DIRECTIONAL PARETO EFFICIENCY: CONCEPTS AND OPTIMALITY CONDITIONS

by

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Abstract: We introduce and study a notion of directional Pareto minimality with respect to a set that generalizes the classical concept of Pareto efficiency. Then we give separate necessary and sufficient conditions for the newly introduced efficiency and several situations concerning the objective mapping and the constraints are considered. In order to investigate different cases, we adapt some well-known constructions of generalized differentiation and the connections with some recent directional regularities come naturally into play. As a consequence, several techniques from the study of genuine Pareto minima are considered in our specific situation.

Keywords: directional Pareto minimality \cdot optimality conditions \cdot directional tangent cones \cdot directional regularity

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1 Introduction and notation

This paper has two main motivations. On one hand, we are aiming at continuing the effort made by several authors in the last decade to investigate directional phenomena in mathematical programming and, on the other hand, we show the power of several tools related to directional regularities that have been developed recently. For detailed accounts on these topics we refer the reader to the following works and references therein: [15], [16], [1], [10], [12].

In this work, inspired by some ideas coming in vector optimization problems from location theory where some directions are privileged with respect to the others, we present a notion of directional minimality for mappings and we illustrate by examples its relevance even for the case of real-valued functions. Then, we observe on the simplest case of real-valued functions of a real variable that the natural necessary optimality conditions are given by the Fermat Theorem at an endpoint of an interval. This gives us the impetus to consider far-reaching generalization of this case, namely, problems where the objective is a set-valued map and the constraint is given by

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means of an inverse image of a cone through another set-valued map. For the study of this general case, we introduce an adapted tangent cone, along with several directional regularity properties of the involved maps, and this approach allows us to derive necessary optimality conditions that, in turn, generalize the prototype of Fermat Theorem at an endpoint of an interval. Furthermore, we present as well optimality conditions in terms of tangent limiting cones and coderivatives. Both on primal and dual spaces we have under consideration several situations concerning the objective and constraint mappings with their specific techniques of study, among which we mention generalized constraint qualification conditions, Gerstewitz scalarization, openness vs. minimality paradigm, Clarke penalization, extremal principle. Some results are dedicated to the sufficient optimality for sets and a brief discussion of this concept reveals the similarities and the differences with respect to the known situation of Pareto efficiency.

The paper is organized as follows. First of all, we introduce the notation we use and then we present the concepts of directional minimality we study in this work. The definitions of these notions along with some comparisons and examples are the subjects of the second section. The main section of the paper is the third one, and it deals with optimality conditions for the above introduced concepts, being, in turn, divided into two subsections. Firstly, we derive optimality conditions using tangent cones and to this aim we adapt a classical concept of the Bouligand tangent cone and Bouligand derivative of a set-valued map. Using some directional metric regularities, we get several assertions concerning these objects and this allows us to present necessary optimality conditions for a wide range of situations going from problems governed by set-valued mappings having generalized inequalities constraints to fully smooth constrained problems. Secondly, we deal with optimality conditions using normal limiting cones and, again, we consider several types of problems. In this process of getting necessary optimality conditions we adapt several techniques from classical vector optimization. Moreover, some generalized convex cases are considered in order to obtain sufficient optimality conditions. The last section deals with Pareto directional minima for sets. We emphasize the fact that even if the directional Pareto efficiency appears naturally in the case of mappings, it can be considered as well for sets and in this respect we present the corresponding concepts and we discuss it by means of some examples and optimality conditions in terms of the modified tangent cones. Several conclusions of this work are collected in a short section that ends the paper.

Throughout this paper, we assume that X, Y and Z are normed vector spaces over the real field \mathbb{R} and on a product of normed vector spaces we consider the sum norm, unless otherwise stated. By $B(x,\varepsilon)$ we denote the open ball with center x and radius $\varepsilon > 0$ and by B_X the open unit ball of X. In the same manner, $D(x,\varepsilon)$ and D_X denote the corresponding closed balls. The symbol S_X stands for the unit sphere of X. By the symbol X^* we denote the topological dual of X, while w^* stands for the weak^{*} topology on X^* .

Let $F: X \rightrightarrows Y$ be a set-valued map. As usual, the graph of F is

$$\operatorname{Gr} F := \left\{ (x, y) \in X \times Y \mid y \in F(x) \right\},\$$

and the inverse of F is the set-valued map $F^{-1}: Y \rightrightarrows X$ given by $(y, x) \in \operatorname{Gr} F^{-1}$ iff $(x, y) \in \operatorname{Gr} F$. Consider a nonempty subset A of X. Then the image of A through F is

$$F(A) := \{ y \in Y \mid \exists x \in A : y \in F(x) \}$$

and the distance function associated to A is $d_A: X \to \mathbb{R}$ given by

$$d_A(x) = d(x, A) := \inf_{a \in A} ||x - a||.$$

The topological interior, topological closure, the convex hull and conic hull of A are denoted, respectively, by int A, cl A, conv A, cone A. The negative polar of A is

$$A^{-} := \{ x^{*} \in X^{*} \mid x^{*} (a) \le 0, \forall a \in A \}.$$

2 The concepts under study

Let $K \subset Y$ be a proper (that is, $K \neq \{0\}, K \neq Y$) convex cone (we do not suppose that K is pointed, in general). For such a cone, its positive dual cone is

$$K^{+} := \{ y^{*} \in Y^{*} \mid y^{*} (y) \ge 0, \forall y \in K \}.$$

Take $F : X \rightrightarrows Y$ as a set-valued mapping, and let us consider the following geometrically constrained optimization problem with multifunctions:

(P) minimize F(x), subject to $x \in A$,

where $A \subset X$ is a closed nonempty set.

Usually, the minimality is understood in the Pareto sense given by the next definition.

Definition 2.1 A point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local Pareto minimum point for F on A if there exists a neighborhood U of \overline{x} such that

$$(F(U \cap A) - \overline{y}) \cap -K \subset K. \tag{2.1}$$

The vectorial notion described by (2.1) covers as well the situation where f is a function (in which case $\overline{y} = f(\overline{x})$ will not be mentioned) and the situation of classical local minima in scalar case (in which case we drop the label "Pareto"). If K is pointed (that is, $K \cap -K = \{0\}$) then (2.1) reduces to

$$(F(U \cap A) - \overline{y}) \cap -K \subset \{0\}.$$

Definition 2.2 If int $K \neq \emptyset$, the point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local weak Pareto minimum point for F on A if there exists a neighborhood U of \overline{x} such that

$$(F(U \cap A) - \overline{y}) \cap -\operatorname{int} K = \emptyset.$$

Let $L \subset S_X$ be a nonempty closed set. Then it is not difficult to see that cone L is closed as well. Indeed, let us consider a sequence $(u_n) \subset \operatorname{cone} L$ converging towards $u \in X$. We have to show that $u \in \operatorname{cone} L$. The case $u = 0 \in \operatorname{cone} L$ is clear. Otherwise, there are some sequences $(t_n) \subset (0, \infty)$ and $(\ell_n) \subset L$ such that $u_n = t_n \ell_n$ for every n. If $(t_n) \to 0$ (on a subsequence), the boundedness of (ℓ_n) leads to u = 0, a situation avoided at this stage. If (t_n) is unbounded, then again the relation $||t_n \ell_n|| \to ||u||$ leads to a contradiction. So, on a subsequence, $(t_n) \to t > 0$ which means, by the closedness of L, that $\ell_n = t_n^{-1} t_n \ell_n \to t^{-1} u \in L$, therefore $u \in \operatorname{cone} L$, as claimed.

The main purpose of this paper is to introduce and to study the following concept.

Definition 2.3 One says that $(\overline{x}, \overline{y}) \in \text{Gr } F \cap (A \times Y)$ is a local directional Pareto minimum point for F on A with respect to (the set of directions) L if there exists a neighborhood U of \overline{x} such that

$$(F(U \cap A \cap (\overline{x} + \operatorname{cone} L)) - \overline{y}) \cap -K \subset K.$$
(2.2)

If one compares this relation to (2.1), then one observes that this concept corresponds to the situation where the restriction has the special form (depending on the reference point) $A \cap$ $(\overline{x} + \operatorname{cone} L)$. Of course, when A = X in (2.2) then one says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local directional Pareto minimum point for F with respect to L. Now, the concept of local directional Pareto maximum is obtained in an obvious way.

If int $K \neq \emptyset$, one defines as well the weak counterpart of the above notion.

Definition 2.4 One says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local weak directional Pareto minimum point for F on A with respect to (the set of directions) L if there exists a neighborhood U of \overline{x} such that

$$(F(U \cap A \cap (\overline{x} + \operatorname{cone} L)) - \overline{y}) \cap -\operatorname{int} K = \emptyset.$$

In all these notions, if one takes U = X, then we get the corresponding global concepts.

Remark 2.5 If $L_1, L_2 \subset S_X$ are nonempty closed subsets such that $L_1 \subset L_2$, then a local directional Pareto minimum point for F with respect to L_2 is a local directional Pareto minimum point for F with respect to L_1 .

It is obvious that (2.1) implies (2.2), but the converse is not true. To justify the latter affirmation, let us consider the following simple scalar example (when the output space is \mathbb{R} we always consider $K := [0, \infty)$).

Example 2.6 Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Then every $\overline{x} \in \mathbb{R}$ is local directional minimum for f with respect to $L := \{+1\}$, but it is not a local minimum for f.

Moreover, the minimality concept introduced here covers some interesting situations described by the next examples.

Example 2.7 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 - y^2$. It is well known that (0, 0) is a critical saddle point, whence it is not a minimum point. However, it is a directional minimum point for f with respect to $L = \{-1, 1\} \times \{0\}$ since for every $(x, y) \in (0, 0) + \operatorname{cone} L = \mathbb{R} \times \{0\}$, one has $f(x, y) \ge f(0, 0)$. Similarly, (0, 0) is a directional maximum point for f with respect to $L = \{0\} \times \{-1, 1\}$.

Example 2.8 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 - y^3$. Again, (0, 0) is a critical saddle point. It is now easy to see that it is, however, a directional minimum point for f with respect to $L = \{-1, 1\} \times \{0\}$ and to respect to $L = \{0\} \times \{-1\}$.

The next example emphasizes that there are points which are not directional minima with respect to any nonempty closed set $L \subset S_X$. This applies also for critical points of smooth functions.

Example 2.9 Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \sin\frac{1}{x}, \text{ if } x \neq 0\\ 0, \text{ if } x = 0 \end{cases}$$

Then $\overline{x} = 0$ is not directional minimum for f neither for $L := \{-1\}$, nor for $L := \{+1\}$. In the same manner, $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x}, \text{ if } x \neq 0\\ 0, \text{ if } x = 0 \end{cases}$$

is differentiable at $\overline{x} = 0$, $f'(\overline{x}) = 0$, but \overline{x} is not a directional minimum for f.

The next example underlines the idea that for every prescribed set of directions one can define functions that achieve directional minimum with respect to the given set.

Example 2.10 Let $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$ and $L := \{(\cos \theta, \sin \theta) \mid \theta_1 \leq \theta \leq \theta_2\}$. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \left(\theta_2 - \arctan \frac{y}{x} \right) \left(\arctan \frac{y}{x} - \theta_1 \right), \text{ if } x \neq 0 \text{ and } (x > 0 \text{ or } y \ge 0) \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \text{ and } y < 0 \end{cases}$$

Then it is not difficult to see that (0,0) is directional minimum for f with respect to L.

Using these basic examples of scalar-valued functions, we are able to easily build examples for vector-valued maps. Here are two such examples.

Example 2.11 Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x^2 - y^2, x^2 - y^3)$. Consider $K := \mathbb{R}^2_+$. Then (0,0) is a directional minimum for f with respect to $L := \{(1,0)\} \subset S_{\mathbb{R}^2}$.

Example 2.12 Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by f(x) = (2x, x) and $K = \operatorname{cone} \operatorname{conv} \{(1, 0), (1, 1)\}$. It is easy to see that $\overline{x} := 0$ is a directional minimum for f with respect to $L := \{+1\}$, but \overline{x} is not a local Pareto minimum point for f.

The concepts introduced in this section are studied in the sequel from the point of view of optimality conditions.

3 Optimality conditions for directional minima

In order to start with the necessary optimality conditions for directional minima, let us to observe that the obvious prototype for such an investigation is the Fermat Theorem for derivable real-valued functions with one variable at interval endpoints: if $f : [a, b] \to \mathbb{R}$ is a function for which a is local minimum point (that is, a directional minimum with respect to $L := \{+1\}$), and f is derivable at a, then $f'(a) \ge 0$, and, similarly, if b is a minimum point for f (that is, a directional minimum with respect to $L := \{-1\}$), and f is derivable at b, then $f'(b) \le 0$.

We approach this issue from two points of view, namely, making use of tangent cones (which are objects of generalized differentiation on primal spaces) and of normal cones (constructions that are defined on dual spaces).

3.1 Optimality conditions using tangent cones

Let us consider now several concepts that will help us in studying optimality conditions for the directional minima.

Definition 3.1 Let $A \subset X$ be a nonempty set and $L \subset S_X$ be a nonempty closed set. Then the Bouligand tangent cone to A at $\overline{x} \in A$ with respect to L is the set

$$T_B^L(A,\overline{x}) := \left\{ u \in X \mid \exists (u_n) \stackrel{\text{cone}\,L}{\longrightarrow} u, \exists (t_n) \stackrel{(0,\infty)}{\longrightarrow} 0 \text{ such that for all } n, \ \overline{x} + t_n u_n \in A \right\},$$

where $(u_n) \xrightarrow{\operatorname{cone} L} u$ means $(u_n) \longrightarrow u$ and $(u_n) \subset \operatorname{cone} L$, and similarly for $(t_n) \xrightarrow{(0,\infty)} 0$.

Obviously, this is a adaptation of the concept of Bouligand tangent cone to A at \overline{x} defined as

$$T_B(A,\overline{x}) := \left\{ u \in X \mid \exists (u_n) \to u, \exists (t_n) \xrightarrow{(0,\infty)} 0 \text{ such that for all } n, \ \overline{x} + t_n u_n \in A \right\}.$$

Some remarks are in order.

Remark 3.2 As the usual Bouligand tangent cone, the set $T_B^L(A, \overline{x})$ is a closed cone: the proof of this assertion can be made directly as for the classical concept (see [3]) or by observing that

$$T_B^L(A,\overline{x}) = T_B(A \cap (\overline{x} + \operatorname{cone} L), \overline{x}).$$

In view of the fact that cone L is closed, one has that $T^L_B(A, \overline{x}) \subset \text{cone } L$. Moreover,

$$T_B^L(A,\overline{x}) \subset T_B(A,\overline{x}) \cap T_B(\overline{x} + \operatorname{cone} L,\overline{x}) = T_B(A,\overline{x}) \cap \operatorname{cone} L.$$

However, the inclusion above does not hold as equality, in general. To see this, consider the set $A \subset X := \mathbb{R}^2$ as the plane domain bounded by the curve (the cardioid), which has the parametric representation

$$\begin{cases} x = -2\cos t + \cos 2t + 1\\ y = 2\sin t - \sin 2t \end{cases}, \ t \in [0, 2\pi],$$

 $\overline{x} := (0,0), L := \{(-1,0)\}$ and observe that $T_B(A,\overline{x}) = X$ and $T_B(A \cap (\overline{x} + \operatorname{cone} L), \overline{x}) = \{\overline{x}\}.$ Another useful and easy-to-see inclusion is

$$\operatorname{cl}(T_B(A,\overline{x}) \cap \operatorname{int} \operatorname{cone} L) \subset T_B^L(A,\overline{x})$$

Definition 3.3 Let $F: X \rightrightarrows Y$ be a set-valued map, $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ and $L \subset S_X$, $M \subset S_Y$ be nonempty closed sets. The Bouligand derivative of F at $(\overline{x}, \overline{y})$ with respect to L and M is the set-valued map $D_B^{L,M}F(\overline{x},\overline{y}): X \rightrightarrows Y$ defined by the relation $v \in D_B^{L,M}F(\overline{x},\overline{y})(u)$ iff there are $(u_n) \xrightarrow{\operatorname{cone} L} u, (v_n) \xrightarrow{\operatorname{cone} M} v, (t_n) \xrightarrow{(0,\infty)} 0$ such that for all n,

$$\overline{y} + t_n v_n \in F(\overline{x} + t_n v_n).$$

Clearly,

$$\operatorname{Gr} D_B^{L,M} F(\overline{x}, \overline{y}) \subset \operatorname{cone} L \times \operatorname{cone} M$$

Again, this is an adaptation of the well-known Bouligand derivative of F at $(\overline{x}, \overline{y})$, which is the set-valued map $D_B F(\overline{x}, \overline{y}) : X \Longrightarrow Y$ defined by

$$\operatorname{Gr} D_B F(\overline{x}, \overline{y}) := T_B \left(\operatorname{Gr} F, (\overline{x}, \overline{y}) \right).$$

Other derivability objects in primal spaces that can be adapted in directional setting in a similar manner are the Ursescu (adjacent) tangent cone and the Ursescu (adjacent) derivative (see [5]), and the Dini lower derivative of F at $(\overline{x}, \overline{y})$, which is the multifunction $D_D F(\overline{x}, \overline{y})$ from X into Y given, for every $u \in X$, by

$$D_D F(\overline{x}, \overline{y})(u) = \{ v \in Y \mid \forall (t_n) \xrightarrow{(0,\infty)} 0, \forall (u_n) \to u, \exists (v_n) \to v, \\ \forall n \in \mathbb{N}, \overline{y} + t_n v_n \in F(\overline{x} + t_n u_n) \}.$$

When F := f is a single-valued map, for simplicity, we write $D_B^{L,M} f(\overline{x})$ for $D_B^{L,M} f(\overline{x}, \overline{y})$, and similarly for D_D .

We present now the first result of this work.

Proposition 3.4 In the above notation, if int $K \neq \emptyset$ and $(\overline{x}, \overline{y}) \in \text{Gr } F$ is a local weak directional Pareto minimum point for F on A with respect to L then

$$D_D F(\overline{x}, \overline{y})(T_B^L(A, \overline{x})) \cap -\operatorname{int} K = \emptyset.$$

Moreover, if A = X, then

$$D_B^{L,S_Y}F(\overline{x},\overline{y})(X)\cap -\operatorname{int} K = \emptyset$$

Proof. We prove only the second part, since the first part, on one hand, is similar, and, on the other hand, it follows from the definitions and [5, Theorem 3.1]. Take $u \in X$. If $u \notin \operatorname{cone} L$ then $D_B^{L,S_Y} F(\overline{x}, \overline{y})(u) = \emptyset$ and there is nothing to prove. If $u \in \operatorname{cone} L$, suppose, by way of contradiction, that there is $k \in -\operatorname{int} K$ such that

$$k \in D_B^{L,S_Y} F(\overline{x}, \overline{y})(u).$$

According to the definition of $D_B^{L,S_Y}F(\overline{x},\overline{y})$, this means that there exist $(t_n) \xrightarrow{(0,\infty)} 0, (u_n) \xrightarrow{\operatorname{cone} L} u,$ $(k_n) \to k$ such that for all n,

$$\overline{y} + t_n k_n \in F(\overline{x} + t_n u_n),$$

that is,

$$t_n k_n \in F(\overline{x} + t_n u_n) - \overline{y}.$$

But, for *n* large enough, $\overline{x} + t_n u_n$ is close enough to \overline{x} and belongs as well to $\overline{x} + \operatorname{cone} L$. Then, for such *n*, taking into account the minimality of $(\overline{x}, \overline{y})$, one gets $t_n k_n \notin -\operatorname{int} K$ which contradicts the fact that $k_n \to k \in -\operatorname{int} K$.

In [9], by means of a special type of minimal time function, several directional regularity properties for set-valued maps are introduced and studied. In order to further investigate the directional minima we need to briefly point out the main aspects concerning the minimal time function and some related directional metric regularity.

Consider $\emptyset \neq L \subset S_X$ and $\emptyset \neq \Omega \subset X$. Then the function

$$T_L(x,\Omega) := \inf \{ t \ge 0 \mid \exists u \in L : x + tu \in \Omega \}$$

= inf $\{ t \ge 0 \mid (x + tL) \cap \Omega \neq \emptyset \}$ (3.1)

is called the directional minimal time function with respect to L.

Remark that, if $L = S_X$, then $T_L(\cdot, \Omega) = d(\cdot, \Omega)$. Moreover, we add the convention that $T_L(x, \emptyset) = \infty$ for every x and we denote in what follows $T_L(x, \{u\})$ by $T_L(x, u)$. Obviously, $T_L(x, u) < +\infty$ is equivalent to $T_L(x, u) = ||u - x||$ and $u - x \in \text{cone } L$.

Let $F: X \rightrightarrows Y$ be a set-valued mapping and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F, \emptyset \neq L \subset S_X, \emptyset \neq M \subset S_Y.$

What we need in the sequel is the following concept of directional calmness. One says that F is directionally calm at $(\overline{x}, \overline{y})$ with respect to L and M if there are $\alpha > 0$ and some neighborhoods U of \overline{x} and V of \overline{y} such that for every $x \in U$,

$$\sup_{y \in F(x) \cap V} T_M(y, F(\overline{x})) \le \alpha T_L(\overline{x}, x).$$
(3.2)

We use the convention $\sup_{x\in\emptyset} T_L(x,\Omega) := 0$ for every nonempty set $\Omega \subset X$.

As usual (see [4, Section 3H]), for a calmness concept for F, it is natural to have a metric subregularity notion such that the former property for F^{-1} to be equivalent to the latter property

for F. In our setting, this corresponding concept reads as follows: one says that F is directionally metric subregular at $(\overline{x}, \overline{y})$ with respect to L and M if there exist $\alpha > 0$ and some neighborhoods U of \overline{x} and V of \overline{y} such that for every $x \in U$,

$$T_L(x, F^{-1}(\overline{y})) \le \alpha T_M(\overline{y}, F(x) \cap V).$$
(3.3)

The expected equivalence is described in the following result.

Proposition 3.5 The set-valued map F is directionally metric subregular at $(\overline{x}, \overline{y})$ with respect to L and M iff F^{-1} is directionally calm at $(\overline{y}, \overline{x})$ with respect to M and L.

Proof. Suppose first that F is directionally metric subregular at $(\overline{x}, \overline{y})$ with respect to L and M. Then, there exist $\alpha > 0, U \in \mathcal{V}(\overline{x})$ and $V \in \mathcal{V}(\overline{y})$ such that for every $x \in U$ relation (3.3) holds. Let $y \in V$. If $T_M(\overline{y}, y) = +\infty$, there is nothing to prove. Suppose that $T_M(\overline{y}, y) < +\infty$, which means that $y - \overline{y} \in \text{cone } M$. Consider $x \in U$ with $y \in F(x)$, i.e., $x \in F^{-1}(y) \cap U$. Then, by hypothesis,

$$T_L(x, F^{-1}(\overline{y})) \le \alpha T_M(\overline{y}, F(x) \cap V) \le \alpha T_M(\overline{y}, y),$$

so,

$$\sup_{x \in F^{-1}(y) \cap U} T_L(x, F^{-1}(\overline{y})) \le \alpha T_M(\overline{y}, y),$$

for all $y \in V$, whence the conclusion.

For the converse, suppose that F^{-1} is directionally calm at $(\overline{y}, \overline{x})$ with respect to M and L. Therefore, there exist $\alpha > 0$, $U \in \mathcal{V}(\overline{x})$ and $V \in \mathcal{V}(\overline{y})$ such that for every $y \in V$

$$\sup_{x \in F^{-1}(y) \cap U} T_L(x, F^{-1}(\overline{y})) \le \alpha T_M(\overline{y}, y).$$

Take $x \in U$. Again, if $T_M(\overline{y}, F(x) \cap V) = +\infty$, the desired inequality holds. Suppose that $T_M(\overline{y}, F(x) \cap V) < +\infty$, which means that for any $\varepsilon > 0$ there exist $u_{\varepsilon} \in M$ and $y_{\varepsilon} \in F(x) \cap V$ such that

$$\overline{y} + (T_M(\overline{y}, F(x) \cap V) + \varepsilon) u_{\varepsilon} = y_{\varepsilon}.$$

Therefore, $y_{\varepsilon} - \overline{y} \in \operatorname{cone} M$, $x \in F^{-1}(y_{\varepsilon}) \cap U$ and from the hypothesis,

$$T_L(x, F^{-1}(\overline{y})) \le \alpha T_M(\overline{y}, y_{\varepsilon}) = \alpha \|y_{\varepsilon} - \overline{y}\| \le \alpha \left(T_M(\overline{y}, F(x) \cap V) + \varepsilon\right).$$

Passing to the limit as $\varepsilon \to 0$ we get the conclusion.

Now, we use the directional calmness for getting an evaluation of the directional Bouligand tangent cone to a value of a set-valued mapping in terms of the image of 0 through the directional Bouligand derivative of the same application.

Proposition 3.6 Let $F : X \rightrightarrows Y$ be a set-valued mapping, $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, and $\emptyset \neq L \subset S_X$, $\emptyset \neq M \subset S_Y$ be closed sets. Then

$$T_B^M(F(\overline{x}), \overline{y}) \subset D_B^{L,M}F(\overline{x}, \overline{y})(0).$$

Moreover, if F is directionally calm at $(\overline{x}, \overline{y})$ with respect to L and M, and cone M is convex, then the equality holds.

Proof. Take $v \in T_B^M(F(\overline{x}), \overline{y})$. According to the definition, there are $(v_n) \xrightarrow{\operatorname{cone} M} v, (t_n) \xrightarrow{(0,\infty)} 0$ such that for all n,

$$\overline{y} + t_n v_n \in F(\overline{x}) = F(\overline{x} + t_n \cdot 0),$$

which clearly implies that $v \in D_B^{L,M} F(\overline{x}, \overline{y})(0)$. For the opposite inclusion, take $v \in D_B^{L,M} F(\overline{x}, \overline{y})(0)$ meaning that there are $(u_n) \xrightarrow{\operatorname{cone} L} 0, (v_n) \xrightarrow{\operatorname{cone} M} 0$ $v, (t_n) \xrightarrow{(0,\infty)} 0$ such that for all n,

$$\overline{y} + t_n v_n \in F(\overline{x} + t_n u_n).$$

But, the assumed calmness of F and the fact that $(t_n u_n) \subset \operatorname{cone} L$, mean that, for a positive α and for all n large enough,

$$T_M(\overline{y} + t_n v_n, F(\overline{x})) \le \alpha T_L(\overline{x}, \overline{x} + t_n u_n) = \alpha t_n \|u_n\|,$$

that is

$$\inf\{\tau \ge 0 \mid \exists w_n \in M \text{ such that for all } n, \ \overline{y} + t_n v_n + \tau w_n \in F(\overline{x})\} \le \alpha t_n \|u_n\|.$$

Therefore, for every n (large enough) there are $w_n \in M$ and $\tau_n \ge 0$ such that $\beta_n := \overline{y} + t_n v_n + \tau_n w_n \in \mathbb{R}$ $F(\overline{x})$ and $\tau_n < \alpha t_n ||u_n|| + t_n^2$. So, for every n,

$$\|\beta_n - (\overline{y} + t_n v_n)\| = \tau_n < \alpha t_n \|u_n\| + t_n^2,$$

whence

$$\left\|\frac{1}{t_n}(\beta_n - \overline{y}) - v_n\right\| < \alpha \|u_n\| + t_n,$$

which gives

$$\frac{1}{t_n}(\beta_n - \overline{y}) \to v.$$

Taking into account the convexity of cone M, for every n,

$$\beta_n - \overline{y} = t_n v_n + \tau_n w_n \in \operatorname{cone} M + \operatorname{cone} M = \operatorname{cone} M.$$

Summing up,

$$\frac{1}{t_n}(\beta_n - \overline{y}) \stackrel{\text{cone}\,M}{\longrightarrow} v,$$

whence $v \in T_B^M(F(\overline{x}), \overline{y})$.

Consider now the situation when $G: X \Rightarrow Z$ is a set-valued map, $Q \subset Z$ is a closed convex and pointed cone and the set of restrictions for (P) is $A := \{x \in X \mid 0 \in G(x) + Q\}$. This is a standard situation which encompasses the classical case where one has equalities and inequalities constraints. The following result holds.

Proposition 3.7 Let $\emptyset \neq L \subset S_X$, $\emptyset \neq N \subset S_Z$ be closed sets, take $\overline{x} \in A$ (meaning that there is $\overline{z} \in G(\overline{x}) \cap -Q$, and define the set-valued map $\mathcal{E}_G: X \rightrightarrows Z, \mathcal{E}_G(x) = G(x) + Q$. Suppose that \mathcal{E}_G is directionally metric subregular at $(\overline{x}, 0)$ with respect to L and N. If cone L is convex then $u \in T_B^L(A,\overline{x})$ iff $0 \in D_B^{L,N} \mathcal{E}_G(\overline{x},0)(u)$. Moreover, if $Q \cap S_Z \subset N$ and cone N is convex then for every $u \in X$,

$$D_B^{L,N}G(\overline{x},\overline{z})(u) + Q \subset D_B^{L,N}\mathcal{E}_G(\overline{x},0)(u).$$

Proof. We remark that $A = \mathcal{E}_G^{-1}(0)$, whence, by Propositions 3.5 and 3.6,

$$T_B^L(A,\overline{x}) = D_B^{N,L} \mathcal{E}_G^{-1}(0,\overline{x})(0),$$

whence $u \in T_B^L(A, \overline{x})$ iff $u \in D_B^{N,L} \mathcal{E}_G^{-1}(0, \overline{x})(0)$ iff $0 \in D_B^{L,N} \mathcal{E}_G(\overline{x}, 0)(u)$.

Now, for the second part, take $w \in D_B^{L,N}G(\overline{x},\overline{z})(u) + Q$. Then there exist $q \in Q$ and $(u_n) \xrightarrow{\text{cone } L} u, (w_n) \xrightarrow{\text{cone } N} w - q, (t_n) \xrightarrow{(0,\infty)} 0$ such that for all n,

$$\overline{z} + t_n w_n \in G(\overline{x} + t_n u_n),$$

whence

$$t_n(w_n+q) \in G(\overline{x}+t_nu_n) - \overline{z} + t_nq \subset \mathcal{E}_G(\overline{x}+t_nu_n).$$

But, $w_n + q \to w$ and for every $n, w_n + q \in \operatorname{cone} N + Q \subset \operatorname{cone} N$, whence $w \in D_B^{L,N} \mathcal{E}_G(\overline{x}, 0)(u)$.

Proposition 3.8 Suppose that int $K \neq \emptyset$ and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local weak directional Pareto minimum point for F on $A := \mathcal{E}_G^{-1}(0)$ with respect to a closed nonempty set $L \subset S_X$. Consider $\overline{z} \in G(\overline{x}) \cap -Q$ and $\emptyset \neq N \subset S_Z$ a closed set. Moreover, suppose that $Q \cap S_Z \subset N$, cone L and cone N are convex, and \mathcal{E}_G is directionally metric subregular at $(\overline{x}, 0)$ with respect to L and N. Then

$$\left\{ (v,w) \mid \exists u \in X, v \in D_D F(\overline{x}, \overline{y})(u), w \in D_B^{L,N} G(\overline{x}, \overline{z})(u) \right\} \cap (-\operatorname{int} K \times -Q) = \emptyset.$$

Proof. The result follows by using successively Propositions 3.4 and 3.7.

Let us to specialize, in two steps, the ideas above to the classical smooth case of optimization problems with single-valued maps. First, suppose that F := f and G := g are continuously Fréchet differentiable functions. Then taking a point $\overline{x} \in A = \mathcal{E}_G^{-1}(0)$ it is easy to see that for all $u \in X$, $D_D f(\overline{x})(u) = \{\nabla f(\overline{x})(u)\}$, while

$$D_B^{L,S_Z}g(\overline{x})(u) = \begin{cases} \{\nabla g(\overline{x})(u)\}, \text{ if } u \in \operatorname{cone} L\\ \emptyset, \text{ if } u \notin \operatorname{cone} L. \end{cases}$$

Then we get the following Fritz John and Karush-Kuhn-Tucker type result.

Theorem 3.9 Suppose that int $K \neq \emptyset$ and $\overline{x} \in A := \mathcal{E}_g^{-1}(0)$ is a local weak directional Pareto minimum point for f on A with respect to L. Moreover, suppose that cone L is convex, and \mathcal{E}_g is directionally metric subregular at $(\overline{x}, 0)$ with respect to L and S_Z . Then, in either of the following conditions:

(i) int $Q \neq \emptyset$ or int $\{(\nabla f(\overline{x})(u), \nabla g(\overline{x})(u)) \mid u \in \operatorname{cone} L\} \neq \emptyset;$

(ii) Y and Z are finite dimensional spaces,

there exist $y^* \in K^+$, $z^* \in Q^+$, $(y^*, z^*) \neq 0$ such that for every $u \in \text{cone } L$,

$$\left(y^* \circ \nabla f(\overline{x}) + z^* \circ \nabla g(\overline{x})\right)(u) \ge 0.$$

If, moreover, there exists $u \in \operatorname{cone} L$ such that $\nabla g(\overline{x})(u) \in \operatorname{int} Q \neq \emptyset$ or $\nabla g(\overline{x})(\operatorname{cone} L) = Z$ then $y^* \neq 0$.

Proof. According to Proposition 3.8 and the subsequent discussion,

$$\{(\nabla f(\overline{x})(u), \nabla g(\overline{x})(u)) \mid u \in \operatorname{cone} L\} \cap (-\operatorname{int} K \times -Q) = \emptyset.$$

Notice that $\{(\nabla f(\overline{x})(u), \nabla g(\overline{x})(u)) \mid u \in \text{cone } L\}$ is a convex set and both (i) and (ii) ensure the possibility to apply a separation result for convex sets. Therefore, there exist $y^* \in Y^*$, $z^* \in Z^*$, $(y^*, z^*) \neq 0$ such that for every $u \in \text{cone } L$, $k \in \text{int } K$, $q \in Q$, one has

$$(y^* \circ \nabla f(\overline{x}) + z^* \circ \nabla g(\overline{x}))(u) \ge -y^*(k) - z^*(q).$$

Standard arguments yield $y^* \in K^+$, $z^* \in Q^+$ and

$$(y^* \circ \nabla f(\overline{x}) + z^* \circ \nabla g(\overline{x}))(u) \ge 0.$$

for every $u \in \operatorname{cone} L$.

If one supposes that $y^* = 0$ then the relation above and the either of the final assumptions give $z^* = 0$, which contradicts $(y^*, z^*) \neq 0$.

A similar but different result could be done taking into account the special structure of this case, using directly Proposition 3.4, and some results one can find in literature concerning the calculus of Bouligand tangent cone to the counter image of a set through a differentiable mapping. Let us recall some facts from [8]. Let $f: X \to Y$ be a function and $D \subset X$ be a nonempty closed set. One says that f is metrically subregular at $(\overline{x}, f(\overline{x})) \in D \times Y$ with respect to D if there exist s > 0, $\mu > 0$ s.t. for every $u \in B(\overline{x}, s) \cap D$

$$d(u, f^{-1}(f(\overline{x})) \cap D) \le \mu \left\| f(\overline{x}) - f(u) \right\|.$$

In fact, the above notion coincides with that of calmness of the set-valued map $y \Rightarrow f^{-1}(y) \cap D$ at $(f(\overline{x}), \overline{x})$ (see, for instance, [4, Section 3H]). One of the main results in [8] reads as follows.

Theorem 3.10 Let X, Y be Banach spaces, $D \subset X, E \subset Y$ be closed sets, $\varphi : X \to Y$ be a continuously Fréchet differentiable map and $\overline{x} \in D \cap \varphi^{-1}(E)$. Suppose that $\psi : X \times Y \to Y$, $\psi(x, y) := \varphi(x) - y$ is metrically subregular at $(\overline{x}, \varphi(\overline{x}), 0)$ with respect to $D \times E$. Then

$$T_U(D,\overline{x}) \cap \nabla \varphi(\overline{x})^{-1}(T_B(E,\varphi(\overline{x}))) \subset T_B(D \cap \varphi^{-1}(E),\overline{x})$$

where $T_U(D, \overline{x})$ denotes the Ursescu tangent cone to D at \overline{x} , that is,

$$T_U(D,\overline{x}) := \left\{ u \in X \mid \forall (t_n) \xrightarrow{(0,\infty)} 0, \exists (u_n) \to u \text{ such that for all } n, \ \overline{x} + t_n u_n \in D \right\}.$$

Coming back to our case, we have X := X, Y := Z, $D := \overline{x} + \operatorname{cone} L$, E := -Q, $\varphi := g$. We have seen that $T_B^L(g^{-1}(-Q), \overline{x}) = T_B((\overline{x} + \operatorname{cone} L) \cap g^{-1}(-Q)), \overline{x})$. With these identifications, we get the next result.

Theorem 3.11 Suppose that X, Z are Banach spaces, int $K \neq \emptyset$ and $\overline{x} \in g^{-1}(-Q)$ is a local weak directional Pareto minimum point for f on $g^{-1}(-Q)$ with respect to L. Moreover, suppose that $\psi : X \times Z \to Z$, $\psi(x, z) := g(x) - z$ is metrically subregular at $(\overline{x}, g(\overline{x}), 0)$ with respect to $(\overline{x} + \operatorname{cone} L) \times -Q$. Then for all $u \in \operatorname{cone} L$ with $\nabla g(\overline{x})(u) \in T_B(-Q, g(\overline{x}))$,

$$\nabla f(\overline{x})(u) \notin -\operatorname{int} K$$

Proof. According to Theorem 3.10,

$$T_U(\overline{x} + \operatorname{cone} L, \overline{x}) \cap \nabla g(\overline{x})^{-1}(T_B(-Q, g(\overline{x}))) \subset T_B^L(g^{-1}(-Q), \overline{x}),$$

whence

cone
$$L \cap \nabla g(\overline{x})^{-1}(T_B(-Q, g(\overline{x}))) \subset T_B^L(g^{-1}(-Q), \overline{x}).$$

By Proposition 3.4,

$$\nabla f(\overline{x})(u) \notin -\operatorname{int} K$$

for all $u \in \operatorname{cone} L \cap \nabla g(\overline{x})^{-1}(T_B(-Q, g(\overline{x})))$, whence the conclusion.

Furthermore, we consider the case where $Y = \mathbb{R}^k$ $(k \ge 1)$, $Z = \mathbb{R}^p$ $(p \ge 1)$, $Q = \mathbb{R}^m_+ \times \{0\}^n$ with m + n = p, and f, g are Fréchet differentiable. This means that we are dealing with a vectorial optimization problem with finitely many inequalities and equalities constraints. Let us denote by μ_i with $i \in \overline{1, m}$ the first m coordinates functions of g and by ν_j with $j \in \overline{1, n}$ the next n coordinates functions of g.

For the next step of our approach, we use the Gerstewitz functional in the special case when the ordering cone has nonempty interior. The next result combines [11, Theorem 2.3.1] and [6, Lemma 2.1].

Theorem 3.12 Let $K \subset Y$ be a closed convex cone with nonempty interior. Then for every $e \in \text{int } K$ the functional $s_{K,e} : Y \to \mathbb{R}$ given by

$$s_{K,e}(y) = \inf\{\lambda \in \mathbb{R} \mid \lambda e \in y + K\}$$
(3.4)

is convex continuous and for every $\lambda \in \mathbb{R}$,

$$\{y \in Y \mid s_{K,e}(y) < \lambda\} = \lambda e - \operatorname{int} K, \text{ and } \{y \in Y \mid s_{K,e}(y) = \lambda\} = \lambda e - \operatorname{bd} K.$$
(3.5)

Moreover, $s_{K,e}$ is sublinear, K-monotone, and for every $u \in Y$, the Fenchel (convex) subdifferential $\partial s_{K,e}(u)$ is nonempty and

$$\partial s_{K,e}(u) = \{ v^* \in K^+ \mid v^*(e) = 1, v^*(u) = s_{K,e}(u) \}.$$
(3.6)

In this notation we have the next result.

Theorem 3.13 Suppose that X is a Banach space, int $K \neq \emptyset$ and $\overline{x} \in g^{-1}(-Q)$ is a local weak directional Pareto minimum point for f on $g^{-1}(-Q)$ with respect to L. Suppose that:

(i) cone L is convex;

(ii) $\psi: X \times Z \to Z$, $\psi(x, z) := g(x) - z$ is metrically subregular at $(\overline{x}, g(\overline{x}), 0)$ with respect to $(\overline{x} + \operatorname{cone} L) \times -Q$;

(*iii*) $\nabla \nu (\overline{x}) (X) = \mathbb{R}^n$, where $\nu := (\nu_1, \nu_2, ..., \nu_n)$;

(iv) there exists $\overline{u} \in \text{int cone } L$ such that $\nabla \mu_i(\overline{x})(\overline{u}) < 0$ for any $i \in I(\overline{x}) := \{i \in \overline{1, m} \mid \mu_i(\overline{x}) = 0\}$ and $\nabla \nu(\overline{x})(\overline{u}) = 0$.

Then there exist $y^* \in K^+ \setminus \{0\}$, $\lambda_i \geq 0$ for $i \in \overline{1, m}$ and $\tau_j \in \mathbb{R}$ for $j \in \overline{1, n}$ such that

$$0 \in y^* \circ \nabla f(\overline{x}) + \sum_{i=1}^m \lambda_i \nabla \mu_i(\overline{x}) + \sum_{j=1}^n \tau_j \nabla \nu_j(\overline{x}) + L^-$$
(3.7)

and

$$\lambda_i \mu_i(\overline{x}) = 0, \forall i \in \overline{1, m}.$$
(3.8)

Proof. Clearly, in this case $u \in \nabla g(\overline{x})^{-1}(T_B(-Q, g(\overline{x})))$ amounts to say that $\nabla \mu_i(\overline{x})(u) \leq 0$ for any $i \in I(\overline{x})$ and $\nabla \nu_j(\overline{x})(u) = 0$ for any $j \in \overline{1, n}$.

Using Theorem 3.11 (all its assumptions hold) we get that

$$\nabla f(\overline{x})(u) \notin -\operatorname{int} K$$

for all $u \in \operatorname{cone} L$ with $\nabla \mu_i(\overline{x})(u) \leq 0$ for any $i \in I(\overline{x})$, and $\nabla \nu_j(\overline{x})(u) = 0$ for any $j \in \overline{1, n}$.

We conclude that $s_{K,e}(\nabla f(\overline{x})(u)) \ge 0$ for all u satisfying the above conditions and this means that u = 0 is a minimum point for the scalar problem

$$\min s_{K,e} \left(\nabla f(\overline{x})(u) \right) \text{ s.t. } u \in \operatorname{cone} L, \nabla \mu_i(\overline{x})(u) \le 0, \forall i \in I(\overline{x}), \nabla \nu_j(\overline{x})(u) = 0, \forall j \in \overline{1, n}.$$

Since cone L is convex, this is a convex problem, whence, from [19, Theorem 2.9.6], there exist $\lambda_i \geq 0$ for $i \in I(\overline{x})$ and $\tau_j \in \mathbb{R}$ for $j \in \overline{1, n}$ such that

$$0 \in \partial \left(s_{K,e} \circ \nabla f(\overline{x}) + \iota_{\operatorname{cone} L} + \sum_{i \in I(\overline{x})} \lambda_i \nabla \mu_i(\overline{x}) + \sum_{j=1}^n \tau_j \nabla \nu_j(\overline{x}) \right) (0),$$

where ι denotes the indicator function. Finally, using (3.6), and taking $\lambda_i := 0$ for $i \in \overline{1, m} \setminus I(\overline{x})$, we get the existence of $y^* \in K^+ \setminus \{0\}$ such that

$$0 \in y^* \circ \nabla f(\overline{x}) + \sum_{i=1}^m \lambda_i \nabla \mu_i(\overline{x}) + \sum_{j=1}^n \tau_j \nabla \nu_j(\overline{x}) + L^{-1}$$

and

$$\lambda_i \mu_i(\overline{x}) = 0, \forall i \in \overline{1, m},$$

whence the conclusion.

Remark 3.14 Observe that in the simplest case of a derivable real-valued function $f : \mathbb{R} \to \mathbb{R}$, if \overline{x} is a directional minimum with respect to $L := \{+1\}$ (without constraints) the above theorem reduces to $-f(\overline{x}) \in L^-$ which is exactly $f'(\overline{x}) \ge 0$, as discussed before.

Our aim now is to derive sufficient conditions for a point $\overline{x} \in g^{-1}(-Q)$ to be a local weak directional Pareto minimum point. In order to formulate such conditions we use, besides the convexity notion for scalar functions, a generalized convexity concept. Namely, we use the following well-known concept: one says that $F: X \rightrightarrows Y$ is K-convex if for any $\lambda \in (0, 1)$, and any $x, y \in X$, one has

$$\lambda F(x) + (1 - \lambda)F(y) \subset F(\lambda x + (1 - \lambda)y) + K.$$

Proposition 3.15 Suppose that X is a Banach space, int $K \neq \emptyset$, cone L is convex, f is K-convex, $\mu_i, i \in \overline{1, m}$, are convex and $\nu_j, j \in \overline{1, n}$, are affine. If there exist $(\lambda, \tau) \in \mathbb{R}^m_+ \times \mathbb{R}^n$ and $y^* \in K^+ \setminus \{0\}$ such that (3.7) and (3.8) hold, then \overline{x} is a global weak directional Pareto minimum point for f on $g^{-1}(-Q)$ with respect to L.

Proof. By relation (3.7), we immediately get that

$$0 \in \nabla \left(y^* \circ f + \sum_{i=1}^m \lambda_i \mu_i + \sum_{j=1}^n \tau_j \nu_j \right) (\overline{x}) + N (\operatorname{cone} L, 0)$$
$$= \nabla \left(y^* \circ f + \sum_{i=1}^m \lambda_i \mu_i + \sum_{j=1}^n \tau_j \nu_j \right) (\overline{x}) + N (\overline{x} + \operatorname{cone} L, \overline{x})$$

Consider the convex optimization problem

$$\min\left(\left(y^{*}\circ f\right)(x)+\sum_{i=1}^{m}\lambda_{i}\mu_{i}\left(x\right)+\sum_{j=1}^{n}\tau_{j}\nu_{j}\left(x\right)\right), \quad x\in\overline{x}+\operatorname{cone} L.$$
(3.9)

We hence obtain, by virtue of [19, Theorem 2.9.1], that \overline{x} is a global minimum point for the above problem. Note that, for all feasible points $x \in g^{-1}(-Q)$, we have

$$\sum_{i=1}^{m} \lambda_{i} \mu_{i}(x) + \sum_{j=1}^{n} \tau_{j} \nu_{j}(x) = \sum_{i=1}^{m} \lambda_{i} \mu_{i}(x) \le 0.$$

Using (3.8), it follows that, given any $x \in (\overline{x} + \operatorname{cone} L) \cap g^{-1}(-Q)$,

$$(y^* \circ f)(x) \ge (y^* \circ f)(\overline{x})$$

that is

$$y^*\left(f\left(x\right) - f\left(\overline{x}\right)\right) \ge 0.$$

Now, since $y^* \in K^+ \setminus \{0\}$, the inequality above gives $f(x) - f(\overline{x}) \notin -int K$, i.e., the conclusion.

3.2 Optimality conditions using normal cones

In order to tackle the question of optimality conditions for directional minima in terms of generalized differentiation objects in dual spaces, we recall some notions and results concerning Fréchet and limiting (Mordukhovich) generalized differentiation (see [14] for details).

Consider S a nonempty subset of a Banach space X and $x \in S$. Then for every $\varepsilon \ge 0$, the set of ε -normals to S at x is defined by

$$\widehat{N}_{\varepsilon}(S, x) = \left\{ x^* \in X^* \mid \limsup_{\substack{u \stackrel{S}{\to} x}} \frac{x^*(u - x)}{\|u - x\|} \le \varepsilon \right\},$$

where $u \xrightarrow{S} x$ means that $u \to x$ and $u \in S$. The set $\widehat{N}_0(S, x)$ is denoted by $\widehat{N}(S, x)$ and it is called the Fréchet normal cone to S at x.

Let $\overline{x} \in S$. The Mordukhovich normal cone to S at \overline{x} is given by

$$N(S,\overline{x}) = \{x^* \in X^* \mid \exists \varepsilon_n \xrightarrow{(0,\infty)} 0, x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}_{\varepsilon_n}(S, x_n), \forall n \in \mathbb{N}\}.$$

Up to the end of this section, we consider that all the involved spaces are Asplund, unless otherwise stated. In this context, if $S \subset X$ is closed around \overline{x} , the formula for the Mordukhovich normal cone takes the following form:

$$N(S,\overline{x}) = \{x^* \in X^* \mid \exists x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}(S, x_n), \forall n \in \mathbb{N}\}.$$

For the set-valued map $F: X \rightrightarrows Y$, its Fréchet coderivative at $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is the set-valued map $\widehat{D}^*F(\overline{x}, \overline{y}): Y^* \rightrightarrows X^*$ given by

$$\widehat{D}^*F(\overline{x},\overline{y})(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}(\operatorname{Gr} F, (\overline{x},\overline{y}))\}.$$

In the same way, the Mordukhovich coderivative of F at $(\overline{x}, \overline{y})$ is the set-valued map $D^*F(\overline{x}, \overline{y})$: $Y^* \rightrightarrows X^*$ given by

$$D^*F(\overline{x},\overline{y})(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in N(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}.$$

As usual, when F = f is a function, since $\overline{y} \in F(\overline{x})$ means $\overline{y} = f(\overline{x})$, we write $\widehat{D}^* f(\overline{x})$ for $\widehat{D}^* f(\overline{x}, \overline{y})$, and similarly for D^* .

Notice that for a convex set $S \subset X$ one has that

$$N(S,\overline{x}) = \{x^* \in X^* \mid x^*(x - \overline{x}) \le 0, \forall x \in S\}$$

and this cone coincides with the negative polar of $T_B(S, \overline{x})$.

If $S \subset X$ is closed around $\overline{x} \in S$, one says that S is sequentially normally compact (SNC, for short) at \overline{x} if

$$\left[x_n \stackrel{S}{\to} \overline{x}, x_n^* \stackrel{w^*}{\to} 0, x_n^* \in \widehat{N}(S, x_n)\right] \Rightarrow x_n^* \to 0.$$

In the case where S = C is a closed convex cone, the (SNC) property at 0 is equivalent to

$$\left[(x_n^*) \subset C^+, x_n^* \stackrel{w^*}{\to} 0 \right] \Rightarrow x_n^* \to 0.$$

In particular, if int $C \neq \emptyset$, then C is (SNC) at 0.

Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be finite at $\overline{x} \in X$ and lower semicontinuous around \overline{x} ; the Fréchet subdifferential of f at \overline{x} is defined by

$$\widehat{\partial}f(\overline{x}) = \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\},\$$

where epi f denotes the epigraph of f. The Mordukhovich subdifferential of f at \overline{x} is given by

$$\partial f(\overline{x}) = \{ x^* \in X^* \mid (x^*, -1) \in N(\operatorname{epi} f, (\overline{x}, f(\overline{x}))) \}.$$

It is well-known that if f is a convex function, then $\widehat{\partial}f(\overline{x})$ and $\partial f(\overline{x})$ coincide with the Fenchel subdifferential. However, in general, $\widehat{\partial}f(\overline{x}) \subset \partial f(\overline{x})$, and the following generalized Fermat rule holds: if $\overline{x} \in X$ with $f(\overline{x}) < +\infty$ is a local minimum point for $f: X \to \mathbb{R} \cup \{+\infty\}$, then $0 \in \widehat{\partial}f(\overline{x})$.

Consider now some subsets $C_1, ..., C_k$ of X $(k \in \mathbb{N} \setminus \{0, 1\})$. Take $\overline{x} \in C_1 \cap ... \cap C_k$ and suppose that all the sets $C_i, i \in \overline{1, k}$ are closed around \overline{x} . One says that $C_1, ..., C_k$ are allied at \overline{x} if for every $(x_{in}) \xrightarrow{C_i} \overline{x}, x_{in}^* \in \widehat{N}(C_i, x_{in}), i \in \overline{1, k}$, the relation $(x_{1n}^* + ... + x_{kn}^*) \to 0$ implies $(x_{in}^*) \to 0$ for every $i \in \overline{1, k}$. The concept of alliedness was introduced by Penot and his coauthors in [17] and [13] in order to get a calculus rule for the Fréchet normal cone to the intersection of sets. More precisely, if the subsets $C_1, ..., C_k$ are allied at \overline{x} , then there exists r > 0 such that, for every $\varepsilon > 0$ and every $x \in [C_1 \cap ... \cap C_k] \cap B_X(\overline{x}, r)$, there exist $x_i \in C_i \cap B_X(x, \varepsilon), i \in \overline{1, k}$ such that

$$\hat{N}(C_1 \cap \ldots \cap C_k, x) \subset \hat{N}(C_1, x_1) + \ldots + \hat{N}(C_k, x_k) + \varepsilon D_{X^*}.$$

In what follows we use the results concerning the theory of generalized differentiation built on these objects directly at the places we need them, without separate quotation.

We discuss next a concept of directional openness at the reference point of a certain multifunction. We recall that the classical concept of openness proven to be useful for the announced aim by means of the incompatibility between this property and the Pareto minimality (see, e.g., [7] for details). In fact, the directional openness we consider here is related to several other notions introduced in [9], and to the concept of directional calmness already used in the previous subsection.

Consider a multifunction $F : X \Longrightarrow Y$, a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, and $\emptyset \neq L \subset S_X$, $\emptyset \neq M \subset S_Y$. One says F is directionally open at $(\overline{x}, \overline{y})$ with respect to L and M if for any $\varepsilon > 0$, there exists r > 0 such that

$$B(\overline{y},r) \cap [\overline{y} - \operatorname{cone} M] \subset F(B(\overline{x},\varepsilon) \cap [\overline{x} + \operatorname{cone} L]).$$

When F is single-valued, for simplicity, we sometimes omit \overline{y} in the definition above and we say that F is directionally open at \overline{x} , instead of directionally open at $(\overline{x}, f(\overline{x}))$.

Proposition 3.16 If $(\overline{x}, \overline{y}) \in \text{Gr } F$ is a local directional Pareto minimum point for F with respect to L, then for every $C \subset S_Y$ with $C \cap (K \setminus -K) \neq \emptyset$, the set-valued map $\mathcal{E}_F : X \rightrightarrows Y$, given by $\mathcal{E}_F(x) := F(x) + K$ is not directionally open at $(\overline{x}, \overline{y})$ with respect to L and C. In particular, F is not directionally open at $(\overline{x}, \overline{y})$ with respect to L and C.

Proof. Suppose, by contradiction, that for $\varepsilon > 0$ involved in the definition of the minimality of $(\overline{x}, \overline{y})$, there exists r > 0 such that

$$B(\overline{y},r) \cap [\overline{y} - \operatorname{cone} C] \subset \mathcal{E}_F(B(\overline{x},\varepsilon) \cap [\overline{x} + \operatorname{cone} L]).$$

By subtracting \overline{y} on both sides, according to the hypothesis, one has that

$$[B(0,r) \cap -C] \cap -K \subset [\mathcal{E}_F(B(\overline{x},\varepsilon) \cap [\overline{x} + \operatorname{cone} L]) - \overline{y}] \cap -K$$
$$= [F(B(\overline{x},\varepsilon) \cap [\overline{x} + \operatorname{cone} L]) + K - \overline{y}] \cap -K \subset K$$

Passing to the conic hull, this yields

$$-\operatorname{cone} C \cap -K \subset K,$$

which contradicts the fact that $C \cap (K \setminus -K) \neq \emptyset$. So \mathcal{E}_F is not directionally open at $(\overline{x}, \overline{y})$ with respect to L and C. Since $F(x) \subset \mathcal{E}_F(x)$ for any x, the same conclusion holds for F as well. \Box

Before obtaining necessary optimality conditions, we remark that a converse of Proposition 3.16 can be done if one considers a (generalized) convex framework.

Proposition 3.17 Suppose that F is K-convex and for every $u \in K \cap S_Y$, \mathcal{E}_F is not directionally open with respect to L and $M := \{u\}$ at $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Then $(\overline{x}, \overline{y})$ is a local directional Pareto minimum point of F with respect to L.

Proof. Suppose, by contradiction, that $(\overline{x}, \overline{y})$ is not a local directional Pareto minimum point of F with respect to L. Then for every r > 0, there is $y_r \in F(B(\overline{x}, r) \cap [\overline{x} + \operatorname{cone} L]) \cap [\overline{y} - K]$ such that $y_r \notin \overline{y} + K$. Denote $\overline{k} := \overline{y} - y_r \in K \setminus -K$ and consider $x_r \in B(\overline{x}, r) \cap (\overline{x} + \operatorname{cone} L)$ such that $y_r \in F(x_r)$.

Moreover, since \mathcal{E}_F is not directionally open with respect to L and $\{\overline{k}\}$ at $(\overline{x}, \overline{y})$, it follows that there is r > 0 such that, for every $\varepsilon > 0$ small enough, there is $y_{\varepsilon} \in B(\overline{y}, \varepsilon) \cap [\overline{y} - \operatorname{cone} \overline{k}] \subset [\overline{y}, y_r]$ such that $y_{\varepsilon} \notin \mathcal{E}_F(B(\overline{x}, r) \cap [\overline{x} + \operatorname{cone} L])$ (hence, in particular, $y_{\varepsilon} \neq \overline{y}$ and $y_{\varepsilon} \neq y_r$).

Then, there is $\lambda \in (0, 1)$ such that

$$y_{\varepsilon} = \lambda \overline{y} + (1 - \lambda)y_r \in \lambda F(\overline{x}) + (1 - \lambda)F(x_r) \subset F(\lambda \overline{x} + (1 - \lambda)x_r) + K$$

= $\mathcal{E}_F(\lambda \overline{x} + (1 - \lambda)x_r) = \mathcal{E}_F(\overline{x} + (1 - \lambda)(x_r - \overline{x})) \subset \mathcal{E}_F(B(\overline{x}, r) \cap [\overline{x} + \operatorname{cone} L]),$

a contradiction.

Now, we use Proposition 3.16 to get optimality conditions.

Theorem 3.18 Suppose that X and Y are finite dimensional spaces, $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local directional Pareto minimum point for F with respect to L, cone L is convex, $u \in \operatorname{int} K \cap S_Y$, and the set-valued map $\mathcal{E}_F : X \rightrightarrows Y$ has closed graph and is Lipschitz-like around $(\overline{x}, \overline{y})$. Then there exist $x^* \in X^*, y^* \in K^+$ with $x^*(\ell) \ge 0$ for all $\ell \in L$, $y^*(u) = 1$ and

$$x^* \in D^* \mathcal{E}_F(\overline{x}, \overline{y})(y^*).$$

Proof. According to Proposition 3.16, \mathcal{E}_F is not directionally open at $(\overline{x}, \overline{y})$ with respect to L and $\{u\}$ (in the sense of [9, Definition 2.2]) and, therefore, the sufficient condition for directional openness from [9, Theorem 4.3] does not hold. This means that for all natural numbers $n \neq 0$, there exist $x_n^* \in X^*, y_n^* \in Y^*$, $(x_n, y_n) \xrightarrow{\text{Gr } F} (\overline{x}, \overline{y})$ such that $y_n^*(u) = 1, n^{-1} > -x_n^*(\ell)$ for all $\ell \in L$ and $x_n^* \in \hat{D}^* \mathcal{E}_F(x_n, y_n)(y_n^*)$. Now [7, Lemma 3.2] ensures that $y_n^* \in K^+$ for any n. This, together with the condition $u \in \text{int } K$ imply, by using [11, Lemma 2.2.17], that the sequence (y_n^*) is bounded. The assumed Lipschitz property of \mathcal{E}_F ensures, by means of [14, Theorem 1.43], that the sequence (x_n^*) is bounded too. Therefore, we can suppose, without loss of generality, that both these sequences are convergent to some $x^* \in X^*$ and $y^* \in K^+$, respectively. Passing to the limit in the relations satisfied by (x_n^*) and (y_n^*) we get, $x^*(\ell) \geq 0$ for all $\ell \in L$, $y^*(u) = 1$ and $x^* \in D^* \mathcal{E}_F(\overline{x}, \overline{y})(y^*)$, that is the conclusion. \Box

Remark 3.19 Observe that, in the case $L = S_X$ (that is Pareto minimality) the necessary optimality condition given by the previous result is the generalized Fermat rule (see [7, Theorem 3.11]): there exists $y^* \in K^+ \setminus \{0\}$ with

$$0 \in D^* \mathcal{E}_F(\overline{x}, \overline{y})(y^*).$$

We tackle now the case of constrained problems and we have the following result.

Theorem 3.20 Let $A \subset X$ and $L \subset S_X$ be nonempty closed sets and $F : X \rightrightarrows Y$ be a set-valued map with $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ such that $\operatorname{Gr} F$ is closed around $(\overline{x}, \overline{y})$. Suppose that the following assertions hold:

(i) F is Lipschitz-like around $(\overline{x}, \overline{y})$;

(ii) $K \setminus -K \neq \emptyset$ and K is (SNC) at 0;

(iii) the sets A and \overline{x} + cone L are allied at \overline{x} .

If $(\overline{x}, \overline{y})$ is a local directional Pareto minimum point for F on A with respect to the set of directions L, then there exists $y^* \in K^+ \setminus \{0\}$ such that

$$0 \in D^*F(\overline{x}, \overline{y})(y^*) + N(A, \overline{x}) + N(\operatorname{cone} L, 0)$$

Proof. From the hypothesis, there exists a neighborhood $U \in \mathcal{V}(\overline{x})$ such that

$$(F(U \cap A \cap (\overline{x} + \operatorname{cone} L)) - \overline{y}) \cap -K \subset K$$
(3.10)

and there exists $c \in Y$ such that $c \in K \setminus -K$. Consider the following two sets:

$$A_1 = \operatorname{Gr} F$$

and

$$A_2 = [(\overline{x} + \operatorname{cone} L) \cap A] \times (\overline{y} - K)$$

We want to prove that the system $\{A_1, A_2, (\overline{x}, \overline{y})\}$ is an extremal system in $X \times Y$ (see [14, Definition 2.1]). For this, since the sets A_1 and A_2 are closed around $(\overline{x}, \overline{y}) \in A_1 \cap A_2$, it is sufficient to show the existence of a sequence $((x_n, y_n))_n \subset X \times Y$ such that $(x_n, y_n) \to (0, 0)$ and

$$A_1 \cap (A_2 - (x_n, y_n)) \cap (U \times Y) = \emptyset,$$

for all large $n \in \mathbb{N}$. Consider $(x_n, y_n) = (0, \frac{c}{n})$ with $n \in \mathbb{N} \setminus \{0\}$ and suppose, by contradiction, that there exist $x \in (\overline{x} + \operatorname{cone} L) \cap A \cap U$ and $y \in F(x) \cap (\overline{y} - K - \frac{c}{n}) \subset F(x) \cap (\overline{y} - K)$, whence $y - \overline{y} \in (F(x) - \overline{y}) \cap -K$. Now, using (3.10) we get that $y - \overline{y} \in K$ and since $\overline{y} - y - \frac{c}{n} \in K$ we arrive at $-c \in K$, a contradiction. Thus, $\{A_1, A_2, (\overline{x}, \overline{y})\}$ is an extremal system in $X \times Y$ and since $X \times Y$ is an Asplund space we can apply the approximate extremal principle to this system (see, [14, Theorem 2.20]). Therefore, for every $n \in \mathbb{N} \setminus \{0\}$, there exist $(x_n^1, y_n^1) \in \operatorname{Gr} F \cap D((\overline{x}, \overline{y}), \frac{1}{n})$, $x_n^2 \in (\overline{x} + \operatorname{cone} L) \cap A \cap D(\overline{x}, \frac{1}{n})$, $y_n^2 \in (\overline{y} - K) \cap D(\overline{y}, \frac{1}{n})$, $x_n^{1*} \in X^*, x_n^{2*} \in X^*, y_n^{1*} \in Y^*, y_n^{2*} \in Y^*$ such that

$$(x_n^{1*}, y_n^{1*}) \in \widehat{N} \left(\operatorname{Gr} F, (x_n^1, y_n^1) \right) + \frac{1}{n} D_{X^* \times Y^*}, x_n^{2*} \in \widehat{N} \left((\overline{x} + \operatorname{cone} L) \cap A, x_n^2 \right) + \frac{1}{n} D_{X^*}, y_n^{2*} \in \widehat{N} \left(\overline{y} - K, y_n^2 \right) + \frac{1}{n} D_{Y^*} = -\widehat{N} \left(K, \overline{y} - y_n^2 \right) + \frac{1}{n} D_{Y^*}$$

and

$$x_n^{1*} + x_n^{2*} = 0, \ y_n^{1*} + y_n^{2*} = 0, \ \left\| \left(x_n^{1*}, y_n^{1*} \right) \right\| + \left\| \left(x_n^{2*}, y_n^{2*} \right) \right\| = 1.$$
(3.11)

Therefore, there exist $(u_n^{1*}, v_n^{1*}) \in \frac{1}{n} D_{X^* \times Y^*}, u_n^{2*} \in \frac{1}{n} D_{X^*}$ and $v_n^{2*} \in \frac{1}{n} D_{Y^*}$ such that $x_n^{1*} - u_n^{1*} \in \widehat{D}^*F(x_n^1, y_n^1)(v_n^{1*} - y_n^{1*}), x_n^{2*} - u_n^{2*} \in \widehat{N}((\overline{x} + \operatorname{cone} L) \cap A, x_n^2)$ and $y_n^{2*} - v_n^{2*} \in -\widehat{N}(K, \overline{y} - y_n^2) \subset K^+$. Using relation (3.11) we obtain that the sequences $(x_n^{1*}), (x_n^{2*}), (y_n^{1*})$ and (y_n^{2*}) are bounded, and since X and Y are Asplund spaces, there exist $x_1^* \in X^*, x_2^* \in X^*, y_1^* \in Y^*$ and $y_2^* \in Y^*$ such that $x_n^{1*} \xrightarrow{w^*} x_1^*, x_n^{2*} \xrightarrow{w^*} x_2^*, y_n^{1*} \xrightarrow{w^*} y_1^*, y_n^{2*} \xrightarrow{w^*} y_2^*$. Obviously, $x_1^* + x_2^* = 0$ and $y_1^* + y_2^* = 0$.

Now, if $y_1^* = 0$, then $y_2^* = 0$, whence $y_n^{2*} - v_n^{2*} \stackrel{w^*}{\to} 0$ and using the (SNC) assumption we have that $y_n^{2*} - v_n^{2*} \rightarrow 0$, whence $y_n^{2*} \rightarrow 0$, so $y_n^{1*} \rightarrow 0$. Taking into account that F is Lipschitz-like around $(\overline{x}, \overline{y})$ and using [14, Theorem 1.43], we obtain that $x_n^{1*} - u_n^{1*} \rightarrow 0$ and since $u_n^{1*} \rightarrow 0$, we have that $x_n^{1*} \rightarrow 0$. Using again (3.11) we obtain that $x_n^{2*} \rightarrow 0$, which contradicts the fact that $y_n^{2*} \rightarrow 0$ and $\|(x_n^{2*}, y_n^{2*})\| = \frac{1}{2}$. Hence $y_1^* \neq 0$. Moreover, since $y_1^* + y_2^* = 0$, $y_n^{2*} - v_n^{2*} \stackrel{w^*}{\rightarrow} y_2^*$, $y_n^{2*} - v_n^{2*} \subset K^+$ and K^+ is weakly-star closed, we obtain that $-y_1^* = y_2^* \in K^+$.

Further, using the hypothesis (iii), for every *n* large enough, we get that there exist $l_n \in (\overline{x} + \operatorname{cone} L) \cap B\left(x_n^2, \frac{1}{n}\right), a_n \in A \cap B\left(x_n^2, \frac{1}{n}\right)$ such that

$$x_n^{2*} \in \widehat{N}\left(\left(\overline{x} + \operatorname{cone} L\right) \cap A, x_n^2\right) \subset \widehat{N}\left(\overline{x} + \operatorname{cone} L, l_n\right) + \widehat{N}\left(A, a_n\right) + \frac{1}{n}D_{X^*},$$

whence, there exist $a_n^* \in \widehat{N}(A, a_n)$, $l_n^* \in \widehat{N}(\overline{x} + \operatorname{cone} L, l_n)$ such that $a_n^* + l_n^* - x_n^{2*} \to 0$. Further, we prove that (a_n^*) or (l_n^*) is bounded. Suppose by contradiction that both sequences are unbounded. It follows that for every n, there is k_n sufficiently large such that

$$n < \min\left\{ \left\| a_{k_n}^* \right\|, \left\| l_{k_n}^* \right\| \right\}.$$
(3.12)

For simplicity we denote the subsequences $(a_{k_n}^*)$, $(l_{k_n}^*)$ by (a_n^*) , (l_n^*) , respectively. Now, since $a_n^* \in \widehat{N}(A, a_n)$, $l_n^* \in \widehat{N}(\overline{x} + \operatorname{cone} L, l_n)$ we obtain that

$$\frac{1}{n}a_n^* \in \widehat{N}(A, a_n),$$

$$\frac{1}{n}l_n^* \in \widehat{N}(\overline{x} + \operatorname{cone} L, l_n) = \widehat{N}(\operatorname{cone} L, l_n - \overline{x}).$$

Since

$$\frac{1}{n} \|a_n^* + l_n^*\| \le \frac{1}{n} \|a_n^* + l_n^* - x_n^{2*}\| + \frac{1}{n} \|x_n^{2*}\|,$$

we obtain that $\frac{1}{n}(a_n^* + l_n^*) \to 0$, so using again the hypothesis of alliedness we obtain that $\frac{1}{n}a_n^* \to 0$ and $\frac{1}{n}l_n^* \to 0$, which is in contradiction with relation (3.12). Consequently, we obtain that $(a_n^*), (l_n^*) \subset X^*$ are bounded, thus there exist $a^*, l^* \in X^*$ such that $a_n^* \xrightarrow{w^*} a^*$ and $l_n^* \xrightarrow{w^*} l^*$, so $x_2^* = a^* + l^* \in N(A, \overline{x}) + N(\operatorname{cone} L, 0)$. Now, observe from above that $x_1^* \in D^*F(\overline{x}, \overline{y})(y_2^*)$, with $y_2^* \in K^+ \setminus \{0\}$ and since $x_1^* + x_2^* = 0$, we get that $0 \in D^*F(\overline{x}, \overline{y})(y_2^*) + N(A, \overline{x}) + N(\operatorname{cone} L, 0)$ with $y_2^* \in K^+ \setminus \{0\}$, i.e., the conclusion.

We end this section by considering the situation where the objective map is a single-valued mapping. Consider $f: X \to \mathbb{R}$ a real-valued function, take $A \subset X$ and $L \subset S_X$ nonempty closed sets. In order to obtain necessary condition for directional Pareto minimum in the nonsmooth case, we make use of the penalty function method.

Proposition 3.21 Let $\overline{x} \in A$ be a local directional minimum for f on A with respect to L. Suppose that f is Lipschitz continuous around \overline{x} , and cone L is convex. In addition, suppose that $N(A, \overline{x}) \cap (-L^{-}) = \{0\}$ and that either A or $\overline{x} + \text{cone } L$ is (SNC) at \overline{x} . Then one has

$$0 \in \partial f\left(\overline{x}\right) + N\left(A, \overline{x}\right) + L^{-}$$

Proof. According to the definition of directional minima, \overline{x} is a local solution of the constrained optimization problem

$$\min f(x), \quad x \in \Omega \tag{3.13}$$

where $\Omega := A \cap (\overline{x} + \operatorname{cone} L)$. Then, following the well-known Clarke penalization, \overline{x} a solution of the unconstrained optimization problem

$$\min f(x) + kd(x, \Omega), \quad x \in X,$$

where k > 0 is the Lipschitz modulus of f. By the generalized Fermat rule and the sum rule for limiting subdifferential, one has

$$0 \in \partial \left(f + kd\left(\cdot, \Omega\right)\right)(\overline{x}) \subset \partial f\left(\overline{x}\right) + k\partial d\left(\cdot, \Omega\right)(\overline{x})$$
$$\subset \partial f\left(\overline{x}\right) + N\left(\Omega, \overline{x}\right).$$

Observe that $N(\overline{x} + \operatorname{cone} L, \overline{x}) = N(\operatorname{cone} L, 0) = L^{-}$ and now we can use [14, Corollary 3.5] since, according to our assumptions, both normal qualification condition and the required (SNC) property hold. Then this allow us to write that

$$N(\Omega,\overline{x}) \subset N(A,\overline{x}) + N(\overline{x} + \operatorname{cone} L,\overline{x}) = N(A,\overline{x}) + L^{-},$$

and the conclusion follows.

Now, we make one step forward by considering the vectorial optimization problem

$$\min f(x), \quad x \in A, \tag{3.14}$$

where $f: X \to Y$ is a vector-valued function and $A \subset X$ is a closed set. As before, the ordering cone on Y is K.

Consider the following vectorial Lipschitz property for f: following [18], one says that f is K-Lipschitz around $\overline{x} \in X$ of rank $\ell_f > 0$ if there exist a neighborhood U of \overline{x} and an element $e \in K \cap S_Y$ such that for every $x', x'' \in U$

$$f(x'') - f(x') + \ell_f ||x'' - x'|| e \in K.$$

We record the following result.

Theorem 3.22 Let $\overline{x} \in A$ be a local directional Pareto minimum for f on A with respect to $L \subset S_X$. Suppose that:

(i) f is K-Lipschitz around \overline{x} of rank ℓ_f and let e be the element in $K \cap S_Y$ given by the Lipschitz property of f;

(ii) K is (SNC) at 0;

(iii) cone L is convex, $N(A, \overline{x}) \cap (-L^-) = \{0\}$ and that either A or \overline{x} + cone L is (SNC) at \overline{x} . Then for every $\ell > \ell_f$, there exist $y^* \in K^+ \setminus \{0\}$ and $x^* \in D^*f(\overline{x})(y^*)$ such that

 $-x^* \in N(A, \overline{x}) + L^- \text{ and } ||x^*|| \le ly^*(e).$

Proof. Again, directional Pareto minimality of \overline{x} means that \overline{x} is a Pareto minimum for f on $A \cap [\overline{x} + \operatorname{cone} L]$. We use now a vectorial variant of Clarke penalization (see [18, Theorem 3.2 (i)]) to deduce that, for every $\ell > \ell_f$, \overline{x} is an unconstrained Pareto minimum for the function $f(\cdot) + \ell d(\cdot, A \cap [\overline{x} + \operatorname{cone} L]) e$. We can now use the method from [2, Theorem 3.11] to deduce that for every $l > l_f$, there exist $y^* \in K^+ \setminus \{0\}, x^* \in D^* f(\overline{x})(y^*)$ such that

$$-x^* \in N(A \cap [\overline{x} + \operatorname{cone} L], \overline{x})$$

and $||x^*|| \leq ly^*(e)$. Using again [14, Corollary 3.5], we have

$$-x^* \in N(A, \overline{x}) + L^-,$$

and this is the conclusion.

4 Pareto directional minima for sets

As made clear in Definition 2.3 and the subsequent comments, the notion of directional Pareto minimum is motivated by the case of (generalized) mappings. However, it is possible to define such a notion for sets as well. In order to point out this aspect of directional minimality, in this section we define some appropriate notions and we give, only briefly, some examples and optimality conditions for them.

Consider, as above, a closed nonempty set $L \subset S_X$ and take now K as a proper closed convex cone in X.

Definition 4.1 Let $M \subset X$ be a nonempty set. One says that $\overline{x} \in M$ is a local directional Pareto minimum point for M with respect to L if

$$(M \cap (\overline{x} + \operatorname{cone} L) - \overline{x}) \cap -K \subset K.$$

$$(4.1)$$

If int $K \neq \emptyset$, one says that $\overline{x} \in M$ is a weak directional Pareto minimum for M if

$$(M \cap (\overline{x} + \operatorname{cone} L) - \overline{x}) \cap -\operatorname{int} K = \emptyset.$$

$$(4.2)$$

It is simple to see that relation (4.1) is equivalent to

$$(M - \overline{x}) \cap \operatorname{cone} L \cap -K \subset K,$$

while relation (4.2) actually means

$$(M - \overline{x}) \cap \operatorname{cone} L \cap -\operatorname{int} K = \emptyset.$$

Therefore, (4.1) is relevant only if $\operatorname{cone} L \cap -K \neq \{0\}$, while for (4.2) it is important to have $\operatorname{cone} L \cap -\operatorname{int} K \neq \emptyset$.

Now, we give an example that justify the above notions of Pareto minimum.

Example 4.2 Let γ be a closed curve described by the following two parametric equations

$$\begin{cases} x(t) = 2 + 2\cos t (1 - \sin t) \\ y(t) = \sin t (1 - \cos t) \end{cases}, \quad t \in [0, 2\pi], \end{cases}$$

 $\overline{\gamma} = \operatorname{int} \gamma \cup \operatorname{bd} \gamma$ and the half-plane $H := \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y \geq -x\}$. Take $K = \mathbb{R}^2_+, \overline{x} := (0, 0)$ and the directions set $L := \{(\cos t, \sin t) \mid t \in (\pi, 1.25\pi)\}$. Now, consider $M := H \cup (\overline{\gamma} \cap -H)$ as a closed subset of $X := \mathbb{R}^2$. Observing that $(M - \overline{x}) \cap \operatorname{cone} L \cap -K = \{(0, 0)\} \subset K$ and $(M - \overline{x}) \cap -K$ has points that are not in $K \setminus \{(0, 0)\}$, for instance those one that are on γ and have negative x-coordinate, we get that \overline{x} is a local directional Pareto minimum point for M with respect to L, but not a local Pareto minimum point for M. Similarly, we have $(M - \overline{x}) \cap \operatorname{cone} L \cap -\operatorname{int} K = \emptyset$ and $(M - \overline{x}) \cap -\operatorname{int} K \neq \emptyset$, so there exists local weak directional Pareto minimum points, that are not local weak Pareto minimum points.

In the notation of Definition 4.1, the following optimality conditions hold.

Theorem 4.3 Suppose that cone $L \cap -$ int $K \neq \emptyset$.

(i) If $\overline{x} \in M$ is a weak directional Pareto minimum for M with respect to L then

$$T_B^L(M,\overline{x}) \cap -\operatorname{int} K = \emptyset.$$

(ii) If for $\overline{x} \in M$ one has

$$T_B^L(\operatorname{cl}(M+K),\overline{x})) \cap -\operatorname{int} K = \emptyset,$$

then \overline{x} is a weak directional Pareto minimum for M with respect to L.

Proof. (i) Suppose that there exists $u \in T_B^L(M, \overline{x})) \cap -\operatorname{int} K$, meaning that $u \in -\operatorname{int} K$ and there are $(u_n) \xrightarrow{\operatorname{cone} L} u, (t_n) \xrightarrow{(0,\infty)} 0$ such that for all $n, \overline{x} + t_n u_n \in M$. Clearly, for n large enough,

 $t_n u_n \in (M - \overline{x}) \cap \operatorname{cone} L \cap -\operatorname{int} K,$

which contradicts the minimality assumption.

(ii) Suppose, again by way of contradiction, that there exists $x \in M$ such that $x - \overline{x} \in \text{cone } L \cap -\text{int } K$. Consider $(t_n) \xrightarrow{(0,\infty)} 0$. Then, for every n large enough,

$$\overline{x} + t_n(x - \overline{x}) = x + (1 - t_n)(\overline{x} - x) \in (M + \operatorname{int} K) \cap (\overline{x} + \operatorname{cone} L),$$

whence using the fact that cl(M + int K) = cl(M + K) (which, in turn, is easy to prove using the closedness and the convexity of K which ensures K = cl int K) one can write:

$$-\operatorname{int} K \ni x - \overline{x} \in T_B((M + \operatorname{int} K) \cap (\overline{x} + \operatorname{cone} L), \overline{x})$$

= $T_B(\operatorname{cl} [(M + \operatorname{int} K) \cap (\overline{x} + \operatorname{cone} L)], \overline{x})$
 $\subset T_B(\operatorname{cl} (M + \operatorname{int} K) \cap (\overline{x} + \operatorname{cone} L), \overline{x})$
= $T_B(\operatorname{cl} (M + K) \cap (\overline{x} + \operatorname{cone} L), \overline{x}) = T_B^L(\operatorname{cl} (M + K), \overline{x})),$

and this is in contradiction with the hypothesis.

Theorem 4.4 Suppose that cone $L \cap -K \neq \{0\}$. If for $\overline{x} \in M$ one has

$$T_B^L(\operatorname{cl}(M+K),\overline{x})) \cap -K \subset K,$$

then \overline{x} is a directional Pareto minimum for M with respect to L.

Proof. The proof is similar to that of Theorem 4.3 (ii).

5 Conclusions

The directional efficiencies introduced in this paper generalize in a meaningful way the classical situation of Pareto optimality and require non-trivial adaptations of the usual techniques of investigation used in the latter case. Besides the results of this paper, we think that our approach opens new possibilities to model directional situations, especially arising in vector optimization problems dealing with location issues. We consider that our concept here introduced is able to capture the situation where some directions are more important than the others (hence which can be dropped) in the possible models under consideration. Another possible continuation for theoretical investigation of directional efficiency is to devise an adapted (directional) normal limiting cone with respect to a set of directions and to use it in order to write down more specific optimality conditions for our concept. All these ideas will be topics for future research.

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References

- S. Alzorba, C. Günther, N. Popovici, C. Tammer, A new algorithm for solving planar multiobjective location problems involving the Manhattan norm, European Journal of Operational Research, 258 (2017), 35–46.
- [2] M. Apetrii, M. Durea, R. Strugariu, A new penalization tool in scalar and vector optimizations, Nonlinear Analysis: Theory, Methods and Applications, 107 (2014), 22–33.
- [3] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser, Basel (1990).
- [4] A.L. Dontchev, R.T. Rockafellar, Implicit Functions and Solution Mappings, Springer, Berlin, 2009.
- [5] M. Durea, First and second order optimality conditions for set-valued optimization problems, Rendiconti del Circolo Matematico di Palermo, 53 (2004), 451–468.
- [6] M. Durea, Estimations of the Lagrange multipliers' norms in set-valued optimization, Pacific Journal of Optimization, 2 (2006), 487–501.
- [7] M. Durea, R. Strugariu, On some Fermat rules for set-valued optimization problems, Optimization, 60 (2011), 575–591.
- [8] M. Durea, R. Strugariu, Calculus of tangent sets and derivatives of set-valued maps under metric subregularity conditions, Journal of Global Optimization, 56 (2013), 587–603.
- [9] M. Durea, M. Panțiruc, R. Strugariu, A new type of directional regularity for mappings and applications to optimization, SIAM Journal on Optimization, 27 (2017), 1204–1229.
- [10] H. Gfrerer, On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs, Set-Valued and Variational Analysis, 21 (2013), 151–176.
- [11] A. Göpfert, H. Riahi, C. Tammer, C. Zălinescu, Variational Methods in Partially Ordered Spaces, Springer, Berlin, 2003.
- [12] V.N. Huynh, M. Théra, Directional metric regularity of multifunctions, Mathematics of Operations Research, 40 (2015), 969–991.
- [13] S. Li, J.-P. Penot, X. Xue, *Codifferential calculus*, Set-Valued and Variational Analysis, 19 (2011), 505–536.
- [14] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, Vol. I: Basic Theory, Vol. II: Applications, Springer, Grundlehren der mathematischen Wissenschaften (A Series of Comprehensive Studies in Mathematics), Vol. 330 and 331, Berlin, 2006.
- [15] N.M. Nam, B.S. Mordukhovich, An Easy Path to Convex Analysis and Applications, Morgan & Claypool, 2013.
- [16] N.M. Nam, C. Zălinescu, Variational analysis of directional minimal time functions and applications to location problems, Set-Valued and Variational Analysis, 21 (2013), 405–430.
- [17] J.-P. Penot, Cooperative behavior of functions, relations and sets, Mathematical Methods of Operations Research, 48 (1998), 229–246.

- [18] J.J. Ye, The exact penalty principle, Nonlinear Analysis: Theory Methods and Applications, 75 (2012), 1642–1654.
- [19] C. Zălinescu, Convex Analysis in General Vector Spaces, World Scientific, Singapore, 2002.