Fully Piecewise Linear Vector Optimization Problems

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Abstract: We distinguish two kinds of piecewise linear functions and provide an interesting representation for a piecewise linear function between two normed spaces. Based on such a representation, we study a fully piecewise linear vector optimization problem with the objective and constraint functions being piecewise linear. To solve this problem, we divide it into some linear subproblems and structure a dimensional reduction method. Under some mild assumptions, we prove that its Pareto (resp. weak Pareto) solution set is the union of finitely many generalized polyhedra (resp. polyhedra), each of which is contained in a Pareto (resp. weak Pareto) face of some linear subproblem. Our main results are even new in the linear case and further generalize Arrow, Barankin and Blackwell's classical results on linear vector optimization problems in the framework of finite dimensional spaces.

Keywords: Polyhedron, Piecewise linear function, Pareto solution, Weak Pareto solution

1 Introduction

Vector optimization has been recognized to be useful in theory and application and extensively studied (see books [7, 12, 13, 22] and the references therein). However, the study of nonlinear vector optimization is far from systemic (possibly because the vector ordering is much more complicated than the scalar one). On the other hand, linear vector optimization has been well studied (cf. [1–3, 6, 7, 12, 13, 15, 20, 22] and the references therein). In particular, in the finite dimensional case, Arrow, Barankin and Blackwell in their pioneering paper [2] proved that both the Pareto solution set and the weak Pareto solution set of a linear vector optimization problem are the unions of finitely many faces of its feasible set (also see [12, Theorem 3.3] and [13, Theorems 4.1.20 and 4.3.8]). The linearity assumption is quite restrictive in both theory and application. To overcome the restriction of linearity, one sometimes adopts the piecewise linear functions (cf. [5, 9, 10, 14, 21, 23, 24]). The family of all piecewise linear functions is much larger than that of all linear functions and there exists a wide class of functions that can be approximated by piecewise linear functions. Moreover, one may use piecewise linear functions generated by finitely many test data to establish mathematical models for some practical problems. Therefore, from the viewpoint of theoretical interest as well as for applications, it is important to study piecewise linear problems. Given two normed spaces X and Y, we consider two kinds of piecewise linear functions from X to Y, denoted respectively by $\mathcal{PL}_1(X,Y)$ and $\mathcal{PL}(X,Y)$. A function f is in $\mathcal{PL}_1(X,Y)$ if and only if its graph is the union of finitely many polyhedra in the product $X \times Y$

(cf. [11,24] and the references therein), while f is in $\mathcal{PL}(X,Y)$ if and only if there exist polyhedra P_i in X, continuous linear operators T_i from X to Y and b_i in Y ($i=1,\cdots,m$) such that

$$X=\bigcup_{i=1}^{m}P_i \;\text{ and }\; f(x)=T_i(x)+b_i \quad \forall x\in P_i,\ i=1,\cdots,m.$$

The family $\mathcal{PL}(X,Y)$ is also known in the literature (cf. [9,21,23]). Clearly, a continuous linear operator from X to Y is always contained in $\mathcal{PL}(X,Y)$; however if Y is infinite dimensional then every linear operator from X to Y must not be in $\mathcal{PL}_1(X,Y)$. This motivates us to study the relationship between $\mathcal{PL}_1(X,Y)$ and $\mathcal{PL}(X,Y)$. To do this, we first consider polyhedra in normed spaces. In Section 2, we provide several properties on polyhedra in normed spaces. In particular, with the help of the notion of a prime generator group of a polyhedron (cf. [4,8,19]), we establish some results on the maximal faces of a polyhedron, which not only play a key role in the proof of the main theorem on piecewise linear functions but also should be valuable by themselves. In Section 3, we prove that $\mathcal{PL}_1(X,Y)$ is always contained in $\mathcal{PL}(X,Y)$ and that

$$\dim(Y)<\infty \Leftrightarrow \mathcal{PL}_1(X,Y)=\mathcal{PL}(X,Y) \;\text{ and }\; \dim(Y)=\infty \Leftrightarrow \mathcal{PL}_1(X,Y)=\emptyset.$$

As one of the mains results, we prove by using the Fubini theorem on Lebesgue's measure that each f in $\mathcal{PL}(X,Y)$ can be represented as the sum of a linear operator on an infinite dimensional subspace and a piecewise linear function on a finite dimensional subspace.

In the case where the objective is piecewise linear but the constraint is linear, several authors considered a partially piecewise linear vector optimization problem and proved that the weak Pareto solution set of such a partially piecewise linear problem is the union of finitely many polyhedra, while its Pareto solution set is the union of generalized polyhedra (cf. [5,9,21–23] and the references therein). In Section 4, we consider a fully piecewise linear vector optimization problem with both the objective and the constraint being piecewise linear. Based on the main results in Section 3, we reduce a fully piecewise linear vector optimization problem (in brief (PLP)) to several linear subproblems in the finite dimensional framework and prove that the weak Pareto solution set (resp. Pareto solution set) of (PLP) is not only equal to the union of finitely many polyhedra (resp. generalized polyhedra) $\Lambda_1,\cdots,\Lambda_\nu$ but also each Λ_i is contained in a weak Pareto face (resp. Pareto face) of some linear subproblem. Moreover, we provide procedures to obtain exact formulas for optimal value sets and (weak) Pareto solution sets of (PLP), which is new even in the framework of finite dimensional spaces and in the linear constraint case.

2 Polyhedra in Normed Spaces

Let Z be a normed space with the dual space Z^*. Recall (cf. [16]) that a subset P of Z is a (convex) polyhedron if there exist $u_1^*,\cdots,u_m^*\in Z^*$ and $s_1,\cdots,s_m\in\mathbb{R}$ such that

$$P=\{x\in Z:\ \langle u_i^*,x\rangle\le s_i,\ i=1,\cdots,m\}. \tag{1}$$

An exposed face of P is a set F such that $F=\{u\in P:\ \langle x^*,u\rangle=\sup\limits_{x\in P}\langle x^*,x\rangle\}$ for some $x^*\in Z^*$ (cf. [16, P.162]). It is known that each polyhedron has finitely many exposed faces. Recall

([21, 24]) that a subset $\tilde{P}$ of Z is a generalized polyhedron if there exist a polyhedron P in Z, $v_1^*, \cdots, v_k^* \in Z^*$ and $t_1, \cdots, t_k \in \mathbb{R}$ such that $\tilde{P} = P \cap \{z \in Z : \ \langle v_i^*, z\rangle < t_i, \ 1 \leq i \leq k\}$. Given $z^* \in Z^* \setminus \{0\}$, let $\mathcal{N}(z^*)$ denote the null space of z^*, that is, $\mathcal{N}(z^*) := \{z \in Z : \langle z^*, z\rangle = 0\}$.

Recall that a normed space Z is a direct sum of two closed subspaces Z_1 and Z_2, denoted by $Z = Z_1 \oplus Z_2$, if $Z_1 \cap Z_2 = \{0\}$ and $Z = Z_1 + Z_2$; and codimension $\mathrm{codim}(Z_1)$ of Z_1 is defined as dimension $\dim(Z_2)$ of Z_2. It is well-known that $\mathrm{codim}(\mathcal{N}(z^*)) = 1$ for a nonzero linear functional z^* on Z. It is easy to verify that if $Z = Z_1 \oplus Z_2$ then for each $z \in Z$ there exists a unique $(z_1, z_2) \in Z_1 \times Z_2$ such that $z = z_1 + z_2$ and the projection mapping $\Pi_{Z_2} : Z = Z_1 \oplus Z_2 \to Z_2$ is linear, where

$$\Pi_{Z_2}(z_1 + z_2) := z_2 \quad \forall (z_1, z_2) \in Z_1 \times Z_2. \tag{2}$$

It is known that if Q is a polyhedron in $Z_1 \oplus Z_2$ then $\Pi_{Z_2}(Q)$ is a polyhedron in Z_2 (cf. [16, Theorem 19.3] and the following Proposition 2.1).

For a convex set C in Z, let $\mathrm{int}(C)$ (resp. $\mathrm{rint}(C)$) denote the interior (relative interior) of C. It is known that if $\dim(Z) < \infty$ and $C \neq \emptyset$ then $\mathrm{rint}(C) \neq \emptyset$. Throughout, let $\mathbb{N}$ denote the set of all natural numbers. For $m \in \mathbb{N}$, let $\overline{1m}$ denote the set $\{1, \cdots, m\}$. Now we provide some results on polyhedra which are useful for our analysis later.

Proposition 2.1. *Let $(z_1^*, s_1), \cdots, (z_m^*, s_m) \in Z^* \times \mathbb{R}$ and $P := \{z \in Z : \ \langle z_i^*, z\rangle \leq s_i \ \ \forall i \in \overline{1m}\}$. Let Z_1 and Z_2 be two closed subspaces of Z such that*

$$Z_1 \subset \bigcap_{i=1}^{m} \mathcal{N}(z_i^*), \ \dim(Z_2) = \mathrm{codim}(Z_1) < \infty \ \text{ and } \ Z = Z_1 \oplus Z_2. \tag{3}$$

Then

$$P = Z_1 + \hat{P} \ \text{ and } \ \mathrm{rint}(P) = Z_1 + \mathrm{rint}(\hat{P}), \tag{4}$$

where $\hat{P} := \{z \in Z_2 : \ \langle z_i^*, z\rangle \leq s_i, \ i = 1, \cdots, m\}$.

The first equality in (4) is a slight variant of [24, Lemma 2.1] and can be proved similar to the proof of [24, Lemma 2.1], while the second equality in (4) is immediate from the following observation: the affine subspace $\mathrm{aff}(Z_1 + \hat{P})$ is equal to $Z_1 + \mathrm{aff}(\hat{P})$ (thanks to (3) and the definition of $\hat{P}$).

From Proposition 2.1, one can see that many properties on polyhedra established in the finite dimensional case also hold in the infinite dimensional one. In particular, the following corollaries are consequences of Proposition 2.1 and [16, Corollary 6.5.1].

Corollary 2.1. *Let $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\}$ and P be as in Proposition 2.1. Then* $\mathrm{rint}(P) = \{z \in Z : \ \langle u_i^*, z\rangle < s_i, \ i \in \overline{1n} \setminus \bar{I}_P\} \cap \bigcap_{i \in \bar{I}_P} F_i$*, where $\bar{I}_P := \{i \in \overline{1n} : \ \langle u_i^*, z\rangle = s_i$ for all $z \in P\}$ and $F_i := \{z \in Z : \ \langle u_i^*, z\rangle = s_i\}$.*

Corollary 2.2. *Let Z_1 and Z_2 be two closed subspaces of Z such that $Z = Z_1 \oplus Z_2$ and $\dim(Z_2) < \infty$. Let $\hat{P}$ be a polyhedron in Z_2 and $\hat{F}$ be a subset of $\hat{P}$. Then $\hat{F}$ is an exposed face of $\hat{P}$ if and only if $Z_1 + \hat{F}$ is an exposed face of the polyhedron $Z_1 + \hat{P}$ in Z.*

The following proposition is known and useful for us (cf. [24, Lemma 2.2]).

Proposition 2.2. *Let P_1 and P_2 be two polyhedra (resp. generalized polyhedra) in Z. Then P_1+P_2 and $P_1 \cap P_2$ are polyhedra (resp. generalized polyhedra).*

Note that a closed subspace of Z is not necessarily a polyhedron in Z. In fact, it is easy to verify that a closed subspace E of Z is a polyhedron in Z if and only if its codimension $\mathrm{codim}(E)$ is finite. Note that if E is a closed subspace of Z with $\mathrm{codim}(E) < +\infty$ and if H is a subspace of E then $E + H$ is a closed subspace of Z with $\mathrm{codim}(E + H) < +\infty$. The following proposition can be easily proved.

Proposition 2.3. *Let Z be a normed space, E be a closed subspace of Z with $\mathrm{codim}(E) < +\infty$, and let H be a subspace of Z. Then the following statements hold:*
(i) $E + H + \hat{P}$ is a polyhedron in Z for each polyhedron $\hat{P}$ in some finite dimensional subspace of Z.
(ii) $H + P$ is a polyhedron for each polyhedron P in Z.

The following lemma is useful in the proofs of some main results.

Lemma 2.1. *Let $C_1, \cdots, C_m$ be closed sets in a normed space Z such that $B(x_0, r_0) \subset \bigcup_{i=1}^{m} C_i$ for some $x_0 \in Z$ and $r_0 > 0$. Then there exists $i_0 \in \overline{1m}$ such that $B(x_0, r_0) \cap \mathrm{int}(C_{i_0}) \neq \emptyset$.*

Proof. By the assumption, $B(x_0, r_0) \setminus \bigcup_{i=1}^{m-1} C_i$ is open, and $B(x_0, r_0) \setminus \bigcup_{i=1}^{m-1} C_i \subset B(x_0, r_0) \cap \mathrm{int}(C_m)$. Hence either $B(x_0, r_0) \cap \mathrm{int}(C_m) \neq \emptyset$ or $B(x_0, r_0) \subset \bigcup_{i=1}^{m-1} C_i$, which implies clearly that the conclusion holds. The proof is complete. □

With the help of Lemma 2.1, we can prove the following interesting proposition.

Proposition 2.4. *Let C be a convex set in a normed space Z and let $F_1, \cdots, F_\nu$ be exposed faces of a polyhedron P in Z such that $C \subset \bigcup_{j=1}^{\nu} F_j$. Then there exists $j_0 \in \overline{1\nu}$ such that $C \subset F_{j_0}$.*

Proof. By Proposition 2.1, there exist two closed subspaces Z_1 and Z_2 of Z and a polyhedron $\hat{P}$ in Z_2 such that (3) and (4) hold. Hence, by Corollary 2.2, there exists an exposed face $\hat{F}_j$ of $\hat{P}$ such that $F_j = Z_1 + \hat{F}_j$ $(j \in \overline{1\nu})$, and so $C \subset \bigcup_{j=1}^{\nu}(Z_1 + \hat{F}_j)$. It follows from (2) and (3) that $\Pi_{Z_2}(C) \subset \bigcup_{j=1}^{\nu} \hat{F}_j$. Thus, it suffices to show that $\Pi_{Z_2}(C) \subset \hat{F}_{j_0}$ for some $j_0 \in \overline{1\nu}$. To prove this, take $(\hat{u}_j^*, \alpha_j) \in Z_2^* \times \mathbb{R}$ such that

$$\alpha_j = \sup_{x_2 \in \hat{P}} \langle \hat{u}_j^*, x_2 \rangle \quad \text{and} \quad \hat{F}_j = \{x_2 \in \hat{P} : \langle \hat{u}_j^*, x_2 \rangle = \alpha_j\} \quad \forall j \in \overline{1\nu}. \tag{5}$$

Since $\Pi_{Z_2}(C)$ is a convex set in the finite dimensional space Z_2, $\mathrm{rint}(\Pi_{Z_2}(C)) \neq \emptyset$. Take $\hat{x} \in \mathrm{rint}(\Pi_{Z_2}(C))$ and let $Z_3 := \mathrm{span}(\Pi_{Z_2}(C) - \hat{x})$. Then Z_3 is a subspace of Z_2 and there exists $\delta > 0$ such that $\hat{x} + B_{Z_3}(0, \delta) \subset \Pi_{Z_2}(C) \subset \bigcup_{j=1}^{\nu} \hat{F}_j$, that is, $B_{Z_3}(0, \delta) \subset \bigcup_{j=1}^{\nu}(\hat{F}_j - \hat{x}) \cap Z_3$. Hence, by Lemma 2.1, there exist $z_3 \in B_{Z_3}(0, \delta)$, $\varepsilon \in (0, +\infty)$ and $j_0 \in \overline{1\nu}$ such that $z_3 + B_{Z_3}(0, \varepsilon) \subset (\hat{F}_{j_0} - \hat{x}) \cap Z_3$. Letting

$\hat{u} := \hat{x} + z_3$, one has $\hat{u} + B_{Z_3}(0, \varepsilon) \subset \hat{F}_{j_0}$. This and (5) imply that $\langle \hat{u}^*_{j_0}, \hat{v} \rangle = 0$ for all $\hat{v} \in B_{Z_3}(0, \varepsilon)$ and so $\langle \hat{u}^*_{j_0}, \hat{v} \rangle = 0$ for all $\hat{v} \in Z_3$. Hence, $\Pi_{Z_2}(C) \subset \hat{x} + Z_3 = \hat{u} + Z_3 \subset \{x_2 \in Z_2 : \langle \hat{u}^*_{j_0}, x_2 \rangle = \alpha_{j_0}\}$. Noting that $\Pi_{Z_2}(C) \subset \bigcup_{j=1}^{\nu} \hat{F}_j \subset \hat{P}$, it follows that

$$\Pi_{Z_2}(C) \subset \hat{P} \cap \{x_2 \in Z_2 : \langle \hat{u}^*_{j_0}, x_2 \rangle = \alpha_{j_0}\} = \hat{F}_{j_0}.$$

□

We also need the following proposition.

Proposition 2.5. *Let P_i be polyhedra in a normed space Z such that* $\mathrm{int}(P_i) \neq \emptyset$ *($i = 1, \cdots, m$). Then there exist polyhedra Q_j in Z with* $\mathrm{int}(Q_j) \neq \emptyset$ *($j = 1, \cdots, \nu$) such that $\bigcup_{i=1}^{m} P_i = \bigcup_{j=1}^{\nu} Q_j$ and* $\mathrm{int}(Q_j) \cap Q_{j'} = \emptyset$ *for all $j, j' \in \overline{1\nu}$* with $j \neq j'$.

Proof. The conclusion holds clearly when $m = 1$. Given a natural number n, suppose that the conclusion holds when $m = n$. Let $P_1, \cdots, P_n, P_{n+1}$ be arbitrary $n+1$ polyhedra in Z such that each $\mathrm{int}(P_i)$ is nonempty. Then, by induction, it suffices to show that there exist polyhedra Q_j in Z with $\mathrm{int}(Q_j) \neq \emptyset$ ($j = 1, \cdots, \nu$) such that $\bigcup_{i=1}^{n+1} P_i = \bigcup_{j=1}^{\nu} Q_j$ and $\mathrm{int}(Q_j) \cap Q_{j'} = \emptyset$ for all $j, j' \in \overline{1\nu}$ with $j \neq j'$. To do this, take polyhedra $H_1, \cdots, H_l$ in Z such that

$$\bigcup_{i=1}^{n} P_i = \bigcup_{i=1}^{l} H_i,\ \mathrm{int}(H_i) \neq \emptyset \ \text{ and } \ H_i \cap \mathrm{int}(H_{i'}) = \emptyset \quad \forall i, i' \in \overline{1l} \text{ with } i \neq i'. \tag{6}$$

If $\mathrm{int}(P_{n+1}) \subset \bigcup_{i=1}^{l} H_i$, then $P_{n+1} \subset \bigcup_{i=1}^{l} H_i$ and so $\bigcup_{i=1}^{n+1} P_i = \bigcup_{i=1}^{l} H_i$; hence the conclusion is trivially true. Next suppose that $\mathrm{int}(P_{n+1}) \not\subset \bigcup_{i=1}^{l} H_i$. For each $i \in \overline{1l}$, take $x^*_{ij} \in Z^* \setminus \{0\}$ and $t_{ij} \in \mathbb{R}$ ($j = 1, \cdots, \kappa_i$) such that $H_i = \bigcap_{j=1}^{\kappa_i} \{x \in Z : \langle x^*_{ij}, x \rangle \leq t_{ij}\}$. Then

$$Z \setminus H_i = \bigcup_{j=1}^{\kappa_i} \{x \in Z : \langle x^*_{ij}, x \rangle > t_{ij}\} = \bigcup_{k=1}^{\kappa_i} \Lambda^i_k \cap \{x \in Z : \langle x^*_{ik}, x \rangle > t_{ik}\},$$

where $\Lambda^i_1 := Z$ and $\Lambda^i_k := \bigcap_{j=1}^{k-1} \{x \in Z : \langle x^*_{ij}, x \rangle \leq t_{ij}\}$ for $k = 2, \cdots, \kappa_i$. Since $\mathrm{int}(P_{n+1}) \setminus H_i = \mathrm{int}(P_{n+1}) \cap (Z \setminus H_i)$,

$$\mathrm{int}(P_{n+1}) \setminus H_i = \bigcup_{k=1}^{\kappa_i} \mathrm{int}(P_{n+1}) \cap \Lambda^i_k \cap \{x \in Z : \langle x^*_{ik}, x \rangle > t_{ik}\}. \tag{7}$$

For each $k \in \overline{1\kappa_i}$, let $Q^i_k := P_{n+1} \cap \Lambda^i_k \cap \{x \in Z : \langle x^*_{ik}, x \rangle \geq t_{ik}\}$. Then each Q^i_k is a polyhedron in Z and, by Corollary 2.1,

$$\mathrm{int}(Q^i_k) = \mathrm{int}(P_{n+1}) \cap \mathrm{int}(\Lambda^i_k) \cap \{z \in Z : \langle x^*_{ik}, z \rangle > t_{ik}\}. \tag{8}$$

Since $\mathrm{int}(\Lambda_k^i) = \{z \in Z : \langle x_{ij}^*, x\rangle < t_{ij},\ j = 1, \cdots, k-1\}$,

$$Q_k^i \cap \mathrm{int}(Q_{k'}^i) = \emptyset \quad \forall k, k' \in \overline{1\kappa_i} \text{ with } k \neq k'. \tag{9}$$

Let $\Gamma := \left\{(k_1, \cdots, k_l) \in \overline{1\kappa_1} \times \cdots \times \overline{1\kappa_l} : \bigcap_{i=1}^{l} \mathrm{int}(Q_{k_i}^i) \neq \emptyset\right\}$ and $Q_{(k_1,\cdots,k_l)} := \bigcap_{i=1}^{l} Q_{k_i}^i$ for all $(k_1, \cdots, k_l) \in \Gamma$. Then, each $Q_{(k_1,\cdots,k_l)}$ is a polyhedron in Z with $\mathrm{int}(Q_{(k_1,\cdots,k_l)}) = \bigcap_{i=1}^{l} \mathrm{int}(Q_{k_i}^i)$ (thanks to Corollary 2.1). Hence, by (9),

$$Q_{(k_1,\cdots,k_l)} \cap \mathrm{int}(Q_{(k_1',\cdots,k_l')}) = \emptyset \tag{10}$$

for all $(k_1, \cdots, k_l) \in \Gamma$ and all $(k_1', \cdots, k_l') \in \Gamma \setminus \{(k_1, \cdots, k_l)\}$. Let

$$\widetilde{Q}_k^i := \mathrm{int}(P_{n+1}) \cap \Lambda_k^i \cap \{x \in Z : \langle x_{ik}^*, x\rangle > t_{ik}\}.$$

Then $\mathrm{int}(P_{n+1}) \setminus H_i = \bigcup_{k=1}^{\kappa_i} \widetilde{Q}_k^i$ (by (7)) and $\mathrm{cl}(\widetilde{Q}_k^i) \subset Q_k^i$. For any $(k_1, \cdots, k_l) \in I_1 \times \cdots \times I_l$, it is easy from (8) to verify that $\bigcap_{i=1}^{l} \widetilde{Q}_{k_i}^i \neq \emptyset$ if and only if $\bigcap_{i=1}^{l} \mathrm{int}(Q_{k_i}^i) \neq \emptyset$. Noting that $P_{n+1} \setminus H_i \subset \mathrm{cl}(\mathrm{int}(P_{n+1}) \setminus H_i)$, by (7) and the definition of $Q_{(k_1,\cdots,k_l)}$, one has

$$\begin{aligned} P_{n+1} \setminus \bigcup_{i=1}^{l} H_i &= \bigcap_{i=1}^{l} (P_{n+1} \setminus H_i) \\ &\subset \bigcup_{(k_1,\cdots,k_l)\in\Gamma} \bigcap_{i=1}^{l} \mathrm{cl}(\widetilde{Q}_{k_i}^i) \subset \bigcup_{(k_1,\cdots,k_l)\in\Gamma} Q_{(k_1,\cdots,k_l)} \subset P_{n+1}. \end{aligned}$$

It follows from (6) that $\bigcup_{i=1}^{n+1} P_i = \left(\bigcup_{i=1}^{l} H_i\right) \cup \left(\bigcup_{(k_1,\cdots,k_l)\in\Gamma} Q_{(k_1,\cdots,k_l)}\right)$. By (10) and (6), this shows that the conclusion also holds when $m = n + 1$. □

For $(u_1^*, s_1), \cdots, (u_n^*, s_n) \in Z^* \times \mathbb{R}$ and $P = \{z \in Z : \langle u_i^*, z\rangle \leq s_i,\ i \in \overline{1n}\}$, we say that (u_i^*, s_i) is a redundant generator of P if $P = \{z \in Z : \langle u_j^*, z\rangle \leq s_j,\ j \in \overline{1n} \setminus \{i\}\}$ (cf. [8,19]). For convenience, we adopt the following notion.

Definition 2.1 *We say that $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\} \subset Z^* \times \mathbb{R}$ is a prime generator group of a polyhedron P in a normed space Z if*

$$P = \{z \in Z : \langle u_i^*, z\rangle \leq s_i,\ i \in \overline{1n}\} \tag{11}$$

and (u_i^*, s_i) is not a redundant generator of P for all $i \in \overline{1n}$.

Every polyhedron has a prime generator group (cf. [4,19]). It is clear that if $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\} \subset Z^* \times \mathbb{R}$ is a prime generator group of P then

$$P \neq \{z \in Z : \langle u_i^*, z\rangle \leq s_i,\ i \in \overline{1n} \setminus \{j\}\} \quad \forall j \in \overline{1n}. \tag{12}$$

In the remainder of this paper, we assume that every polyhedron P of Z is not equal to Z. So, it is clear that $u_i^* \neq 0$ for all $i \in \overline{1n}$ whenever $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\}$ is a prime generator group of P. Moreover, we have the following lemma.

Lemma 2.2. *Let* $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\}$ *be a prime generator group of a polyhedron* P *in a normed space* Z *and suppose that* $\mathrm{int}(P) \neq \emptyset$*. Then* $u_{j_1}^*$ *and* $u_{j_2}^*$ *are linearly independent whenever* $j_1 \in \overline{1n}$ *and* $j_2 \in \overline{1n} \setminus \{j_1\}$ *satisfy* $\langle u_{j_1}^*, \bar{x}\rangle = s_{j_1}$ *and* $\langle u_{j_2}^*, \bar{x}\rangle = s_{j_2}$ *for some* $\bar{x} \in P$*.*

Proof. Suppose to the contrary that there exist $j_1 \in \overline{1n}$, $j_2 \in \overline{1n} \setminus \{j_1\}$, $\bar{x} \in P$ and $\alpha \in \mathbb{R} \setminus \{0\}$ such that $\langle u_{j_1}^*, \bar{x}\rangle = s_{j_1}$, $\langle u_{j_2}^*, \bar{x}\rangle = s_{j_2}$ and $u_{j_2}^* = \alpha u_{j_1}^*$. Take $x_0 \in \mathrm{int}(P)$ and $r > 0$ such that $B(x_0, r) \subset P$. Then $\langle u_{j_1}^*, x\rangle \leq s_{j_1}$ and $\langle u_{j_2}^*, x\rangle \leq s_{j_2}$ for all $x \in B(x_0, r)$. It follows that $\alpha > 0$ and

$$\{x \in Z : \ \langle u_{j_1}^*, x\rangle \leq s_{j_1}\} = \{x \in Z : \ \langle u_{j_2}^*, x\rangle \leq s_{j_2}\}.$$

Thus, $P = \{x \in Z : \ \langle u_i^*, x\rangle \leq s_i, \ i \in \overline{1n} \setminus \{j_1\}\}$, contradicting (12). □

Lemma 2.3. *Let* $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\}$ *be a prime generator group of a polyhedron* P *in a normed space* Z*. Then, for each* $j \in \overline{1n}$*,*

$$F_j(P) := P \cap \{x \in Z : \ \langle u_j^*, x\rangle = s_j\} \neq \emptyset. \tag{13}$$

Lemma 2.3 is immediate from Definition 2.1. The following two lemmas will be quite useful in the proof of our main result.

Lemma 2.4. *Let* $\{(u_1^*, s_1), \cdots, (u_n^*, s_n)\}$ *be a prime generator group of a polyhedron* P *in a normed space* Z*. Let* $F_j(P)$ *be as in (13) and*

$$F_j^\circ(P) := \{z \in Z : \ \langle u_j^*, z\rangle = s_j \text{ and } \langle u_i^*, z\rangle < s_i, \ i \in \overline{1n} \setminus \{j\}\} \tag{14}$$

for all $j \in \overline{1n}$*. Then the following statements are equivalent:*
(i) $\mathrm{int}(P) \neq \emptyset$*.*
(ii) $F_j(P) = \mathrm{cl}(F_j^\circ(P))$ *for all* $j \in \overline{1n}$*.*
(iii) $F_j^\circ(P) \neq \emptyset$ *for all* $j \in \overline{1n}$*.*
(iv) $F_{j_0}^\circ(P) \neq \emptyset$ *for some* $j_0 \in \overline{1n}$*.*

Proof. First suppose that (i) holds. Then, by Corollary 2.1, there exists $x_0 \in Z$ such that $\langle u_i^*, x_0\rangle < s_i$ for all $i \in \overline{1n}$. For each $j \in \overline{1n}$, by (12), there exists $v \in Z$ such that $\langle u_j^*, v\rangle > s_j$ and $\langle u_i^*, v\rangle \leq s_i$ for all $i \in \overline{1n} \setminus \{j\}$. It follows that there exists $\lambda_0 \in (0, \ 1)$ such that

$$\langle u_j^*, \lambda_0 x_0 + (1-\lambda_0)v\rangle = s_j \ \text{ and } \ \langle u_i^*, \lambda_0 x_0 + (1-\lambda_0)v\rangle\rangle < s_i \quad \forall i \in \overline{1n} \setminus \{j\}.$$

Therefore, $\frac{kx}{1+k} + \frac{\lambda_0 x_0 + (1-\lambda_0)v}{k+1} \in F_j^\circ(P)$ for all $(x, k) \in F_j(P) \times \mathbb{N}$. Letting $k \to \infty$, it follows that $x \in \mathrm{cl}(F_j^\circ(P))$ for all $x \in F_j(P)$, that is, $F_j(P) \subset \mathrm{cl}(F_j^\circ(P))$. Since the converse inclusion holds trivially, this shows implication (i)⇒(ii). Since (ii)⇒(iii) is immediate from Lemma 2.3 and (iii)⇒(iv) is trivial, it suffices to show (iv)⇒(i). To prove this, let $\bar{x} \in F_{j_0}^\circ(P)$, that is, $\langle u_{j_0}^*, \bar{x}\rangle = s_{j_0}$ and $\langle u_i^*, \bar{x}\rangle < s_i$ for all $i \in \overline{1n} \setminus \{j_0\}$. Taking $h \in Z$ with $\langle u_{j_0}^*, h\rangle < 0$ (thanks to $u_{j_0}^* \neq 0$), it follows that there exists $t > 0$ sufficiently small such that $\langle u_k^*, \bar{x} + th\rangle < s_k$ for all $k \in \overline{1n}$. This shows that $\bar{x} + th \in \mathrm{int}(P)$, and hence (iv)⇒(i) holds. The proof is complete. □

Lemma 2.5. *Let P_1 and P_2 be two polyhedra in a normed space Z such that* $\mathrm{int}(P_1) \cap P_2 = \emptyset$, *and let $\{(u_{ij}^*, s_{ij}) : \ j = 1, \cdots, n_i\} \subset Z^* \times \mathbb{R}$ be a prime generator group of P_i $(i = 1, 2)$. Then for any $(j_1, j_2) \in \overline{1n_1} \times \overline{1n_2}$ and $x_0 \in F_{j_1}^\circ(P_1) \cap F_{j_2}^\circ(P_2)$ there exists $r > 0$ such that $\mathcal{N}(u_{1j_1}^*) = \mathcal{N}(u_{2j_2}^*)$ and*

$$F_{j_1}^\circ(P_1) \cap B_Z(x_0, r) = F_{j_2}^\circ(P_2) \cap B_Z(x_0, r) = (x_0 + \mathcal{N}(u_{1j_1}^*)) \cap B_Z(x_0, r), \tag{15}$$

where $B_Z(x_0, r) := \{x \in Z : \ \|x - x_0\| < r\}$ and $F_{j_1}^\circ(P_1)$ is as in (14).

Proof. Let $(j_1, j_2) \in \overline{1n_1} \times \overline{1n_2}$ and $x_0 \in F_{j_1}^\circ(P_1) \cap F_{j_2}^\circ(P_2)$. Then $x_0 \in P_1 \cap P_2$. Since $\mathrm{int}(P_1) \cap P_2 = \emptyset$, the separation theorem implies that there exists $v^* \in Z^* \setminus \{0\}$ such that $\langle v^*, x_0 \rangle = \inf\limits_{x \in P_1} \langle v^*, x \rangle = \sup\limits_{x \in P_2} \langle v^*, x \rangle$. Noting that

$$F_{j_1}^\circ(P_1) \cap B_Z(x_0, r) = (x_0 + \mathcal{N}(u_{1j_1}^*)) \cap B_Z(x_0, r) \subset P_1 \tag{16}$$

and

$$F_{j_2}^\circ(P_2) \cap B_Z(x_0, r) = (x_0 + \mathcal{N}(u_{2j_2}^*)) \cap B_Z(x_0, r) \subset P_2 \tag{17}$$

for some $r > 0$ (thanks to the definitions of $F_{j_1}^\circ(P_1)$ and $F_{j_2}^\circ(P_2)$), it follows that $\langle v^*, x_0 \rangle = \inf\limits_{x \in (x_0 + \mathcal{N}(u_{1j_1}^*)) \cap B_Z(x_0, r)} \langle v^*, x \rangle = \sup\limits_{x \in (x_0 + \mathcal{N}(u_{2j_2}^*)) \cap B_Z(x_0, r)} \langle v^*, x \rangle$. Hence $\inf\limits_{x \in \mathcal{N}(u_{1j_1}^*) \cap B_Z(0, r)} \langle v^*, x \rangle = \sup\limits_{x \in \mathcal{N}(u_{2j_2}^*) \cap B_Z(0, r)} \langle v^*, x \rangle = 0$, and so

$$\mathcal{N}(v^*) = \mathcal{N}(u_{1j_1}^*) = \mathcal{N}(u_{2j_2}^*)$$

because v^* is linear and both $\mathcal{N}(u_{1j_1}^*)$ and $\mathcal{N}(u_{2j_2}^*)$ are maximal linear subspaces of Z. This, together with (16) and (17), implies that (15) holds. □

3 Piecewise Linear Vector-Valued Functions

In this section, we will distinguish $\mathcal{PL}_1(X, Y)$ and $\mathcal{PL}(X, Y)$ and consider the structure of a piecewise linear function.

Proposition 3.1. *Let X and Y be normed spaces. Then the following statements hold.*
(i) The space $\mathcal{L}(X, Y)$ of all continuous linear operators between X and Y is always contained in $\mathcal{PL}(X, Y)$.
(ii) $\mathcal{PL}_1(X, Y) \neq \emptyset$ if and only if $\dim(Y) < \infty$. *Consequently, if Y is infinite dimensional, then every linear function from X to Y is not in $\mathcal{PL}_1(X, Y)$.*
(iii) $\mathcal{PL}_1(X, Y) = \mathcal{PL}(X, Y)$ when $\dim(Y) < \infty$.

Proof. Since (i) is trivial and the sufficiency part of (ii) is a straightforward consequence of (i) and (iii), it suffices to show (iii) and the necessity part of (ii). First suppose that $\mathcal{PL}_1(X, Y) \neq \emptyset$, and let f be an element in $\mathcal{PL}_1(X, Y)$. Then there exist finitely many polyhedra $\Lambda_1, \cdots, \Lambda_k$ in the product $X \times Y$ such that

$$\mathrm{gph}(f) = \bigcup_{i=1}^{k} \Lambda_i \ \text{ and } \ X = \bigcup_{i=1}^{k} \Lambda_i|_X, \tag{18}$$

where $\Lambda_i|_X := \{x \in X : (x, y) \in \Lambda_i \text{ for some } y \in Y\}$ is the projection of Λ_i into X. Given an $i \in \overline{1k}$, by Proposition 2.1, there exist two closed subspaces $X_i, \tilde{X}_i$ of X and two closed subspaces $Y_i, \tilde{Y}_i$ of Y such that

$$X \times Y = (X_i \times Y_i) \oplus (\tilde{X}_i \times \tilde{Y}_i), \quad \mathrm{codim}(X_i \times Y_i) = \dim(\tilde{X}_i \times \tilde{Y}_i) < \infty, \tag{19}$$

$$\Lambda_i = X_i \times Y_i + \tilde{\Lambda}_i, \tag{20}$$

where $\tilde{\Lambda}_i$ is a polyhedron in $\tilde{X}_i \times \tilde{Y}_i$. Since f is a single-valued function, it follows from (18) that $Y_i = \{0\}$ and $\tilde{Y}_i = Y$. Hence Y is finite dimensional, and the necessity part of (ii) is proved. Next we prove $f \in \mathcal{PL}(X, Y)$. To prove this, we only need to show that there exist $T_i \in \mathcal{L}(X, Y)$ and $b_i \in Y$ such that

$$f(x) = T_i(x) + b_i \quad \forall x \in \Lambda_i|_X. \tag{21}$$

Since every convex set in a finite-dimensional space has a nonempty relative interior, $\mathrm{rint}(\tilde{\Lambda}_i) \neq \emptyset$. Take a point $(\tilde{a}_i, \tilde{b}_i)$ in $\mathrm{rint}(\tilde{\Lambda}_i)$. Thus, $\tilde{a}_i \in \mathrm{rint}(\tilde{\Lambda}_i|_{\tilde{X}_i})$, and $E_i := \mathbb{R}_+(\tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i)$ and $Z_i := \mathbb{R}_+(\tilde{\Lambda}_i - (\tilde{a}_i, \tilde{b}_i))$ are linear subspaces of $\tilde{X}_i$ and $\tilde{X}_i \times \tilde{Y}_i$, respectively. Noting that $\tilde{\Lambda}_i \subset \mathrm{gph}(f)$, define $\hat{T}_i : \tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i \to \tilde{Y}_i$ such that

$$\hat{T}_i(u_i) := f(u_i + \tilde{a}_i) - f(\tilde{a}_i) = f(u_i + \tilde{a}_i) - \tilde{b}_i \quad \forall u_i \in \tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i. \tag{22}$$

Then $\mathrm{gph}(\hat{T}_i) = \tilde{\Lambda}_i - (\tilde{a}_i, \tilde{b}_i)$. Let $\tilde{T}_i : E_i \to \tilde{Y}_i$ be such that

$$\tilde{T}_i(tu_i) := t\hat{T}_i(u_i) \quad \forall (t, u_i) \in \mathbb{R}_+ \times (\tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i).$$

It is easy to verify that $\tilde{T}_i$ is well-defined and its graph is just the linear subspace $Z_i = \mathbb{R}_+(\tilde{\Lambda}_i - (\tilde{a}_i, \tilde{b}_i))$, and so $\tilde{T}_i$ is linear. Hence there exist $e_j \in Y$ and $e^*_{ij} \in E^*_i$ $(j = 1, \cdots, p)$ such that $e_1, \cdots, e_p$ are linearly independent and $\tilde{T}_i(x) = \sum_{j=1}^{p} \langle e^*_{ij}, x\rangle e_j$ for all $x \in E_i$. For each $j \in \overline{1p}$, let $\tilde{e}^*_{ij} : X_i + E_i \to \mathbb{R}$ be such that $\langle \tilde{e}^*_{ij}, u + v\rangle = \langle e^*_{ij}, v\rangle$ for all $(u, v) \in X_i \times E_i$. Then, by (19) and $E_i \subset \tilde{X}_i$, $\tilde{e}^*_{ij}$ is a linear functional on $X_i + E_i$, and its null space

$$\mathcal{N}(\tilde{e}^*_{ij}) := \{x \in X_i + E_i : \langle \tilde{e}^*_{ij}, x\rangle = 0\} = X_i + \{v \in E_i : \langle e^*_{ij}, v\rangle = 0\}.$$

Since X_i is a closed subspace of X and $\dim(E_i) < \infty$, it follows that $\mathcal{N}(\tilde{e}^*_{ij})$ is a closed subspace of X. Hence $\tilde{e}^*_{ij}$ is a continuous linear functional on $X_i + E_i$ (thanks to [17, Theorem 1.18]). By the Hahn-Banach theorem, there exists $x^*_{ij} \in X^*$ such that $x^*_{ij}|_{X_i + E_i} = \tilde{e}^*_{ij}$. Let $T_i(x) := \sum_{j=1}^{p} \langle x^*_{ij}, x\rangle e_j$ for all $x \in X$. Then $T_i \in \mathcal{L}(X, Y)$,

$$\mathcal{N}(T_i) \supset \bigcap_{j=1}^{p} \mathcal{N}(x^*_{ij}) \supset X_i \quad \text{and} \quad T_i|_{\tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i} = \tilde{T}_i|_{\tilde{\Lambda}_i|_{\tilde{X}_i} - \tilde{a}_i} = \hat{T}_i. \tag{23}$$

Let x be an arbitrary element in $\Lambda_i|_X$ and take $y \in Y$ such that $(x, y) \in \Lambda_i$. Then, by (19) and (20), there exist $x_i \in X_i$ and $\tilde{x}_i \in \tilde{\Lambda}_i|_{\tilde{X}_i}$ such that $(\tilde{x}_i, y) \in \tilde{\Lambda}_i$ and $(x, y) = (x_i + \tilde{x}_i, y)$ (because $Y_i = \{0\}$). Hence, by (22) and (23), one has

$$f(x) = f(\tilde{x}_i) = y = \hat{T}_i(\tilde{x}_i - \tilde{a}_i) + \tilde{b}_i = T_i(\tilde{x}_i - \tilde{a}_i) + \tilde{b}_i = T_i(x) - T_i(\tilde{a}_i) + \tilde{b}_i.$$

This shows that (21) holds with $b_i = -T_i(\tilde{a}_i) + \tilde{b}_i$ and so $f \in \mathcal{PL}(X,Y)$. Therefore, $\mathcal{PL}_1(X,Y) \subset \mathcal{PL}(X,Y)$.

Now suppose that $\dim(Y) < \infty$. To prove the converse inclusion $\mathcal{PL}_1(X,Y) \supset \mathcal{PL}(X,Y)$, let $f \in \mathcal{PL}(X,Y)$. Then there exist $P_i \in \mathcal{P}(X)$, $T_i \in \mathcal{L}(X,Y)$ and $b_i \in Y$ $(i = 1, \cdots, m)$ such that

$$X = \bigcup_{i=1}^{m} P_i \ \text{ and } \ f(x) = T_i(x) + b_i \quad \forall x \in P_i \text{ and } \forall i \in \overline{1m}. \tag{24}$$

By $\dim(Y) < \infty$, there exist $y_1^*, \cdots, y_q^* \in Y^*$ such that $Y^* = \operatorname{span}\{y_1^*, \cdots, y_q^*\}$. For any $x \in X$, since

$$\begin{aligned} T_i(x) = y \quad &\Longleftrightarrow \quad [\langle y^*, T_i(x)\rangle = \langle y^*, y\rangle \ \ \forall y^* \in Y^*] \\ &\Longleftrightarrow \quad [\langle y_j^*, T_i(x)\rangle = \langle y_j^*, y\rangle, \ j = 1, \cdots, q], \end{aligned}$$

$$T_i(x) = y \Longleftrightarrow [\langle T_i^*(y_j^*), x\rangle = \langle y_j^*, y\rangle, \ j = 1, \cdots, q].$$

Hence $\operatorname{gph}(T_i) = \{(x,y) \in X \times Y : \ \langle T_i^*(y_j^*), x\rangle - \langle y_j^*, y\rangle = 0, \ j = 1, \cdots, q\}$, and so $\operatorname{gph}(T_i)$ is a polyhedron of $X \times Y$. Noting (by (24)) that

$$\operatorname{gph}(f) = \bigcup_{i=1}^{m} (\operatorname{gph}(T_i) + (0, b_i)) \cap (P_i \times Y),$$

it follows that $\operatorname{gph}(f)$ is the union of finitely many polyhedra in $X \times Y$. Therefore, $f \in \mathcal{PL}_1(X,Y)$. The proof of (iii) is complete. □

Given $f \in \mathcal{PL}(X,Y)$, there exist $(P_1, T_1, b_1), \cdots, (P_m, T_m, b_m)$ in the product $\mathcal{P}(X) \times \mathcal{L}(X,Y) \times Y$ such that (24) holds. For $i \in \overline{1m}$, since each polyhedron is closed, the first equality of (24) implies that $\operatorname{int}(P_i) \supset X \setminus \bigcup_{j \in \overline{1m} \setminus \{i\}} P_j$ and so $X = \bigcup_{j \in \overline{1m} \setminus \{i\}} P_j$ whenever $\operatorname{int}(P_i) = \emptyset$. Hence, without loss of generality, we can assume that each P_i in (24) has a nonempty interior. Moreover, we assume without loss of generality that there exists $k \in \overline{1m}$ satisfying the following property:

$$(T_i, b_i) \neq (T_{i'}, b_{i'}) \quad \forall i, i' \in \overline{1k} \text{ with } i \neq i' \tag{25}$$

and for each $j \in \overline{1m}$ there exists $i \in \overline{1k}$ such that $(T_j, b_j) = (T_i, b_i)$. For each $i \in \overline{1k}$, let

$$I_i := \{j \in \overline{1m} : (T_j, b_j) = (T_i, b_i)\} \ \text{ and } \ Q_i := \bigcup_{j \in I_i} P_j. \tag{26}$$

Then $X = \bigcup_{i \in \overline{1k}} Q_i$, $X \neq \bigcup_{i \in \overline{1k}, i \neq j} Q_i$ and $f|_{Q_j} = T_j|_{Q_j} + b_j$ for all $j \in \overline{1k}$. We claim that

$$\operatorname{int}(Q_i) \cap \operatorname{int}(Q_{i'}) = \emptyset \quad \forall i, i' \in \overline{1k} \text{ with } i \neq i'. \tag{27}$$

Indeed, if this is not the case, there exist $i, i' \in \overline{1k}$ with $i \neq i'$, $x \in X$ and $r > 0$ such that $B(x,r) \subset Q_i \cap Q_{i'}$, and so $f(x) = T_i(u) + b_i = T_{i'}(u) + b_{i'}$ for all $u \in B(x,r)$. Since T_i and $T_{i'}$ are linear, it follows that $(T_i, b_i) = (T_{i'}, b_{i'})$, contradicting (25). Hence (27) holds. Since each Q_i is closed, (27) can be rewritten as

$$Q_i \cap \operatorname{int}(Q_{i'}) = \emptyset \quad \forall i, i' \in \overline{1k} \text{ with } i \neq i'.$$

Therefore, by Proposition 2.5, we have the following result.

Proposition 3.2. *For each $f \in \mathcal{PL}(X,Y)$ there exist $(P_i, T_i, b_i) \in \mathcal{P}(X)\times\mathcal{L}(X,Y)\times Y$ ($i = 1, \cdots, m$) such that*

$$X = \bigcup_{i=1}^{m} P_i, \ \operatorname{int}(P_i) \neq \emptyset, \ P_i \cap \operatorname{int}(P_j) = \emptyset \ \ \forall i, j \in \overline{1m} \text{ with } i \neq j, \tag{28}$$

and

$$f|_{P_i} = T_i|_{P_i} + b_i \quad \forall i \in \overline{1m}, \tag{29}$$

that is, $f(x) = T_i x + b_i$ for all $i \in \overline{1m}$ and all $x \in P_i$.

Let $\{\Lambda_1, \cdots, \Lambda_m, \Lambda'_1, \cdots, \Lambda'_{m'}\} \subset \mathcal{P}(X)$ be such that $X = \bigcup_{i=1}^{m} \Lambda_i = \bigcup_{j=1}^{m'} \Lambda'_j$. Then $X = \bigcup_{(i,j)\in\overline{1m}\times\overline{1m'}} \Lambda_i \cap \Lambda'_j$. Setting $I = \{(i,j) \in \overline{1m}\times\overline{1m'} : \ \operatorname{int}(\Lambda_i \cap \Lambda'_j) \neq \emptyset\}$, one has $X = \bigcup_{(i,j)\in I} \Lambda_i \cap \Lambda'_j$. This yields the following proposition.

Proposition 3.3. *For any $f, f' \in \mathcal{PL}(X,Y)$ there exist $(P_i, T_i, b_i), (P'_i, T'_i, b'_i) \in \mathcal{P}(X)\times\mathcal{L}(X,Y)\times Y$ ($i = 1, \cdots, m$) such that (28) holds and*

$$f|_{P_i} = T_i|_{P_i} + b_i \ \text{ and } \ f'|_{P_i} = T'_i|_{P_i} + b'_i \quad \forall i \in \overline{1m}. \tag{30}$$

Now we are ready to establish the main result in this section.

Theorem 3.1. *Let $(P_1, T_1, b_1), \cdots, (P_m, T_m, b_m) \in \mathcal{P}(X)\times\mathcal{L}(X,Y)\times Y$ and $f \in \mathcal{PL}(X,Y)$ be such that (28) and (29) hold. Then $X_f := \operatorname{lin}\left(\bigcap_{i=1}^{m} P_i\right)$ is a closed subspace of X with $\operatorname{codim}\left(X_f\right) < \infty$ and*

$$T_1(x) = \cdots = T_m(x) \quad \forall x \in X_f.$$

Proof. For each $i \in \overline{1m}$, take a prime generator group $\{(x^*_{ij}, t_{ij}) : \ j \in \overline{1\nu_i}\}$ of P_i. Then,

$$P_i = \{x \in X : \ \langle x^*_{ij}, x\rangle \leq t_{ij}, \ j \in \overline{1\nu_i}\} \tag{31}$$

and

$$P_i \neq \{x \in X : \ \langle x^*_{ij}, x\rangle \leq t_{ij}, \ j \in \overline{1\nu_i} \setminus \{j'\}\} \quad \forall j' \in \overline{1\nu_i}. \tag{32}$$

Hence, $X_1 := X_f = \bigcap_{i\in\overline{1m}} \bigcap_{j\in\overline{1\nu_i}} \mathcal{N}(x^*_{ij})$ is a closed subspace of X with $\operatorname{codim}(X_1) \leq \sum_{i=1}^{m} \nu_i$, and so there exists a closed subspace X_2 of X such that

$$X = X_1 \oplus X_2 \ \text{ and } \ \operatorname{codim}(X_1) = \dim(X_2) < \infty. \tag{33}$$

Let

$$\hat{P}_i := \{x \in X_2 : \ \langle x^*_{ij}, x\rangle \leq t_{ij}, \quad j \in \overline{1\nu_i}\}. \tag{34}$$

Then, by (31) and Proposition 2.1,

$$P_i = X_1 + \hat{P}_i \quad \text{and} \quad \operatorname{int}(P_i) = X_1 + \operatorname{int}_{X_2}(\hat{P}_i). \tag{35}$$

It follows from (28) and (33) that

$$\mathrm{int}_{X_2}(\hat{P}_i) \neq \emptyset \ \text{ and } \ \hat{P}_i \cap \mathrm{int}_{X_2}(\hat{P}_{i'}) = \emptyset \ \ \forall i, i' \in \overline{1m} \text{ with } i \neq i'. \tag{36}$$

Fix two arbitrary distinct indices i and i' in $\overline{1m}$. We only need to prove $T_i|_{X_1} = T_{i'}|_{X_1}$. To prove this, let

$$F_j^\circ(P_i) := \left\{x \in X : \ \langle x_{ij}^*, x\rangle = t_{ij} \text{ and } \langle x_{il}^*, x\rangle < t_{il} \text{ for all } l \in \overline{1\nu_i} \setminus \{j\}\right\}$$

and

$$F_j^\circ(\hat{P}_i) := \{x \in X_2 : \ \langle x_{ij}^*, x\rangle = t_{ij} \text{ and } \langle x_{il}^*, x\rangle < t_{il} \text{ for all } l \in \overline{1\nu_i} \setminus \{j\}\}. \tag{37}$$

Then, by the definition of X_1 and (33), $F_j^\circ(P_i) = X_1 + F_j^\circ(\hat{P}_i)$ and $F_j^\circ(\hat{P}_i) \neq \emptyset$ (thanks to Lemma 2.4). Take $(\bar{u}, \bar{u}') \in \mathrm{int}_{X_2}(\hat{P}_i) \times \mathrm{int}_{X_2}(\hat{P}_{i'})$ and $u^* \in X_2^* \setminus \{0\}$ such that $\langle u^*, \bar{u}' - \bar{u}\rangle \neq 0$. Then there exists $\delta > 0$ such that

$$\bar{u} + B_{X_3}(0,\delta) \subset \mathrm{int}_{X_2}(\hat{P}_i) \ \text{ and } \ \bar{u}' + B_{X_3}(0,\delta) \subset \mathrm{int}_{X_2}(\hat{P}_{i'}), \tag{38}$$

where $X_3 := \mathcal{N}(u^*) = \{x \in X_2 : \ \langle u^*, x\rangle = 0\}$. Hence

$$\dim(X_3) = \dim(X_2) - 1, \ \ X_2 = X_3 \oplus \mathbb{R}(\bar{u}' - \bar{u}) \tag{39}$$

and

$$\mathrm{int}_{X_2}\big([\bar{u}, \ \bar{u}'] + B_{X_3}(0,\delta)\big) = (\bar{u}, \ \bar{u}) + B_{X_3}(0,\delta) \neq \emptyset, \tag{40}$$

where $[\bar{u}, \ \bar{u}'] := \{\bar{u} + t(\bar{u}' - \bar{u}) : \ 0 \leq t \leq 1\}$ and $(\bar{u}, \ \bar{u}') := \{\bar{u} + t(\bar{u}' - \bar{u}) : \ 0 < t < 1\}$. For each $z \in B_{X_3}(0,\delta)$, let

$$I_z := \{i \in \overline{1m} : \ \hat{P}_i \cap (z + [\bar{u}, \ \bar{u}']) \text{ contains at least two points}\}$$

and $I_z^\circ := \{i \in \overline{1m} : \ \mathrm{int}_{X_2}(\hat{P}_i) \cap (z + [\bar{u}, \ \bar{u}']) \neq \emptyset\}$. Then $I_z^\circ \subset I_z$, and $\hat{P}_i \cap (z + [\bar{u}, \ \bar{u}'])$ contains at most an element for all $i \in \overline{1m} \setminus I_z$. Noting that $X_2 = \bigcup\limits_{i \in \overline{1m}} \hat{P}_i$ (thanks to (33)—(35) and (28)), it follows that

$$z + [\bar{u}, \ \bar{u}'] = \bigcup_{i \in I_z} \hat{P}_i \cap (z + [\bar{u}, \ \bar{u}']) \quad \forall z \in B_{X_3}(0,\delta). \tag{41}$$

Regarding X_2 as the Euclidean space $\mathbb{R}^{\dim(X_2)}$ (without loss of generality), let μ_{X_2} and μ_{X_3} denote the Lebesgue measures on X_2 and X_3, respectively. Setting $E_0 := \{z \in B_{X_3}(0,\delta) : \ I_z^\circ \neq I_z\}$, we claim that $\mu_{X_3}(E_0) = 0$. To prove this, let z be an arbitrary element in E_0. Then there exists $i_z \in I_z$ such that $i_z \notin I_z^\circ$, and so $\hat{P}_{i_z} \cap (z + [\bar{u}, \ \bar{u}']) \subset \hat{P}_{i_z} \setminus \mathrm{int}_{X_2}(\hat{P}_{i_z})$. Since $\hat{P}_{i_z} \setminus \mathrm{int}_{X_2}(\hat{P}_{i_z})$ is the union of finitely many faces of $\hat{P}_{i_z}$, it follows from Proposition 2.4 that there exists a face of $\hat{P}_{i_z}$ containing the convex set $\hat{P}_{i_z} \cap (z + [\bar{u}, \ \bar{u}'])$, that is, there exists $v^* \in X_2^* \setminus \{0\}$ such that

$$\hat{P}_{i_z} \cap (z + [\bar{u}, \ \bar{u}']) \subset \{x_2 \in \hat{P}_{i_z} : \ \langle v^*, z_2\rangle = \sup_{x \in \hat{P}_{i_z}} \langle v^*, x\rangle\} = \hat{P}_{i_z} \cap (v + \mathcal{N}(v^*))$$

for some $v \in X_2$ with $\langle v^*, v\rangle = \sup\limits_{x \in \hat{P}_{i_z}} \langle v^*, x\rangle$. Since $\hat{P}_{i_z} \cap (z + [\bar{u}, \ \bar{u}'])$ is a segment containing at least two points (thanks to the definition of I_z) and v^* is linear, the entire segment $z + [\bar{u}, \ \bar{u}']$ is contained in the hyperplane $v + \mathcal{N}(v^*)$. Noting that each $\hat{P}_i$ (as a polyhedron in X_2) has finitely many faces, it follows that there exist $v_1^*, \cdots, v_q^* \in X_2^* \setminus \{0\}$ and $v_1, \cdots, v_q \in X_2$ such that $z + [\bar{u},\ \bar{u}'] \subset \bigcup_{k=1}^{q} (v_k + \mathcal{N}(v_k^*))$ for all $z \in E_0$, that is, $E_0 + [\bar{u},\ \bar{u}'] \subset \bigcup_{k=1}^{q} (v_k + \mathcal{N}(v_k^*))$. Since each $\mathcal{N}(v_k^*)$ is of dimension $\dim(X_2) - 1$, $\mu_{X_2}(E_0 + [\bar{u},\ \bar{u}']) \le \mu_{X_2}\left(\bigcup_{k=1}^{q} (v_k + \mathcal{N}(v_k^*))\right) \le \sum_{k=1}^{q} \mu_{X_2}(v_k + \mathcal{N}(v_k^*)) = 0$. Thus, by (39) and the Fubini Theorem on measure, one has $\mu_{X_3}(E_0) = 0$. Next, let $z \in B_{X_3}(0, \delta) \setminus E_0$. Then $I_z = I_z^\circ$. Thus, by (41) and the definition of I_z°,

$$z + [\bar{u},\ \bar{u}'] = \bigcup_{\kappa \in I_z^\circ} \hat{P}_\kappa \cap (z + [\bar{u},\ \bar{u}']) \ \text{ and } \ \mathrm{int}_{X_2}(\hat{P}_\kappa) \cap (z + [\bar{u},\ \bar{u}']) \neq \emptyset \quad \forall \kappa \in I_z^\circ.$$

Hence, by (36), $\mathrm{int}_{X_2}(\hat{P}_\kappa) \cap (z + [\bar{u},\ \bar{u}'])$ and $\mathrm{int}_{X_2}(\hat{P}_{\kappa'}) \cap (z + [\bar{u},\ \bar{u}'])$ are two disjoint open segments in $z + [\bar{u},\ \bar{u}']$ for any $\kappa, \kappa' \in I_z^\circ$ with $\kappa \neq \kappa'$. Noting that $z + \bar{u} \in \mathrm{int}_{X_2}(\hat{P}_i)$ and $z + \bar{u}' \in \mathrm{int}_{X_2}(\hat{P}_{i'})$ (by (38)), it follows that there exist $\iota_0^z, \iota_1^z, \cdots, \iota_{\gamma_z}^z \in \overline{1m}$ and $\lambda_0^z, \lambda_1^z, \cdots, \lambda_{\gamma_z}^z \in [0,\ 1)$ such that

$$I_z = I_z^\circ = \{\iota_0^z, \iota_1^z, \cdots, \iota_{\gamma_z}^z\},\ \iota_0^z = i,\ \iota_{\gamma_z}^z = i',\ \lambda_0^z = 0,\ \lambda_{k-1}^z < \lambda_k^z, \tag{42}$$

$$z + \bar{u} + [0,\ \lambda_1^z)(\bar{u}' - \bar{u}) = (z + [\bar{u},\ \bar{u}']) \cap \mathrm{int}_{X_2}(\hat{P}_i),$$
$$z + \bar{u} + (\lambda_{\gamma_z}^z,\ 1](\bar{u}' - \bar{u}) = (z + [\bar{u},\ \bar{u}']) \cap \mathrm{int}_{X_2}(\hat{P}_{i'}),$$
$$z + \bar{u} + [\lambda_{k-1}^z,\ \lambda_k^z](\bar{u}' - \bar{u}) = (z + [\bar{u},\ \bar{u}']) \cap \hat{P}_{\iota_{k-1}^z}$$

and

$$z + \bar{u} + (\lambda_{k-1}^z,\ \lambda_k^z)(\bar{u}' - \bar{u}) = (z + [\bar{u},\ \bar{u}']) \cap \mathrm{int}_{X_2}(\hat{P}_{\iota_{k-1}^z})$$

for all $k \in \overline{1\gamma_z}$. Therefore

$$z + \bar{u} + \lambda_k^z(\bar{u}' - \bar{u}) \in \hat{P}_{\iota_{k-1}^z} \cap \hat{P}_{\iota_k^z} \quad \forall k \in \overline{1\gamma_z}. \tag{43}$$

This and (36) imply that $z + \bar{u} + \lambda_k^z(\bar{u}' - \bar{u}) \notin \mathrm{int}_{X_2}(\hat{P}_{\iota_{k-1}^z}) \cup \mathrm{int}_{X_2}(\hat{P}_{\iota_k^z})$ for all $k \in \overline{1\gamma_z}$. Letting

$$J_{(z,k)}^- := \{j \in \overline{1\nu_{\iota_{k-1}^z}} :\ \langle x_{\iota_{k-1}^z j}^*, z + \bar{u} + \lambda_k^z(\bar{u}' - \bar{u})\rangle = t_{\iota_{k-1}^z j}\} \tag{44}$$

and

$$J_{(z,k)} := \{j \in \overline{1\nu_{\iota_k^z}} :\ \langle x_{\iota_k^z j}^*, z + \bar{u} + \lambda_k^z(\bar{u}' - \bar{u})\rangle = t_{\iota_k^z j}\}, \tag{45}$$

it follows from (34) and Corollary 2.1 that $J_{(z,k)}^- \neq \emptyset$ and $J_{(z,k)} \neq \emptyset$ for all $k \in \overline{1\gamma_z}$. We claim that there exist $\bar{z} \in B_{X_3}(0, \delta) \setminus E_0$ and $(j_k^-, j_k) \in \overline{1\nu_{\iota_{k-1}^{\bar{z}}}} \times \overline{1\nu_{\iota_k^{\bar{z}}}}$ such that

$$J_{(\bar{z},k)}^- = \{j_k^-\} \ \text{ and } \ J_{(\bar{z},k)} = \{j_k\} \quad \forall k \in \overline{1\gamma_{\bar{z}}}. \tag{46}$$

Indeed, if this is not the case, for each $z \in B_{X_3}(0, \delta) \setminus E_0$ there exists $k \in \overline{1\gamma_z}$ such that either $J_{(z,k)}^-$ or $J_{(z,k)}$ contains at least two elements; we assume without loss of generality that there exist $k \in \overline{1\gamma_z}$ and $j_1, j_2 \in J_{(z,k)}$ such that $j_1 \neq j_2$. Then, by (43) and (45),

$$z + \bar{u} + \lambda_k^z(\bar{u}' - \bar{u}) \in \{x \in \hat{P}_{\iota_k^z} :\ \langle x_{\iota_k^z j_1}^*, x\rangle = t_{\iota_k^z j_1} \ \text{ and } \ \langle x_{\iota_k^z j_2}^*, x\rangle = t_{\iota_k^z j_2}\}. \tag{47}$$

Since $\{(x^*_{\iota^z_k 1}, t_{\iota^z_k 1}), (x^*_{\iota^z_k 2}, t_{\iota^z_k 2}), \cdots, (x^*_{\iota^z_k \nu_{\iota^z_k}}, t_{\iota^z_k \nu_{\iota^z_k}})\}$ is a prime generator group of $P_{\iota^z_k}$ and $\mathrm{int}(P_{\iota^z_k})$ is nonempty, this and Lemma 2.2 imply that $x^*_{\iota^z_k j_1}$ and $x^*_{\iota^z_k j_2}$ are linearly independent. Hence $\mathrm{codim}(\mathcal{N}(x^*_{\iota^z_k j_1}) \cap \mathcal{N}(x^*_{\iota^z_k j_2})) = 2$. Noting that X_1 is a subspace of $\mathcal{N}(x^*_{\iota^z_k j_1}) \cap \mathcal{N}(x^*_{\iota^z_k j_2})$, it follows from (33) that $X_2 \cap \mathcal{N}(x^*_{\iota^z_k j_1}) \cap \mathcal{N}(x^*_{\iota^z_k j_2})$, as a linear subspace of X_2, is of codimension 2. Hence, by (47), $\hat{P}_{\iota^z_k}$ has a face $\hat{F}$ such that $z + \bar{u} + \lambda^z_k(\bar{u}' - \bar{u}) \in \hat{F}$, $\dim(\hat{F}) \le \dim(X_2) - 2$ and so $z + [\bar{u},\ \bar{u}'] \subset \hat{F} - (\bar{u} + \lambda^z_k(\bar{u}' - \bar{u})) + [\bar{u},\ \bar{u}'] \subset \hat{F} + [\bar{u} - \bar{u}',\ \bar{u}' - \bar{u}]$. Since each polyhedron has finitely many faces (cf. [10, 16]) , there exist finitely many linear subspaces $S_1, \cdots, S_l$ of X_2 and $\omega_1, \cdots, \omega_l \in X_2$ such that $\dim(S_j) \le \mathrm{dom}(X_2) - 2$ $(j = 1, \cdots, l)$ and $z + [\bar{u},\ \bar{u}'] \subset \bigcup_{j=1}^{l} (S_j + \omega_j + [\bar{u} - \bar{u}',\ \bar{u}' - \bar{u}])$ for all $z \in B_{X_3}(0, \delta) \setminus E_0$. This means that $(B_{X_3}(0, \delta) \setminus E_0) + [\bar{u},\ \bar{u}'] \subset \bigcup_{j=1}^{l} (S_j + \omega_j + [\bar{u} - \bar{u}',\ \bar{u}' - \bar{u}])$ and so

$$\mu_{X_2}((B_{X_3}(0, \delta) \setminus E_0) + [\bar{u},\ \bar{u}']) \le \sum_{j=1}^{l} \mu_{X_2}(S_j + \omega_j + [\bar{u} - \bar{u}',\ \bar{u}' - \bar{u}]) = 0.$$

Thus, by (39) and the Fubini theorem, $\mu_{X_3}(B_{X_3}(0, \delta) \setminus E_0) = 0$. Hence $\mu_{X_3}(E_0) \ge \mu_{X_3}(B_{X_3}(0, \delta)) > 0$, contradicting $\mu_{X_3}(E_0) = 0$. This shows that (46) holds, that is, there exist $\bar{z} \in B_{X_3}(0, \delta) \setminus E_0$ and $(j^-_k, j_k) \in \overline{1\nu_{\iota^{\bar{z}}_{k-1}}} \times \overline{1\nu_{\iota^{\bar{z}}_k}}$ such that

$$\bar{x}_k := \bar{z} + \bar{u} + \lambda^{\bar{z}}_k(\bar{u}' - \bar{u}) \in F^\circ_{j^-_k}(\hat{P}_{\iota^{\bar{z}}_{k-1}}) \cap F^\circ_{j_k}(\hat{P}_{\iota^{\bar{z}}_k}) \quad \forall k \in \overline{1\gamma_{\bar{z}}}.$$

Noting that $F^\circ_{j^-_k}(P_{\iota^{\bar{z}}_{k-1}}) = X_1 + F^\circ_{j^-_k}(\hat{P}_{\iota^{\bar{z}}_{k-1}})$ and $F^\circ_{j_k}(P_{\iota^{\bar{z}}_k}) = X_1 + F^\circ_{j_k}(\hat{P}_{\iota^{\bar{z}}_k})$, one has $\bar{x}_k \in F^\circ_{j^-_k}(P_{\iota^{\bar{z}}_{k-1}}) \cap F^\circ_{j_k}(P_{\iota^{\bar{z}}_k})$ for all $k \in \overline{1\gamma_{\bar{z}}}$. It follows from Lemma 2.5 that for each $k \in \overline{1\gamma_{\bar{z}}}$, $\mathcal{N}_k := \mathcal{N}(x^*_{\iota^{\bar{z}}_{k-1} j^-_k}) = \mathcal{N}(x^*_{\iota^{\bar{z}}_k j_k})$ and

$$F^\circ_{j^-_k}(P_{\iota^{\bar{z}}_{k-1}}) \cap B_X(\bar{x}_k, r_k) = F^\circ_{j_k}(P_{\iota^{\bar{z}}_k}) \cap B_X(\bar{x}_k, r_k) = (\bar{x}_k + \mathcal{N}_k) \cap B_X(\bar{x}_k, r_k)$$

for some $r_k > 0$. Hence $(\bar{x}_k + \mathcal{N}_k) \cap B_X(\bar{x}_k, r_k) \subset P_{\iota^{\bar{z}}_{k-1}} \cap P_{\iota^{\bar{z}}_k}$ for all $k \in \overline{1\gamma_{\bar{z}}}$. This and (29) imply that

$$T_{\iota^{\bar{z}}_{k-1}}|_{(\bar{x}_k + \mathcal{N}_k) \cap B_X(\bar{x}_k, r_k)} + b_{\iota^{\bar{z}}_{k-1}} = T_{\iota^{\bar{z}}_k}|_{(\bar{x}_k + \mathcal{N}_k) \cap B_X(\bar{x}_k, r_k)} + b_{\iota^{\bar{z}}_k} \quad \forall k \in \overline{1\gamma_{\bar{z}}}.$$

Since $\mathcal{N}_k$ is a maximal subspace of X and both $T_{\iota^{\bar{z}}_{k-1}}$ and $T_{\iota^{\bar{z}}_k}$ are linear, $T_{\iota^{\bar{z}}_{k-1}}|_{\mathcal{N}_k} = T_{\iota^{\bar{z}}_k}|_{\mathcal{N}_k}$ for all $k \in \overline{1\gamma_{\bar{z}}}$. Noting that $X_1 \subset \mathcal{N}_k$ (thanks to the definitions of $\mathcal{N}_k$ and X_1), it follows that $T_{\iota^{\bar{z}}_{k-1}}|_{X_1} = T_{\iota^{\bar{z}}_k}|_{X_1}$ for all $k \in \overline{1\gamma_{\bar{z}}}$, and so $T_i|_{X_1} = T_{\iota^{\bar{z}}_0}|_{X_1} = T_{\iota^{\bar{z}}_{\gamma_{\bar{z}}}}|_{X_1} = T_{i'}|_{X_1}$ (thanks to (42)). □

Theorem 3.2. *Let $f \in \mathcal{PL}(X, Y)$, and let X_1 and X_2 be closed subspaces of X such that*

$$X_1 \subset X_f, \quad X = X_1 \oplus X_2,\ \mathrm{codim}(X_1) = \dim(X_2) < \infty, \tag{48}$$

where X_f is as in Theorem 3.1. Then there exist $T \in \mathcal{L}(X_1, Y)$, a finite dimensional subspace $\hat{Y}$ of Y and $g \in \mathcal{PL}(X_2, \hat{Y})$ such that

$$f(x_1 + x_2) = T(x_1) + g(x_2) \quad \forall (x_1, x_2) \in X_1 \times X_2. \tag{49}$$

Proof. Take $(P_1, T_1, b_1), \cdots, (P_m, T_m, b_m) \in \mathcal{P}(X) \times \mathcal{L}(X, Y) \times Y$ such that (28) and (29) hold. Then, by Theorem 3.1,

$$T_1(x) = \cdots = T_m(x) \quad \forall x \in X_f. \tag{50}$$

Define $T : X_1 \to Y$ as $T(x) := T_1(x)$ for all $x \in X_1$. Clearly, $T \in \mathcal{L}(X_1, Y)$. For each $i \in \overline{1m}$, by (48) and Proposition 2.1, take a polyhedron $\hat{P}_i$ in X_2 such that $P_i = X_1 + \hat{P}_i$. It follows from (28) that $X_2 = \bigcup\limits_{i=1}^{m} \hat{P}_i$. Moreover, since $X_1 \subset X_f$, (29) and (50) imply that

$$f(x_1 + x_2) = T_i(x_1 + x_2) + b_i = T_i(x_1) + T_i(x_2) + b_i = T(x_1) + T_i(x_2) + b_i \tag{51}$$

for all $(x_1, x_2) \in X_1 \times \hat{P}_i$ and $i \in \overline{1m}$. Hence $f(x_2) = T_i(x_2) + b_i$ for all $i \in \overline{1m}$ and $x_2 \in \hat{P}_i$. It follows that

$$T_i(x_2) + b_i = T_{i'}(x_2) + b_{i'} \quad \forall i, i' \in \overline{1m} \text{ and } \forall x_2 \in \hat{P}_i \cap \hat{P}_{i'}.$$

Let $\hat{Y} := \operatorname{span}\left(\bigcup\limits_{i=1}^{m}(T_i(X_2) + b_i)\right)$, and let $g : X_2 \to \hat{Y}$ be such that $g(x_2) := T_i(x_2) + b_i$ for all $x \in \hat{P}_i$ and $i \in \overline{1m}$. Then $\hat{Y}$ is a subspace of Y with $\dim(\hat{Y}) \leq m\dim(X_2) + 1 < +\infty$ and g is well defined. It is easy from (51) to verify that (49) holds. □

4 Dimensional Reduction Method to Solve (PLP)

In this section, we always assume that X, Y are normed spaces and that $f \in \mathcal{PL}(X, Y)$ and $\varphi_j \in \mathcal{PL}(X, \mathbb{R}$ $(j \in \overline{1l})$. Consider the following fully piecewise linear vector optimization problem

(PLP) $\qquad C - \text{Min} f(x)$ subject to $\varphi_j(x) \leq 0,\ j = 1, \cdots, l.$

Based on Proposition 2.5 and its proof (an elementary method), we can select $P_i \in \mathcal{P}(X)$, $T_i \in \mathcal{L}(X, Y)$, $u_{ik}^*, x_{ij}^* \in X^* = \mathcal{L}(X, \mathbb{R})$, $b_i \in Y$ and $t_{ik}, c_{ij} \in \mathbb{R}$ $(i = 1, \cdots, m,\ k = 1, \cdots, q_i$ and $j = 1, \cdots, l)$ such that

$$P_i = \{x \in X : \langle u_{ik}^*, x\rangle \leq t_{ik} \ \ \forall k \in \overline{1q_i}\}, \tag{52}$$

$$X = \bigcup_{i=1}^{m} P_i,\ \operatorname{int}(P_i) \neq \emptyset,\ P_i \cap \operatorname{int}(P_{i'}) = \emptyset \quad \forall i, i' \in \overline{1m} \text{ with } i \neq i' \tag{53}$$

and

$$f|_{P_i} = T_i|_{P_i} + b_i \ \text{ and } \ \varphi_j|_{P_i} = x_{ij}^*|_{P_i} - c_{ij} \quad \forall (i, j) \in \overline{1m} \times \overline{1l}. \tag{54}$$

For each $i \in \overline{1m}$, let

$$A_i := \{x \in P_i : \langle x_{ij}^*, x\rangle \leq c_{ij} \ \ \forall j \in \overline{1l}\}. \tag{55}$$

For a closed convex cone C in Y, let $\leq_C$ denote the preorder induced by C in Y, that is, for $y_1, y_2 \in Y$, $y_1 \leq_C y_2 \Leftrightarrow y_2 - y_1 \in C$. When the interior $\operatorname{int}(C)$ of C is nonempty, $y_1 <_C y_2$ is defined as $y_2 - y_1 \in \operatorname{int}(C)$. For a subset Ω of Y and a point ω in Ω, we say that ω is a Pareto efficient point of Ω (with respect to C), denoted by $\omega \in E(\Omega, C)$, if there is no element $v \in \Omega \setminus \{\omega\}$ such that $v \leq_C \omega$. In the case where $\operatorname{int}(C) \neq \emptyset$, we say that ω is a weak Pareto efficient point of Ω, denoted by $\omega \in \mathrm{WE}(\Omega, C)$, if there is no element $v \in \Omega$ such that $v <_C \omega$.

Let A denote the feasible set of fully piecewise linear vector optimization problem (PLP), that is,

$$A := \{x \in X : \ \varphi_1(x) \leq 0, \cdots, \varphi_l(x) \leq 0\}. \tag{56}$$

We say that $\bar{x} \in A$ is a Pareto (resp. weak Pareto) solution of (PLP) if $f(\bar{x}) \in \mathrm{E}(f(A), C)$ (resp. $f(\bar{x}) \in \mathrm{WE}(f(A), C)$). Now we provide the procedures to obtain exact formulas for optimal value sets and solution sets of fully piecewise linear vector optimization problem (PLP):
Step 1 (Decomposing the space X): Let

$$X_1 := \bigcap_{i=1}^{m} \bigcap_{(j,k)\in\overline{1l}\times\overline{1q_i}} \mathcal{N}(x_{ij}^*) \cap \mathcal{N}(u_{ik}^*), \tag{57}$$

where $\mathcal{N}(x_{ij}^*) := \{x \in X : \langle x_{ij}^*, x\rangle = 0\}$ is the null space of x_{ij}^*; namely, X_1 is the solution space of the following system of linear equations

$$\langle u_{ik}^*, x\rangle = \langle x_{ij}^*, x\rangle = 0, \ i = 1, \cdots, m, \ j = 1, \cdots, l, \ k = 1, \cdots, q_i.$$

Take a maximal linearly independent subset $\{e_1^*, \cdots, e_\nu^*\}$ of the finite set $\{u_{ik}^*, x_{ij}^* : \ i \in \overline{1m}, \ j \in \overline{1l}, \ k \in \overline{1q_i}\}$. Then, for each $\iota \in \overline{1\nu}$, the following system of linear equations

$$\langle e_\iota^*, x\rangle = 1 \quad \text{and} \quad \langle e_{\iota'}^*, x\rangle = 0 \ \ \forall \iota' \in \overline{1\nu} \setminus \{\iota\}$$

is solvable; take a solution h_ι of this system of linear equations. Let

$$X_2 := \operatorname{span}\{h_1, \cdots, h_\nu\} = \left\{\sum_{\iota=1}^{\nu} t_\iota h_\iota : \ t_1, \cdots, t_\nu \in \mathbb{R}\right\} \tag{58}$$

(in particular, $X_2 = \operatorname{span}\{e_1^*, \cdots, e_\nu^*\}$ when X is a Hilbert space). Then

$$X = X_1 + X_2 \quad \text{and} \quad X_1 \cap X_2 = \{0\}. \tag{59}$$

Step 2 (Constructing finite dimensional subspace Z of Y): Thanks to Theorem 3.1 and (59),

$$\hat{T} := T_1|_{X_1} = T_2|_{X_1} = \cdots = T_m|_{X_1}. \tag{60}$$

Let D denote the finite set $\bigcup_{i=1}^{m}\{T_i(h_1), \cdots, T_i(h_\nu), b_i\}$ and take $u_1, \cdots, u_\varsigma$ in D with ς being the maximal integer such that $u_1 \in D \setminus \hat{T}(X_1)$,

$$u_2 \in D \setminus (\hat{T}(X_1) + \operatorname{span}\{u_1\}), \cdots, u_\varsigma \in D \setminus (\hat{T}(X_1) + \operatorname{span}\{u_1, \cdots, u_{\varsigma-1}\}),$$

where X_1 and $h_1, \cdots, h_\nu$ are as in Step 1. Let

$$Z := \operatorname{span}\{u_1, \cdots, u_\varsigma\}. \tag{61}$$

Clearly, Z is a subspace of Y such that

$$\dim(Z) = \varsigma \quad \text{and} \quad \hat{T}(X_1) \cap Z = \{0\}. \tag{62}$$

Let Π_Z denote the projection from $\hat{T}(X_1) \oplus Z$ onto Z, that is,

$$\Pi_Z(y+z) := z \quad \forall (y,z) \in \hat{T}(X_1) \times Z, \tag{63}$$

and let C_Z be a convex cone in the finite dimensional space Z defined by

$$C_Z := \Pi_Z((\hat{T}(X_1) \oplus Z) \cap C). \tag{64}$$

Step 3 (Exact formulas for weak Pareto optimal value set and weak Pareto set of (PLP)): For each $i \in \overline{1m}$, let

$$\hat{A}_i := \{x_2 \in \hat{P}_i : \langle x^*_{ij}, x_2 \rangle \le c_{ij} \ \ \forall j \in \overline{1l}\}, \tag{65}$$

where $\hat{P}_i := \{x_2 \in X_2 : \langle u^*_{ik}, x_2 \rangle \le t_{ik} \ \ \forall k \in \overline{1q_i}\}$. The weak Pareto optimal value set $\mathrm{WE}(f(A), C)$ and weak Pareto solution set S^w of (PLP) can be formulized as follows:
(i) If $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) = \emptyset$ then $\mathrm{WE}(f(A), C) = f(A)$ and $S^w = A$.
(ii) If $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) \neq \emptyset$ then

$$\mathrm{WE}(f(A), C) = \hat{T}(X_1) + \bigcup_{i=1}^{m} \hat{V}^w_i \ \text{ and } \ S^w = X_1 + \bigcup_{i=1}^{m} \hat{A}_i \cap (\Pi_Z \circ T_i)^{-1}(\hat{V}^w_i - \Pi_Z(b_i)),$$

where $\hat{V}^w_i := \Pi_Z(T_i(\hat{A}_i) + b_i) \setminus \left(\bigcup_{i' \in \overline{1m}} \Pi_Z(T_{i'}(\hat{A}_{i'}) + b_{i'}) + (\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) \right)$ (thanks to Theorems 4.1 and 4.3 and Lemma 4.1).

Similarly, with Corollary 4.1 and Proposition 4.2 replacing Theorems 4.1 and 4.3, we can also obtain the formulas for the Pareto optimal value set and Pareto solution set of (PLP). Based on the above procedures, we establish the structure theorems for Pareto solution sets and optimal value sets of (PLP).

To prove the main results in this section, we need the following lemma.

Lemma 4.1. *Suppose that* $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C)$ *is nonempty. Then*

$$\mathrm{int}_Z(C_Z) = \Pi_Z((\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C)), \tag{66}$$

where $\mathrm{int}_Z(C_Z)$ *denotes the interior of* C_Z *in the subspace* Z.

Proof. By the assumption, take $(\bar{x}_1, \bar{z}) \in X_1 \times Z$ and $r > 0$ such that

$$\hat{T}(\bar{x}_1) + \bar{z} + rB_{\hat{T}(X_1)\oplus Z} \subset (\hat{T}(X_1) \oplus Z) \cap C. \tag{67}$$

Noting that the projection Π_Z is an open mapping from $\hat{T}(X_1) \oplus Z$ to Z, (64) implies that $\mathrm{int}_Z(C_Z) \supset \Pi_Z((\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C))$. Hence it suffices to show the converse inclusion. To do this, let $z \in \mathrm{int}_Z(C_Z)$. Then there exists $\sigma > 0$ sufficiently small such that $z + \sigma(z - \bar{z}) \in C_Z$, that is, $\hat{T}(x_1) + z + \sigma(z - \bar{z}) \in (\hat{T}(X_1) \oplus Z) \cap C$ for some $x_1 \in X_1$. It follows from (67) and the convexity of C that $\frac{\hat{T}(x_1)+z+\sigma(z-\bar{z})}{1+\sigma} + \frac{\sigma(\hat{T}(\bar{x}_1)+\bar{z}+rB_{\hat{T}(X_1)\oplus Z})}{1+\sigma} \subset (\hat{T}(X_1) \oplus Z) \cap C$, and so $\hat{T}(\frac{x_1+\sigma\bar{x}_1}{1+\sigma}) + z + \frac{\sigma r B_{\hat{T}(X_1)\oplus Z}}{1+\sigma} \subset (\hat{T}(X_1) \oplus Z) \cap C$. Hence, by (64) and (63),

$$C_Z \supset \Pi_Z\left(\hat{T}(\frac{x_1 + \sigma\bar{x}_1}{1+\sigma}) + z + \frac{\sigma r B_{\hat{T}(X_1)\oplus Z}}{1+\sigma} \right) = z + \frac{\sigma r \Pi_Z(B_{\hat{T}(X_1)\oplus Z})}{1+\sigma}.$$

This shows that $z \in \mathrm{int}_Z(C_Z)$ (because Π_Z is an open mapping from $\hat{T}(X_1) \oplus Z$ onto Z). The proof is complete. □

Define $\hat{f} : X_2 \to Z$ by

$$\hat{f}(x_2) := (\Pi_Z \circ f)(x_2) = \Pi_Z(f(x_2)) \quad \forall x_2 \in X_2.$$

Then, $\hat{f}$ is a piecewise linear function between the finite dimensional spaces X_2 and Z. For each $i \in \overline{1m}$, recall that $\hat{P}_i := \{x_2 \in X_2 : \langle u_{ik}^*, x_2\rangle \le t_{ik} \quad \forall k \in \overline{1q_i}\}$. By (52), (53), (57) and (58), one has $X_2 = \bigcup\limits_{i\in\overline{1m}} \hat{P}_i$ and $P_i = X_1 + \hat{P}_i$ ($\forall i \in \overline{1m}$). From (54), (60) and the definition of Z (see (61)), it is easy to verify that

$$f(X) = \bigcup_{i\in\overline{1m}} (\hat{T}(X_1) + T_i(\hat{P}_i) + b_i) \subset \hat{T}(X_1) + Z = \hat{T}(X_1) \oplus Z$$

and $f(x_1+x_2) = \hat{T}(x_1)+f(x_2) \in \hat{T}(X_1)+f(x_2)$ for all $(x_1, x_2) \in X_1\times X_2$. Noting that $f(x_2)-\hat{f}(x_2) = f(x_2) - \Pi_Z(f(x_2)) \in \hat{T}(X_1)$, it follows that

$$f(x_1 + x_2) \in \hat{T}(X_1) + \hat{f}(x_2) \quad \forall (x_1, x_2) \in X_1 \times X_2. \tag{68}$$

To solve the original piecewise linear problem (PLP), consider the following piecewise linear problem in the framework of finite dimensional spaces:

$$(\widehat{\text{PLP}}) \qquad C_Z - \min \hat{f}(x_2) \quad \text{subject to } x_2 \in X_2 \text{ and } \varphi_1(x_2) \le 0, \cdots, \varphi_l(x_2) \le 0.$$

Let $\hat{A}$ denote the feasible set of $(\widehat{\text{PLP}})$. Then the feasible set A of (PLP) is equal to $X_1 + \hat{A}$.

Next we establish the relationship between the weak Pareto optimal value set and weak Pareto solution set (resp. the Pareto solution set) of (PLP) and that of $(\widehat{\text{PLP}})$.

Theorem 4.1. *Let S^w and $\hat{S}^w$ denote the weak Pareto solution sets of piecewise linear problems* (PLP) *and* $(\widehat{\text{PLP}})$*, respectively. The following statements hold:*
(i) If $(\hat{T}(X_1) \oplus Z) \cap \text{int}(C) = \emptyset$ then $\text{WE}(f(A), C) = f(A)$ and $S^w = A$.
(ii) If $(\hat{T}(X_1) \oplus Z) \cap \text{int}(C) \neq \emptyset$ then

$$\text{WE}(f(A), C) = \hat{T}(X_1) + \text{WE}(\hat{f}(\hat{A}), C_Z) \text{ and } S^w = X_1 + \hat{S}^w. \tag{69}$$

Proof. First suppose that $(\hat{T}(X_1) \oplus Z) \cap \text{int}(C) = \emptyset$. Then, since $\hat{T}(X_1) \oplus Z$ is a linear subspace of Y, $(\hat{T}(X_1) \oplus Z) \cap ((\hat{T}(X_1) \oplus Z) - \text{int}(C)) = \emptyset$. Noting that

$$f(A) = f(X_1 + \hat{A}) = \hat{T}(X_1) + \hat{f}(\hat{A}) \subset \hat{T}(X_1) \oplus Z \tag{70}$$

(thanks to (68)), it follows that $f(A) \cap (f(A) - \text{int}(C)) = \emptyset$. This shows that $\text{WE}(f(A), C) = f(A)$ and $S^w = A$. Next suppose that $(\hat{T}(X_1) \oplus Z) \cap \text{int}(C) \neq \emptyset$. Then, by Lemma 4.1, $\text{int}_Z(C_Z) = \Pi_Z(\hat{T}(X_1)\oplus Z)\cap\text{int}(C))$. Since Π_Z is the projection from $\hat{T}(X_1)\oplus Z$ to Z, $\hat{T}(X_1)+E = \hat{T}(X_1)+\Pi_Z(E)$ for any set E in $\hat{T}(X_1) \oplus Z$. Hence, by Lemma 4.1,

$$\begin{aligned} \hat{T}(X_1) + (\hat{T}(X_1) \oplus Z) \cap \text{int}(C) &= \hat{T}(X_1) + \Pi_Z((\hat{T}(X_1) \oplus Z) \cap \text{int}(C)) \\ &= \hat{T}(X_1) + \text{int}_Z(C_Z). \end{aligned}$$

Noting that $\mathrm{WE}(\Omega, C) = \Omega \setminus (\Omega + \mathrm{int}(C))$ for any set in Y, it follows from (70) that

$$
\begin{aligned}
\mathrm{WE}(f(A), C) &= (\hat{T}(X_1) + \hat{f}(\hat{A})) \setminus (\hat{T}(X_1) + \hat{f}(\hat{A}) + \mathrm{int}(C)) \\
&= (\hat{T}(X_1) + \hat{f}(\hat{A})) \setminus (\hat{T}(X_1) + \hat{f}(\hat{A}) + (\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C)) \\
&= (\hat{T}(X_1) + \hat{f}(\hat{A})) \setminus (\hat{T}(X_1) + \hat{f}(\hat{A}) + \mathrm{int}_Z(C_Z)) \\
&= \hat{T}(X_1) + \hat{f}(\hat{A}) \setminus (\hat{f}(\hat{A}) + \mathrm{int}_Z(C_Z)) \\
&= \hat{T}(X_1) + \mathrm{WE}(\hat{f}(\hat{A}), C_Z).
\end{aligned}
$$

This shows the first equality of (69). To prove the second equality of (69), let $x_2 \in \hat{S}^w$. Then $x_2 \in \hat{A}$ and $\hat{f}(x_2) \in \mathrm{WE}(\hat{f}(\hat{A}), C_Z)$. Hence,

$$X_1 + x_2 \subset X_1 + \hat{A} = A \text{ and } f(X_1 + x_2) = \hat{T}(X_1) + \hat{f}(x_2) \subset \mathrm{WE}(f(A), C)$$

(thanks to (68) and the first equality of (69)). It follows that $X_1 + x_2 \subset S^w$ and so $X_1 + \hat{S}^w \subset S^w$. Conversely, let $x \in S^w$. Then there exists $(x_1, x_2) \in X_1 \times \hat{A}$ such that $x = x_1 + x_2$ and $f(x_1 + x_2) \in \mathrm{WE}(f(A), C) = \hat{T}(X_1) + \mathrm{WE}(\hat{f}(\hat{A}), C_Z)$. Noting that $f(x_1 + x_2) \in f(X_1 + x_2) = \hat{T}(X_1) + \hat{f}(x_2)$ and $\hat{T}(X_1) \cap Z = \{0\}$, one has $\hat{f}(x_2) \in \mathrm{WE}(\hat{f}(\hat{A}), C_Z)$. Hence $x_2 \in \hat{S}^w$ and $x = x_1 + x_2 \in X_1 + \hat{S}^w$. Hence the second equality of (69) holds. □

Theorem 4.2. *Let $(x_1, x_2) \in X_1 \times \hat{A}$. Then $f(x_1 + x_2) \in \mathrm{E}(f(A), C)$ if and only if $\hat{f}(x_2) \in \mathrm{E}(\hat{f}(\hat{A}), C_Z)$ and $C_Z = C \cap (\hat{T}(X_1) \oplus Z)$.*

Proof. By (68), $f(A) = f(X_1 + \hat{A}) = \hat{T}(X_1) + \hat{f}(\hat{A})$. Hence

$$f(A) - f(x_1 + x_2) = \hat{T}(X_1) + \hat{f}(\hat{A}) - \hat{f}(x_2).$$

Noting that $\hat{f}(\hat{A}) - \hat{f}(x_2) \subset Z$, it follows that

$$(f(A) - f(x_1 + x_2)) \cap -C = (\hat{T}(X_1) + \hat{f}(\hat{A}) - \hat{f}(x_2)) \cap -(C \cap (\hat{T}(X_1) \oplus Z)).$$

Thus, from the definitions of the projection $\Pi_Z : \hat{T}(X_1) \oplus Z \to Z$ (see (63)), it is easy to verify that

$$(f(A) - f(x_1 + x_2)) \cap -C = \Pi_1(C \cap (\hat{T}(X_1) \oplus Z)) + (\hat{f}(\hat{A}) - \hat{f}(x_2)) \cap -C_Z,$$

where $\Pi_1(y + z) = y$ for all $(y, z) \in \hat{T}(X_1) \oplus Z$. Therefore, $f(x_1 + x_2) \in \mathrm{E}(f(A), C)$ is equivalent to

$$\Pi_1(C \cap (\hat{T}(X_1) \oplus Z)) + (\hat{f}(\hat{A}) - \hat{f}(x_2)) \cap -C_Z = \{0\}.$$

Since $\hat{T}(X_1) \cap Z = \{0\}$, it follows that $f(x_1 + x_2) \in \mathrm{E}(f(A), C)$ if and only if

$$\Pi_1(C \cap (\hat{T}(X_1) \oplus Z)) = (\hat{f}(\hat{A}) - \hat{f}(x_2)) \cap -C_Z = \{0\},$$

namely $C_Z = C \cap (\hat{T}(X_1) \oplus Z)$ and $\hat{f}(x_2) \in E(\hat{f}(\hat{A}), C_Z)$. The proof is complete. □

The following corollary is a consequence of Theorem 4.2.

Corollary 4.1. *Let $\hat{S}$ denote the Pareto solution set of piecewise linear problem* ($\widehat{\text{PLP}}$)*. The following statements hold:*
(i) If $C_Z \neq C \cap (\hat{T}(X_1) \oplus Z)$ then $S = \emptyset$.
(ii) If $C_Z = C \cap (\hat{T}(X_1) \oplus Z)$ then

$$S = X_1 + \hat{S} \ \text{ and } \ \mathrm{E}(f(A), C) = \hat{T}(X_1) + \mathrm{E}(\hat{f}(\hat{A}), C_Z).$$

Remark 4.1 *By Corollary 4.1(i) and Theorem 4.1(i), piecewise linear problem*

(PLP) has no Pareto solution when $C_Z \neq C \cap (\hat{T}(X_1) \oplus Z)$, and the weak Pareto solution set of (PLP) is just the entire feasible set A of (PLP) when $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) = \emptyset$. Therefore, we only need to consider the Pareto solution set and the weak Pareto solution of (PLP) when $C_Z = C \cap (\hat{T}(X_1) \oplus Z)$ and $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) \neq \emptyset$, respectively.

To establish formulas for the solution sets of fully piecewise linear problem (PLP), for $i \in \overline{1m}$, consider the following linear subproblems

$$(\text{LP})_i \qquad C - \min T_i x + b_i \ \text{ subject to } x \in A_i.$$

In the framework of finite dimensional spaces, we also consider the following linear subproblem

$$(\widehat{\text{LP}})_i \qquad C_Z - \min \Pi_Z(T_i x + b_i) \ \text{ subject to } x \in \hat{A}_i,$$

where $\hat{A}_i$ is as in (65).

By Theorem 4.1 and Corollary 4.1 (with linear problems $(\text{LP})_i$ and $(\widehat{\text{LP}})_i$ replacing respectively piecewise linear problems (PLP) and ($\widehat{\text{PLP}}$)), we have the following result (thanks to $A_i = X_1 + \hat{A}_i$).

Proposition 4.1. *For each $i \in \overline{1m}$, let S_i (resp. S_i^w) and $\hat{S}_i$ (resp. $\hat{S}_i^w$) denote the Pareto solution sets (resp. weak Pareto solution sets) of linear problem $(LP)_i$ and $(\widehat{\text{LP}})_i$, respectively. The following statements hold:*
(i) $S_i = \emptyset$ if $C_Z \neq C \cap (\hat{T}(X_1) \oplus Z)$.
(ii) $S_i = X_1 + \hat{S}_i$ if $C_Z = C \cap (\hat{T}(X_1) \oplus Z)$.
(iii) $S_i^w = A_i$ if $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) = \emptyset$.
(iv) $S_i^w = X_1 + \hat{S}_i^w$ if $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) \neq \emptyset$.

The following theorem provides formulas for the weak Pareto solution set and weak Pareto optimal value set for piecewise linear problem ($\widehat{\text{PLP}}$). Recall that a weak Pareto face (resp. Pareto face) F of linear problem $(\text{LP})_i$ is a face of A_i such that each point in F is a weak Pareto solution (resp. Pareto solution) of $(\text{LP})_i$. We also need the following theorem by Arrow, Barankin and Blackwell [2].

Theorem ABB. *Let $X = \mathbb{R}^p$, $Y = \mathbb{R}^q$ and $C = \mathbb{R}^q_+$. Suppose that f and each φ_j in (PLP) are linear. Then the Pareto solution set and weak Pareto solution set of (PLP) are the unions of finitely many faces of the feasible set A.*

Theorem 4.3. *For each $i \in \overline{1m}$, let*

$$\hat{V}_i^w := \Pi_Z\left(T_i(\hat{A}_i) + b_i\right) \setminus \left(\hat{f}(\hat{A}) + \mathrm{int}_Z(C_Z)\right) \tag{71}$$

and

$$\breve{S}_i := \hat{A}_i \cap (\Pi_Z \circ T_i)^{-1}\left(\hat{V}_i^w - \Pi_Z(b_i)\right). \tag{72}$$

Suppose that $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) \neq \emptyset$*. Then the following statements hold:*
(i) $\hat{S}^w = \bigcup_{i\in\bar{I}} \breve{S}_i$ *and* $\mathrm{WE}\left(\hat{f}(\hat{A}), C_Z\right) = \bigcup_{i\in\bar{I}} \hat{V}_i^w$*, where* $I := \{i \in \overline{1m} : \ \hat{A}_i \neq \emptyset\}$*.*
(ii) If, in addition, the ordering cone C in Y is assumed to be polyhedral, then for each $i \in \bar{I}$ *there exist finitely many polyhedra* $\hat{P}_{i1}, \cdots, \hat{P}_{iq_i}$ *in* X_2 *and faces* $\hat{F}_{i1}, \cdots, \hat{F}_{iq_i}$ *of* $\hat{A}_i$ *such that* $\breve{S}_i = \bigcup_{j=1}^{q_i} \hat{P}_{ij}$ *and* $\hat{P}_{ij} \subset \hat{F}_{ij} \subset \hat{S}_i^w$ *for all* $j \in \overline{1q_i}$*. Consequently,* $\hat{S}^w$ *is the union of finitely many polyhedra in* X_2*, each one of which is contained in a weak Pareto face of some linear subproblem* $(\widehat{\mathrm{LP}})_i$*.*

Proof. Let i be an arbitrary element in $\bar{I}$. Then, since $\hat{f}(\hat{x}) = \Pi_Z(T_i(\hat{x})) + \Pi_Z(b_i)$ for all $\hat{x} \in \hat{A}_i$, $\hat{A}_i \cap (\Pi_Z \circ T_i)^{-1}(\hat{V}_i^w - \Pi_Z(b_i)) = \hat{A}_i \cap \hat{f}^{-1}(\hat{V}_i^w)$. Hence, by (71) and (72),

$$\breve{S}_i = \hat{A}_i \cap \hat{f}^{-1}(\hat{V}_i^w) \quad \text{and} \quad \hat{V}_i^w = \hat{f}(\breve{S}_i). \tag{73}$$

Thus, to prove (i), it suffices to show that $\breve{S}_i = \hat{S}^w \cap \hat{A}_i$ (because $\hat{A} = \bigcup_{i\in\bar{I}} \hat{A}_i$ and $\hat{f}(\hat{S}^w) = \mathrm{WE}(\hat{f}(\hat{A}), C_Z)$). To do this, let $\hat{a}_i \in \hat{S}^w \cap \hat{A}_i$. Then $\hat{f}(\hat{a}_i) \in \mathrm{WE}(\hat{f}(\hat{A}), C_Z)$, that is, $\hat{f}(\hat{a}_i) \notin \hat{f}(\hat{A}) + \mathrm{int}_Z(C_Z)$. Since

$$\hat{f}(\hat{a}_i) = \Pi_Z(T_i(\hat{a}_i) + b_i) \in \Pi_Z(T_i(\hat{A}_i) + b_i),$$

this and (71) imply that $\hat{f}(\hat{a}_i) \in \hat{V}_i^w$. Hence $\hat{a}_i \in \breve{S}_i$ (thanks to (73)). This shows that $\hat{S}^w \cap \hat{A}_i \subset \breve{S}_i$. Conversely, let $\hat{a}_i \in \breve{S}_i$. Then, by (72), $\hat{a}_i \in \hat{A}_i$, $\Pi_Z(T_i(\hat{a}_i)) \in \hat{V}_i^w - \Pi_Z(b_i)$ and so $\hat{f}(\hat{a}_i) \in \hat{V}_i^w$. Since $\hat{f}(\hat{a}_i) \in \hat{f}(\hat{A}_i) = \Pi_Z(T_i(\hat{A}_i) + b_i)$, $\hat{f}(\hat{a}_i) \notin \hat{f}(\hat{A}) + \mathrm{int}_Z(C_Z)$ (thanks to (71)). Noting that $\hat{A}_i \subset \hat{A}$, it follows that $\hat{f}(\hat{a}_i) \in \hat{f}(\hat{A}) \backslash (\hat{f}(\hat{A}) + \mathrm{int}_Z(C_Z)) = \mathrm{WE}(\hat{f}(\hat{A}), C_Z)$, and so $\hat{a}_i \in \hat{A}_i \cap \hat{f}^{-1}(\mathrm{WE}(\hat{f}(\hat{A}), C_Z)) = \hat{A}_i \cap \hat{S}^w$. This shows that $\breve{S}_i \subset \hat{A}_i \cap \hat{S}^w$. Therefore, $\breve{S}_i = \hat{A}_i \cap \hat{S}^w$. The proof of (i) is complete.

To prove (ii), suppose that the ordering cone C is polyhedral. Then, since the projection mapping $\Pi_Z : \hat{T}(X_1) \oplus Z \to Z$ is a linear operator and since Z is finite dimensional, $C_Z = \Pi_Z((\hat{T}(X_1) \oplus Z) \cap C)$ is a polyhedral cone in Z (thanks to [16, Theorem 19.3] and Proposition 2.1). On the other hand, by the assumption that $(\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C) \neq \emptyset$, Lemma 4.1 implies that

$$\mathrm{int}_Z(C_Z) = \Pi_Z((\hat{T}(X_1) \oplus Z) \cap \mathrm{int}(C)) \neq \emptyset.$$

Since $\Pi_Z(T_j(\hat{A}_j) + b_j)$ and C_Z are polyhedra in the finite dimensional space Z, their sum $\Pi_Z(T_j(\hat{A}_j) + b_j) + C_Z$ is a polyhedron in Z and so is closed. Hence $\Pi_Z(T_j(\hat{A}_j) + b_j) + C_Z = \mathrm{cl}(\Pi_Z(T_j(\hat{A}_j) + b_j) + \mathrm{int}_Z(C_Z))$. Noting that $\Pi_Z(T_j(\hat{A}_j) + b_j) + \mathrm{int}_Z(C_Z)$ is open in Z, it follows that

$$\mathrm{int}_Z(\Pi_Z(T_j(\hat{A}_j) + b_j) + C_Z) = \Pi_Z(T_j(\hat{A}_j) + b_j) + \mathrm{int}_Z(C_Z).$$

Thus, by Proposition 2.1, there exist $(z_{j1}^*, r_{j1}), \cdots, (z_{jq_j}^*, r_{jq_j})$ in $Z^* \times \mathbb{R}$ such that

$$\Pi_Z(T_j(\hat{A}_j) + b_j) + \mathrm{int}_Z(C_Z) = \{z \in Z : \ \langle z_{jk}^*, z\rangle < r_{jk}, \ k = 1 \cdots, q_j\}. \tag{74}$$

Since $\hat{A} = \bigcup_{j\in\bar{I}} \hat{A}_j$, it follows from (71) that

$$\begin{aligned}
\hat{V}_i^w &= \Pi_Z(T_i(\hat{A}_i)+b_i) \setminus \left(\bigcup_{j\in\bar{I}} (\hat{f}(\hat{A}_j) + \mathrm{int}_Z(C_Z) \right) \\
&= \Pi_Z(T_i(\hat{A}_i)+b_i) \setminus \left(\bigcup_{j\in\bar{I}} (\Pi_Z(T_j(\hat{A}_j) + b_j) + \mathrm{int}_Z(C_Z) \right) \\
&= \Pi_Z(T_i(\hat{A}_i)+b_i) \setminus \left(\bigcup_{j\in\bar{I}} \bigcap_{k=1}^{q_j} \{z \in Z : \ \langle z_{jk}^*, z\rangle < r_{jk}\} \right) \\
&= \bigcap_{j\in\bar{I}} \bigcup_{k=1}^{q_j} \left(\Pi_Z(T_i(\hat{A}_i)+b_i) \setminus \{z \in Z : \ \langle z_{jk}^*, z\rangle < r_{jk}\} \right) \\
&= \bigcap_{j\in\bar{I}} \bigcup_{k=1}^{q_j} \left(\Pi_Z(T_i(\hat{A}_i)+b_i) \cap \{z \in Z : \ \langle z_{jk}^*, z\rangle \geq r_{jk}\} \right).
\end{aligned}$$

Since $\bar{I}$ is a subset of $\overline{1m}$, we assume without loss of generality that there exists $n \in \overline{1m}$ such that $\bar{I} = \overline{1n}$. For any $(k_1, \cdots, k_n) \in \overline{1q_1} \times \cdots \times \overline{1q_n}$, let

$$Q^i_{(k_1,\cdots,k_n)} := \bigcap_{j=1}^{n} \left(\Pi_Z(T_i(\hat{A}_i)+b_i) \cap \{z \in Z : \ \langle z_{jk_j}^*, z\rangle \geq r_{jk_j}\} \right).$$

Then, each $Q^i_{(k_1,\cdots,k_n)}$ is a polyhedron in Z and

$$\hat{V}_i^w = \bigcup_{(k_1,\cdots,k_n)\in\Pi_i} Q^i_{(k_1,\cdots,k_n)}, \tag{75}$$

where $\Pi_i := \{(k_1, \cdots, k_n) \in \overline{1q_1} \times \cdots \times \overline{1q_n} : \ Q^i_{(k_1,\cdots,k_n)} \neq \emptyset\}$. Let

$$\hat{P}^i_{(k_1,\cdots,k_n)} := \hat{A}_i \cap (\Pi_Z \circ T_i)^{-1}(Q^i_{(k_1,\cdots,k_n)} - \Pi_Z(b_i)) \quad \forall (k_1, \cdots, k_n) \in \Pi_i.$$

Then each $\hat{P}^i_{(k_1,\cdots,k_n)}$ is a polyhedron in the finite dimensional space X_2 and

$$\check{S}_i = \hat{A}_i \cap (\Pi_Z \circ T_i)^{-1}(\hat{V}_i^w - \Pi_Z(b_i)) = \bigcup_{(k_1,\cdots,k_n)\in\Pi_i} \hat{P}^i_{(k_1,\cdots,k_n)}. \tag{76}$$

Thus, to prove (ii), it suffices to show that for each $(k_1, \cdots, k_n) \in \Pi_i$ there exists a face $\hat{F}$ of $\hat{A}_i$ such that $\hat{P}^i_{(k_1,\cdots,k_n)} \subset \hat{F} \subset \hat{S}_i^w$. By Theorem ABB (applied to linear problem $(\widehat{\mathrm{LP}})_i$), there exist finitely many faces $\hat{F}_{i1} \cdots, \hat{F}_{i\nu_i}$ of $\hat{A}_i$ such that $\hat{S}_i^w = \bigcup_{j=1}^{\nu_i} \hat{F}_{ij}$. Noting that each $\hat{P}^i_{(k_1,\cdots,k_n)}$ is contained in $\hat{S}_i^w$ (thanks to (i) and (76)), it follows from Proposition 2.4 that $\hat{P}^i_{(k_1,\cdots,k_n)} \subset \hat{F}_{ij'}$ for some $j' \in \overline{1\nu_i}$. The proof is complete. □

Formulas (i) and (ii) in Step 3 of the procedure provided in Section 1 are immediate from Theorems 4.1 and 4.3.

To establish formulas for the Pareto solution set and Pareto optimal value set, we need the following lemma, which is a variant of a formula appearing in the proof of [24, Theorem 3.4].

Lemma 4.2. *Let $B_1, \cdots, B_m$ be subsets of Y. Then*

$$\mathrm{E}\left(\bigcup_{i\in\overline{1m}} B_i, C\right) = \bigcup_{i\in\overline{1m}} \bigcap_{j\in\overline{1m}} (\mathrm{E}(B_i, C) \setminus ((B_j + C) \setminus \mathrm{E}(B_j, C))).$$

Proof. Let $B := \bigcup_{i\in\overline{1m}} B_i$ and $E_i := \bigcap_{j\in\overline{1m}} (\mathrm{E}(B_i, C) \setminus ((B_j + C) \setminus \mathrm{E}(B_j, C)))$ for all $i \in \overline{1m}$. We need to show $\mathrm{E}(B, C) = \bigcup_{i=1}^{m} E_i$. For each $y' \in \mathrm{E}(B, C)$, there exists $i' \in \overline{1m}$ such that $y' \in B_{i'}$ and so $y' \in E(B_{i'}, C)$. Since $(B_j + C) \cap \mathrm{E}(B, C) \subset \mathrm{E}(B_j, C)$ for all $j \in \overline{1m}$, $y' \in \mathrm{E}(B_j, C)$ for all $j \in \overline{1m}$ with $y' \in B_j + C$. It follows that $y' \notin (B_j + C) \setminus \mathrm{E}(B_j, C)$ for all $j \in \overline{1m}$. Hence $y' \in E(B_{i'}, C) \setminus ((B_j + C) \setminus \mathrm{E}(B_j, C)))$ for all $j \in \overline{1m}$, that is, $y' \in E_{i'}$. This shows that $\mathrm{E}(B, C) \subset \bigcup_{i\in\overline{1m}} E_i$. Conversely, let $y \in \bigcup_{i=1}^{m} E_i$. Then there exists $i_0 \in \overline{1m}$ such that $y \in E_{i_0}$. Let $z \in B \cap (y - C)$. We only need to show $z = y$. Take $j \in \overline{1m}$ such that $z \in B_j$. It follows that $z \in B_j \cap (y - C)$. Noting that $E_{i_0} \subset \mathrm{E}(B_{i_0}, C)$, it is clear that $z = y$ if $j = i_0$. Now suppose that $j \neq i_0$. By the definition of E_{i_0}, one has $y \in E(B_{i_0}, C) \setminus ((B_j + C) \setminus \mathrm{E}(B_j, C))$, and so $y \notin (B_j + C) \setminus \mathrm{E}(B_j, C)$. Since $y \in z + C \subset B_j + C$, $y \in \mathrm{E}(B_j, C)$. Hence $\{y\} = B_j \cap (y - C) \ni z$. This shows that $y = z$. The proof is complete. □

Proposition 4.2. *Let $\hat{S}$ and $\hat{S}_i$ ($i \in \bar{I} := \{i \in \overline{1m} : \hat{A}_i \neq \emptyset\}$) denote the Pareto solution set of piecewise linear problem $(\widehat{\mathrm{PLP}})$ and linear subproblem $(\widehat{\mathrm{LP}})_i$, respectively. Suppose that the ordering cone C is polyhedral. Then there exist finitely many generalized polyhedra $\hat{F}_1, \cdots, \hat{F}_p$ in X_2 such that the following statements hold:*
(i) $\hat{S} = \bigcup_{k=1}^{p} \hat{F}_k$.
(ii) For each $k \in \overline{1p}$ there exist $i \in \bar{I}$ and a face $\hat{F}$ of $\hat{A}_i$ such that $\hat{F}_k \subset \hat{F} \subset \hat{S}_i$.

Proof. For each $i \in \bar{I}$, let $\tilde{S}_i := \hat{A}_i \cap \hat{S}$. Then $\hat{S} = \bigcup_{i\in\bar{I}} \tilde{S}_i$, and $\tilde{S}_i$ is clearly contained in the Pareto solution set $\hat{S}_i$ of linear subproblem $(\widehat{\mathrm{LP}})_i$. Thus, by Theorem ABB and Proposition 2.4, it suffices to show that there exist finitely many generalized polyhedra $\hat{G}_{i1}, \cdots, \hat{G}_{i\nu_i}$ in X_2 such that $\tilde{S}_i = \bigcup_{k=1}^{\nu_i} \hat{G}_{ik}$. Noting that $\hat{f}|_{\hat{A}_i} = \Pi_Z \circ f|_{\hat{A}_i} = \Pi_Z \circ T_i|_{\hat{A}_i} + \Pi_Z(b_i)$, one has

$$\tilde{S}_i = \hat{A}_i \cap \hat{f}^{-1}(\mathrm{E}(\hat{f}(\hat{A}), C_Z)) = \hat{A}_i \cap (\Pi_Z \circ T_i)^{-1}(\mathrm{E}(\hat{f}(\hat{A}), C_Z) - \Pi_Z(b_i)). \tag{77}$$

Since C is a polyhedral cone in Y, $C \cap (\hat{T}(X_1) \oplus Z)$ is a polyhedral cone in $\hat{T}(X_1) \oplus Z$. Hence $C_Z = \Pi_Z(C \cap (\hat{T}(X_1) \oplus Z))$ is a polyhedral cone in the finite dimensional space Z. It follows that $B_j + C_Z$ is a polyhedron in Z and $\mathrm{E}(B_j, C_Z) = \mathrm{E}(B_j + C_Z, C_Z)$ is the union of finitely many polyhedra in Z for each $j \in \bar{I}$ (thanks to Theorem ABB), where $B_j := \Pi_Z(T_j(\hat{A}_j) + b_j)$. Hence $E_i := \bigcap_{j\in\bar{I}} \mathrm{E}(B_i, C_Z) \setminus (B_j + C_Z) \setminus \mathrm{E}(B_j, C_Z))$ is the union of finitely many generalized polehedra in Z for all $i \in \bar{I}$. Since

$$\hat{f}(\hat{A}) = \bigcup_{i\in\bar{I}} \hat{f}(\hat{A}_i) = \bigcup_{i\in\bar{I}} B_i,$$

This and Lemma 4.2 imply that $\mathrm{E}(\hat{f}(\hat{A}), C_Z) = \bigcup_{i\in\bar{I}} E_i$ and so $\mathrm{E}(\hat{f}(\hat{A}), C_Z)$ is the union of finitely many generalized polyhedra in Z. Thus, by (77), for each $i \in \bar{I}$ there exist finitely many generalized polyhedra $\hat{G}_{i1}, \cdots, \hat{G}_{i\nu_i}$ in X_2 such that $\tilde{S}_i = \bigcup_{k=1}^{\nu_i} \hat{G}_{ik}$. The proof is complete. □

Based on Corollary 4.1 and Proposition 4.2 (and its proof), we can establish exact formulas for the Pareto solution set and Pareto optimal value set of fully piecewise linear vector optimization problem (PLP).

The following corollary establishes the structure of the weak Pareto solution set and Pareto solution set for (PLP).

Corollary 4.2. *Let S^w and S be the weak Pareto solution set and Pareto solution set of fully piecewise linear vector optimization problem (PLP), respectively. Suppose that the ordering cone C is polyhedral. Then the following statements hold:*
(i) There exist finitely many polyhedra $F_1, \cdots, F_p$ in X such that $S^w = \bigcup_{k=1}^{p} F_k$ and each F_k is contained in a weak Pareto face of some linear subproblem (LP)$_i$.
(ii) There exist finitely many generalized polyhedra $F_1, \cdots, F_p$ in X such that $S = \bigcup_{k=1}^{p} F_k$ and F_k is contained in a Pareto face of some linear subproblem (LP)$_i$.

Corollary 4.2 is immediate from Theorem 4.1, Corollary 4.1, Propositions 4.1, 4.2 and 4.3 and Corollary 2.2.

Dropping the polyhedral assumption on the ordering cone C but imposing the C-convexity assumption on $f(A)$, we have the following structure theorem on the weak Pareto solution set of (PLP), which generalizes and improves the corresponding result established by Arrow et al. [2] in the finite dimensional and linear case.

Theorem 4.4. *Let C be a convex cone in Y such that $f(A)$ is C-convex, that is, $f(A) + C$ is a convex subset of Y. Then there exist finitely many polyhedra $F_1, \cdots, F_p$ in X satisfying the following properties:*
(i) $S^w = \bigcup_{k=1}^{p} F_k$.
(ii) For each k there exists $i \in \bar{I}$ such that F_k is a face of A_i and $F_k \subset S^w_i$, where $\bar{I} := \{i \in \overline{1m} : A_i \neq \emptyset\}$ and S^w_i is the weak Pareto solution set of linear subproblem (LP)$_i$.
Consequently each F_k is just a weak Pareto face of linear subproblem (LP)$_i$ for some $i \in \bar{I}$.

Proof. Let $x \in A$. Then $x \in S^w$ if and only if $f(A) \cap (f(x) - \mathrm{int}(C)) = \emptyset$, which is equivalent to $(f(A) + C) \cap (f(x) - \mathrm{int}(C)) = \emptyset$. Thus, by the separation theorem and the convexity of $f(A) + C$,

$x \in S^w$ if and only if there exists $c^* \in C^+ \setminus \{0\}$ such that $\langle c^*, f(x)\rangle = \inf\limits_{u\in A}\langle c^*, f(u)\rangle$. Let

$$S^w(c^*) := \left\{x \in A : \langle c^*, f(x)\rangle = \inf_{u\in A}\langle c^*, f(u)\rangle\right\} \quad \forall c^* \in C^+ \setminus \{0\}$$

and $C^+(f, A) := \{c^* \in C^* \setminus \{0\} : \ S^w(c^*) \neq \emptyset\}$. Then, since the feasible set A of (PLP) is equal to $\bigcup\limits_{i\in\overline{1m}} A_i$, one has $S^w = \bigcup\limits_{c^*\in C^+(f,A)} S^w(c^*) = \bigcup\limits_{c^*\in C^+(f,A)} \bigcup\limits_{i\in\Lambda(c^*)} S^w(c^*) \cap A_i$, where $\Lambda(c^*) := \{i \in \bar{I} : S^w(c^*) \cap A_i \neq \emptyset\}$. On the other hand, for $c^* \in C^+(f, A)$ and $i \in \Lambda(c^*)$,

$$\begin{aligned} S^w(c^*) \cap A_i &= \{x \in A_i : \langle c^*, f(x) = \min_{u\in A_i}\langle c^*, f(u)\rangle\} \\ &= \{x \in A_i : \langle c^*, T_i x + b_i\rangle = \min_{u\in A_i}\langle c^*, T_i u + b_i\rangle\} \\ &= \{x \in A_i : \langle c^*, T_i x\rangle = \min_{u\in A_i}\langle c^*, T_i u\rangle\} \\ &= \{x \in A_i : \langle T_i^*(c^*), x\rangle = \min_{u\in A_i}\langle T_i^*(c^*), u\rangle\} \end{aligned}$$

(thanks to (54) and (55)) is a face of A_i and a subset of the weak Pareto solution set of linear subproblem $(\mathrm{LP})_i$. Since every polyhedron only has finitely many faces, there exist $c_1^*, \cdots, c_p^* \in C^+(f, A)$ such that

$$S^w = \bigcup_{c^*\in C^+(f,A)} \bigcup_{i\in\Lambda(c^*)} S^w(c^*) \cap A_i = \bigcup_{k=1}^{p} \bigcup_{i\in\Lambda(c_k^*)} S^w(c_k^*) \cap A_i.$$

The proof is complete. □

Remark 4.2 If $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, then each set in Y is trivially C-convex. The C-convexity of the set $f(A)$ is equivalent to that the function f is C-convexlike on A(cf. [18]).

In the framework of locally convex topological vector spaces, Luan [9] considered the structure of solution sets for (PLP) when each φ_j is linear. It is worth mentioning that our all results are still valid when X and Y are locally convex topological vector spaces. In fact, noting that Proposition 2.1 is valid when Z is a locally convex topological vector space and that the subspace Z_2 in Proposition 2.1 is finite-dimensional, Z_2 can be regarded as a normed space (because Z_2 and $\mathbb{R}^{\dim(Z_2)}$ are linearly isomorphic and topologically homeomorphous (cf. [17]).

5 Conclusions

In this paper, we clarify relations between two kinds of piecewise linear functions. In particular, it is proved that there does not exist a piecewise (single-valued) function into an infinite dimensional space such that its graph is the union of finitely many polyhedra. With the help of the Fubini theorem on measure theory, we prove in a constructive way that every piecewise linear function between two infinite dimensional spaces can be represented as the sum of a linear operator on an infinite dimensional subspace and a piecewise linear function on a finite dimensional subspace. Based on such a representation, we provide a dimensional reduction method for solving a fully piecewise linear vector optimization problem (PLP). Using such a reduction method, we establish procedures to obtain exact formulas for (weak) optimal value sets and (weak) Pareto solution sets of (PLP), which requires only to solve two systems of linear equations and to compute the projections to the solution space of a system of linear equations. As a consequence, the weak Pareto solution set of (PLP) is proved to be the union of finitely many polyhedra, each of which is contained in a weak Pareto face of some linear subproblem. This is new even in the case where every constraint function is linear.

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