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Abstract We study an n-player game with random payoffs and continuous strategy sets. The payoff function of each player is defined by its expected value and the strategy set of each player is defined by a joint chance constraint. The random constraint vectors defining the joint chance constraint are dependent and follow elliptically symmetric distributions. The Archimedean copula is

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used to capture the dependence among random constraint vectors. We propose a reformulation of the joint chance constraint of each player. Under mild assumptions on the players' payoff functions and 1-dimensional spherical distribution functions, we show that there exists a Nash equilibrium of the game.

Keywords Chance-constrained game · Elliptical distribution · Nash equilibrium · Archimedean copulas.

Mathematics Subject Classification (2000) MSC 90C15 · 90C25 · 90C59

1 Introduction

The publication of the seminal book Researches into the Mathematical Principles of the Theory of Wealth by Cournot in 1838 was the trigger for the widespread use of the equilibrium under market conditions [5]. Later, the saddle point equilibrium for a two player zero-sum game of von Neumann [17] follows on this work. In 1950, Nash [15] showed that for a finite strategic game there exists an equilibrium point, know as Nash equilibrium, from which there is no incentive for any player to deviate unilaterally. Despite its practical limitation, the general strategic games are extensively studied in the literature [1, 6, 7]. The theory of Nash equilibrium in deterministic setup faces challenges especially when it comes to deal with real applications with random payoffs and strategy sets. The most commonly used tool to deal with random payoffs is the expectation function [21] which is more appropriate for risk neutral cases. The risk averse games are studied by considering an alternative payoff criterion based on risk measure CVaR [13, 21] and chance constraint programming [23, 25]. In [23], the authors studied a finite strategic game where the payoff vector of each player is elliptically distributed, and showed the existence of a Nash equilibrium. The equivalence between the Nash equilibrium of chance-constrained games (CCGs for short) in [23] and the global optimal solutions of a certain mathematical program is stated in [25].

In the above referred games, the players payoff functions are random while the strategy sets are deterministic. However, the strategy sets containing chance constraints are often considered in various applications, e.g., risk constraints in portfolio optimization problem [11] and resource constraints in stochastic shortest path problem [4]. Recently, the games with chance constraint based strategy sets are introduced in the literature [18, 19, 20, 26, 27]. Singh and Lisser [26] considered a 2-player zero-sum game with individual chance constraints and showed that a saddle point equilibrium problem is equivalent to a primal-dual pair of second order cone programs when the random constraint vectors follow elliptically symmetric distribution. Singh et al. [27] considered an n-player general-sum game with individual chance constraints under elliptically symmetric distributions and showed that a Nash equilibrium problem is equivalent to the global optimization of a nonlinear optimization problem. In the wake of these results, Peng et al. [19] showed the existence of Nash equilibrium for the n-player general-sum games where the strategy profile set of each player is defined by a joint chance constraint, and the random constraint vectors are either independently normally distributed or follow a mixture of elliptical distributions [20]. When the probability distributions are not completely known and belong to a given distributional uncertainty set, Peng et al. [18] formulated the chance constraints of each player as distributional robust joint chance constrained problem. They consider several uncertainty sets, namely a density based uncertainty set and four two-moments based uncertainty sets where one of them has a nonnegative support. They show that there exists a Nash equilibrium of a distributionally robust joint chance constrained game for each uncertainty set.

In [18, 19, 20], the authors assume that the random constraint vectors are independently distributed. However, the random variables are usually dependent in real world applications. In order to study the dependence structure of random variable, the concept of copula was introduced by Abe Sklar in 1959 [28], as a solution to a probability problem stated by Maurice Fréchet in the context of random metric spaces. Copulas are functions used to distinguish the marginal distributions from a given dependent structure based multivariate distribution. Henrion and Strukgarek [9] introduced the notion of log-exp concavity of copula to investigate the convexity of elliptically dependent distributed joint chance constraints. We refer the reader to [16] for a detailed introduction to the theory of copulas.

In this paper, we extend the results of [19, 20] to the general case where the payoff function is random and the strategy profile set of each player is defined by elliptically distributed dependent joint chance constraints. We derive a new reformulation of joint chance constraint with dependent random constraint vectors and show that there exists a Nash equilibrium of the game under mild conditions on the payoff functions.

The rest of the paper is organized as follows. Section 2 contains the definition of an n-player CCG. In Section 3, we prove the existence of a Nash equilibrium of the CCG under elliptical distributions. We conclude the paper in Section 4.

2 The model and preliminary results

2.1 Chance-constrained game

We consider an n-player CCG, where $H = \{1, 2, .., n\}$ is the set of players. Let $S^i \subset \mathbb{R}^{d_i}$ be the strategy set of player i which is a non-empty, convex and compact set. For each $i \in H$, S^{-i} denotes the set of strategy vectors of all players j, $j \neq i$. A strategy profile $x = (x^1, x^2, \dots, x^n) \in S$ is represented as (x^i, x^{-i}) where x^i denotes the strategy of player i and x^{-i} denotes the vector of strategies of the players other than player i. The strategy set of player i, $i \in H$, is further restricted by the following joint chance constraint

$$\mathbb{P}(V^i x^i \leq D^i) \geq \alpha_i, \tag{1}$$

where $\alpha_i \in [0, 1]$ is a given probability level, $D^i = (D^{i,1}, ..., D^{i,K_i})^T \in \mathbb{R}^{K_i}$ is a deterministic vector and $V^i = [V^{i,1}, ..., V^{i,K_i}]^T$ is a $K_i \times d_i$ random matrix, where $V^{i,k}$ denotes the kth row of matrix V^i; T denotes the transposition. Let $J^i = \{1, 2, \dots, K_i\}$ denotes the index set of ith player's constraints. The feasible strategy set of player i is defined as

$$S^i_{\alpha_i} = \left\{x^i \in S^i \mid \mathbb{P}(V^i x^i \leq D^i) \geq \alpha_i\right\}.$$

We assume that for each $i \in H$, $S^i_{\alpha_i}$ is a non-empty set. Let $\alpha = (\alpha_i)_{i\in H}$ be the confidence level vector and $S_\alpha = \prod_{i=1}^n S^i_{\alpha_i}$ be the set of all feasible strategy profiles. The payoff function of each player is defined using random variables. For each $x \in S_\alpha$, the payoff of player i is given by $f_i(x, \zeta)$ where ζ is an m-dimensional random vector. We use expected value approach to model the payoff function of each player. Therefore, the payoff function of player i is given by

$$p_i(x) = \mathbb{E}[f_i(x, \zeta)], \ \forall\, x \in S_\alpha. \tag{2}$$

We assume that the CCG is of complete information, i.e., the payoff function, the strategy set of each player, and the confidence level vector α are known to all the players.

Definition 2.1 A strategy profile y^* is a Nash equilibrium of the CCG at confidence level vector α if for each $i \in H$

$$p_i(y^{i*}, y^{-i*}) \geq p_i(x^i, y^{-i*}), \ \forall\, x^i \ \in S^i_{\alpha_i}.$$

Assumption 1. *For each player i, $i \in H$, the following conditions hold.*

1. *$f_i(., x^{-i}, \zeta)$ is a concave function of x^i for every $(x^{-i}, \zeta) \in S^{-i} \times \mathbb{R}^m$.*
2. *$f_i(\cdot)$ is a continuous function.*
3. *$p_i(x)$ is finite valued for every $x \in S$.*

Under Assumption 1, the payoff function of player i defined by (2) is a continuous function of x and it is a concave function of x^i for every x^{-i} [20].

2.2 Basic concepts and known results

In this section, we present some basic definitions and important results which are used in our subsequent analysis.

Definition 2.2 An n-dimensional random vector X follows a spherical distribution if there exists a function $\Psi : \mathbb{R} \to \mathbb{R}$ such that the characteristic function $\phi_X(t)$ of X is given by

$$\phi_X(t) = \mathbb{E}(e^{it^T X}) = \Psi(t^T t).$$

The function Ψ is called a characteristic generator of the spherical distribution.

Definition 2.3 An n-dimensional random vector U follows an elliptical distribution with location parameter μ, positive definite scale matrix Σ and characteristic generator Ψ, i.e., $U \sim Ellip(\mu, \Sigma, \Psi)$, if we have the following representation

$$U \stackrel{\mathrm{d}}{=} \mu + AX,$$

where X follows a spherical distribution with a characteristic generator Ψ, $A \in \mathbb{R}^{n\times n}$ such that $AA^T = \Sigma$ and $\mu \in \mathbb{R}^n$; $\stackrel{\mathrm{d}}{=}$ implies that the both sides have the same distribution.

Not all elliptical distributions have a probability density function. Whenever it exists, it has the form

$$f_U(z) = \frac{c}{\sqrt{\det(\Sigma)}} g_s\left(\sqrt{(z-\mu)^T \Sigma^{-1}(z-\mu)}\right),$$

where g_s is a nonnegative function called *radial density* and $c > 0$ is a normalization factor which makes f_U a probability density function.

Definition 2.4 (Definition 2.2, [9]) A real function $f : \mathbb{R} \to \mathbb{R}$ is r-decreasing for some real number $r \in \mathbb{R}$, if f is continuous on $(0, +\infty)$ and there exists some strictly positive real number t^* such that the function $t \mapsto t^r f(t)$ is strictly decreasing on $(t^*, +\infty)$.

Definition 2.5 The cumulative distribution function (CDF) $F : \mathbb{R} \to [0, 1]$ of an 1-dimensional real-valued random variable X has r-decreasing density if the probability density function of X is r-decreasing for some real number $r \in \mathbb{R}$.

Table 1 presents some 1-dimensional spherical distributions with r-decreasing densities for some values of r and their thresholds t^*.

Table 1 List of selected 1-dimensional spherical distributions with r-decreasing density and their thresholds t^*.

Distribution	Radial density	r	t^*
Normal	$e^{-\frac{1}{2}u^2}$	$r > 0$	$\sqrt{r}$
t	$\left(1 + \frac{1}{\nu}u^2\right)^{-(1+\nu)/2}$, $\nu > 0, \nu$ integer	$0 < r < \nu + 1$	$\sqrt{\frac{r\nu}{\nu+1-r}}$
Laplace	$e^{-\lvert u \rvert}$	$r > 0$	$\frac{r}{\sqrt{2}}$
Kotz type	$u^{2(N-1)}e^{-qu^{2s}}$, $q, s > 0, N > \frac{1}{2}$	$r > 2(1 - N)$	$\sqrt[2s]{\frac{2(N-1)+r}{2qs}}$
Pearson type VII	$\left(1 + \frac{u^2}{m}\right)^{-N}$, $m > 0, N > \frac{1}{2}$	$0 < r < 2N$	$\sqrt{\frac{rm}{2N-r}}$

Definition 2.6 A function $C : [0,1]^K \to [0,1]$ is a K-dimensional copula if C is a joint CDF of a K-dimensional random vector, on the unit cube $[0,1]^K$, whose marginals are uniformly distributed on $[0,1]$.

Proposition 2.1 (Sklar's Theorem) *Given $F : \mathbb{R}^K \to [0,1]$ is a joint CDF of a K-dimensional random vector and $F_1, ..., F_K$ are the marginal CDFs, respectively. Then, there exists a K-dimensional copula C such that*

$$F(z) = C\left(F_1(z_1), ..., F_K(z_K)\right).$$

Moreover, if F_i is continuous for any $i = 1, ..., K$, then C is uniquely given by

$$C(u) = F\left(F_1^{(-1)}(u_1), ..., F_K^{(-1)}(u_K)\right).$$

Proposition 2.2 (Fréchet-Hoeffding upper bound) *For any K-dimensional copula C and K-dimensional vector $u = (u_1, ..., u_K) \in [0,1]^K$, we have*

$$C(u) \leq \min_{k=1,...,K} u_k.$$

Definition 2.7 A K-dimensional copula C is strictly Archimedean if there exists a continuous and strictly decreasing function $\psi : (0,1] \to [0,+\infty)$, such that $\psi(1) = 0$, $\lim_{t\to 0} \psi(t) = +\infty$, and for any K-dimensional vector $u = (u_1, ..., u_K) \in [0,1]^K$, we have

$$C(u) = \psi^{(-1)}\left(\sum_{i=1}^{K} \psi(u_i)\right).$$

The function ψ is called a generator of the copula C.

Table 2 presents a selection of some strictly Archimedean copulas with their generators.

Table 2 Different types of strictly Archimedean copulas.

Type of copula	Parameter θ	Generator $\psi_\theta(t)$
Independent	-	-log(t)
Gumbel-Hougaard	$\theta \geq 1$	$[-\log(t)]^\theta$
Frank	$\theta > 0$	$-\log\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$
Clayton	$\theta > 0$	$\frac{1}{\theta}(t^\theta - 1)$
Joe	$\theta \geq 1$	$-\log[1-(1-t)^\theta]$

Definition 2.8 A function $f : \mathbb{R} \to \mathbb{R}$ is K-monotonic on an open interval $I \subseteq \mathbb{R}$ for some positive integer $K \geq 2$, if the following three conditions hold:

1. f is differentiable up to the order $(K-2)$ on I,
2. The derivatives of f are satisfied by
$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0, \quad 0 \leq k \leq K-2,$$
for all $t \in I$,
3. The function $t \mapsto (-1)^{K-2} \frac{d^{K-2}}{dt^{K-2}} f(t)$ is nonincreasing and convex on I.

Proposition 2.3 (Theorem 2.2, [14]) *Given $\psi : (0,1] \to [0,+\infty)$ is a strictly decreasing function such that $\psi(1) = 0$ and $\lim_{t\to 0} \psi(t) = +\infty$. Then, ψ is the generator of a K-dimensional strictly Archimedean copula if and only if the inverse function $\psi^{(-1)}$ is K-monotonic on $(0,+\infty)$ and continuous on $[0,+\infty)$.*

Definition 2.9 A function $f : Q \to (0, +\infty)$ is r-concave on a convex set $Q \subset \mathbb{R}^s$ for a given $r \in (-\infty, +\infty)$ if for any $x, y \in Q$ and $\alpha \in [0, 1]$,

$$f(\alpha x + (1-\alpha)y) \geq [\alpha f(x)^r + (1-\alpha) f(y)^r]^{\frac{1}{r}}, \text{ when } r \neq 0,$$

$$f(\alpha x + (1-\alpha)y) \geq f(x)^\alpha f(y)^{1-\alpha}, \text{ otherwise.}$$

3 Existence of Nash equilibrium

For each $i \in H$, we assume that the random constraint vector $V^{i,k} \sim Ellip(\mu^{i,k}, \Sigma^{i,k}, \Psi^{i,k})$, $k \in J^i$. Let $\lambda_{i,k,min}$ be the smallest eigenvalue of the positive definite matrix $\Sigma^{i,k}$. Define, $\tilde{S}^i_{\alpha_i} = S^i_{\alpha_i} \setminus \{0\}$, then for $x^i \in \tilde{S}^i_{\alpha_i}$, let

$$\begin{aligned} \xi^{i,k}(x^i) &= \frac{(V^{i,k})^T x^i - (\mu^{i,k})^T x^i}{\sqrt{(x^i)^T \Sigma^{i,k} x^i}}. \\ g^{i,k}(x^i) &= \frac{D^{i,k} - (\mu^{i,k})^T x^i}{\sqrt{(x^i)^T \Sigma^{i,k} x^i}}. \end{aligned} \tag{3}$$

It is well known that $\xi^{i,k}(x^i)$ follows 1-dimensional spherical distribution with characteristic generator $\Psi^{i,k}$ [8]. Using the above mentioned notations, the constraint (1) can be written as

$$\mathbb{P}\left\{\xi^{i,k}(x^i) \leq g^{i,k}(x^i),\ k \in J^i\right\} \geq \alpha_i. \tag{4}$$

By Proposition 2.1, (4) can be equivalently written as

$$C^i_{x^i}\left((F^{i,1} \circ g^{i,1})(x^i), ..., (F^{i,K_i} \circ g^{i,K_i})(x^i)\right) \geq \alpha_i, \tag{5}$$

where $C^i_{x^i}$ is the K_i-dimensional copula of the random vector $(\xi^{i,k}(x^i))_{k=1}^{K_i}$ and $F^{i,k}$ is the cumulative distribution function of $\xi^{i,k}(x^i)$; $\circ$ denotes the function composition.

Assumption 2. *There exists a K_i-dimensional copula C^i such that $C^i_{x^i} = C^i$ for all $x^i \in S^i$, and C^i is a K_i-dimensional strictly Archimedean copula with a generator ψ_i such that the inverse function $\psi_i^{(-1)}$ is 4-monotonic on $(0, +\infty)$.*

Remark 3.1 The 4-monotonicity of $\psi_i^{(-1)}$ ensures that $\psi_i^{(-1)}$ is twice differentiable. It follows from Proposition 2.3 that 4-monotonicity condition holds if $K_i \geq 4$.

Under Assumption 2, we can equivalently write (5) as

$$C^i[(F^{i,1} \circ g^{i,1})(x^i), ..., (F^{i,K_i} \circ g^{i,K_i})(x^i)] \geq \alpha_i. \tag{6}$$

Proposition 3.1 *If $x^i \in \tilde{S}^i_{\alpha_i}$ and Assumption 2 holds, the joint chance constraint* (1) *is equivalent to*

$$\begin{cases} (i)\ (F^{i,k} \circ g^{i,k})(x^i) \geq \psi_i^{(-1)}(y_{i,k}\psi_i(\alpha_i)),\ k \in J^i. \\ (ii)\ \displaystyle\sum_{k \in J^i} y_{i,k} = 1,\ y_{i,k} \geq 0,\ k \in J^i. \end{cases} \tag{7}$$

Proof. Let $x^i \in \tilde{S}^i_{\alpha_i}$. Under Assumption 2, the joint chance constraint (1) is equivalent to (6). It is enough to show the equivalence between (6) and (7). Since, C^i is strictly Archimedean copula, (6) is equivalent to

$$\sum_{k \in J^i} (\psi_i \circ F^{i,k} \circ g^{i,k})(x^i) \leq \psi_i(\alpha_i). \tag{8}$$

Define K_i-dimensional vector $y_i = (y_{i,1}, ..., y_{i,K_i}) \in [0,1]^{K_i}$ such that

$$y_{i,k} = \frac{(\psi_i \circ F^{i,k} \circ g^{i,k})(x^i)}{\sum_{j \in J^i} (\psi_i \circ F^{i,j} \circ g^{i,j})(x^i)},\ k \in J^i.$$

From non-increasing property of ψ_i^{-1}, it follows that (x^i, y_i) satisfies (7). Conversely, we assume (x^i, y_i) satisfies (7). By adding all the constraints (i) of (7) after applying $\psi_i(\cdot)$ on both sides, we can say that x^i satisfies (8) which is equivalent to (6). □

The convexity of the feasible strategy set $S^i_{\alpha_i}$ plays a very important role in showing the existence of Nash equilibrium. We show that there exists $\alpha_i^* \in [0,1]$ such that $S^i_{\alpha_i}$ is a convex set for all $\alpha_i \in (\alpha_i^*, 1]$. For each $i \in H$, define an index set $I^{(i)} = \{k \in J^i \mid \mu^{i,k} \neq 0\}$ and a set of real numbers $\{r^{i,k} \mid k \in J^i\}$ such that

$$\left.\begin{aligned} &r^{i,k} > 1, \text{ if } k \in I^{(i)}, \\ &r^{i,k} = 1, \text{ if } k \notin I^{(i)}. \end{aligned}\right\} \tag{9}$$

Lemma 3.1 *Let Assumption 2 holds and for each $k \in J^i$, the CDF $F^{i,k}$ has $(r^{i,k}+1)$-decreasing density with a threshold $t^*_{i,k}$, where $r^{i,k}$ is defined by* (9) *and $t^*_{i,k}$ refers to Definition 2.4. Then, $S^i_{\alpha_i}$ is a convex set for all $\alpha_i \in (\alpha_i^*, 1]$, where*

$$\alpha_i^* := \max\left\{\frac{1}{2}, \max_{k \in I^{(i)}} F^{i,k}\left(\frac{r^{i,k}+1}{r^{i,k}-1}\lambda_{i,k,min}^{-\frac{1}{2}}||\mu^{i,k}||\right), \max_{k \in J^i} F^{i,k}(t^*_{i,k})\right\}. \tag{10}$$

In order to prove Lemma 3.1, we need the three following lemmas.

Lemma 3.2 *Let $\alpha_i \in (\frac{1}{2}, 1]$ and $x^i \in \tilde{S}^i_{\alpha_i}$. Then, $D^{i,k} > (\mu^{i,k})^T x^i$ for all $k \in J^i$.*

Proof. The proof is given in 4. □

Lemma 3.3 *Let $r^{i,1}, ..., r^{i,K_i}$ be the real numbers defined by (9) and for each $k \in I^{(i)}$, define*

$$\Omega^{i,k} := \left\{ x^i \in S^i \mid D^{i,k} - \mu_{i,k}^T x^i > \frac{r^{i,k}+1}{r^{i,k}-1} \lambda_{i,k,min}^{-\frac{1}{2}} ||\mu_{i,k}|| \sqrt{(x^i)^T \Sigma^{i,k} x^i} \right\}.$$

Then,

$$Conv(\tilde{S^i_{\alpha_i}}) \subset \bigcap_{k \in I^{(i)}} \Omega^{i,k},$$

for all $\alpha_i \in (\alpha_i^{**}, 1]$, where

$$\alpha_i^{**} = \max\left\{ \frac{1}{2}, \max_{k \in I^{(i)}} F^{i,k} \left(\frac{r^{i,k}+1}{r^{i,k}-1} \lambda_{i,k,min}^{-\frac{1}{2}} ||\mu^{i,k}|| \right) \right\}, \tag{11}$$

and $Conv$ represents the convex hull. Moreover, for any convex subset $Q^{i,k}$ of $\bigcap_{k \in I^{(i)}} \Omega^{i,k}$ such that $0 \notin Q^{i,k}$, $g^{i,k}(x^i)$ is defined and $(-r^{i,k})$-concave on $Q^{i,k}$ for all $k \in J^i$

Proof. The proof is given in 4. □

Lemma 3.4 *Let Assumption 2 holds. Then, $\psi_i^{(-1)}(y_{i,k}\psi_i(\alpha_i))$ is a convex function of $y_{i,k}$ for all $\alpha_i \in [0, 1]$.*

Proof. The proof is given in 4. □

We present the proof of Lemma 3.1 using the results of Lemma 3.2, Lemma 3.3 and Lemma 3.4.

Proof of Lemma 3.1. Let $\alpha_i \in (\alpha_i^*, 1]$, $\lambda \in [0, 1]$ and $z_1, z_2 \in S^i_{\alpha_i}$. We need to show that $\lambda z_1 + (1 - \lambda) z_2 \in S^i_{\alpha_i}$.

Case 1: Let $z_1 = 0$ or $z_2 = 0$. Without loss of generality, we assume that $z_2 = 0$. This gives $D^{i,k} \geq 0$ for all $k \in J^i$, which in turn implies that

$$\mathbb{P}(V^i \lambda z_1 \leq D^i) \geq \mathbb{P}(V^i z_1 \leq D^i) \geq \alpha_i.$$

Hence, $\lambda z_1 + (1-\lambda) z_2 \in S^i_{\alpha_i}$.

Case 2: Let $z_1 \neq 0$, $z_2 \neq 0$ and $\lambda z_1 + (1-\lambda) z_2 = 0$. In this case, $z_2 = \frac{-\lambda}{1-\lambda} z_1 \in \tilde{S}^i_{\alpha_i}$ and $z_1 \in \tilde{S}^i_{\alpha_i}$. It follows from Lemma 3.2 that

$$(\mu^{i,k})^T z_1 > \frac{\lambda - 1}{\lambda} D^{i,k}, \ (\mu^{i,k})^T z_1 < D^{i,k}, \ \forall \ k \in J^i.$$

This implies that $D^{i,k} \geq 0$ for all $k \in J^i$. Therefore, $\lambda z_1 + (1-\lambda) z_2 = 0 \in S^i_{\alpha_i}$.

Case 3: Let $z_1 \neq 0$, $z_2 \neq 0$ and $0 \in Seg(z_1, z_2)$, where $Seg(z_1, z_2) = \{z_1 + l(z_2 - z_1), 0 \leq l \leq 1\}$. Then, the points on the line segment $Seg(z_1, z_2)$ are either belong to $Seg(z_1, 0)$ or $Seg(0, z_2)$. It follows from Case 1 that $Seg(z_1, 0)$ and $Seg(0, z_2)$ are subset of $S^i_{\alpha_i}$. Therefore, $\lambda z_1 + (1-\lambda) z_2 \in S^i_{\alpha_i}$ for all $\lambda \in [0, 1]$.

Case 4: Let $z_1 \neq 0$, $z_2 \neq 0$ such that $0 \notin Seg(z_1, z_2)$. It is clear that $Seg(z_1, z_2) \subset Conv(\tilde{S}^i_{\alpha_i})$. From Lemma 3.3, $g^{i,k}(\cdot)$ is defined and $(-r^{i,k})$-concave on $Seg(z_1, z_2)$. Therefore,

$$g^{i,k}(\lambda z_1 + (1-\lambda) z_2) \geq \left(\lambda (g^{i,k}(z_1))^{-r^{i,k}} + (1-\lambda)(g^{i,k}(z_2))^{-r^{i,k}} \right)^{-\frac{1}{r^{i,k}}}. \tag{12}$$

Since, $z_1 \in \tilde{S}^i_{\alpha_i}$, from Lemma 3.2 $g^{i,k}(z_1) > 0$ and it follows from (6) that

$$C^i[(F^{i,1} \circ g^{i,1})(z_1), ..., (F^{i,K_i} \circ g^{i,K_i})(z_1)] > \alpha_i^*. \tag{13}$$

By using Proposition 2.2 and the definition of α_i^* from (10), we get

$$F^{i,k}(g^{i,k}(z_1)) > \alpha_i^* \geq F^{i,k}(t_{i,k}^*). \tag{14}$$

This implies that

$$0 < g^{i,k}(z_1)^{-r^{i,k}} < (t_{i,k}^*)^{-r^{i,k}}.$$

Similarly,

$$0 < g^{i,k}(z_2)^{-r^{i,k}} < (t_{i,k}^*)^{-r^{i,k}}.$$

By applying the non-decreasing function $F^{i,k}(\cdot)$ on both side of (12), we can write

$$\begin{aligned}&(F^{i,k} \circ g^{i,k})(\lambda z_1 + (1-\lambda)z_2) \geq \\ &F^{i,k}\left(\left(\lambda(g^{i,k}(z_1))^{-r^{i,k}} + (1-\lambda)(g^{i,k}(z_2))^{-r^{i,k}}\right)^{-\frac{1}{r^{i,k}}}\right).\end{aligned} \tag{15}$$

Since, $F^{i,k}(\cdot)$ has $(r^{i,k}+1)$-decreasing density, from Lemma 3.1 of [9], the function $t \mapsto F^{i,k}\left(t^{-\frac{1}{r^{i,k}}}\right)$ is concave on $(0, (t_{i,k}^*)^{-r^{i,k}})$. Therefore, we can write

$$\begin{aligned}&F^{i,k}\left(\left(\lambda(g^{i,k}(z_1))^{-r^{i,k}} + (1-\lambda)(g^{i,k}(z_2))^{-r^{i,k}}\right)^{-\frac{1}{r^{i,k}}}\right) \\ &\geq \lambda\left((F^{i,k} \circ g^{i,k})(z_1)\right) + (1-\lambda)\left((F^{i,k} \circ g^{i,k})(z_2)\right).\end{aligned} \tag{16}$$

From (15) and (16), we have

$$\begin{aligned}&(F^{i,k} \circ g^{i,k})(\lambda z_1 + (1-\lambda)z_2) \geq \\ &\lambda\left((F^{i,k} \circ g^{i,k})(z_1)\right) + (1-\lambda)\left((F^{i,k} \circ g^{i,k})(z_2)\right).\end{aligned} \tag{17}$$

This implies that the composition function $(F^{i,k} \circ g^{i,k})(\cdot)$ is a concave function over $Seg(z_1, z_2)$. It follows from Lemma 3.4 that $\psi_i^{(-1)}(y_{i,k}\psi_i(\alpha_i))$ is a convex function of $y_{i,k}$. Because $z_1, z_2 \in \tilde{S}^i_{\alpha_i}$ and from Proposition 3.1, $\tilde{S}^i_{\alpha_i}$ and (7) are equivalent, then there exists vectors $(y^1_{i,k})_{k\in J^i}$ and $(y^2_{i,k})_{k\in J^i}$ such that $\left(z_1, (y^1_{i,k})_{k\in J^i}\right)$ and $\left(z_2, (y^2_{i,k})_{k\in J^i}\right)$ are feasible points of (7). Using the fact that $(F^{i,k} \circ g^{i,k})(\cdot)$ is a concave function and $\psi_i^{(-1)}(\cdot)$ is a convex function, we can say that the convex combination of points $\left(z_1, (y^1_{i,k})_{k\in J^i}\right)$ and $\left(z_2, (y^2_{i,k})_{k\in J^i}\right)$ is also a feasible point of (7). Again from the equivalence of $\tilde{S}^i_{\alpha_i}$ and (7), $\lambda z_1 + (1-\lambda) z_2 \in \tilde{S}^i_{\alpha_i}$ which in turn implies that $\lambda z_1 + (1-\lambda) z_2 \in S^i_{\alpha_i}$.

□

Next, we prove that $S^i_{\alpha_i}$ is a closed set.

Lemma 3.5 *The probability function $x^i \mapsto \mathbb{P}(V^i x^i \leq D^i)$ used in the joint chance constraint* (1) *of player i is continuous on $\tilde{S}^i$, where $\tilde{S}^i = S^i \setminus \{0\}$.*

Proof. We can write $\mathbb{P}(V^i x^i \leq D^i)$ as follows

$$\mathbb{P}(V^i x^i \leq D^i) = \mathbb{E}\left(\mathbb{I}_{\{V^i x^i \leq D^i\}}\right) = \mathbb{E}\left(\prod_{k\in J^i} \mathbb{I}_{\{(V^{i,k})^T x^i \leq D^{i,k}\}}\right),$$

where $\mathbb{I}_A$ denotes the indicator function of an event A. Given $x^i \in \tilde{S}^i$ and a sequence $x^i_j \in \tilde{S}^i$ such that $x^i_j \to x^i$ when $j \to +\infty$. For each $k \in J^i$, let $A^{i,k} = \left\{\omega \mid (V^{i,k}(\omega))^T x^i > D^{i,k}\right\}$ and $B^{i,k} = \left\{\omega \mid (V^{i,k}(\omega))^T x^i < D^{i,k}\right\}$. For $\omega \in A^{i,k}$, we have $\mathbb{I}_{\{(V^{i,k})^T x^i \leq D^{i,k}\}}(\omega) = 0$. Since, $x^i_j \to x^i$, there exists a positive integer $N(\omega)$ such that for all $j > N(\omega)$, we have $(V^{i,k}(\omega))^T x^i_j > D^{i,k}$. In other words, $\mathbb{I}_{\left\{(V^{i,k})^T x^i_j \leq D^{i,k}\right\}}(\omega) = 0$, for all $j > N(\omega)$. Hence,

$$\mathbb{I}_{\left\{(V^{i,k})^T x^i_j \leq D^{i,k}\right\}} \to \mathbb{I}_{\{(V^{i,k})^T x^i \leq D^{i,k}\}} \text{ on } A^{i,k}, \quad j \to +\infty.$$

Similarly,

$$\mathbb{I}_{\left\{(V^{i,k})^T x_j^i \leq D^{i,k}\right\}} \to \mathbb{I}_{\{(V^{i,k})^T x^i \leq D^{i,k}\}} \text{ on } B^{i,k}, \quad j \to +\infty.$$

Define $C^{i,k} = \left\{\omega \mid (V^{i,k}(\omega))^T x^i = D^{i,k}\right\}$. Using the notations from (3), $C^{i,k} = \left\{\omega \mid \xi^{i,k}(x^i)(\omega) = g^{i,k}(x^i)\right\}$. Since, $\xi^{i,k}(x^i)$ follows an 1-dimensional real continuous distribution with a density function, $\mathbb{P}(\xi^{i,k}(x^i) = c) = 0$, for any $c \in \mathbb{R}$. In other words, $C^{i,k}$ is a negligible set. Note that $A^{i,k} \cup B^{i,k} \cup C^{i,k} = \Omega$, where Ω is the sample space. Therefore, for each $k \in J^i$, $\mathbb{I}_{\left\{(V^{i,k})^T x_j^i \leq D^{i,k}\right\}} \to \mathbb{I}_{\{(V^{i,k})^T x^i \leq D^{i,k}\}}$ almost everywhere which in turn implies that $\mathbb{I}_{\left\{V^i x_j^i \leq D^i\right\}} \to \mathbb{I}_{\{V^i x^i \leq D^i\}}$ almost everywhere. Moreover, $\mathbb{I}_{\left\{V^i x_j^i \leq D^i\right\}}$ is upper bounded by a positive integrable function $\mathbb{I}_\Omega$. Then, it follows from dominated convergence theorem that $\mathbb{E}(\mathbb{I}_{\left\{V^i x_j^i \leq D^i\right\}}) \to \mathbb{E}(\mathbb{I}_{\{V^i x^i \leq D^i\}})$ as $j \to +\infty$. Therefore, $\mathbb{P}(V^i x^i \leq D^i)$ is a continuous function on $\tilde{S}^i$. □

Lemma 3.6 *The feasible strategy set $S^i_{\alpha_i}$ of player i is a closed set.*

Proof. Given $x^i \in S^i$ and a sequence $x^i_j \in S^i_{\alpha_i}$ such that $x^i_j \to x^i$ when $j \to +\infty$. If $x^i \neq 0$, the proof follows from Lemma 3.5. If $x^i = 0$, we need to prove that $0 \in S^i_{\alpha_i}$. Let $0 \notin S^i_{\alpha_i}$ and $x^i_j \neq 0$ for all $j \in \mathbb{N}$. Then, there exists $k^* \in J^i$ such that $D^{i,k^*} < 0$. For each $j \in \mathbb{N}$, we have

$$\mathbb{P}(V^i x^i_j \leq D^i) \leq \mathbb{P}((V^{i,k^*})^T x^i_j \leq D^{i,k^*}).$$

It follows from the Cauchy-Schwarz inequality that $\mathbb{P}((V^{i,k^*})^T x_j^i \leq D^{i,k^*}) \leq \mathbb{P}(-||V^{i,k^*}|| \times ||x_j^i|| \leq D^{i,k^*}) = \mathbb{P}(||V^{i,k^*}|| \times ||x_j^i|| \geq -D^{i,k^*})$. Hence,

$$\begin{aligned}\mathbb{P}(V^i x_j^i \leq D^i) &\leq \mathbb{P}(||V^{i,k^*}|| \times ||x_j^i|| \geq -D^{i,k^*}).\\ &= \mathbb{P}\left(||V^{i,k^*}|| \geq \frac{-D^{i,k^*}}{||x_j^i||}\right). \end{aligned} \tag{18}$$

As $-D^{i,k^*} > 0$ and $x_j^i \to 0$, we deduce that $\mathbb{P}\left(||V^{i,k^*}|| \geq \frac{-D^{i,k^*}}{||x_j^i||}\right) \to 0$. Then, from (18), we have

$$\mathbb{P}(V^i x_j^i \leq D^i) \to 0, \ \ j \to +\infty. \tag{19}$$

However, as $x_j^i \in S_{\alpha_i}^i$, we deduce that $\mathbb{P}(V^i x_j^i \leq D^i) \geq \alpha_i$, for all $j \in \mathbb{N}$ which contradicts (19). Therefore, $0 \in S_{\alpha_i}^i$. □

The feasible strategy set $S_{\alpha_i}^i$ is a compact set because, from Lemma 3.6, it is a closed subset of the compact set S^i. Finally, we show that there exists a Nash equilibrium of the CCG.

Theorem 3.1 *Consider an n-player CCG defined in Section 2.1, where*

1. *Assumptions 1, 2 hold.*
2. *For each $i \in H$ and $k \in J^i$, $V^{i,k} \sim Ellip(\mu^{i,k}, \Sigma^{i,k}, \Psi^{i,k})$, where $\Sigma^{i,k}$ is a positive definite matrix.*
3. *For each $i \in H$ and $k \in J^i$, suppose CDF $F^{i,k}(\cdot)$ has $(r^{i,k}+1)$-decreasing density with a threshold $t_{i,k}^*$, where $r^{i,k}$ is defined by (9) and $t_{i,k}^*$ refers to Definition 2.4.*

Then, there exists a Nash equilibrium of the CCG for any $\alpha \in (\alpha_1^*, 1] \times ... \times (\alpha_n^*, 1]$, where α_i^*, $i \in H$, is defined by (10).

Proof. Let $\alpha \in (\alpha_1^*, 1] \times ... \times (\alpha_n^*, 1]$. Under Assumption 1, the payoff function $p_i(x^i, x^{-i})$ is a concave function of x^i, for every $x^{-i} \in S^{-i}$, and a continuous function of x. It follows from Lemma 3.1 that the feasible strategy set $S^i_{\alpha_i}$, $i \in H$, is a convex set for all $\alpha_i \in (\alpha_i^*, 1]$. For each $i \in H$, $S^i_{\alpha_i}$ is a compact set. Then, the existence of a Nash equilibrium of the CCG follows from Theorem 4 of [7]. □

4 Conclusion

In this paper, we studied an n-player non-cooperative CCG where the strategy sets of each player is given by joint chance constraint with dependent random constraint vectors which follow elliptically symmetric distributions. We propose a new reformulation of the joint chance constraints based on the family of Archimedean copulas. We assume that 1-dimensional spherical distribution function in the reformulation has r-decreasing densities for some values of r. This condition is satisfied by a list of prominent 1-dimensional spherical distributions. Under mild conditions on the payoff functions, we show that there exists a Nash equilibrium of the CCG.

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Appendices

Appendix A: Proof of Lemma 3.2

Let $x^i \in \tilde{S}^i_{\alpha_i}$. By applying Proposition 2.2 on (6), we get

$$(F^{i,k} \circ g^{i,k})(x^i) \geq \alpha_i > \frac{1}{2}. \tag{20}$$

Since, $F^{i,k}$ is the CDF of an 1-dimensional real-valued random variable which is symmetric at 0, $F^{i,k}(0) = \frac{1}{2}$. From (20) we get $g^{i,k}(x^i) > 0$ which in turn implies that $D^{i,k} - (\mu^{i,k})^T x^i > 0$.

Appendix B: Proof of Lemma 3.3

Let $k \in I^{(i)}$ and $x^i \in \tilde{S}^i_{\alpha_i}$. By applying Proposition 2.2 on (6), we get

$$(F^{i,k} \circ g^{i,k})(x^i) \geq \alpha_i. \tag{21}$$

From the definition of α_i^{**} given in (11), we have

$$F^{i,k}(g^{i,k})(x^i)) > \alpha_i^{**} \geq F^{i,k}\left(\frac{r^{i,k}+1}{r^{i,k}-1}\lambda_{i,k,min}^{-\frac{1}{2}}||\mu^{i,k}||\right). \tag{22}$$

Since, $F^{i,k}(\cdot)$ is a non-decreasing function, from (22) we have

$$D^{i,k} - (\mu^{i,k})^T x^i > \frac{r^{i,k}+1}{r^{i,k}-1}\lambda_{i,k,min}^{-\frac{1}{2}}||\mu^{i,k}||\sqrt{(x^i)^T \Sigma^{i,k} x^i}. \tag{23}$$

Therefore, $\tilde{S}^i_{\alpha_i} \subset \bigcap_{k\in I^{(i)}} \Omega^{i,k}$. For each $k \in I^{(i)}$, $\Omega^{i,k}$ is a convex set which implies that $Conv(\tilde{S}^i_{\alpha_i}) \subset \bigcap_{k\in I^i} \Omega^{i,k}$. We prove the second part of Lemma 3.3 by considering two cases as below:

Case 1: Let $k \notin I^{(i)}$, then $\mu^{i,k} = 0$. From the definition of α_i^{**}, we have $\alpha_i > \frac{1}{2}$. From Lemma 3.2, $D^{i,k} > 0$. In this case, the proof follows directly from Lemma 3 of [2].

Case 2: Let $k \in I^i$. It follows from Lemma 2 of [2] that the function

$$f^{i,k}(x^i) = \left(\frac{\sqrt{(x^i)^T \Sigma^{i,k} x^i}}{D^{i,k} - (\mu^{i,k})^T x^i}\right)^{r^{i,k}}.$$

is defined and a convex function on $\bigcap_{k\in I^{(i)}} \Omega^{i,k}$. Therefore, for any $y, z \in Q^{i,k}$ and $\lambda \in [0,1]$, we have

$$f^{i,k}[\lambda y + (1-\lambda)z] \le \lambda f^{i,k}(y) + (1-\lambda) f^{i,k}(z). \tag{24}$$

Note that $g^{i,k}(x^i) = \left(f^{i,k}(x^i)\right)^{\frac{-1}{r^{i,k}}}$ on $Q^{i,k}$. From (24), we can write

$$g^{i,k}[\lambda y + (1-\lambda)z] \ge \left(\lambda (g^{i,k}(y))^{-r^{i,k}} + (1-\lambda)(g^{i,k}(z))^{-r^{i,k}}\right)^{\frac{-1}{r^{i,k}}}.$$

Hence, $g^{i,k}$ is defined and $(-r^{i,k})$-concave on $Q^{i,k}$.

Appendix C: Proof of Lemma 3.4

Let $U(y_{i,k}) = \psi_i^{(-1)}(y_{i,k}\psi_i(\alpha_i))$. If $\psi_i(\alpha_i) = 0$, the proof is trivial because $U(y_{i,k}) = 1$, for all $y_{i,k} \in [0,1]$. Let $\psi_i(\alpha_i) > 0$. The second-order differentiation of $U(y_{i,k})$ is given by

$$\frac{d^2}{dy_{i,k}^2} U(y_{i,k}) = [\psi_i(\alpha_i)]^2 \times \left(\psi_i^{(-1)}\right)'' (y_{i,k}\psi_i(\alpha_i)),$$

for all $y_{i,k} \in (0,1]$. Since, $\psi_i^{(-1)}$ is 4-monotonic on $(0,+\infty)$, $\left(\psi_i^{(-1)}\right)'' \ge 0$ on $(0,+\infty)$. This implies $\frac{d^2}{dy_{i,k}^2} U(y_{i,k}) \ge 0$ for all $y_{i,k} \in (0,1]$. Therefore, $U(y_{i,k})$ is a convex function of $y_{i,k}$ on $(0,1]$. The convexity of U on $[0,1]$ follows from the continuity of U at 0.

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