convex.*

Proof. Set $K = \text{epi } f$, by [14, Lemma 2.1] we have that rec(epi f) = epi f_q^∞ is convex. Thus, f_q^∞ is convex. ∎

Remark 3.4 *The λ-asymptotic function $g := f^\infty(\cdot;\lambda)$ satisfies*

$$S_0(g) = \operatorname{rec}(S_\lambda(f)).$$

Indeed, $u \in \operatorname{rec}(S_\lambda(f))$ is equivalent to $x + tu \in S_\lambda(f)$ for all $x \in S_\lambda(f)$ and all $t > 0$. This is equivalent to $f(x + tu) \leq \lambda,\ \forall\, x \in S_\lambda(f),\ \forall\, t > 0$, that is,

$$\sup_{x\in S_\lambda(f)} \sup_{t>0} \frac{f(x+tu)-\lambda}{t} \leq 0.$$

This is turn means $u \in S_0(g)$.

As seen in Proposition 3.6, the r-asymptotic function f^r is very particular when the function f is proper, convex and lsc. However, in some situations, even in this case, f^r gives us information about the behavior of the function at the infinity while other asymptotic functions fail to do so.

Example 3.3 *Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be the convex function given by $f(x) = -\sqrt{x}$ for $x \geq 0$, and $f(x) = +\infty$ otherwise. Here*

$$f^\infty(u) = f_q^\infty(u) = f^\infty(u;\lambda) = 0,\ \ u \geq 0,$$

and no information about the unboundedness from below of f was detected.

On the other hand, for $u > 0$ we have $f^r(u) = -\infty$. Which means that f is not bounded from below.

4 Applications in Optimization

In this section, applications for quasiconvex optimization problems are given. We analyze the link between our new results with previous ones for the convex case. We also show that our new asymptotic function has some properties than previous quasiconvex asymptotic functions do not have.

The next proposition is straightforward since f and f^r has the same infimum.

Proposition 4.1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. Then f is bounded from below iff $f^r > -\infty$.*

A characterization result for boundedness from below for convex functions using first and second order asymptotic functions can be found in [6, Section 3.3].

The r-asymptotic function characterizes the boundedness of a quasiconvex function as the next proposition shows. For the convex case, a related result is [2, Proposition 2.5.5].

Proposition 4.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. Then f is bounded iff f^r is real-valued.*

Proof. If f is bounded, then obviously f^r is real-valued, by formula (3.4). Conversely, assume that f^r is real-valued, then f is bounded from below by the previous proposition. To show that f is bounded from above, we observe that since f^r is real-valued, $\mathbb{R}^n = \bigcup_{k\in\mathbb{N}} S_k(f^r)$. As the sets $S_k(f^r)$ are closed, by Baire's theorem there exists $k_0 \in \mathbb{N}$ such that the interior of $S_{k_0}(f^r)$ is nonempty. Thus, there exist $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $B(x_0,\varepsilon) \subseteq S_{k_0}(f^r)$. Now, let $m \in \mathbb{N}$ be such that $m > \max\{f^r(-x_0), k_0\}$. Then $-x_0 \in S_m(f^r)$ and $B(x_0,\varepsilon) \subseteq S_m(f^r)$, thus $\mathrm{co}\,(\{-x_0\} \cup B(x_0,\varepsilon)) \subseteq S_m(f^r)$. It follows that $0 \in \mathrm{int}\ S_m(f^r)$ and since $S_m(f^r)$ is a cone, $S_m(f^r) = \mathbb{R}^n$. Since $S_m(f^r) = (S_m(f))^\infty$, then $S_m(f) = \mathbb{R}^n$, so f is bounded from above by m and the result follows. $\blacksquare$

Remark 4.1 *We notice that the previous proposition does not hold for the q-asymptotic function. In fact, set $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \min\{\sqrt{|x|}, 3\}$, which is continuous, bounded and quasiconvex. Here $f_q^\infty(u) = +\infty$ for all $u \neq 0$. On the other hand, for the function $f(x) = |x|$, the function f_q^∞ is real valued, but f is unbounded.*

The next result provides a characterization of the nonempiness and compactness of the solution set of a lsc quasiconvex function.

Theorem 4.1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. Then the following assertions are equivalent.*

(a) $\mathrm{argmin}_{\mathbb{R}^n}\ f$ *is nonempty and compact.*

(b) $\mathrm{argmin}_{\mathbb{R}^n}\ f^r$ *is nonempty and compact.*

(c) $f^r(u) > f^r(0)$ *for all* $u \neq 0$.

Proof. Obviously (c) implies (b). If (b) holds and $u_0 \in \mathrm{argmin}_{\mathbb{R}^n}\ f^r$, then $tu_0 \in \mathrm{argmin}_{\mathbb{R}^n}\ f^r$ for all $t > 0$ since f^r is 0-homogeneous. Hence necessarily $u_0 = 0$, so (c) holds.

$(c) \Rightarrow (a)$: If (a) does not hold, then there exists a sequence (x_k) with $f(x_k) \to \inf_{\mathbb{R}^n} f$ and $\|x_k\| \to +\infty$. By selecting a subsequence if necessary, we may assume that $\frac{x_k}{\|x_k\|} \to u$. For every $\lambda > \inf_{\mathbb{R}^n} f$ we have that $x_k \in S_\lambda(f)$ for k large enough, so $u \in (S_\lambda(f))^\infty = S_\lambda(f^r)$, that is, $f^r(u) \leq \lambda$. Hence, $f^r(u) \leq \inf_{\mathbb{R}^n} f = f^r(0)$, contradicting (c).

$(a) \Rightarrow (c)$: Suppose for the contrary that (c) does not hold. Then there exists $u \neq 0$ with $f^r(u) \leq f^r(0) = \inf_{\mathbb{R}^n} f$. Then $u \in S_{\inf f}(f^r) = (S_{\inf f}(f))^\infty$. Choose $x \in \mathrm{argmin}_{\mathbb{R}^n}\ f$, then $x \in S_{\inf f}(f)$. Thus for every $t > 0$, we have that $x + tu \in S_{\inf f}(f)$, that is, $x + tu \in \mathrm{argmin}_{\mathbb{R}^n}\ f$. Which contradicts the compactness of $\mathrm{argmin}_{\mathbb{R}^n}\ f$. $\blacksquare$

Remark 4.2 *Since for a proper, lsc and convex function $f^\infty(0) = 0$, the previous characterization for quasiconvexity is similar to the characterization* (2.5).

In the next example, we study the quasiconvex quadratic case, that is the case when the function f is given by $f(x) = \frac{1}{2}\langle x, Ax\rangle + \langle a, x\rangle + \alpha$, where $A \in \mathbb{R}^{n\times n}$ is symmetric, a belongs to $\mathbb{R}^n$ and α belongs to $\mathbb{R}$. To that end, we first recall that whenever f is quadratic, f is convex on $\mathbb{R}^n$ iff f is quasiconvex on $\mathbb{R}^n$ (see [3, Theorem 6.3.1]). Thus, a quadratic function f can be quasiconvex without being convex, only if its domain is a proper subset K of $\mathbb{R}^n$. We say that f is *merely quasiconvex* on K if f is a quasiconvex function without being convex on K [3, page 120]. A necessary condition for a quadratic f to be merely quasiconvex is the existence of exactly one simple negative eigenvalue of A (see [3, Remark 6.3.1]). The properties of quasiconvex quadratic functions are investigated in depth in [3, Chapter 6].

Example 4.1 *Let K be a nonempty closed convex and proper subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ a quasiconvex quadratic function $f(x) = \frac{1}{2}\langle x, Ax\rangle + \langle a, x\rangle + \alpha$. As usual, we extend f to the whole $\mathbb{R}^n$ by setting $f(x) = +\infty$ for $x \notin K$.*

Observe that if $x \in K$ and $u \in K^\infty$, then

$$f(x + tu) = f(x) + t\langle\nabla f(x), u\rangle + \frac{1}{2}t^2\langle u, Au\rangle.$$

Accordingly, by Proposition (3.2),

$$f^r(u) = \inf_{x\in K}\lim_{t\to+\infty}\left(f(x) + t\left\langle\nabla f(x), u\right\rangle + \frac{1}{2}t^2\left\langle u, Au\right\rangle\right).$$

- *If $\langle u, Au\rangle > 0$ then the limit equals $+\infty$ for all $x \in K$, so $f^r(u) = +\infty$.*
- *If $\langle u, Au\rangle < 0$, then it is clear that $f^r(u) = -\infty$, so $\inf f = -\infty$.*
- *If $\langle u, Au\rangle = 0$, then we have the following cases:*
  - (i) *If $\langle\nabla f(x), u\rangle < 0$ for some $x \in K$, then $f^r(u) = -\infty$.*
  - (ii) *If $\langle\nabla f(x), u\rangle > 0$ for some x, then the limit is $+\infty$. These x can be omitted from the calculation of the infimum.*

Thus, f^r is given by the following formula:

$$f^r(u) = \begin{cases} +\infty, & \text{if } \langle u, Au\rangle > 0,\\ -\infty, & \text{if } \langle u, Au\rangle < 0,\\ -\infty, & \text{if } \langle u, Au\rangle = 0 \text{ and } u \notin \{\nabla f(K)\}^*,\\ \inf\limits_{x\in K,\ \langle\nabla f(x),u\rangle=0} f(x), & \text{if} \langle u, Au\rangle = 0 \text{ and } u \in \{\nabla f(K)\}^*. \end{cases}$$

Remark 4.3

(i) *Characterizations for the nonempiness and compactness of the solution set for quasiconvex quadratic functions are well-known. See for instance [8, Theorem 4.6] where the authors use the q-asymptotic function.*

(ii) *The term $\langle u, Au\rangle$ is exactly the second order asymptotic function $f^{\infty\infty}$ introduced in [8] (see [8, Example 3.5]).*

We provide another classical example for a class of nonconvex functions.

Example 4.2 *Consider the affine functions* $h(x) = \langle a, x\rangle + \alpha$ *and* $g(x) = \langle b, x\rangle + \beta$ *with* $a, b \in \mathbb{R}^n$ *and* $\alpha, \beta \in \mathbb{R}$, *and the closed convex set* $K := \{x \in \mathbb{R}^n : \ g(x) \geq 1\}$.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be the linear fractional function*

$$f(x) = \begin{cases} \dfrac{h(x)}{g(x)}, & \text{if } x \in K, \\ +\infty, & \text{if } x \notin K. \end{cases}$$

It is well-known that f *is semistrictly quasiconvex on* K *(see [3, 13]) and* $K^\infty = \{u \in \mathbb{R}^n : \ \langle b, u\rangle \geq 0\}$. *Notice that*

$$f^r(u) = \inf_{x\in K} \lim_{t\to+\infty} f(x + tu) = \inf_{x\in K} \lim_{t\to+\infty} \frac{h(x) + t\langle a, u\rangle}{g(x) + t\langle b, u\rangle}. \tag{4.1}$$

We have the cases:

(*i*) *If* $\langle b, u\rangle > 0$ *then it is easy to see that* $f^r(u) = \frac{\langle a,u\rangle}{\langle b,u\rangle}$.

(*ii*) *If* $\langle b, u\rangle < 0$ *then for* t *sufficiently large,* $x + tu \notin K$ *so* $f(x + tu) = +\infty$. *In this case,* $f^r(u) = +\infty$.

(*iii*) *If* $\langle b, u\rangle = 0$ *then again we have three cases: For* $\langle a, u\rangle > 0$ *we find from* (4.1) *that* $f^r(u) = +\infty$. *For* $\langle a, u\rangle < 0$ *we find* $f^r(u) = -\infty$. *Finally, for* $\langle a, u\rangle = 0$, *relation* (4.1) *gives*

$$f^r(u) = \inf_{x\in K} \frac{h(x)}{g(x)} = \inf_{x\in K} f(x) = f^r(0).$$

Before we introduce our next proposition, we remind that for a proper lsc convex function, $f^\infty(0) = 0$ so $f^\infty(u) \leq 0$ is equivalent to $f^\infty(u) \leq f^\infty(0)$. Also, for proper lsc quasiconvex functions, $f^r(0) = \inf f = \inf f^r$ so $f^r(u) \leq f^r(0)$ is equivalent to $f^r(u) = f^r(0)$.

Proposition 4.3 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper, lsc and quasiconvex and* $u \in \mathbb{R}^n$. *Then* $f^r(u) = f^r(0)$ *iff for every* $x \in \mathrm{dom} f$, *the function* $t \to f(x + tu), \ t > 0$ *is nonincreasing.*

Proof. For $u = 0$ it is obvious, so we assume that $u \neq 0$.

($\Rightarrow$) Take $x \in \mathrm{dom}\ f$. Since $f^r(u) = f^r(0) = \inf f$, we have $f^r(u) \leq f(x)$ so $u \in S_{f(x)}(f^r) = \big(S_{f(x)}(f)\big)^\infty$. From $x \in S_{f(x)}(f)$, for every $t \geq 0$ we have that $x + tu \in S_{f(x)}(f)$, that is, $f(x + tu) \leq f(x)$. Thus, for every $x \in \mathrm{dom}\ f$ and every $t > 0$ we have $f(x + tu) \leq f(x)$. For every $t' > t > 0$, set $x' = x + tu$ and $t'' = t' - t$. Then we have $f(x' + t''u) \leq f(x')$, so $f(x + t'u) \leq f(x + tu)$. Consequently, the function $t \to f(x + tu), \ t > 0$ is nonincreasing.

($\Leftarrow$) Assume that for each $x \in \text{dom } f$, the function $t \to f(x+tu),\ t > 0$ is nonincreasing. Suppose to the contrary that $f^r(u) > f^r(0)$. As $f^r(0) = \inf f$, we can choose $x \in \text{dom } f$ with $f(x) < f^r(u)$. Then

$$f(x) < f^r(u) \leq \lim_{t \to +\infty} f(x+tu).$$

It follows that $t \to f(x+tu),\ t > 0$ cannot be nonincreasing, a contradiction. ■

Remark 4.4 *If f is proper, lsc, convex, then $f^\infty(u) \leq 0$ iff for one (equivalently, for every) $x \in \text{dom } f$, $\lim_{t \to +\infty} f(x+tu) < +\infty$; see [2, Theorem 2.5.2] and [11, Theorem 8.6]. Note that for a convex function, or more generally for a quasiconvex function, $t \to f(x+tu),\ t > 0$ is monotone for large values of t, so the limit always exist and the $\liminf$ or $\limsup$ used in [2, Theorem 2.5.2] and [11, Theorem 8.6] are not needed. The last assertion is also equivalent to the fact that for each $x \in \text{dom } f$, the function $t \to f(x+tu),\ t > 0$ is nonincreasing.*

In contrast to convex functions, if f is quasiconvex, it is possible that $t \to f(x+tu),\ t > 0$ is nonincreasing only for some $x \in \text{dom} f$. For example, consider the quasiconvex function $f(x) = \min\{\|x\| - 1, 0\}$, $x \in \mathbb{R}^2$. If e_1 and e_2 are the usual basis vectors, then $t \to f(e_2 + te_1)$ is nonincreasing, while $t \to f(-e_1 + te_1)$ is not.

Now, we will recall [6, Theorem 3.1]. To that end, we first recall the following class of functions which includes those functions that are convex or coercive.

Definition 4.1 *([6, Definition 3.1]) A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ it is said to be in $\mathcal{C}$ if for all $x \in \text{dom } f$ and $u \in (\text{dom } f)^\infty$, the function $s \mapsto f(x+su)$, $s > 0$, is either unbounded from above or non-increasing.*

Now the mentioned Theorem.

Theorem 4.2 *([6, Theorem 3.1]) Let $F = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, be a vector function with each f_j, $j = 1, 2, \ldots, m$, being a finite-valued continuous, semistrictly quasiconvex function belonging to $\mathcal{C}$, and let $K \subseteq \mathbb{R}^n$ be closed and convex. Assume that*

$$L_j := \{u \in K^\infty :\ (f_j)_q^\infty(u) \leq 0\}, \tag{4.2}$$

is a linear subspace for all $j \in \{1, 2, \ldots, m\}$. Then $F(K)$ is closed.

Using the r-asymptotic function, Proposition 4.2 and 4.3, we can rewrite the previous theorem as follows.

Corollary 4.1 *Let $F := (f_1, f_2, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with each f_j, $j = 1, 2, \ldots, m$, being a continuous and semistrictly quasiconvex function. Let $K \subseteq \mathbb{R}^n$ be a closed convex set. Assume that for every $u \in K^\infty$, $f^r(u) = +\infty$ or $f^r(u) = f^r(0)$, and*

$$(L_j)^r := \{u \in K^\infty :\ f^r(u) = f^r(0)\}, \tag{4.3}$$

is a linear subspace for all $j \in \{1, 2, \ldots, m\}$. Then $F(K)$ is a closed set.

Notice that, the same result was written only in terms of the r-asymptotic function and no class of functions was used.

Acknowledgements

For the second author, this research was partially supported by Conicyt–Chile under project Fondecyt Postdoctorado 3160205. Part of this work was carried out when the second author was visiting the Departament d'Economia i d'Historia Economica, Univeritat Autónoma de Barcelona, Barcelona, Spain, during November of 2016. The author wishes to thanks the Department for its hospitality. (**I (Nicolas) also have to mention my stay in Concepción**)

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