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Indefinite Abstract Splines with a Quadratic Constraint

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Abstract

We study an extension to Krein spaces of the abstract interpolating spline problem in Hilbert spaces, introduced by M. Atteia. This is a quadratically constrained quadratic programming problem, where the objective function is not convex, while the equality constraint is sign indefinite. We characterize the existence of solutions and, if there are any, we describe the set of solutions as the union of a family of affine manifolds parallel to a fixed subspace, which depend on the original data.

Keywords Abstract splines · Krein spaces · Quadratically constrained quadratic programming · Linear quadratic regulator

Mathematics Subject Classification 46C20 · 47B50 · 65D07 · 65D10

1 Introduction

The aim of this work is to study an extension to Krein spaces of the abstract interpolating spline problem in Hilbert spaces, introduced by Atteia [1] and extended by several authors; see, for instance, [2–7].

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In this setting, the interpolating spline problem becomes a quadratically constrained quadratic programming (QCQP) problem, where the objective function is not convex, while the equality constraint is sign indefinite.

The non-convexity of the objective function arises from it being defined through an indefinite inner product. If this inner product is (semidefinite) positive, then the objective function is convex. Such QCQP problems, especially in finite dimensional settings, have been extensively considered in the literature; see [8–13]. Usually, duality concepts and variational methods are applied to characterize and compute global minimizers. If the equality constraint is sign definite, then the objective function is minimized over an affine manifold. This version of the indefinite abstract interpolating spline problem was previously studied in [14,15]. Also, interpolating spline functions in indefinite metric spaces have been studied in [16,17] to solve certain learning theory problems. Although the problems presented there are different from those studied in this work, they are closely related.

As a motivation to studying indefinite splines, an example of the classical finite time horizon linear quadratic regulator [18,19] is presented in Sect. 2. There, the control function is selected among those satisfying an indefinite quadratic constraint imposed on the output samples and the problem is restated in terms of an indefinite abstract interpolating spline problem.

In Sect. 3, the indefinite abstract interpolating spline problem is introduced, and necessary and sufficient conditions for the existence of solutions are given. Moreover, it is shown that this problem is equivalent to a dual maximization problem, and the set of solutions is the union of a family of affine manifolds.

Section 4 starts with another version of the S-Lemma (or Farkas Lemma) [20, 21], which allows one to translate the indefinite abstract interpolating spline problem into a proximinality problem. Finally, sufficient conditions for the existence of spline interpolants to every element of the vector space are provided.

Familiarity with operator theory on Krein spaces is assumed. However, the terminology, notations and some results on this topic are explained in "Appendix."

2 Motivation: The Finite Time Horizon Linear Quadratic Regulator

A motivation for the problem studied in the following sections is the next (finite time horizon) control theory problem: given the state of a linear system $\mathbf{x}(t) \in \mathbb{R}^n$, analyse how this state evolves over a time interval $[0, T]$, where the governing dynamics are given by the following time-invariant system:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) &= \mathbf{c}^T \mathbf{x}(t), \end{aligned} \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $\mathbf{b} \in \mathbb{R}^n$, $\mathbf{c} \in \mathbb{R}^n$ and y being the measurable output. The goal is to find a control signal $u \in L^2[0, T]$, which suitably changes the trajectory of the system according to some criterion.

Setting $\mathbf{x}(0) = 0$ for the sake of simplicity, the solution to (1) is given by

$$\mathbf{x}(t) = \int_0^t e^{A(t-s)}\mathbf{b}u(s)\mathrm{d}s, \quad t \in [0, T].$$

Suppose that the output y is sampled at $0 < t_1 < t_2 < \cdots < t_m \leq T$. Defining the set of functions $(\eta_k)_{k=1}^m \subseteq L^2[0, T]$ as

$$\eta_k(t) := \mathbf{c}^T e^{A(t_k-t)}\mathbf{b}\chi_{[0,t_k]}(t), \quad t \in [0, T], \quad k = 1, 2, ..., m,$$

the output samples can be expressed as

$$y(t_k) = \int_0^{t_k} \mathbf{c}^T e^{A(t_k-t)}\mathbf{b}u(t)\mathrm{d}t = \int_0^T \eta_k(t)u(t)\mathrm{d}t = \langle u, \eta_k \rangle_2, \quad k = 1, 2, ..., m,$$

where $\langle \cdot, \cdot \rangle_2$ stands for the usual inner product on $L^2[0, T]$.

Define the operator $V : L^2[0, T] \to \mathbb{R}^m$ by

$$Vu = \begin{bmatrix} \langle u, \eta_1 \rangle_2 \\ \langle u, \eta_2 \rangle_2 \\ \vdots \\ \langle u, \eta_m \rangle_2 \end{bmatrix}, \quad u \in L^2[0, T],$$

and assume that V is surjective. Now, the objective is to drive the output samples so that they "resemble" in some sense a reference output vector $\mathbf{z}_0 \in \mathbb{R}^m$. Thus, the following condition is established:

$$(Vu - \mathbf{z}_0)^T W(Vu - \mathbf{z}_0) = 0, \tag{2}$$

where $W \in \mathbb{R}^{m\times m}$ represents a convenient weight matrix, which is assumed to be nonsingular, symmetric, and not definite but indefinite; see [22–24]. Defining an indefinite inner product $[\,\cdot, \cdot\,]$ in $\mathbb{R}^m$ by

$$\big[\mathbf{x}, \mathbf{y}\big] = \mathbf{y}^T W\mathbf{x}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^m,$$

(2) can be alternatively expressed as

$$[\,Vu - \mathbf{z}_0, Vu - \mathbf{z}_0\,] = 0.$$

Finally, the control signal u is selected so that it minimizes a cost functional, as well as satisfying the interpolation condition (2). For the energy cost functional,

$$J(u) = \int_0^T u^2(t)\mathrm{d}t = \|u\|_2^2, \quad u \in L^2[0, T],$$

the problem becomes

$$\min_{u\in L^2[0,T]} \|u\|_2, \quad \text{subject to } [\, Vu - \mathbf{z}_0, Vu - \mathbf{z}_0\,] = 0.$$

A similar formulation can be found in [25] for automated guided vehicles.

More generally, an indefinite cost functional can be considered:

$$J(u) = \int_0^T Q(t)u^2(t)\mathrm{d}t, \quad u \in L^2[0,T],$$

where $Q(t)$ represents a cost weight function, which is assumed to be indefinite; see [26–28]. Defining the indefinite inner product $[\,\cdot,\cdot\,]_Q$ in $L^2[0,T]$ as

$$[\,x, y\,]_Q = \int_0^T y(t)Q(t)x(t)\mathrm{d}t, \quad x, y \in L^2[0,T],$$

the problem becomes

$$\min_{u\in L^2[0,T]} [\,u,u\,]_Q\,, \quad \text{subject to } [\, Vu - \mathbf{z}_0, Vu - \mathbf{z}_0\,] = 0.$$

The above problems are particular cases of the indefinite abstract interpolating spline problem studied in the following section.

3 Indefinite Abstract Interpolating Splines: Theoretical Approach

From now on, $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ denotes a complex (separable) Hilbert space, and $\mathcal{K}$ and $\mathcal{E}$ denote Krein spaces. The notation $[\,\cdot,\cdot\,]$ will be used for the inner products on $\mathcal{K}$ and $\mathcal{E}$, using $[\,\cdot,\cdot\,]_{\mathcal{K}}$ and $[\,\cdot,\cdot\,]_{\mathcal{E}}$ to emphasize the Krein space considered, if necessary. Also, $\mathcal{L}(\mathcal{H},\mathcal{K})$ is the space of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H},\mathcal{H})$ stands for the algebra of bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$, then $R(T)$ indicates the range of T and $N(T)$ its nullspace.

Throughout this paper $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$ and $V \in \mathcal{L}(\mathcal{H},\mathcal{E})$ are surjective operators.

The aim of this section is to study the following problem:

Problem 1 Given $z_0 \in \mathcal{E}$, analyse the existence of

$$\min_{x\in\mathcal{H}} [\,Tx, Tx\,]\,, \quad \text{subject to } [\, Vx - z_0, Vx - z_0\,] = 0,$$

and if the minimum exists, find the set of arguments at which it is attained.

Definition 3.1 A vector $\tilde{x} \in \mathcal{H}$ is an *indefinite abstract spline or* (T,V)*-interpolant* to $z_0 \in \mathcal{E}$, if it is a solution to Problem 1. The set of (T,V)-interpolants to z_0 is denoted by $\mathcal{S}_{z_0}$.

Fix $x_0 \in \mathcal{H}$ such that $Vx_0 = z_0$ and denote by $\mathcal{C}_V$ the set of neutral elements of the quadratic form $x \mapsto [\, Vx, Vx\,]$, i.e.

$$\mathcal{C}_V = \{\, u \in \mathcal{H} : [\, Vu, Vu\,] = 0\,\}.$$

Then, $x \in \mathcal{H}$ satisfies $[\, Vx - z_0, Vx - z_0\,] = 0$ if and only if $x \in x_0 + \mathcal{C}_V$, and thus, $\mathcal{S}_{z_0}$ is a subset of $x_0 + \mathcal{C}_V$. As a result, Problem 1 can be restated in the following way:

Problem 2 Given $x_0 \in \mathcal{H}$, analyse the existence of

$$\min_{y \in \mathcal{C}_V} [\, T(x_0 + y), T(x_0 + y)\,],$$

and if the minimum exists, find the set of arguments at which it is attained.

One of the main difficulties in tackling Problem 2 is that $\mathcal{C}_V$ is not a convex set. Moreover, the convex hull of $\mathcal{C}_V$ is the complete Hilbert space $\mathcal{H}$. Thus, replacing $\mathcal{C}_V$ by its convex hull trivializes the problem.

If $V^\# V$ is a positive (or negative) semidefinite operator in $\mathcal{H}$, then $\mathcal{C}_V$ coincides with $N(V)$, and Problem 1 becomes the interpolation problem studied in [15]. But, if $V^\# V$ is *indefinite*, the set $\mathcal{C}_V$ is strictly larger than $N(V)$. Therefore, from now on $V^\# V$ is assumed to be indefinite, i.e. neither positive nor negative semidefinite.

Necessary and sufficient conditions for the existence of a solution to Problem 1 are presented below.

Proposition 3.1 *Given $z_0 \in \mathcal{E}$, let $x_0 \in \mathcal{H}$ be such that $z_0 = Vx_0$. Then, $\mathcal{S}_{z_0} \neq \varnothing$ if and only if $T(\mathcal{C}_V)$ is a non-negative set of $\mathcal{K}$ and there exists $y_0 \in \mathcal{C}_V$ such that for all $y \in \mathcal{C}_V$,*

$$|\,[\, Tx_0, Ty\,]\,|^2 \le [\, Ty_0, Ty_0\,]\,[\, Ty, Ty\,], \tag{3}$$

with equality when $y = y_0$.

In this case, $x_0 + y_0 \in \mathcal{S}_{z_0}$ if and only if $y_0 \in \mathcal{C}_V$ satisfies (3) and

$$[\, T(x_0 + y_0), Ty_0\,] = 0. \tag{4}$$

Proof Suppose that $\mathcal{S}_{z_0} \neq \varnothing$, i.e. assume that there exists $y_0 \in \mathcal{C}_V$ such that

$$[\, T(x_0 + y_0), T(x_0 + y_0)\,] \le [\, T(x_0 + y), T(x_0 + y)\,], \quad \text{for every } y \in \mathcal{C}_V.$$

Then, for every $y \in \mathcal{C}_V$,

$$[\, Ty, Ty\,] + 2\,\mathrm{Re}\,[\, Tx_0, Ty\,] - [\, Ty_0, Ty_0\,] - 2\,\mathrm{Re}\,[\, Tx_0, Ty_0\,] \ge 0. \tag{5}$$

If $y \in \mathcal{C}_V$ and $t \in \mathbb{R}$, then $ty \in \mathcal{C}_V$. Therefore, replacing y by ty in (5) we have

$$at^2 + bt + c \ge 0 \quad \text{for every } t \in \mathbb{R}, \tag{6}$$

where $a = [\,Ty, Ty\,]$, $b = 2\,\mathrm{Re}\,[\,Tx_0, Ty\,]$, and $c = -\,[\,Ty_0, Ty_0\,] - 2\,\mathrm{Re}\,[\,Tx_0, Ty_0\,]$. But (6) holds if and only if $a \geq 0$ and $b^2 - 4ac \leq 0$. Since $y \in \mathcal{C}_V$ was arbitrary, it follows that $T(\mathcal{C}_V)$ is a non-negative set of $\mathcal{K}$, and for every $y \in \mathcal{C}_V$,

$$\left(\mathrm{Re}\,[\,Tx_0, Ty\,]\right)^2 \leq (-\,[\,Ty_0, Ty_0\,] - 2\,\mathrm{Re}\,[\,Tx_0, Ty_0\,])\,[\,Ty, Ty\,]. \tag{7}$$

Setting $y = y_0$ in (7) yields

$$([\,Ty_0, Ty_0\,] + \mathrm{Re}\,[\,Tx_0, Ty_0\,])^2 \leq 0.$$

Thus, $\mathrm{Re}\,[\,Tx_0, Ty_0\,] = -\,[\,Ty_0, Ty_0\,]$, and it follows that

$$\left(\mathrm{Re}\,[\,Tx_0, Ty\,]\right)^2 \leq [\,Ty_0, Ty_0\,]\,[\,Ty, Ty\,], \quad \text{for every}\, y \in \mathcal{C}_V.$$

Now, for a fixed $y \in \mathcal{C}_V$, let $\theta \in [0, 2\pi)$ be such that $[\,Tx_0, Ty\,] = e^{i\theta}\,|\,[\,Tx_0, Ty\,]\,|$, and set $v := e^{i\theta} y \in \mathcal{C}_V$. Then, $[\,Tv, Tv\,] = [\,Ty, Ty\,]$ and $\mathrm{Re}\,[\,Tx_0, Tv\,] = |\,[\,Tx_0, Ty\,]\,|$. Hence,

$$|\,[\,Tx_0, Ty\,]\,|^2 \leq [\,Ty_0, Ty_0\,]\,[\,Ty, Ty\,], \quad \text{for every } y \in \mathcal{C}_V. \tag{8}$$

Further, since $\mathrm{Re}\,[\,Tx_0, Ty_0\,] = -\,[\,Ty_0, Ty_0\,]$, setting $y = y_0$ in (8) yields

$$\left(\mathrm{Re}\,[\,Tx_0, Ty_0\,]\right)^2 \leq |\,[\,Tx_0, Ty_0\,]\,|^2 \leq [\,Ty_0, Ty_0\,]^2.$$

Thus, equality in (8) is attained at $y = y_0$, in which case $[\,T(x_0 + y_0), Ty_0\,] = 0$.

Conversely, suppose that $T(\mathcal{C}_V)$ is non-negative and that (3) holds with equality attained at $y = y_0$. Then, $[\,Ty, Ty\,] \geq 0$ for every $y \in \mathcal{C}_V$, and

$$\left(\mathrm{Re}\,[\,Tx_0, Ty\,]\right)^2 \leq |\,[\,Tx_0, Ty\,]\,|^2 \leq [\,Ty_0, Ty_0\,]\,[\,Ty, Ty\,].$$

Let $\theta \in [0, 2\pi)$ be such that $[\,Tx_0, Ty_0\,] = e^{i\theta}\,|\,[\,Tx_0, Ty_0\,]\,|$, and set $v_0 := -e^{i\theta} y_0 \in \mathcal{C}_V$. Then, $[\,Tv_0, Tv_0\,] = [\,Ty_0, Ty_0\,]$ and $[\,Tx_0, Tv_0\,] = -\,|\,[\,Tx_0, Ty_0\,]\,|$. Since $|\,[\,Tx_0, Ty_0\,]\,| = [\,Ty_0, Ty_0\,]$, it follows $[\,T(x_0 + v_0), Tv_0\,] = 0$. For an arbitrary (fixed) vector $y \in \mathcal{C}_V$ define $a = [\,Ty, Ty\,]$, $b = 2\,\mathrm{Re}\,[\,Tx_0, Ty\,]$ and $c = -\,[\,Tv_0, Tv_0\,] - 2\,\mathrm{Re}\,[\,Tx_0, Tv_0\,]$. Then, $a \geq 0$, $b^2 - 4ac \leq 0$, and (6) follows. Equivalently,

$$[\,T(x_0 + v_0), T(x_0 + v_0)\,] \leq [\,T(x_0 + ty), T(x_0 + ty)\,],$$

where $y \in \mathcal{C}_V$ and $t \in \mathbb{R}$. Since $y \in \mathcal{C}_V$ is arbitrary, $x_0 + v_0 \in \mathcal{S}_{z_0}$. □

The following corollary states necessary conditions for the existence of indefinite interpolating splines. Hereafter, $\mathcal{C}_T$ denotes the set of neutral elements of the quadratic form $x \mapsto [\,Tx, Tx\,]$.

Corollary 3.1 *Given $z_0 \in \mathcal{E}$ and $x_0 \in \mathcal{H}$ such that $z_0 = Vx_0$, if $\mathcal{S}_{z_0} \neq \varnothing$, then $T(\mathcal{C}_V)$ is a non-negative set of $\mathcal{K}$ and $x_0 \in T^{\#}T(\mathcal{C}_T \cap \mathcal{C}_V)^{\perp}$.*

Proof The first assertion follows from Proposition 3.1. Also, if $y \in \mathcal{C}_T \cap \mathcal{C}_V$, then (3) becomes

$$\left|\left\langle x_0, T^{\#}Ty \right\rangle\right| = |\,[\,Tx_0, Ty\,]\,| \leq [\,Ty_0, Ty_0\,]^{1/2}\,[\,Ty, Ty\,]^{1/2} = 0,$$

where $y_0 \in \mathcal{C}_V$ is such that $x_0 + y_0 \in \mathcal{S}_{z_0}$. □

The necessary and sufficient conditions for the existence of an indefinite abstract interpolating spline stated in Proposition 3.1 can be alternatively expressed, by transforming the minimization problem into a dual maximization problem. In order to do so, consider the set $\mathcal{D} := \{y \in \mathcal{C}_V : \ [\,Ty, Ty\,] = 1\}$, and the function

$$\psi : \mathcal{D} \to \mathbb{R}^+ \quad \text{defined by} \quad \psi(y) = |[\,Tx_0, Ty\,]|\,, \tag{9}$$

where $x_0 \in T^{\#}T(\mathcal{C}_T \cap \mathcal{C}_V)^{\perp}$.

Proposition 3.2 *Assume that $T(\mathcal{C}_V)$ is a non-negative set of $\mathcal{K}$. Given a vector $x_0 \in T^{\#}T(\mathcal{C}_T \cap \mathcal{C}_V)^{\perp}$, set $z_0 = Vx_0$. The following conditions are equivalent:*

1. $\mathcal{S}_{z_0} \neq \varnothing$;
2. $\max_{y\in\mathcal{D}} \psi(y)$ *is attained.*

Proof If $x_0 \in N(T)$, then $[\,T(x_0 + y), T(x_0 + y)\,] = [\,Ty, Ty\,] \geq 0$. Hence, it is immediate that $\min_{y\in\mathcal{C}_V} [\,T(x_0 + y), T(x_0 + y)\,] = 0$ and $\mathcal{S}_{z_0} = x_0 + \mathcal{C}_T \cap \mathcal{C}_V$. Since $|\,[\,Tx_0, Ty\,]\,| = 0$ for every $y \in \mathcal{C}_V$, the assertion follows.

Now suppose that $x_0 \in T^{\#}T(\mathcal{C}_T \cap \mathcal{C}_V)^{\perp} \backslash N(T)$. Assume that $\mathcal{S}_{z_0} \neq \varnothing$, and let $y_0 \in \mathcal{C}_V$ be such that $x_0 + y_0 \in \mathcal{S}_{z_0}$. For every $y \in \mathcal{C}_V \backslash \mathcal{C}_T$, (3) yields

$$\left|\left[\,Tx_0, T\left(\frac{y}{[\,Ty, Ty\,]^{1/2}}\right)\right]\right|^2 \leq [\,Ty_0, Ty_0\,]\,.$$

By (4), $[\,Ty_0, Ty_0\,] = -\,[\,Tx_0, Ty_0\,]$. Hence, for every $y \in \mathcal{C}_V \backslash \mathcal{C}_T$

$$\left|\left[\,Tx_0, T\left(\frac{y}{[\,Ty, Ty\,]^{1/2}}\right)\right]\right|^2 \leq \left|\left[\,Tx_0, T\left(\frac{y_0}{[\,Ty_0, Ty_0\,]^{1/2}}\right)\right]\right|^2. \tag{10}$$

That is, the maximum is attained at $y_0/\,[\,Ty_0, Ty_0\,]^{1/2}$.

Conversely, suppose that the maximum is attained at $v_0 \in \mathcal{C}_V$. Then, for every $y \in \mathcal{C}_V \backslash \mathcal{C}_T$,

$$\left|\left[\,Tx_0, T\left(\frac{y}{[\,Ty, Ty\,]^{1/2}}\right)\right]\right|^2 \leq |\,[\,Tx_0, Tv_0\,]\,|^2\,. \tag{11}$$

Define $y_0 := -[\,Tx_0, Tv_0\,]\,v_0 \in \mathcal{C}_V$. Then, $[\,Ty_0, Ty_0\,] = |\,[\,Tx_0, Tv_0\,]\,|^2$, and (11) implies that for all $y \in \mathcal{C}_V \backslash \mathcal{C}_T$,

$$|\,[\,Tx_0, Ty\,]\,|^2 \leq [\,Ty_0, Ty_0\,][\,Ty, Ty\,].$$

Since $x_0 \in T^{\#}T(\mathcal{C}_T \cap \mathcal{C}_V)^{\perp}$, the above inequality is also valid for every vector $y \in \mathcal{C}_T \cap \mathcal{C}_V$. Further, since $[\,Tx_0, Ty_0\,] = -|\,[\,Tx_0, Tv_0\,]\,|^2 = -[\,Ty_0, Ty_0\,]$, the assertion follows from Proposition 3.1. □

Remark 3.1 If $\mathcal{E}$ is a Hilbert space and $\mathcal{S}_{z_0} \neq \varnothing$ for every $z \in \mathcal{E}$, it was shown that $\mathcal{S}_{z_0}$ is an affine manifold parallel to $N(T) \cap N(V)$, see [15, Prop. 3.8].

Now, assuming that $\mathcal{E}$ is a Krein space and $V^{\#}V$ is indefinite, observe that if $\tilde{x} \in \mathcal{S}_{z_0}$ for some $z_0 \in \mathcal{E}$, then $\tilde{x} + N(T) \cap N(V) \subseteq \mathcal{S}_{z_0}$. Indeed, if $\tilde{x} \in \mathcal{S}_{z_0}$, then $\tilde{x} = x_0 + y_0$ where $y_0 \in \mathcal{C}_V$ satisfies (3) and $[\,T(x_0 + y_0), Ty_0\,] = 0$. Given $u \in N(T) \cap N(V)$, note that $y := y_0 + u \in \mathcal{C}_V$ also satisfies (3) and

$$[\,T(x_0 + y), Ty\,] = [\,T(x_0 + y_0), Ty_0\,] = 0.$$

Hence, $\tilde{x} + u = x_0 + y \in \mathcal{S}_{z_0}$.

The next example shows that $\mathcal{S}_{z_0}$ need not be an affine manifold parallel to $N(T) \cap N(V)$.

Example 3.1 Consider surjective operators $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $V \in \mathcal{L}(\mathcal{H}, \mathcal{E})$ such that $T^{\#}T = I$ and $V^{\#}V = J$ is a symmetry. Note that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_\pm = N(I \mp J)$. Then, for every $x = x_+ + x_-$ with $x_\pm \in \mathcal{H}_\pm$ we have that $[\,Vx, Vx\,] = \|x_+\|^2 - \|x_-\|^2$. Hence,

$$\mathcal{C}_V = \{\, y = y_+ + y_- :\ y_\pm \in \mathcal{H}_\pm \text{ with } \|y_+\| = \|y_-\|\,\}.$$

It is immediate that $\mathcal{S}_{z_0} = \mathcal{C}_T \cap \mathcal{C}_V = \{\,0\,\}$ if $z_0 = 0$. If $z_0 \neq 0$, let $x_0 \in \mathcal{H}$ be such that $Vx_0 = z_0$. Given $x, y \in \mathcal{H}$, we have that $[\,Tx, Ty\,] = \langle\, x, y\,\rangle$. Thus,

$$\mathcal{D} = \left\{ \frac{1}{\sqrt{2}}\left(\frac{x_+}{\|x_+\|} + \frac{x_-}{\|x_-\|} \right) : x_\pm \in \mathcal{H}_\pm, x_\pm \neq 0 \right\}.$$

Assume that $x_0 = x_{0+} + x_{0-}$ with $x_{0\pm} \in \mathcal{H}_\pm$, and let $y \in \mathcal{C}_V$ be such that $[\,Ty, Ty\,] = \|y\| = 1$. Then, there exists $x_\pm \in \mathcal{H}_\pm$, $x_\pm \neq 0$ such that

$$\begin{aligned} |\,[\,Tx_0, Ty\,]\,| = |\,\langle\, x_0, y\,\rangle\,| &= \frac{1}{\sqrt{2}} \left| \frac{\langle\, x_{0+}, x_+\,\rangle}{\|x_+\|} + \frac{\langle\, x_{0-}, x_-\,\rangle}{\|x_-\|} \right| \\ &\leq \frac{1}{\sqrt{2}}\Big(\|x_{0+}\| + \|x_{0-}\| \Big). \end{aligned} \tag{12}$$

From the proof of Proposition 3.2, it follows that $y_0 \in \mathcal{C}_V$ is such that $x_0 + y_0 \in \mathcal{S}_{z_0}$ if and only if $y_0 = -[\,Tx_0, Tv_0\,]\,v_0 = -\langle\, x_0, v_0\,\rangle\,v_0$, where $v_0 \in \mathcal{C}_V$ satisfies $\|v_0\| = 1$ and

$$\max_{y\in\mathcal{D}} |\langle x_0, y\rangle| = |\langle x_0, v_0\rangle|\,. \tag{13}$$

Assume first that $x_{0\pm} \neq 0$. Then, from (12) it is readily seen that v_0 attains the maximum in (13) if and only if $v_0 = \frac{e^{i\theta}}{\sqrt{2}}\left(\frac{x_{0+}}{\|x_{0+}\|} + \frac{x_{0-}}{\|x_{0-}\|}\right)$ with $\theta \in [0, 2\pi)$. In this case, $\mathcal{S}_{z_0}$ is a singleton, namely

$$\mathcal{S}_{z_0} = \left\{ \frac{1}{2}\left(\left(1 - \frac{\|x_{0-}\|}{\|x_{0+}\|}\right)x_{0+} + \left(1 - \frac{\|x_{0+}\|}{\|x_{0-}\|}\right)x_{0-}\right)\right\}.$$

Now consider the case $x_{0-} = 0$. Then, v_0 attains the maximum in (13) if and only if $v_0 = \frac{e^{i\theta}}{\sqrt{2}}\left(\frac{x_{0+}}{\|x_{0+}\|} + x_-\right)$ where $\theta \in [0, 2\pi)$ and $x_- \in \mathcal{H}_-$ is an arbitrary vector with $\|x_-\| = 1$. Therefore,

$$\mathcal{S}_{z_0} = \left\{ \frac{1}{2}\big(x_{0+} + \|x_{0+}\|\, x_-\big) : x_- \in \mathcal{B}_{\mathcal{H}_-} \right\},$$

where $\mathcal{B}_{\mathcal{H}_-}$ is the unit sphere in $\mathcal{H}_-$. Analogously, if $x_{0+} = 0$ and $\mathcal{B}_{\mathcal{H}_+}$ is the unit sphere in $\mathcal{H}_+$, then

$$\mathcal{S}_{z_0} = \left\{ \frac{1}{2}\big(x_{0-} + \|x_{0-}\|\, x_+\big) : x_+ \in \mathcal{B}_{\mathcal{H}_+} \right\}.$$

In the above example, $N(T) \cap N(V) = \{0\}$ and $\mathcal{S}_{z_0}$ is a singleton (i.e. an affine manifold parallel to $\{0\}$) whenever $x_{0\pm} \neq 0$. Hence, $\mathcal{S}_{z_0}$ is not a singleton only in the case that x_0 is contained in one of the subspaces $\mathcal{H}_\pm$ determined by the symmetry $J = V^\# V$.

The behaviour illustrated in this example occurs generically. The set of interpolating splines is a single affine manifold for all z_0 in an open dense subset of the vector space.

Under the conditions stated in Proposition 3.2, let Θ be the set of points attaining the maximum value of ψ at $\mathcal{D}$:

$$\Theta = \left\{ w \in \mathcal{D} :\ \psi(w) = \max_{y\in\mathcal{D}} \psi(y) \right\}.$$

In the proof of Proposition 3.2, it was shown that if $x_0 \in N(T)$ and $z_0 = V x_0$, then $\mathcal{S}_{z_0} = x_0 + \mathcal{C}_T \cap \mathcal{C}_V$. The next result deals with the other case: $x_0 \notin N(T)$.

Theorem 3.1 *Given $x_0 \in T^\# T(\mathcal{C}_T \cap \mathcal{C}_V)^\perp \setminus N(T)$, set $z_0 = V x_0 \in \mathcal{E}$. Then, $\mathcal{S}_{z_0}$ is the union of a family of affine manifolds parallel to $N(T) \cap N(V)$:*

$$\mathcal{S}_{z_0} = \bigcup_{w\in\Theta} (\, x_0 - [\, T x_0, T w\,]\, w + N(T) \cap N(V)\,)\,. \tag{14}$$

Proof Given $y_0 \in \mathcal{C}_V$, assume that $x_0 + y_0 \in \mathcal{S}_{z_0}$ and $[\, T y_0, T y_0\,] = 0$. By (3) we have that $[\, T x_0, T y\,] = 0$ for every $y \in \mathcal{C}_V$. Hence, $T^\# T x_0 \in (\mathcal{C}_V)^\perp = \{0\}$. But this says that $x_0 \in N(T)$ which contradicts the hypothesis. Therefore, if $x_0 + y_0 \in \mathcal{S}_{z_0}$, then $[\,Ty_0, Ty_0\,] \neq 0$. By (10), the maximum value of ψ over $\mathcal{D}$ is attained at $w_0 := \frac{y_0}{[\,Ty_0,Ty_0\,]^{1/2}}$ and

$$x_0 + y_0 = x_0 + [\,Ty_0, Ty_0\,]^{1/2}\, w_0.$$

By Proposition 3.1, $[\,Tx_0, Ty_0\,] = -[\,Ty_0, Ty_0\,]$. Hence,

$$[\,Tx_0, Tw_0\,] = \frac{[\,Tx_0, Ty_0\,]}{[\,Ty_0, Ty_0\,]^{1/2}} = -[\,Ty_0, Ty_0\,]^{1/2}.$$

Thus, $x_0 + y_0 = x_0 - [\,Tx_0, Tw_0\,]\, w_0$ which is clearly in the right-hand side of (14).

The other inclusion can be traced back to the proof of Proposition 3.2, by considering that, if $y_0 \in \mathcal{C}_V$ is such that $x_0 + y_0 \in \mathcal{S}_{z_0}$, then Remark 3.1 guarantees that the affine manifold $x_0 + y_0 + N(T) \cap N(V)$ is contained in $\mathcal{S}_{z_0}$. □

4 A Sufficient Condition for the Existence of Interpolating Splines

One of the necessary conditions for the existence of indefinite interpolating splines given in Proposition 3.1 is that T maps the set $\mathcal{C}_V$ into the set of non-negative vectors of the Krein space $\mathcal{K}$. The following result can be interpreted as another manifestation of the S-Lemma (or Farkas lemma), see [20,21]. For its proof, see Lemma 1.35 and Corollary 1.36 in [34, Chapter 1, §1].

Proposition 4.1 *Assume that $V^\# V$ is indefinite. Then, the following conditions are equivalent:*

1. *$T(\mathcal{C}_V)$ is a non-negative set of $\mathcal{K}$.*
2. *There exists $\rho \in I\!R$ such that $T^\# T + \rho V^\# V \in \mathcal{L}(\mathcal{H})^+$.*

Given a real constant $\rho \neq 0$, define the following indefinite inner product on $\mathcal{K} \times \mathcal{E}$:

$$[(y,z),(y',z')]_\rho = [y,y']_{\mathcal{K}} + \rho[z,z']_{\mathcal{E}}, \quad y, y' \in \mathcal{K} \text{ and } z, z' \in \mathcal{E}. \tag{15}$$

It is easy to see that $(\mathcal{K}\times\mathcal{E}, [\,\cdot,\cdot\,]_\rho)$ is a Krein space. Define the operator $L : \mathcal{H} \to \mathcal{K}\times\mathcal{E}$ by

$$Lx = (Tx, Vx), \quad x \in \mathcal{H}. \tag{16}$$

Given $x \in \mathcal{H}$ and $(y,z) \in \mathcal{K} \times \mathcal{E}$,

$$\begin{aligned}[\,Lx,(y,z)\,]_\rho &= [\,Tx, y\,]_{\mathcal{K}} + \rho\,[\,Vx, z\,]_{\mathcal{E}} = \left\langle x, T^\# y\right\rangle + \rho\left\langle x, V^\# z\right\rangle \\ &= \left\langle x, T^\# y + \rho V^\# z\right\rangle.\end{aligned}$$

Hence, the adjoint operator of L with respect to the indefinite inner product $[\,\cdot,\cdot\,]_\rho$ in $\mathcal{K} \times \mathcal{E}$ is given by

$$L^\#(y,z) = T^\# y + \rho V^\# z, \quad (y,z) \in \mathcal{K} \times \mathcal{E},$$

and it is immediate that $L^{\#}L = T^{\#}T + \rho V^{\#}V$.

Considering the inner product $[\,\cdot,\cdot\,]_\rho$ given by (15) and the linear operator L defined in (16), the conditions in Proposition 4.1 are also equivalent to the condition, "$R(L)$ is a non-negative subspace of the Krein space $(\mathcal{K} \times \mathcal{E}, [\,\cdot,\cdot\,]_\rho)$".

Assume that there exists $\rho \neq 0$ such that $L^{\#}L = T^{\#}T + \rho V^{\#}V \in \mathcal{L}(\mathcal{H})^+$. By the discussion above, $R(L^{\#}L)$ is a pre-Hilbert space with respect to the inner product defined by:

$$(\,x, y\,) := [\,Lx, Ly\,]_\rho = \left\langle L^{\#}Lx, y \right\rangle, \quad x, y \in R(L^{\#}L).$$

In particular, $(R(L^{\#}L), \|\cdot\|_L)$ is a normed space, where $\|x\|_L := (\,x, x\,)^{1/2}$ for $x \in R(L^{\#}L)$. Let us denote by $\mathcal{B}_L$ the unit sphere in $R(L^{\#}L)$ with respect to the norm $\|\cdot\|_L$:

$$\mathcal{B}_L = \{x \in R(L^{\#}L) : \,(\,x, x\,) = 1\}.$$

Given $x \in R(L^{\#}L)$ and a subset M of $R(L^{\#}L)$, $\mathrm{d}(x, M)$ denotes the distance between x and M with respect to the norm $\|\cdot\|_L$:

$$\mathrm{d}(x, M) = \inf\{\|x - m\|_L : \, m \in M\}.$$

Recall the function ψ defined in (9), and observe that in this case, the domain $\mathcal{D}$ coincides with the intersection between $\mathcal{C}_V$ and the unit sphere $\mathcal{B}_L$ of $R(L^{\#}L)$:

$$\mathcal{D} = \mathcal{C}_V \cap \mathcal{B}_L.$$

Thus, under the additional hypothesis that $T^{\#}Tx_0 \in R(L^{\#}L)$, Proposition 3.2 can be reinterpreted as follows:

Proposition 4.2 *Suppose that there exists $\rho \neq 0$ such that $L^{\#}L \in \mathcal{L}(\mathcal{H})^+$. Given $x_0 \in \mathcal{H}$, assume that there exists a vector $u_0 \in R(L^{\#}L)$ such that $T^{\#}Tx_0 = L^{\#}Lu_0$. If $z_0 = Vx_0 \in \mathcal{E}$, then the following conditions are equivalent:*

1. $\mathcal{S}_{z_0} \neq \varnothing$*;*
2. $\mathrm{d}(u_0, \mathcal{D})$ *is attained.*

Proof In the first place, note that

$$R(L^{\#}L) \subseteq N(L^{\#}L)^{\perp} \subseteq (\mathcal{C}_T \cap \mathcal{C}_V)^{\perp}$$

because $\mathcal{C}_T \cap \mathcal{C}_V \subseteq N(L^{\#}L)$. Hence, if $T^{\#}Tx_0 \in R(L^{\#}L)$, it follows that $T^{\#}Tx_0 \in (\mathcal{C}_T \cap \mathcal{C}_V)^{\perp}$, and consequently $x_0 \in T^{\#}T(\mathcal{C}_T \cap \mathcal{C}_V)^{\perp}$. Therefore, by Proposition 3.2 $\mathcal{S}_{z_0} \neq \varnothing$ if and only if the maximum $\max_{y\in\mathcal{D}} \psi(y) = \max_{y\in\mathcal{D}} |\,[\,Tx_0, Ty\,]\,|$ is attained.

Furthermore, if $y \in \mathcal{D}$, then $[\,Tx_0, Ty\,] = \left\langle L^{\#}Lu_0, y \right\rangle = (\,u_0, y\,)$ and

$$\|u_0 - y\|_L^2 = \|u_0\|_L^2 - 2\,\mathrm{Re}\,(\,u_0, y\,) + \|y\|_L^2 = (\|u_0\|_L^2 + 1) - 2\,\mathrm{Re}\,[\,Tx_0, Ty\,],$$

and the assertion is immediate. □

Observe that $\mathcal{D} = \mathcal{C}_V \cap \mathcal{B}_L$ is not a convex set. It would be natural to replace it by its convex hull in order to transform the indefinite abstract interpolation problem into an optimization problem over a convex set. However, this is not possible. In fact, given an arbitrary vector $x_0 \in \mathcal{H}$, it is easy to see that

$$\sup_{w \in \operatorname{conv}(\mathcal{D})} |\,[\,Tx_0, Tw\,]\,| = \sup_{y \in \mathcal{D}} |\,[\,Tx_0, Ty\,]\,|.$$

On the other hand, it is possible to find $u_0 \in R(L^\# L)$ such that

$$\mathrm{d}(u_0, \operatorname{conv}(\mathcal{D})) < \mathrm{d}(u_0, \mathcal{D}),$$

(for instance, consider $u_0 \in \operatorname{conv}(\mathcal{D}) \setminus \mathcal{D}$). Therefore, the equivalence proved in Proposition 4.2 does not hold if $\mathcal{D}$ is replaced by $\operatorname{conv}(\mathcal{D})$.

Let K denote a non-empty subset of a normed linear space $(X, \|\cdot\|)$. The subset K is called *proximinal* (resp., *Chebyshev*), if for each $x \in X \setminus K$, the set of best approximations to x from K,

$$A_K(x) = \{y \in K : \; \|y - x\| = \mathrm{d}(x, K)\},$$

is non-empty (resp., a singleton).

If X is a reflexive Banach space (for example, a Hilbert space), then every weakly closed subset K of X is proximinal [29, Thm. 4.28]. This is one of the main arguments in the proof of the following theorem.

While $\mathcal{C}_V$ is not weakly compact, it is weakly closed.

Lemma 4.1 *The set $\mathcal{C}_V$ is weakly closed.*

Proof Since $G := V^\# V$ is a self-adjoint operator in $\mathcal{H}$, there exist (unique) closed subspaces $\mathcal{H}_\pm$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus N(V)$, and (unique) invertible operators $G_\pm \in \mathcal{L}(\mathcal{H}_\pm)^+$ such that $G = G_+ - G_-$.

Given $y \in \mathcal{H}$ decomposed as $y = y_+ + y_- + y_0$ with $y_\pm \in \mathcal{H}_\pm$ and $y_0 \in N(V)$, note that

$$[\,Vy, Vy\,] = \langle\, Gy, y\,\rangle = \langle\, G_+ y_+, y_+\,\rangle - \langle\, G_- y_-, y_-\,\rangle = \|G_+^{1/2} y_+\|^2 - \|G_-^{1/2} y_-\|^2.$$

Thus, $y \in \mathcal{C}_V$ if and only if $\|G_+^{1/2} y_+\| = \|G_-^{1/2} y_-\|$.

Let $(y_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_V$ be such that $y_n \overset{w}{\to} y \in \mathcal{H}$, where $y_n = y_n^+ + y_n^- + y_n^0$, $y = y_+ + y_- + y_0$, $y_n^\pm, y_\pm \in \mathcal{H}_\pm$, $y_n^0, y_0 \in N(V)$, and $\|G_+^{1/2} y_n^+\| = \|G_-^{1/2} y_n^-\|$ for every $n \in \mathbb{N}$. It readily follows that $y_n^\pm \overset{w}{\to} y_\pm$ and $y_n^0 \overset{w}{\to} y_0$. If there exists $n_0 \in \mathbb{N}$ such that $y_n^+ = 0$ for every $n \geq n_0$, then $\|G_-^{1/2} y_n^-\| = \|G_+^{1/2} y_n^+\| = 0$ for $n \geq n_0$ and, since $G_-^{1/2}$ is invertible on $\mathcal{H}_-$, it follows that $y_n^- = 0$ for $n \geq n_0$. Therefore, $y_\pm = 0$ and $y = y_0 \in N(V) \subseteq \mathcal{C}_V$. The same holds if we assume that there exists $n_0 \in \mathbb{N}$ such that $y_n^- = 0$ for every $n \geq n_0$.

Further, if for every $n_0 \in \mathbb{N}$ there exists $n \geq n_0$ such that $y_n^{\pm} \neq 0$, there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $y_{n_k}^{\pm} \neq 0$ for every $k \in \mathbb{N}$. Without loss of generality let us assume that $y_n^{\pm} \neq 0$ for every $n \in \mathbb{N}$. Since $G_{\pm}^{1/2}$ is invertible on $\mathcal{H}_{\pm}$, we have that $\|G_{\pm}^{1/2} y_n^{\pm}\| \neq 0$ for every $n \in \mathbb{N}$. By the weak compactness of the unit sphere in $\mathcal{H}_+$, there exists a subsequence $(G_+^{1/2} y_{n_k}^+ / \|G_+^{1/2} y_{n_k}^+\|)_{k\in\mathbb{N}} \subseteq (G_+^{1/2} y_n^+ / \|G_+^{1/2} y_n^+\|)_{n\in\mathbb{N}}$ and $x_+ \in \mathcal{H}_+$ such that $G_+^{1/2} y_{n_k}^+ / \|G_+^{1/2} y_{n_k}^+\| \xrightarrow[k\to\infty]{w} G_+^{1/2} x_+ / \|G_+^{1/2} x_+\|$.

For every $x \in \mathcal{H}_+$,

$$\|G_+^{1/2} y_{n_k}^+\| \left\langle G_+^{-1/2} x, \frac{G_+^{1/2} y_{n_k}^+}{\|G_+^{1/2} y_{n_k}^+\|} \right\rangle = \left\langle x, y_{n_k}^+ \right\rangle \xrightarrow[k\to\infty]{} \langle x, y_+ \rangle ,$$

and

$$\left\langle G_+^{-1/2} x, \frac{G_+^{1/2} y_{n_k}^+}{\|G_+^{1/2} y_{n_k}^+\|} \right\rangle \xrightarrow[k\to\infty]{} \left\langle G_+^{-1/2} x, \frac{G_+^{1/2} x_+}{\|G_+^{1/2} x_+\|} \right\rangle = \left\langle x, \frac{x_+}{\|G_+^{1/2} x_+\|} \right\rangle .$$

Hence, there exists $A_+ \geq 0$ such that $\lim_{k\to\infty} \|G_+^{1/2} y_{n_k}^+\| = A_+$, and $y_+ = A_+(x_+/\|G_+^{1/2} x_+\|)$. Thus, $A_+ = \|G_+^{1/2} y_+\|$.

Applying the same procedure to $(y_{n_k}^-)_{k\in\mathbb{N}}$ in $\mathcal{H}_-$ yields that there exists a subsequence $(y_{n_{k_l}}^-)_{l\in\mathbb{N}}$ of $(y_{n_k}^-)_{k\in\mathbb{N}}$ such that $\lim_{l\to\infty} \|G_-^{1/2} y_{n_{k_l}}^-\| = \|G_-^{1/2} y_-\|$. Since $\|G_+^{1/2} y_{n_{k_l}}^+\| = \|G_-^{1/2} y_{n_{k_l}}^-\|$ for every $l \in \mathbb{N}$, it holds that $\|G_+^{1/2} y_+\| = \|G_-^{1/2} y_-\|$. Thus, $y \in \mathcal{C}_V$, and the assertion is proved. □

In the following, we present a sufficient condition for the existence of indefinite interpolating splines for every $z_0 \in \mathcal{E}$.

Theorem 4.1 *If $R(L)$ is a (closed) uniformly positive subspace of $(\mathcal{K} \times \mathcal{E}, [\cdot, \cdot]_\rho)$ for some $\rho \neq 0$, then $\mathcal{S}_{z_0} \neq \varnothing$ for every $z_0 \in \mathcal{E}$.*

Proof In order to prove the theorem, we apply Proposition 4.2. To this end, we first show that $T^\# T x \in R(L^\# L)$ for every $x \in \mathcal{H}$. By Proposition A.1 in Appendix, $R(L)$ is a regular subspace. Then, for every $(y, z) \in \mathcal{K} \times \mathcal{E}$ there exists (a unique) $x \in \mathcal{H}$ such that $Lx - (y, z) \in R(L)^{[\perp]}$, or equivalently, $L^\# L x = L^\#(y, z)$. Since T and V are surjective, for each $(y, z) \in \mathcal{K} \times \mathcal{E}$ there exist $u, w \in \mathcal{H}$ such that $y = Tu$ and $z = Vw$. Therefore, there exists $x \in \mathcal{H}$ such that

$$T^\# T u + \rho V^\# V w = L^\#(Tu, Vw) = (T^\# T + \rho V^\# V)x,$$

and consequently $R(L^\# L) = R(T^\# T + \rho V^\# V) = R(T^\# T) + R(V^\# V)$. Thus, $R(T^\# T) \subseteq R(L^\# L)$.

Given $z_0 \in \mathcal{E}$, let $x_0, u_0 \in \mathcal{H}$ be such that $V x_0 = z_0$ and $T^\# T x_0 = L^\# L u_0$. Then, $\mathrm{d}(u_0, \mathcal{C}_V \cap \mathcal{B}_L)$ is attained.

To prove this assertion, we claim that $R(L^\# L)$ is closed: in fact, since $R(L)$ is regular it follows that $R(L) + N(L^\#) = R(L) + R(L)^{[\perp]} = \mathcal{K} \times \mathcal{E}$. Hence, $R(L^\# L) = R(L^\#)$, and since $R(L)$ is closed, it holds that $R(L^\# L)$ is closed. Consequently, $(R(L^\# L), (\cdot, \cdot))$ is a Hilbert space. Moreover, the norms $\| \cdot \|$ and $\| \cdot \|_L$ (associated to the inner products $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$, respectively) are equivalent on $R(L^\# L)$.

By Lemma 4.1, $\mathcal{C}_V$ is weakly closed in $(\mathcal{H}, (\cdot, \cdot))$. Since $\mathcal{B}_L$ is weakly compact in $(R(L^\# L), (\cdot, \cdot))$, it follows that $\mathcal{C}_V \cap \mathcal{B}_L$ is also weakly compact (in particular, it is weakly closed). Hence, $\mathcal{C}_V \cap \mathcal{B}_L$ is a proximinal set in $(R(L^\# L), (\cdot, \cdot))$. Since $\mathrm{d}(u_0, \mathcal{C}_V \cap \mathcal{B}_L)$ is attained, Proposition 4.2 ensures that $\mathcal{S}_{z_0} \neq \varnothing$. □

In 1961, Klee [30] asked whether a Chebyshev set in a Hilbert space $\mathcal{H}$ must be convex. If $\mathcal{H}$ is finite-dimensional, then the answer is yes. On the other hand, if $\mathcal{H}$ is an infinite-dimensional Hilbert space, it is known that if K is a weakly closed Chebyshev set in a Hilbert space, then K is convex [31], but so far there is no definitive answer to Klee's question. According to Deutsch [32], this is "... perhaps the major unsolved problem in (abstract) approximation theory today.".

From Example 3.1, $\mathcal{D}$ is not a Chebyshev set.

5 Conclusions

The indefinite abstract interpolation spline problem is a suitable generalization of the classical abstract spline formulation, belonging to the QPQC problems family, and as such, applicable to linear control modelling.

Under a condition that ensures the existence of spline interpolants to every element of the vector space, we show that the set of indefinite abstract interpolating splines is a single affine manifold, in a generic case (in the sense that it corresponds to an open dense subset of the vector space). When this is not the case, the set turns out to be the union of a family of affine manifolds.

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Appendix: Terminology and Notations Related to Krein Spaces

In the following, we present the standard notation and some basic results on indefinite inner product spaces and, in particular, on Krein spaces. For a complete exposition on the subject (and the proofs of the results below) see, for example, [33–37].

An indefinite inner product space $(\mathcal{F}, [\cdot, \cdot])$ is a (complex) vector space $\mathcal{F}$ endowed with a Hermitian sesquilinear form $[\cdot, \cdot] : \mathcal{F} \times \mathcal{F} \to \mathbb{C}$.

Two vectors $x, y \in \mathcal{F}$ are *orthogonal*, denoted by $x[\perp]y$, if $[x, y] = 0$.

If $\mathcal{S}$ is a subset of an indefinite inner product space $\mathcal{F}$, the *orthogonal companion* to $\mathcal{S}$ is defined by

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{F} : [\,x, s\,] = 0 \text{ for every } s \in \mathcal{S}\},$$

and it is always a subspace of $\mathcal{F}$.

Definition A.1 An indefinite inner product space $(\mathcal{H}, [\,\cdot, \cdot\,])$ is a *Krein space*, if it can be decomposed as a direct (orthogonal) sum of a Hilbert space and an anti-Hilbert space, i.e. there exist subspaces $\mathcal{H}_\pm$ of $\mathcal{H}$ such that $(\mathcal{H}_+, [\,\cdot, \cdot\,])$ and $(\mathcal{H}_-, -[\,\cdot, \cdot\,])$ are Hilbert spaces,

$$\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-, \tag{17}$$

and $\mathcal{H}_+$ is orthogonal to $\mathcal{H}_-$ with respect to the indefinite inner product. Sometimes we use the notation $[\,\cdot, \cdot\,]_{\mathcal{H}}$ instead of $[\,\cdot, \cdot\,]$ to emphasize the Krein space considered.

A pair of subspaces $\mathcal{H}_\pm$ as in (17) is called a *fundamental decomposition* of $\mathcal{H}$. Given a Krein space $\mathcal{H}$ and a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$, the direct sum of the Hilbert spaces $(\mathcal{H}_+, [\,\cdot, \cdot\,])$ and $(\mathcal{H}_-, -[\,\cdot, \cdot\,])$ is denoted by $(\mathcal{H}, \langle\,\cdot, \cdot\,\rangle)$.

If $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$ and $\mathcal{H} = \mathcal{H}'_+ \dot{+} \mathcal{H}'_-$ are two different fundamental decompositions of $\mathcal{H}$, then the corresponding associated inner products $\langle\,\cdot, \cdot\,\rangle$ and $\langle\,\cdot, \cdot\,\rangle'$ turn out to be equivalent on $\mathcal{H}$. Therefore, the norm topology on $\mathcal{H}$ does not depend on the chosen fundamental decomposition.

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces, $\mathcal{L}(\mathcal{H}, \mathcal{K})$ stands for the vector space of linear transformations which are bounded with respect to any of the associated Hilbert spaces.

Given $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, the adjoint operator of T is the unique operator $T^\# \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ such that

$$[\,Tx, y\,]_{\mathcal{K}} = [x, T^\# y]_{\mathcal{H}}, \quad \text{for every } x \in \mathcal{H}, y \in \mathcal{K}.$$

We frequently use that if $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $\mathcal{M}$ is a closed subspace of $\mathcal{K}$, then

$$T^\#(\mathcal{M})^{[\perp]} = T^{-1}(\mathcal{M}^{[\perp]}).$$

A vector $x \in \mathcal{F}$ is *positive*, *negative*, or *neutral*, if $[\,x, x\,] > 0$, $[\,x, x\,] < 0$, or $[\,x, x\,] = 0$, respectively. A set $\mathcal{M}$ of $\mathcal{F}$ is *positive* (*negative*) if x is positive (negative) for every $x \in \mathcal{M}$, $x \neq 0$; and it is *non-negative* (*non-positive*) if $[\,x, x\,] \geq 0$ ($[\,x, x\,] \leq 0$) for every $x \in \mathcal{M}$.

A subspace $\mathcal{M}$ of a Krein space is *uniformly positive* if there exists $\alpha > 0$ such that

$$[\,x, x\,] \geq \alpha \|x\|^2 \quad \text{for every } x \in \mathcal{M}.$$

Uniformly negative subspaces are defined in a similar way.

A subspace $\mathcal{M}$ of a Krein space $\mathcal{H}$ is *regular* if $\mathcal{M} + \mathcal{M}^{[\perp]} = \mathcal{H}$, or equivalently, if there exists a projection $Q \in \mathcal{L}(\mathcal{H})$ onto $\mathcal{M}$ such that $Q^\# = Q$. Regular subspaces are closed.

The following proposition shows that closed uniformly definite subspaces are regular subspaces (see [34, Chapter 1, §7]).

Proposition A.1 *Let $\mathcal{M}$ be a subspace of a Krein space $\mathcal{H}$. Then, $\mathcal{M}$ is closed and uniformly positive (negative) if and only if $\mathcal{M}$ is regular and non-negative (non-positive).*

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