# A STRONG CONVERGENCE THEOREM FOR SOLVING AN EQUILIBRIUM PROBLEM AND A FIXED POINT PROBLEM USING THE BREGMAN DISTANCE 

MOSTAFA GHADAMPOUR ${ }^{1}$, EBRAHIM SOORI ${ }^{*, 2}$, RAVI P. AGARWAL ${ }^{3}$, DONAL $\mathrm{O}^{\prime}$ REGAN ${ }^{4}$<br>${ }^{1}$ Department of Mathematics, Payame Noor University P. O. Box 19395-3697, Tehran, Iran.<br>${ }^{2}$ Department of Mathematics, Lorestan University, Lorestan, Khoramabad, Iran.<br>${ }^{3}$ Department of Mathematics Texas A\&M University-Kingsville 700 University Blvd., MSC 172 Kingsville, Texas, USA.<br>${ }^{4}$ School of Mathematics, Statistics, National University of Ireland, Galway, Ireland.


#### Abstract

In this paper, using the Bregman distance, we introduce a new projection-type algorithm for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points. Then the strong convergence of the sequence generated by the algorithm will be established under suitable conditions. Finally, using MATLAB software, we present a numerical example to illustrate the convergence performance of our algorithm.


Keywords: Variational inequality; Bregman nonexpansive mapping; Fixed point problem; Fréchet differentiable; Asymptotical fixed point.

## 1. Introduction

Let $C$ be a nonempty closed and convex subset of a reflexive real Banach space X. Throughout this paper, $X^{*}$ denotes the dual space of $X$. The norm and the duality pairing between $X$ and $X^{*}$ are denoted by $\|$.$\| and \langle.,$.$\rangle , respectively. Now$ $\mathbb{R}$ stands for the set of real numbers. The equilibrium problem for a bifunction $g: C \times C \rightarrow \mathbb{R}$ satisfying the condition $g(x, x)=0$ for every $x \in C$ is stated as follows

Find $y^{*} \in C$ such that $g\left(x, y^{*}\right) \leq 0$,
for all $x \in C$. The set of solutions of (1.1) is denoted by $E P(g)$.

[^0]It is well known that variational inequalities arise in optimal control problems, optimization problems, fixed point problems, partial differential equations, engineering and equilibrium models and hence they have been formulated by many authors in recent years(see [33], [28] ).

Tada and Takahashi [26] proposed the hybrid method for finding the common element of the set of solutions of the monotone equilibrium problem (1.1) and a set of fixed points of a nonexpansive map represented in their algorithm. Anh [3] proposed the hybrid extragradient iteration method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for a pseudomonotone and Lipschitz-type continuous bifunction. Eskandani et al. [14] proposed a hybrid extragradient method and they introduced a new iterative process for approximating a common element of the set of solutions of equilibrium problems involving pseudomonotone bifunctions and the set of common fixed points of a finite family of multi-valued Bregman relatively nonexpansive mappings in Banach spaces. They proved that for any $x \in C$ the mapping $y \rightarrow g(x, y)+D_{f}(x, y)$ has a unique minimizer where $g(x,$.$) is proper, convex, lower$ semicontinuous and $D_{f}$ is the Bregman distance. Also, Jolaoso et al. [16] proved that under some suitable conditions, a point $x^{*} \in E P(g)$ if and only if $x^{*}$ solves the following minimization problem:

$$
\min \left\{\lambda g(x, y)+D_{f}(x, y): y \in C\right\}
$$

In this paper, motivated by the work of Jolaoso et al. [16], we will present a new projection-type algorithm for approximating a common solution of a Bregman nonexpansive mapping which is a solution of (1.1) in the setting of reflexive Banach spaces. Then using MATLAB software, the main result will be illustrated with a numerical example.

## 2. PRELIMINARIES

We present some preliminaries and lemmas which will be used in the next section. Let $f: X \rightarrow(-\infty, \infty]$ be an admissible function, i.e., a proper, convex and lower semicontinuous function. The domain of $f$ is the set $\{x \in X: f(x)<\infty\}$ denoted by $\operatorname{dom} f$. The set of minimizers of f is denoted by $\operatorname{Argmin} f$ and its unique element by $\operatorname{argmin}_{x \in X} f(x)$, if $\operatorname{Argmin} f$ is a singleton. Let $x \in \operatorname{int} \operatorname{dom} f$, for any $y \in X$, the directional derivative of f at $x$ is defined by

$$
\begin{equation*}
f^{\circ}(x, y)=\lim _{t \rightarrow 0} \frac{f(x+t y)-f(x)}{t} \tag{2.1}
\end{equation*}
$$

If the limit (2.1) as $t \rightarrow 0$ exists for each $y$, then $f$ is said to be Gâteaux differentiable at $x$. The function $f$ is said to be Gâteaux differentiable if it is Gâteaux
differentiable for all $x \in \operatorname{int} \operatorname{dom} f$. When the limit as $t \rightarrow 0$ in (2.1) is attained uniformly for any $y \in X$ with $\|y\|=1$, we say that $f$ is Fréchet differentiable at $x$. Finally, $f$ is said to be uniformly Fréchet differentiable on a subset $C$ of $X$ if the limit is attained uniformly at each $x \in C$ and $\|y\|=1$. In this case, the gradient of $f$ at $x$ is the linear function $\nabla f(x)$ which is defined by $\langle y, \nabla f(x)\rangle:=f^{\circ}(x, y)$ for all $y \in X$.

Let $x \in \operatorname{int} \operatorname{dom} f$. The subdifferential of $f$ at $x$ is the convex set defined by

$$
\begin{equation*}
\partial f(x)=\left\{l \in X^{*}: f(x)+\langle y-x, l\rangle \leq f(y), \forall y \in X\right\} \tag{2.2}
\end{equation*}
$$

where the Fenchel conjugate of $f$ is the convex function $f^{*}: X^{*} \rightarrow(-\infty, \infty]$ defined by

$$
f^{*}(l)=\sup \{\langle l, x\rangle-f(x): x \in X\} .
$$

It is well known that by the Young-Fenchel inequality, if $\partial f(x)$ is nonempty, then we have

$$
\langle l, x\rangle \leq f(x)+f^{*}(l), \quad \forall x \in X, l \in X^{*}
$$

and also

$$
f(x)+f^{*}(l)=\langle l, x\rangle \Leftrightarrow l \in \partial f(x)
$$

Let $X$ be a reflexive Banach space. The function $f$ is Legendre if and only if it satisfies the following two conditions:
(L1) int $\operatorname{dom} f \neq \emptyset, f$ is Gâteaux differentiable and $\operatorname{dom} \nabla f=\operatorname{int} \operatorname{dom} f$.
(L2) int $\operatorname{dom} f^{*} \neq \emptyset, f^{*}$ is Gâteaux differentiable and $\operatorname{dom} \nabla f^{*}=\operatorname{int} \operatorname{dom} f^{*}$.
Since $X$ is a reflexive Banach space, $(\partial f)^{-1}=\partial f^{*}$ (see [7, p 83]). Also, we know that $\nabla f=\left(\nabla f^{*}\right)^{-1}$, this together with conditions (L1) and (L2) imply that $\operatorname{ran} \nabla f=\operatorname{dom} \nabla f^{*}=\operatorname{int} \operatorname{dom} f^{*}$ and $\operatorname{ran} \nabla f^{*}=\operatorname{dom} \nabla f=\operatorname{int} \operatorname{dom} f$. In addition, if $X$ is reflexive, then $f$ is Legendre if and only if $f^{*}$ is Legendre (see [5, corollary 5.5]).

Let $f: X \rightarrow(-\infty, \infty]$ be a Gâteaux differentiable function. The bifunction $D_{f}: \operatorname{domf} \times \operatorname{int} \operatorname{domf} \rightarrow[0,+\infty]$ defined by

$$
\begin{equation*}
D_{f}(y, x):=f(y)-f(x)-\langle y-x, \nabla f(x)\rangle, \tag{2.3}
\end{equation*}
$$

is called the Bregman distance with respect to f (see [20]). The Bregman distance does not satisfy the well known properties of a metric. Clearly, $D_{f}(x, x)=0$, but $D_{f}(y, x)=0$ may not imply $x=y$, but when $f$ is Legendre this indeed holds (see [5, Theorem 7.3(vi)]).

The modulus of total convexity at $x \in \operatorname{int} \operatorname{dom} f$ is the function $v_{f}(x,):.[0, \infty) \rightarrow$ $[0, \infty]$, defined by

$$
v_{f}(x, t):=\inf \left\{D_{f}(y, x): y \in \operatorname{dom} f,\|y-x\|=t\right\}
$$

The function $f$ is called totally convex at $x \in \operatorname{int} \operatorname{dom} f$ if $v_{f}(x, t)$ is positive for any $t>0$ [9]. Let $C$ be a nonempty subset of $X$. The modulus of total convexity of $f$ on $C$ is defined by

$$
v_{f}(C, t)=\inf \left\{v_{f}(x, t): x \in C \cap \operatorname{int} \operatorname{dom} f\right\}
$$

The function $f$ is called totally convex on bounded subsets if $v_{f}(C, t)$ is positive for any nonempty and bounded subset $C$ and for any $t>0$.

The following result establishes the characteristic continuity properties for the derivative of a lower semicontinuous convex function.

Proposition 1. [9] Let $f$ be a lower semicontinuous convex function with int $\operatorname{domf} \neq \emptyset$. Then the function $f$ is differentiable at the point $x \in$ int domf if and only if $\partial f(x)$ consists of a single element.

Let $X$ be a Banach space and $f: X \rightarrow(-\infty,+\infty]$ be a convex function. The function $f$ is called sequentially consistent if for any two sequences $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ in $X$, such that the first one is bounded:

$$
\lim _{n \rightarrow \infty} D_{f}\left(y_{n}, x_{n}\right)=0 \Rightarrow \lim _{n \rightarrow \infty}\left\|y_{n}-x_{n}\right\|=0
$$

Lemma 2.1. [22] If $f: X \rightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of $X$, then $\nabla f$ is uniformly continuous on bounded subsets of $X$ from the strong topology of $X$ to the strong topology of $X^{*}$.

Lemma 2.2. [9] If domf contains at least two points, then the function $f$ is totally convex on bounded sets if and only if the function $f$ is sequentially consistent.

Lemma 2.3. [8] Suppose that $f: X \rightarrow(-\infty,+\infty]$ is a Legendre function. The function $f$ is totally convex on bounded subsets if and only if $f$ is uniformly convex on bounded subsets.
Lemma 2.4. [2] Let $f: X \rightarrow Y$ be Gâteaux differentiable at any point of $X$. Given $u, v \in X$ such that $[u, v] \subset X$, then

$$
\|f(u)-f(v)\| \leq \sup \left\{\left\|d_{G} f(w)\right\|: w \in[u, v]\right\}\|u-v\|
$$

where $d_{G} f(w)$ is called the Gâteaux differential of $f$ at $w$.
Lemma 2.5. [24] Let $f: X \rightarrow \mathbb{R}$ be a Legendre function such that $\nabla f^{*}$ is bounded on bounded subsets of int domf*. Let $x_{0} \in X$. If $\left\{D_{f}\left(x_{0}, x_{n}\right)\right\}$ is bounded, then the sequence $\left\{x_{n}\right\}$ is bounded too.

Lemma 2.6. [22] Suppose that $f$ is Gâteaux differentiable and totally convex on int domf. Let $x \in \operatorname{int} \operatorname{domf}$ and $C \subset$ int domf be a nonempty, closed and convex set. If $\hat{x} \in C$, then the following conditions are equivalent:
(i) The vector $\hat{x} \in C$ is the Bregman projection of $x$ onto $C$ with respect to $f$.
(ii) The vector $\hat{x} \in C$ is the unique solution of the variational inequality:

$$
\langle z-y, \nabla f(z)-\nabla f(x)\rangle \leq 0, \quad \forall y \in C
$$

(iii) The vector $\hat{x}$ is the unique solution of the inequality:

$$
D_{f}(y, z)+D_{f}(z, x) \leq D_{f}(y, x), \quad \forall y \in C
$$

The function $V_{f}: X \times X^{*} \rightarrow[0,+\infty]$ is defined by

$$
\begin{equation*}
V_{f}\left(x, x^{*}\right)=f(x)-\left\langle x, x^{*}\right\rangle+f^{*}\left(x^{*}\right), \quad \forall x \in X, x^{*} \in X^{*} \tag{2.4}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
V_{f}\left(x, x^{*}\right)=D_{f}\left(x, \nabla f^{*}\left(x^{*}\right)\right), \quad \forall x \in X, x^{*} \in X^{*} \tag{2.5}
\end{equation*}
$$

Also, by the subdifferential inequality, we obtain

$$
\begin{equation*}
V_{f}\left(x, x^{*}\right)+\left\langle\nabla f^{*}\left(x^{*}\right)-x, y^{*}\right\rangle \leq V_{f}\left(x, x^{*}+y^{*}\right) \tag{2.6}
\end{equation*}
$$

for all $x \in X$ and $x^{*}, y^{*} \in X^{*}[17]$. It is known that $V_{f}$ is convex in the second variable. Hence, for all $z \in X$, we have

$$
\begin{equation*}
D_{f}\left(z, \nabla f^{*}\left(\sum_{i=1}^{N} t_{i} \nabla f\left(x_{i}\right)\right)\right) \leq \sum_{i=1}^{N} t_{i} D_{f}\left(z, x_{i}\right) \tag{2.7}
\end{equation*}
$$

where $\left\{x_{i}\right\} \subset X$ and $\left\{t_{i}\right\} \subset(0,1)$ with $\Sigma_{i=1}^{N} t_{i}=1$.
The Bregman projection $\overleftarrow{\operatorname{Proj}} \underset{C}{f}: \operatorname{int}(\operatorname{dom} f) \rightarrow C$ is defined as the necessarily unique vector $\overleftarrow{\operatorname{Proj}}_{C}^{f}(x) \in C$ that satisfies(see [6])

$$
\begin{equation*}
D_{f}\left(\overleftarrow{\operatorname{Proj}}_{C}^{f}(x), x\right)=\inf \left\{D_{f}(y, x): y \in C\right\} \tag{2.8}
\end{equation*}
$$

Let $X$ be a Banach space. We use $S_{X}$ to denote the unit sphere $S_{X}=\{x \in X$ : $\|x\|=1\}$ and $B_{r}:=\{y \in X:\|y\| \leq r\}$ for all $r>0$, and $B_{r}$ is the closed ball in $X$. Then the function $f: X \rightarrow \mathbb{R}$ is said to be uniformly convex on bounded subsets of $X$ (see [32]) if $\rho_{r}(t)>0$ for all $r, t>0$, where $\rho_{r}:[0, \infty) \rightarrow[0, \infty]$ is defined by

$$
\rho_{r}(t)=\inf _{x, y \in B_{r},\|x-y\|=t, \alpha \in(0,1)} \frac{\alpha f(x)+(1-\alpha) f(y)-f(\alpha x+(1-\alpha) y)}{\alpha(1-\alpha)}
$$

for all $t \geq 0$. The function $\rho_{r}$ is called the gauge of uniform convexity of $f$.
Now, if $f$ is uniformly convex, then the following lemma is known.
Lemma 2.7. [19] Let $X$ be a Banach space, $r>0$ be a constant and $f: X \rightarrow \mathbb{R}$ be a uniformly convex function on bounded subsets of $X$. Then

$$
f\left(\sum_{k=0}^{n} \alpha_{k} x_{k}\right) \leq \sum_{k=0}^{n} \alpha_{k} f\left(x_{k}\right)-\alpha_{i} \alpha_{j} \rho_{r}\left(\left\|x_{i}-x_{j}\right\|\right)
$$

for all $i, j \in\{0,1,2, \ldots, n\}, x_{k} \in B_{r}, \alpha_{k} \in(0,1)$ and $k=0,1,2, \ldots, n$ with $\sum_{k=0}^{n} \alpha_{k}=$ 1 , where $\rho_{r}$ is the gauge of uniform convexity of $f$.

A function $f$ on $X$ is said to be coercive [15] if the sublevel set of $f$ is bounded, equivalently, $\lim _{\|x\| \rightarrow \infty} f(x)=\infty$. A function $f$ on $X$ is said to be strongly coercive [32] if $\lim _{\|x\| \rightarrow \infty} \frac{f(x)}{\|x\|}=\infty$. The function $f$ is also said to be uniformly smooth on
bounded subsets( [32]) if $\lim _{t \rightarrow 0} \frac{\sigma_{r}(t)}{t}=0$ for all $r>0$, where $\sigma_{r}:[0, \infty) \rightarrow[0, \infty]$ is defined by

$$
\sigma_{r}(t)=\sup _{x, y \in B_{r},\|x-y\|=t, \alpha \in(0,1)} \frac{\alpha f(x+(1-\alpha) t y)+(1-\alpha) f(x-\alpha t y)-f(x)}{\alpha(1-\alpha)},
$$

for all $t \geq 0$. We will need the following Propositions.
Proposition 2. [32] Let $f: X \rightarrow \mathbb{R}$ be a convex function which is strongly coercive. Then, the following are equivalent:
(i) $f$ is bounded on bounded subsets and uniformly smooth on bounded subsets of $X$.
(ii) $f$ is Fréchet differentiable and $\nabla f$ is uniformly norm-to-norm continuous on bounded subsets of $X$.
(iii) dom $f^{*}=X^{*}, f^{*}$ is strongly coercive and uniformly convex on bounded subsets of $X^{*}$.

Proposition 3. [32] Let $f: X \rightarrow \mathbb{R}$ be a convex function which is bounded on bounded subsets of $X$. Then the following are equivalent:
(i) $f$ is strongly coercive and uniformly convex on bounded subsets of $X$.
(ii) dom $f^{*}=X^{*}, f^{*}$ is bounded on bounded subsets and uniformly smooth on bounded subsets of $X^{*}$.
(iii) dom $f^{*}=X^{*}, f^{*}$ is Fréchet differentiable and $\nabla f^{*}$ is uniformly norm-tonorm continuous on bounded subsets of $X^{*}$.

Lemma 2.8. [29] Let $C$ be a nonempty convex subset of $X$ and $f: C \rightarrow \mathbb{R}$ be a convex and subdifferentiable function on $C$. Then $f$ attains its minimum at $x \in C$ if and only if $0 \in \partial f(x)+N_{C}(x)$, where $N_{C}(x)$ is the normal cone of $C$ at $x$, that is

$$
N_{C}(x):=\left\{x^{*} \in X^{*}:\left\langle x-y, x^{*}\right\rangle \geq 0, \forall y \in C\right\} .
$$

Lemma 2.9. [12] If $f$ and $g$ are two convex functions on $X$ such that there is $a$ point $x_{0} \in \operatorname{dom} f \cap \operatorname{dom} g$ where $f$ is continuous. Then

$$
\partial(f+g)(x)=\partial f(x)+\partial g(x), \quad \forall x \in X
$$

Let $X$ be a real reflexive Banach space and $C$ be a nonempty, closed and convex subset of $X$. Let $g: C \times C \rightarrow \mathbb{R}$ be a bifunction such that $g(x, x)=0$ for all $x \in C$. The equilibrium problem ( $E P$ ) with respect to $g$ on $C$ is stated as follows:

$$
\begin{equation*}
\text { Find } y^{*} \in C \text { such that } g\left(x, y^{*}\right) \leq 0, \text { for all } x \in C \text {. } \tag{2.9}
\end{equation*}
$$

The solution set of equilibrium problem (2.9) is denoted by $E P(g)$.
Throughout this paper, we assume that $g: C \times C \rightarrow \mathbb{R}$ is a bifunction satisfying the following conditions:
(A1) $g$ is monotone, i.e., $g(x, y)+g(y, x) \leq 0$, for all $x, y \in C$,
(A2) $g$ is pseudomonotone, i.e., $g(x, y) \geq 0 \Rightarrow g(y, x) \leq 0$, for all $x, y \in C$,
(A3) $g$ is a Bregman-Lipschitz-type continuous function, i.e., there exist two positive constants $c_{1}, c_{2}$, such that

$$
g(x, y)+g(y, z) \geq g(x, z)-c_{1} D_{f}(y, x)-c_{2} D_{f}(z, y), \quad \forall x, y, z \in C
$$

where $f: X \rightarrow(-\infty,+\infty]$ is a Legendre function. The constants $c_{1}, c_{2}$ are called Bregman-Lipschitz coefficients with respect to $f$,
(A4) $g$ is weakly continuous on $C \times C$,
(A5) $g(x,$.$) is convex, lower semicontinuous and subdifferentiable on C$ for every fixed $x \in C$,
(A6) $\lim \sup _{t \downarrow 0} g(t x+(1-t) y, z) \leq g(y, z)$, for each $x, y, z \in C$.
Remark 1. [31] Every monotone bifunction on $C$ is pseudo-monotone but the converse is not true. A mapping $A: C \rightarrow X^{*}$ is pseudo-monotone if and only if the bifunction $g(x, y)=\langle A x, y-x\rangle$ is pseudo-monotone on $C$.

We denote the set of fixed points of $S$ by $F(S)$, that is $F(S)=\{x \in C: x \in S x\}$. A point $x \in C$ is called an asymptotic fixed point of $S$ if $C$ contains a sequence $\left\{x_{n}\right\}$ which converges weakly to $x$ such that $\lim _{n \rightarrow \infty}\left\|x_{n}-S x_{n}\right\|=0 . \hat{F}(S)$ denotes the set of asymptotic fixed points of $S$ (see $[11,21]$ ). The mapping $T: C \rightarrow C$ is called Bregman nonexpansive if $D_{f}(T x, T y) \leq D_{f}(x, y)$, for all $x, y \in C$.
Lemma 2.10. [14] Let $C$ be a nonempty closed convex subset of a reflexive Banach space $X$, and $f: X \rightarrow \mathbb{R}$ be a Legendre and strongly coercive function. Suppose that $g: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying A2-A5. For the arbitrary sequences $\left\{x_{n}\right\} \subset C$ and $\left\{\lambda_{n}\right\} \subset(0, \infty)$, let $\left\{y_{n}\right\}$ and $\left\{z_{n}\right\}$ be sequences generated by

$$
\left\{\begin{array}{l}
y_{n}=\operatorname{argmin}\left\{\lambda_{n} g\left(x_{n}, y\right)+D_{f}\left(y, x_{n}\right): y \in C\right\},  \tag{2.10}\\
z_{n}=\operatorname{argmin}\left\{\lambda_{n} g\left(y_{n}, y\right)+D_{f}\left(y, x_{n}\right): y \in C\right\} .
\end{array}\right.
$$

Then, for all $x^{*} \in E P(g)$
$D_{f}\left(x^{*}, z_{n}\right) \leq D_{f}\left(x^{*}, x_{n}\right)-\left(1-\lambda_{n} c_{1}\right) D_{f}\left(y_{n}, x_{n}\right)-\left(1-\lambda_{n} c_{2}\right) D_{f}\left(z_{n}, y_{n}\right)$.
Lemma 2.11. [30] Let $\left\{a_{n}\right\}$ be a sequence of nonnegative real numbers satisfying the inequality:

$$
a_{n+1} \leq\left(1-\alpha_{n}\right) a_{n}+\alpha_{n} \sigma_{n}, \quad \forall \geq 0
$$

where
(a) $\left\{\alpha_{n}\right\} \subset[0,1]$ and $\sum_{n=0}^{\infty} \alpha_{n}=\infty$,
(b) $\lim \sup _{n \rightarrow \infty} \sigma_{n} \leq 0$, or $\sum_{n=0}^{\infty}\left|\alpha_{n} \sigma_{n}\right|<\infty$.

Then, $\lim _{n \rightarrow \infty} a_{n}=0$.
Lemma 2.12. [18] Let $\left\{a_{n}\right\}$ be a sequence of real numbers such that there exists a subsequence $\left\{n_{i}\right\}$ of $\{n\}$ such that $a_{n_{i}}<a_{n_{i}+1}$ for all $i \in \mathbb{N}$. Then, there exists a subsequence $\left\{m_{k}\right\} \subset \mathbb{N}$ such that $m_{k} \rightarrow \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$ :

$$
a_{m_{k}} \leq a_{m_{k}+1} \text { and } a_{k} \leq a_{m_{k}+1}
$$

In fact, $m_{k}=\max \left\{j \leq k: a_{j}<a_{j+1}\right\}$.
We assume that $\varphi: C \times C \rightarrow \mathbb{R}$ is a bifunction satisfying the following conditions:
(B1) $\varphi(x, x)=0$, for all $x \in C$.
(B2) $\varphi$ is monotone, i.e. $\varphi(x, y)+\varphi(y, x) \leq 0$, for all $x, y \in C$.
(B3) $\lim _{t \downarrow 0} \varphi(t z+(1-t) x, y) \leq \varphi(x, y)$, for all $x, y, z \in C$.
(B4) for all $x \in C, y \mapsto \varphi(x, y)$ is convex and lower semicontinuous.
The resolvent of $\varphi$ is the operator $\operatorname{Res}_{\varphi}^{f}: X \rightarrow 2^{C}$ defined by $[13,23]$

$$
\begin{equation*}
\operatorname{Res}_{\varphi}^{f} x=\{z \in C: \varphi(z, y)+\langle\nabla f(z)-\nabla f(x), y-z\rangle \geq 0, \quad \forall y \in C\} \tag{2.11}
\end{equation*}
$$

If $f: X \rightarrow(-\infty, \infty]$ be a Gâteaux differentiable and strongly coercive function and $\varphi$ satisfies conditions B1-B4. Then $\operatorname{dom} \operatorname{Res}_{\varphi}^{f}=X$ (see [23, Lemma 1]).
Lemma 2.13. [23] Suppose that $X$ is a real reflexive Banach space and $C$ a nonempty closed convex subset of $X$. Let $f: X \rightarrow(-\infty, \infty]$ be a Legendre function. If $\varphi$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfies B1-B4, then the followings hold:
(i) $\operatorname{Res}_{\varphi}^{f}$ is single-valued.
(ii) $\operatorname{Res}_{\varphi}^{f}$ is a Bregman firmly nonexpansive operator.
(iii) $F\left(\operatorname{Res}_{\varphi}^{f}\right)=E P(\varphi)$.
(iv) $E P(\varphi)$ is a closed and convex subset of $C$.
(v) $D_{f}(u, x) \geq D_{f}\left(u, \operatorname{Res}_{\varphi}^{f}(x)\right)+D_{f}\left(\operatorname{Res}_{\varphi}^{f}(x), x\right)$, for all $x \in X$ and $u \in$ $F\left(\operatorname{Res}_{\varphi}^{f}\right)$.

## 3. Main Results

Theorem 3.1. Let $C$ be a nonempty closed convex subset of a real reflexive Banach space $X$, and let $f: X \rightarrow \mathbb{R}$ be an admissible, strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $X$. Let $g: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying A1-A5. Assume that $S$ : $C \rightarrow C$ is a Bregman nonexpansive mapping with $\hat{F}(S)=F(S)$. Let $\varphi: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying B1-B4 and $\Omega=F(S) \cap E P(g) \cap F\left(\right.$ Res $\left._{\varphi}^{f}\right) \neq \emptyset$. Suppose that $\left\{x_{n}\right\}$ is a sequence generated by $x_{1} \in C, u \in X$ and

$$
\left\{\begin{array}{l}
y_{n}=\operatorname{argmin}\left\{\lambda_{n} g\left(x_{n}, y\right)+D_{f}\left(y, x_{n}\right): y \in C\right\}, \\
z_{n}=\operatorname{argmin}\left\{\lambda_{n} g\left(y_{n}, y\right)+D_{f}\left(y, x_{n}\right): y \in C\right\}, \\
v_{n}=\nabla f^{*}\left(\delta_{n} \nabla f\left(\operatorname{Res}_{\varphi}^{f} x_{n}\right)+\left(1-\delta_{n}\right) \nabla f\left(\operatorname{Res}_{\varphi}^{f} z_{n}\right)\right), \\
w_{n}=\nabla f^{*}\left(\gamma_{n, 1} \nabla f\left(v_{n}\right)+\gamma_{n, 2} \nabla f\left(z_{n}\right)+\gamma_{n, 3} \nabla f\left(S z_{n}\right)\right),  \tag{3.1}\\
u_{n} \in C \text { such that } \\
\varphi\left(u_{n}, y\right)+\left\langle\nabla f\left(u_{n}\right)-\nabla f\left(w_{n}\right), y-u_{n}\right\rangle \geq 0, \quad \forall y \in C, \\
k_{n}=\nabla f^{*}\left(\beta_{n} \nabla f\left(w_{n}\right)+\left(1-\beta_{n}\right) \nabla f\left(S w_{n}\right)\right), \\
h_{n}=\nabla f^{*}\left(\alpha_{n, 1} \nabla f(u)+\alpha_{n, 2} \nabla f\left(x_{n}\right)+\alpha_{n, 3} \nabla f\left(u_{n}\right)+\alpha_{n, 4} \nabla f\left(k_{n}\right)\right), \\
x_{n+1}=\overleftarrow{\operatorname{Proj}}{ }_{C}^{f} h_{n} .
\end{array}\right.
$$

where $\left\{\delta_{n}\right\} \subset(0,1)$ and $\left\{\alpha_{n, i}\right\}_{i=1}^{4},\left\{\beta_{n}\right\}$ and $\left\{\lambda_{n}\right\}$ satisfy the following conditions:
(i) $\left\{\alpha_{n, i}\right\} \subset(0,1), \sum_{i=1}^{4} \alpha_{n, i}=1, \lim _{n \rightarrow \infty} \alpha_{n, 1}=0, \Sigma_{n=1}^{\infty} \alpha_{n, 1}=\infty, \liminf _{n \rightarrow \infty} \alpha_{n, i} \alpha_{n, j}>$ 0 for all $i \neq j$ and $2 \leq i, j \leq 3$.
(ii) $\left\{\gamma_{n, i}\right\} \subset(0,1), \gamma_{n, 1}+\gamma_{n, 2}+\gamma_{n, 3}=1, \liminf _{n \rightarrow \infty} \gamma_{n, i} \gamma_{n, j}>0$ for all $i \neq j$ and $1 \leq i, j \leq 3$.
(iii) $\left\{\lambda_{n}\right\} \subset[a, b] \subset(0, p)$, where $p=\min \left\{\frac{1}{c_{1}}, \frac{1}{c_{2}}\right\}, c_{1}, c_{2}$ are the BregmanLipschitz coefficients of $g$.
(iv) $\left\{\beta_{n}\right\} \subset(0,1), \liminf _{n \rightarrow \infty} \beta_{n}\left(1-\beta_{n}\right)>0$.

Then, the sequence $\left\{x_{n}\right\}$ converges strongly to $\overleftarrow{\operatorname{Proj}}_{\Omega}^{f} u$.
Proof. First, we show that $\Omega$ is a closed and convex subset of $C$. It follows from [14, Lemma 2.14] that $E P(g)$ is a convex and weakly closed (so closed) subset of $C$. By the conditions (iii) and (iv) of Lemma 2.13, $F\left(\operatorname{Res}_{\varphi}^{f}\right)$ is a closed and convex subset of $C$. Also, similar to [25, Proposition 3.1] $F(S)$ is a closed and convex subset of $C$. Hence, $\Omega$ is closed and convex subset of $C$.

Let $\hat{u}=\overleftarrow{\operatorname{Proj}}_{\Omega}^{f} u$. By (2.7), Lemma 2.10 (v) of Lemma 2.13, we have

$$
\begin{align*}
D_{f}\left(\hat{u}, v_{n}\right) & =D_{f}\left(\hat{u}, \nabla f^{*}\left(\delta_{n} \nabla f\left(\operatorname{Res}_{\varphi}^{f} x_{n}\right)+\left(1-\delta_{n}\right) \nabla f\left(\operatorname{Res}_{\varphi}^{f} z_{n}\right)\right)\right. \\
& \leq \delta_{n} D_{f}\left(\hat{u}, \operatorname{Res}_{\varphi}^{f} x_{n}\right)+\left(1-\delta_{n}\right) D_{f}\left(\hat{u}, \operatorname{Res}_{\varphi}^{f} z_{n}\right) \\
& \leq \delta_{n} D_{f}\left(\hat{u}, x_{n}\right)+\left(1-\delta_{n}\right) D_{f}\left(\hat{u}, z_{n}\right) \\
& \leq D_{f}\left(\hat{u}, x_{n}\right) . \tag{3.2}
\end{align*}
$$

Then from (2.7), (3.2), Lemma 2.10 and the Bregman nonexpansiveness of $S$, we conclude that

$$
\begin{align*}
D_{f}\left(\hat{u}, w_{n}\right) & =D_{f}\left(\hat{u}, \nabla f^{*}\left(\gamma_{n, 1} \nabla f\left(v_{n}\right)+\gamma_{n, 2} \nabla f\left(z_{n}\right)+\gamma_{n, 3} \nabla f\left(S z_{n}\right)\right)\right) \\
& \leq \gamma_{n, 1} D_{f}\left(\hat{u}, v_{n}\right)+\gamma_{n, 2} D_{f}\left(\hat{u}, z_{n}\right)+\gamma_{n, 3} D_{f}\left(\hat{u}, S z_{n}\right) \\
& \leq \gamma_{n, 1} D_{f}\left(\hat{u}, v_{n}\right)+\gamma_{n, 2} D_{f}\left(\hat{u}, z_{n}\right)+\gamma_{n, 3} D_{f}\left(\hat{u}, z_{n}\right) \\
& \leq D_{f}\left(\hat{u}, x_{n}\right) \tag{3.3}
\end{align*}
$$

It follows from (2.7) and the Bregman nonexpansiveness of $S$ that

$$
\begin{align*}
D_{f}\left(\hat{u}, k_{n}\right) & =D_{f}\left(\hat{u}, \nabla f^{*}\left(\beta_{n} \nabla f\left(w_{n}\right)+\left(1-\beta_{n}\right) \nabla f\left(S w_{n}\right)\right)\right) \\
& \leq \beta_{n} D_{f}\left(\hat{u}, w_{n}\right)+\left(1-\beta_{n}\right) D_{f}\left(\hat{u}, S w_{n}\right) \\
& \leq \beta_{n} D_{f}\left(\hat{u}, w_{n}\right)+\left(1-\beta_{n}\right) D_{f}\left(\hat{u}, w_{n}\right) \\
& \leq D_{f}\left(\hat{u}, w_{n}\right) \tag{3.4}
\end{align*}
$$

Therefore, from (3.3),

$$
\begin{equation*}
D_{f}\left(\hat{u}, k_{n}\right) \leq D_{f}\left(\hat{u}, x_{n}\right) \tag{3.5}
\end{equation*}
$$

By (2.11) and the algorithm (4.1), we obtain that $u_{n} \in \operatorname{Res}{ }_{\varphi}^{f} w_{n}$. It follows from Lemma 2.13 that $\operatorname{Res}{ }_{\varphi}^{f}$ is single valued, hence $u_{n}=\operatorname{Res}{ }_{\varphi}^{f} w_{n}$. So from part (v) of Lemma 2.13, we have

$$
\begin{equation*}
D_{f}\left(\hat{u}, u_{n}\right)=D_{f}\left(\hat{u}, \operatorname{Res}_{\varphi}^{f} w_{n}\right) \leq D_{f}\left(\hat{u}, w_{n}\right) \tag{3.6}
\end{equation*}
$$

Then, from (2.7), (3.4), (3.6) and part (iii) of Lemma 2.6, we obtain

$$
\begin{align*}
D_{f}\left(\hat{u}, x_{n+1}\right) & \leq D_{f}\left(\hat{u}, h_{n}\right) \\
& =D_{f}\left(\hat{u}, \nabla f^{*}\left(\alpha_{n, 1} \nabla f(u)+\alpha_{n, 2} \nabla f\left(x_{n}\right)+\alpha_{n, 3} \nabla f\left(u_{n}\right)+\alpha_{n, 4} \nabla f\left(k_{n}\right)\right)\right) \\
& \leq \alpha_{n, 1} D_{f}(\hat{u}, u)+\alpha_{n, 2} D_{f}\left(\hat{u}, x_{n}\right)+\left(\alpha_{n, 3}+\alpha_{n, 4}\right) D_{f}\left(\hat{u}, w_{n}\right) \tag{3.7}
\end{align*}
$$

Therefore, it follows from (3.3) that

$$
\begin{aligned}
D_{f}\left(\hat{u}, x_{n+1}\right) & \leq \alpha_{n, 1} D_{f}(\hat{u}, u)+\left(1-\alpha_{n, 1}\right) D_{f}\left(\hat{u}, x_{n}\right) \\
& \leq \max \left\{D_{f}(\hat{u}, u), D_{f}\left(\hat{u}, x_{n}\right)\right\},
\end{aligned}
$$

hence, by the induction process, we conclude that

$$
D_{f}\left(\hat{u}, x_{n+1}\right) \leq \max \left\{D_{f}(\hat{u}, u), D_{f}\left(\hat{u}, x_{1}\right)\right\}
$$

Now from the above, we have that the sequence $\left\{D_{f}\left(\hat{u}, x_{n}\right)\right\}$ is bounded. From Lemma 2.3, $f$ is a uniformly convex function on bounded subsets. Hence, the condition (i) of Proposition 3 holds, equivalently, the condition (ii) of the proposition holds, i.e., $f^{*}$ is bounded on bounded subsets of $X^{*}$. Thus, $\nabla f^{*}$ is also bounded on bounded subsets of $X^{*}$ (see [4, Proposition 7.8]). From Lemma 2.5, we conclude that the sequence $\left\{x_{n}\right\}$ is bounded. It follows from (3.2), (3.3), (3.6), Lemma 2.10 and boundednes of $\left\{D_{f}\left(\hat{u}, x_{n}\right)\right\}$ that $\left\{D_{f}\left(\hat{u}, z_{n}\right)\right\},\left\{D_{f}\left(\hat{u}, v_{n}\right)\right\},\left\{D_{f}\left(\hat{u}, w_{n}\right)\right\}$ and $\left\{D_{f}\left(\hat{u}, u_{n}\right)\right\}$ are bounded, hence from Lemma 2.5, we conclude that the sequences $\left\{z_{n}\right\},\left\{v_{n}\right\},\left\{w_{n}\right\}$ and $\left\{u_{n}\right\}$ are bounded. Hence $\left\{\nabla f\left(z_{n}\right)\right\},\left\{\nabla f\left(v_{n}\right)\right\}$ and $\left\{\nabla f\left(w_{n}\right)\right\}$ are bounded (see [9, Proposition 1.1.11]). It follows from algorithm (4.1) that $\gamma_{n, 3} \nabla f\left(S z_{n}\right)=\nabla f\left(w_{n}\right)-\gamma_{n, 1} \nabla f\left(v_{n}\right)-\gamma_{n, 2} \nabla f\left(z_{n}\right)$. Then from (ii), $\left\{\nabla f\left(S z_{n}\right)\right\}$ is bounded.

Since $f$ is uniformly Fréchet differentiable and bounded on bounded subsets of $X$, then from Lemma 2.1, $\nabla f$ is uniformly norm-to-norm continuous on bounded subsets of $X$. Also from the assumption that $f$ is a convex and strongly coercive function, we may conclude that the condition (ii) of Proposition 2 holds, equivalently, the condition (iii) of the proposition holds, i.e., $f^{*}$ is uniformly convex on bounded subsets of $X^{*}$. Let $r_{1}=\sup \left\{\left\|\nabla f\left(z_{n}\right)\right\|,\left\|\nabla f\left(S z_{n}\right)\right\|,\left\|\nabla f\left(v_{n}\right)\right\|\right\}$. Since $\left\{\nabla f\left(z_{n}\right)\right\},\left\{\nabla f\left(S z_{n}\right)\right\}$ and $\left\{\nabla f\left(v_{n}\right)\right\}$ are bounded sequences then $r_{1}<\infty$. Hence, by (2.4), (2.5), (3.2), Lemmas 2.7, 2.10 and the Bregman nonexpansiveness of $S$, we have that

$$
\begin{align*}
D_{f}\left(\hat{u}, w_{n}\right)= & V_{f}\left(\hat{u}, \gamma_{n, 1} \nabla f\left(v_{n}\right)+\gamma_{n, 2} \nabla f\left(z_{n}\right)+\gamma_{n, 3} \nabla f\left(S z_{n}\right)\right) \\
= & f(\hat{u})-\left\langle\hat{u}, \gamma_{n, 1} \nabla f\left(v_{n}\right)+\gamma_{n, 2} \nabla f\left(z_{n}\right)+\gamma_{n, 3} \nabla f\left(S z_{n}\right)\right\rangle \\
& +f^{*}\left(\gamma_{n, 1} \nabla f\left(v_{n}\right)+\gamma_{n, 2} \nabla f\left(z_{n}\right)+\gamma_{n, 3} \nabla f\left(S z_{n}\right)\right) \\
\leq & f(\hat{u})-\gamma_{n, 1}\left\langle\hat{u}, \nabla f\left(v_{n}\right)\right\rangle-\gamma_{n, 2}\left\langle\hat{u}, \nabla f\left(z_{n}\right)\right\rangle-\gamma_{n, 3}\left\langle\hat{u}, \nabla f\left(S z_{n}\right)\right\rangle \\
& +\gamma_{n, 1} f^{*}\left(\nabla f\left(v_{n}\right)\right)+\gamma_{n, 2} f^{*}\left(\nabla f\left(z_{n}\right)\right)+\gamma_{n, 3} f^{*}\left(\nabla f\left(S z_{n}\right)\right) \\
& -\gamma_{n, 2} \gamma_{n, 3} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|\right) \\
= & \gamma_{n, 1} V_{f}\left(\hat{u}, \nabla f\left(v_{n}\right)\right)+\gamma_{n, 2} V_{f}\left(\hat{u}, \nabla f\left(z_{n}\right)\right)+\gamma_{n, 3} V_{f}\left(\hat{u}, \nabla f\left(S z_{n}\right)\right) \\
& -\gamma_{n, 2} \gamma_{n, 3} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|\right) \\
= & \gamma_{n, 1} D_{f}\left(\hat{u}, v_{n}\right)+\gamma_{n, 2} D_{f}\left(\hat{u}, z_{n}\right)+\gamma_{n, 3} D_{f}\left(\hat{u}, S z_{n}\right) \\
& -\gamma_{n, 2} \gamma_{n, 3} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|\right) \\
\leq & \gamma_{n, 1} D_{f}\left(\hat{u}, v_{n}\right)+\gamma_{n, 2} D_{f}\left(\hat{u}, z_{n}\right)+\gamma_{n, 3} D_{f}\left(\hat{u}, z_{n}\right) \\
& -\gamma_{n, 2} \gamma_{n, 3} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|\right) \\
\leq & D_{f}\left(\hat{u}, x_{n}\right)-\gamma_{n, 2} \gamma_{n, 3} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|\right), \tag{3.8}
\end{align*}
$$

where $\rho_{r_{1}}^{*}$ is the gauge of uniform convexity of $f^{*}$. Also, by repeating the above process, we obtain

$$
\begin{equation*}
D_{f}\left(\hat{u}, w_{n}\right) \leq D_{f}\left(\hat{u}, x_{n}\right)-\gamma_{n, 1} \gamma_{n, 2} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(v_{n}\right)-\nabla f\left(z_{n}\right)\right\|\right) \tag{3.9}
\end{equation*}
$$

It follows from (3.4) that $\left\{D_{f}\left(\hat{u}, k_{n}\right)\right\}$ is bounded, hence from Lemma 2.5, we conclude that $\left\{k_{n}\right\}$ is bounded. Therefore, $\left\{\nabla f\left(k_{n}\right)\right\}$ is bounded. Hence from $\left(1-\beta_{n}\right) \nabla f\left(S w_{n}\right)=\nabla f\left(k_{n}\right)-\beta_{n} \nabla f\left(w_{n}\right)$, we obtain that $\left\{\nabla f\left(S w_{n}\right)\right\}$ is bounded. In a similar way as above, from (2.4), (2.5), (3.3) and Lemma 2.7, there exists a number $r_{2}=\sup _{n}\left\{\left\|\nabla f\left(w_{n}\right)\right\|,\left\|\nabla f\left(S w_{n}\right)\right\|\right\}$ such that

$$
\begin{equation*}
D_{f}\left(\hat{u}, k_{n}\right) \leq D_{f}\left(\hat{u}, x_{n}\right)-\beta_{n}\left(1-\beta_{n}\right) \rho_{r_{2}}^{*}\left(\left\|\nabla f\left(w_{n}\right)-\nabla f\left(S w_{n}\right)\right\|\right) \tag{3.10}
\end{equation*}
$$

Now, from (3.7) and (3.8), we have

$$
\begin{align*}
D_{f}\left(\hat{u}, x_{n+1}\right) \leq & \alpha_{n, 1} D_{f}(\hat{u}, u)+\alpha_{n, 2} D_{f}\left(\hat{u}, x_{n}\right)+\left(\alpha_{n, 3}+\alpha_{n, 4}\right) D_{f}\left(\hat{u}, x_{n}\right) \\
& -\left(\alpha_{n, 3}+\alpha_{n, 4}\right) \gamma_{n, 2} \gamma_{n, 3} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|\right) \\
= & \alpha_{n, 1} D_{f}(\hat{u}, u)+\left(1-\alpha_{n, 1}\right) D_{f}\left(\hat{u}, x_{n}\right) \\
& -\left(\alpha_{n, 3}+\alpha_{n, 4}\right) \gamma_{n, 2} \gamma_{n, 3} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|\right) . \tag{3.11}
\end{align*}
$$

Also, it is implied from (3.7) and (3.9) that

$$
\begin{align*}
D_{f}\left(\hat{u}, x_{n+1}\right) \leq & \alpha_{n, 1} D_{f}(\hat{u}, u)+\left(1-\alpha_{n, 1}\right) D_{f}\left(\hat{u}, x_{n}\right) \\
& -\left(\alpha_{n, 3}+\alpha_{n, 4}\right) \gamma_{n, 1} \gamma_{n, 2} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(v_{n}\right)-\nabla f\left(z_{n}\right)\right\|\right) \tag{3.12}
\end{align*}
$$

Moreover, from (2.7), (3.3), (3.6), part (iii) of Lemma 2.6 and (3.10), we obtain

$$
\begin{align*}
D_{f}\left(\hat{u}, x_{n+1}\right) \leq & D_{f}\left(\hat{u}, h_{n}\right) \\
\leq & \alpha_{n, 1} D_{f}(\hat{u}, u)+\alpha_{n, 2} D_{f}\left(\hat{u}, x_{n}\right)+\alpha_{n, 3} D_{f}\left(\hat{u}, u_{n}\right)+\alpha_{n, 4} D_{f}\left(\hat{u}, k_{n}\right) \\
\leq & \alpha_{n, 1} D_{f}(\hat{u}, u)+\left(1-\alpha_{n, 1}\right) D_{f}\left(\hat{u}, x_{n}\right) \\
& -\alpha_{n, 4} \beta_{n}\left(1-\beta_{n}\right) \rho_{r_{2}}^{*}\left(\left\|\nabla f\left(w_{n}\right)-\nabla f\left(S w_{n}\right)\right\|\right) \tag{3.13}
\end{align*}
$$

From (2.7), (3.2), Lemma 2.10 and the Bregman nonexpansiveness of $S$, we have

$$
\begin{align*}
D_{f}\left(\hat{u}, w_{n}\right) \leq & \gamma_{n, 1} D_{f}\left(\hat{u}, v_{n}\right)+\gamma_{n, 2} D_{f}\left(\hat{u}, z_{n}\right)+\gamma_{n, 3} D_{f}\left(\hat{u}, S z_{n}\right) \\
\leq & \gamma_{n, 1} D_{f}\left(\hat{u}, x_{n}\right)+\left(\gamma_{n, 2}+\gamma_{n, 3}\right) D_{f}\left(\hat{u}, z_{n}\right) \\
\leq & \gamma_{n, 1} D_{f}\left(\hat{u}, x_{n}\right)+\left(\gamma_{n, 2}+\gamma_{n, 3}\right)\left(D_{f}\left(\hat{u}, x_{n}\right)\right. \\
& \left.-\left(1-\lambda_{n} c_{1}\right) D_{f}\left(y_{n}, x_{n}\right)-\left(1-\lambda_{n} c_{2}\right) D_{f}\left(z_{n}, y_{n}\right)\right) \\
= & D_{f}\left(\hat{u}, x_{n}\right)-\left(\gamma_{n, 2}+\gamma_{n, 3}\right)\left(\left(1-\lambda_{n} c_{1}\right) D_{f}\left(y_{n}, x_{n}\right)\right. \\
& \left.+\left(1-\lambda_{n} c_{2}\right) D_{f}\left(z_{n}, y_{n}\right)\right) . \tag{3.14}
\end{align*}
$$

From (3.7) and (3.14), we have

$$
\begin{aligned}
D_{f}\left(\hat{u}, x_{n+1}\right) \leq & \alpha_{n, 1} D_{f}(\hat{u}, u)+\alpha_{n, 2} D_{f}\left(\hat{u}, x_{n}\right)+\left(\alpha_{n, 3}+\alpha_{n, 4}\right) D_{f}\left(\hat{u}, w_{n}\right) \\
\leq & \alpha_{n, 1} D_{f}(\hat{u}, u)+\alpha_{n, 2} D_{f}\left(\hat{u}, x_{n}\right)+\left(\alpha_{n, 3}+\alpha_{n, 4}\right)\left[D_{f}\left(\hat{u}, x_{n}\right)\right. \\
& \left.-\left(\gamma_{n, 2}+\gamma_{n, 3}\right)\left(\left(1-\lambda_{n} c_{1}\right) D_{f}\left(y_{n}, x_{n}\right)+\left(1-\lambda_{n} c_{2}\right) D_{f}\left(z_{n}, y_{n}\right)\right)\right] \\
= & \alpha_{n, 1} D_{f}(\hat{u}, u)+\left(1-\alpha_{n, 1}\right) D_{f}\left(\hat{u}, x_{n}\right)-\left(\alpha_{n, 3}+\alpha_{n, 4}\right)\left(\gamma_{n, 2}\right. \\
& \left.+\gamma_{n, 3}\right)\left(\left(1-\lambda_{n} c_{1}\right) D_{f}\left(y_{n}, x_{n}\right)+\left(1-\lambda_{n} c_{2}\right) D_{f}\left(z_{n}, y_{n}\right)\right) .
\end{aligned}
$$

Thus it follows that

$$
\begin{align*}
\left(\alpha_{n, 3}+\alpha_{n, 4}\right)\left(\gamma_{n, 2}+\gamma_{n, 3}\right)(1- & \left.\lambda_{n} c_{1}\right) D_{f}\left(y_{n}, x_{n}\right) \\
\leq & \alpha_{n, 1} D_{f}(\hat{u}, u)+D_{f}\left(\hat{u}, x_{n}\right)-D_{f}\left(\hat{u}, x_{n+1}\right) \\
& -\left(\alpha_{n, 3}+\alpha_{n, 4}\right)\left(\gamma_{n, 2}+\gamma_{n, 3}\right)\left(1-\lambda_{n} c_{2}\right) D_{f}\left(z_{n}, y_{n}\right) . \tag{3.15}
\end{align*}
$$

Now by (2.5), (2.6), (2.7), (3.3), (3.5), (3.6) and the part (iii) of Lemma 2.6, we conclude that

$$
\begin{align*}
D_{f}\left(\hat{u}, x_{n+1}\right) \leq & D_{f}\left(\hat{u}, \nabla f^{*}\left(\alpha_{n, 1} \nabla f(u)+\alpha_{n, 2} \nabla f\left(x_{n}\right)+\alpha_{n, 3} \nabla f\left(u_{n}\right)+\alpha_{n, 4} \nabla f\left(k_{n}\right)\right)\right. \\
= & V_{f}\left(\hat{u}, \alpha_{n, 1} \nabla f(u)+\alpha_{n, 2} \nabla f\left(x_{n}\right)+\alpha_{n, 3} \nabla f\left(u_{n}\right)+\alpha_{n, 4} \nabla f\left(k_{n}\right)\right) \\
\leq & V_{f}\left(\hat{u}, \alpha_{n, 1} \nabla f(u)+\alpha_{n, 2} \nabla f\left(x_{n}\right)+\alpha_{n, 3} \nabla f\left(u_{n}\right)+\alpha_{n, 4} \nabla f\left(k_{n}\right)\right. \\
& \left.-\alpha_{n, 1}(\nabla f(u)-\nabla f(\hat{u}))\right)+\alpha_{n, 1}\left\langle h_{n}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle \\
= & V_{f}\left(\hat{u}, \alpha_{n, 2} \nabla f\left(x_{n}\right)+\alpha_{n, 3} \nabla f\left(u_{n}\right)+\alpha_{n, 4} \nabla f\left(k_{n}\right)+\alpha_{n, 1} \nabla f(\hat{u})\right) \\
& +\alpha_{n, 1}\left\langle h_{n}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle \\
= & D_{f}\left(\hat{u}, \nabla f^{*}\left(\alpha_{n, 1} \nabla f(\hat{u})+\alpha_{n, 2} \nabla f\left(x_{n}\right)+\alpha_{n, 3} \nabla f\left(u_{n}\right)+\alpha_{n, 4} \nabla f\left(k_{n}\right)\right)\right) \\
& +\alpha_{n, 1}\left\langle h_{n}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle \\
\leq & \alpha_{n, 1} D_{f}(\hat{u}, \hat{u})+\alpha_{n, 2} D_{f}\left(\hat{u}, x_{n}\right)+\alpha_{n, 3} D_{f}\left(\hat{u}, u_{n}\right)+\alpha_{n, 4} D_{f}\left(\hat{u}, k_{n}\right) \\
& +\alpha_{n, 1}\left\langle h_{n}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle \\
\leq & \alpha_{n, 2} D_{f}\left(\hat{u}, x_{n}\right)+\alpha_{n, 3} D_{f}\left(\hat{u}, x_{n}\right)+\alpha_{n, 4} D_{f}\left(\hat{u}, x_{n}\right) \\
& +\alpha_{n, 1}\left\langle h_{n}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle \\
= & \left(1-\alpha_{n, 1}\right) D_{f}\left(\hat{u}, x_{n}\right)+\alpha_{n, 1}\left\langle h_{n}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle . \tag{3.16}
\end{align*}
$$

Also, from (2.7), we have

$$
\begin{align*}
D_{f}\left(w_{n}, h_{n}\right) \leq & \alpha_{n, 1} D_{f}\left(w_{n}, u\right)+\alpha_{n, 2} D_{f}\left(w_{n}, x_{n}\right)+\alpha_{n, 3} D_{f}\left(w_{n}, u_{n}\right)+\alpha_{n, 4} D_{f}\left(w_{n}, k_{n}\right) \\
\leq & \alpha_{n, 1} D_{f}\left(w_{n}, u\right)+\alpha_{n, 2} D_{f}\left(w_{n}, x_{n}\right)+\alpha_{n, 3} D_{f}\left(w_{n}, u_{n}\right) \\
& +\alpha_{n, 4}\left(\beta_{n} D_{f}\left(w_{n}, w_{n}\right)+\left(1-\beta_{n}\right) D_{f}\left(w_{n}, S w_{n}\right)\right) \\
\leq & \alpha_{n, 1} D_{f}\left(w_{n}, u\right)+\alpha_{n, 2} D_{f}\left(w_{n}, x_{n}\right)+\alpha_{n, 3} D_{f}\left(w_{n}, u_{n}\right) \\
& +\alpha_{n, 4}\left(1-\beta_{n}\right) D_{f}\left(w_{n}, S w_{n}\right) \tag{3.17}
\end{align*}
$$

We now consider the following two possible cases.

## Case1

There exists some $n_{0} \in \mathbb{N}$ such that $\left\{D_{f}\left(\hat{u}, x_{n}\right)\right\}$ is nonincreasing for all $n \geq n_{0}$. Therefore, $\lim _{n \rightarrow \infty} D_{f}\left(\hat{u}, x_{n}\right)$ exists and $D_{f}\left(\hat{u}, x_{n}\right)-D_{f}\left(\hat{u}, x_{n+1}\right) \rightarrow 0$ as $n \rightarrow \infty$. From (3.15), the conditions (i), (ii) and (iii), it follows that $D_{f}\left(y_{n}, x_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$, then by Lemma 2.2 and boundedness of $\left\{x_{n}\right\}$, we obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|y_{n}-x_{n}\right\|=0 \tag{3.18}
\end{equation*}
$$

In a similar way, from (3.15), we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|z_{n}-y_{n}\right\|=0 \tag{3.19}
\end{equation*}
$$

By (3.18) and (3.19), we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|z_{n}-x_{n}\right\|=0 \tag{3.20}
\end{equation*}
$$

It follows from (3.18) that the sequence $\left\{y_{n}\right\}$ is bounded. Next, from (3.11) and the condition (i), we have

$$
\begin{align*}
&\left(\alpha_{n, 3}+\alpha_{n, 4}\right) \gamma_{n, 2} \gamma_{n, 3} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|\right) \\
& \leq \alpha_{n, 1} D_{f}(\hat{u}, u)+\left(1-\alpha_{n, 1}\right) D_{f}\left(\hat{u}, x_{n}\right)-D_{f}\left(\hat{u}, x_{n+1}\right) \\
& \leq \alpha_{n, 1} D_{f}(\hat{u}, u)+D_{f}\left(\hat{u}, x_{n}\right)-D_{f}\left(\hat{u}, x_{n+1}\right) \rightarrow 0, \text { as } n \rightarrow \infty \tag{3.21}
\end{align*}
$$

Hence, it follows from the conditions (i) and (ii) that $\lim _{n \rightarrow \infty} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|\right)=$ 0. Furthermore, we claim that $\lim _{n \rightarrow \infty}\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|=0$. If not, there exist a subsequence $\left\{n_{m}\right\}$ of $\{n\}$ and a positive number $\epsilon_{1}$ such that $\| \nabla f\left(z_{n_{m}}\right)-$ $\nabla f\left(S z_{n_{m}}\right) \| \geq \epsilon_{1}$. Therefore $\rho_{r_{1}}^{*}\left(\left\|\nabla f\left(z_{n_{m}}\right)-\nabla f\left(S z_{n_{m}}\right)\right\|\right) \geq \rho_{r_{1}}^{*}\left(\epsilon_{1}\right)$ for all $m \in \mathbb{N}$, because $\rho_{r_{1}}^{*}$ is nondecreasing. Letting $m \rightarrow \infty$, we conclude that $\rho_{r_{1}}^{*}\left(\epsilon_{1}\right) \leq 0$ which is a contradiction to the uniform convexity of $f^{*}$ on bounded subsets of $X^{*}$. Therefore $\lim _{n \rightarrow \infty}\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|=0$. By Proposition 3 and Lemma 2.3, $\nabla f^{*}$ is uniformly norm-to-norm continuous on bounded subsets of $X^{*}$. Then

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|z_{n}-S z_{n}\right\|=0 \tag{3.22}
\end{equation*}
$$

From (3.12) and (3.13), the conditions (ii), (iv) and a similar technique as in the above, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|v_{n}-z_{n}\right\|=0 \& \lim _{n \rightarrow \infty}\left\|w_{n}-S w_{n}\right\|=0 \tag{3.23}
\end{equation*}
$$

Now, since $\left\{x_{n}\right\}$ is bounded and $X$ is reflexive, then by [27, Theorem 1.2.14], there exists a subsequence $\left\{x_{n_{m}}\right\}$ of $\left\{x_{n}\right\}$ and a point $q \in X$ such that $x_{n_{m}} \rightharpoonup q$. Therefore by (3.20), $z_{n_{m}} \rightharpoonup q$. Using (3.22), we obtain $q \in \hat{F}(S)=F(S)$.

Next, we show that $q \in E P(g)$. For this purpose, let $x, y \in \operatorname{int} \operatorname{dom} f$. We prove that $\nabla f(x)-\nabla f(y) \in \partial_{1} D_{f}(x, y)$ which $\partial_{1} D_{f}(x, y)$ is the subdifferential of $D_{f}(x, y)$ at the first component. We let $\vartheta:=\nabla f(x)-\nabla f(y)$. Note that

$$
\begin{aligned}
0 \geq-D_{f}(z, x)= & f(x)-f(z)-\langle x-z, \nabla f(x)\rangle \\
= & f(x)-f(z)-\langle x-z, \nabla f(y)\rangle-\langle x-z, \vartheta\rangle \\
= & f(x)-f(z)-\langle x-y, \nabla f(y)\rangle-\langle y-z, \nabla f(y)\rangle-\langle x-z, \vartheta\rangle \\
= & f(x)-f(y)-\langle x-y, \nabla f(y)\rangle+f(y)-f(z) \\
& -\langle y-z, \nabla f(y)\rangle-\langle x-z, \vartheta\rangle \\
= & D_{f}(x, y)-D_{f}(z, y)-\langle x-z, \vartheta\rangle,
\end{aligned}
$$

for all $z \in \operatorname{dom} f$. Then

$$
D_{f}(x, y)+\langle z-x, \vartheta\rangle \leq D_{f}(z, y)
$$

for all $z \in \operatorname{dom} f$. Therefore, $\nabla f(x)-\nabla f(y)=\vartheta \in \partial_{1} D_{f}(x, y)$. On the other hand, using the fact that $D_{f}(., x)$ is convex, differentiable and lower semicontinuous on int $\operatorname{dom} f$, it follows from Proposition 1 that $\partial_{1} D_{f}(x, y)=\{\nabla f(x)-\nabla f(y)\}$. Since
$y_{n}=\operatorname{argmin}\left\{\lambda_{n} g\left(x_{n}, y\right)+D_{f}\left(y, x_{n}\right): y \in C\right\}$, from the condition $A 5$, Lemmas 2.8 and 2.9, it follows that

$$
\begin{aligned}
0 \in & \partial\left\{\lambda_{n} g\left(x_{n}, y_{n}\right)+D_{f}\left(y_{n}, x_{n}\right)\right\}+N_{C}\left(y_{n}\right) \\
& =\lambda_{n} \partial_{2} g\left(x_{n}, y_{n}\right)+\partial_{1} D_{f}\left(y_{n}, x_{n}\right)+N_{C}\left(y_{n}\right)
\end{aligned}
$$

Hence, there exist $\zeta_{n} \in \partial_{2} g\left(x_{n}, y_{n}\right)$ and $\eta_{n} \in N_{C}\left(y_{n}\right)$ such that

$$
\begin{equation*}
\lambda_{n} \zeta_{n}+\nabla f\left(y_{n}\right)-\nabla f\left(x_{n}\right)+\eta_{n}=0 . \tag{3.24}
\end{equation*}
$$

Since $\eta_{n} \in N_{C}\left(y_{n}\right)$, then

$$
\begin{equation*}
\left\langle y_{n}-z, \eta_{n}\right\rangle \geq 0, \quad \forall z \in C \tag{3.25}
\end{equation*}
$$

from (3.24) and (3.25), we have

$$
\begin{aligned}
0 \leq\left\langle y_{n}-z\right. & \left.,-\lambda_{n} \zeta_{n}-\nabla f\left(y_{n}\right)+\nabla f\left(x_{n}\right)\right\rangle \\
& =\lambda_{n}\left\langle z-y_{n}, \zeta_{n}\right\rangle-\left\langle y_{n}-z, \nabla f\left(y_{n}\right)-\nabla f\left(x_{n}\right)\right\rangle
\end{aligned}
$$

for all $z \in C$. Therefore

$$
\begin{equation*}
\left\langle y_{n}-z, \nabla f\left(y_{n}\right)-\nabla f\left(x_{n}\right)\right\rangle \leq \lambda_{n}\left\langle z-y_{n}, \zeta_{n}\right\rangle . \tag{3.26}
\end{equation*}
$$

Moreover, since $\zeta_{n} \in \partial_{2} g\left(x_{n}, y_{n}\right)$, then

$$
\begin{equation*}
g\left(x_{n}, y_{n}\right)-g\left(x_{n}, z\right) \leq\left\langle y_{n}-z, \zeta_{n}\right\rangle, \tag{3.27}
\end{equation*}
$$

for all $z \in C$. Now, from (3.26) and (3.27), we have

$$
\frac{1}{\lambda_{n}}\left[\left\langle y_{n}-z, \nabla f\left(y_{n}\right)-\nabla f\left(x_{n}\right)\right\rangle\right] \leq g\left(x_{n}, z\right)-g\left(x_{n}, y_{n}\right),
$$

for all $z \in C$. Now, substituting $n_{m}$ instead of $n$ in the above, we conclude that

$$
\begin{equation*}
\frac{1}{\lambda_{n_{m}}}\left[\left\langle y_{n_{m}}-z, \nabla f\left(y_{n_{m}}\right)-\nabla f\left(x_{n_{m}}\right)\right\rangle\right] \leq g\left(x_{n_{m}}, z\right)-g\left(x_{n_{m}}, y_{n_{m}}\right) \tag{3.28}
\end{equation*}
$$

for all $z \in C$. From (3.18), (3.28), Lemma 2.1, the conditions (iii), $A 4$ and the fact that $x_{n_{m}} \rightharpoonup q$ as $m \rightarrow \infty$, we have that $g(q, z) \geq 0$ for all $z \in C$, then from $A 2$, we have $g(z, q) \leq 0$, for all $z \in C$, i.e., $q \in E P(g)$.

Now, we show that $q \in F\left(\operatorname{Res}_{\varphi}^{f}\right)$. Since $\left\{u_{n}\right\}$ is bounded, hence $\left\{\nabla f\left(u_{n}\right)\right\}$ is bounded. Let $r_{3}=\sup _{n}\left\{\|\nabla f(u)\|,\left\|\nabla f\left(x_{n}\right)\right\|,\left\|\nabla f\left(u_{n}\right)\right\|,\left\|\nabla f\left(k_{n}\right)\right\|\right\}$. Then by (2.4),
(2.5), (3.3), (3.5), (3.6) and Lemmas 2.6, 2.7, we have

$$
\begin{align*}
D_{f}\left(\hat{u}, x_{n+1}\right) \leq & D_{f}\left(\hat{u}, h_{n}\right) \\
= & V_{f}\left(\hat{u}, \alpha_{n, 1} \nabla f(u)+\alpha_{n, 2} \nabla f\left(x_{n}\right)+\alpha_{n, 3} \nabla f\left(u_{n}\right)+\alpha_{n, 4} f\left(k_{n}\right)\right) \\
= & f(\hat{u})-\left\langle\hat{u}, \alpha_{n, 1} \nabla f(u)+\alpha_{n, 2} \nabla f\left(x_{n}\right)+\alpha_{n, 3} \nabla f\left(u_{n}\right)+\alpha_{n, 4} \nabla f\left(k_{n}\right)\right\rangle \\
& +f^{*}\left(\alpha_{n, 1} \nabla f(u)+\alpha_{n, 2} \nabla f\left(x_{n}\right)+\alpha_{n, 3} \nabla f\left(u_{n}\right)+\alpha_{n, 3} \nabla f\left(k_{n}\right)\right) \\
\leq & f(\hat{u})-\alpha_{n, 1}\langle\hat{u}, \nabla f(u)\rangle-\alpha_{n, 2}\left\langle\hat{u}, \nabla f\left(x_{n}\right)\right\rangle-\alpha_{n, 3}\left\langle\hat{u}, \nabla f\left(u_{n}\right)\right\rangle \\
& -\alpha_{n, 4}\left\langle\hat{u}, \nabla f\left(k_{n}\right)\right\rangle+\alpha_{n, 1} f^{*}(\nabla f(u))+\alpha_{n, 2} f^{*}\left(\nabla f\left(x_{n}\right)\right) \\
& +\alpha_{n, 3} f^{*}\left(\nabla f\left(u_{n}\right)\right)+\alpha_{n, 4} f^{*}\left(\nabla f\left(k_{n}\right)\right) \\
& -\alpha_{n, 2} \alpha_{n, 3} \rho_{r_{3}}^{*}\left(\left\|\nabla f\left(x_{n}\right)-\nabla f\left(u_{n}\right)\right\|\right) \\
= & \alpha_{n, 1} V_{f}(\hat{u}, \nabla f(u))+\alpha_{n, 2} V_{f}\left(\hat{u}, \nabla f\left(x_{n}\right)\right)+\alpha_{n, 3} V_{f}\left(\hat{u}, \nabla f\left(u_{n}\right)\right) \\
& +\alpha_{n, 4} V_{f}\left(\hat{u}, \nabla f\left(k_{n}\right)\right)-\alpha_{n, 2} \alpha_{n, 3} \rho_{r_{3}}^{*}\left(\left\|\nabla f\left(x_{n}\right)-\nabla f\left(u_{n}\right)\right\|\right) \\
= & \alpha_{n, 1} D_{f}(\hat{u}, u)+\alpha_{n, 2} D_{f}\left(\hat{u}, x_{n}\right)+\alpha_{n, 3} D_{f}\left(\hat{u}, u_{n}\right)+\alpha_{n, 4} D_{f}\left(\hat{u}, k_{n}\right) \\
& -\gamma_{n, 2} \gamma_{n, 3} \rho_{r_{1}}^{*}\left(\left\|\nabla f\left(z_{n}\right)-\nabla f\left(S z_{n}\right)\right\|\right) \\
\leq & \alpha_{n, 1} D_{f}(\hat{u}, u)+\left(1-\alpha_{n, 1}\right) D_{f}\left(\hat{u}, x_{n}\right) \\
& -\alpha_{n, 2} \alpha_{n, 3} \rho_{r_{3}}^{*}\left(\left\|\nabla f\left(x_{n}\right)-\nabla f\left(u_{n}\right)\right\|\right) . \tag{3.29}
\end{align*}
$$

Hence, it follows from the condition (i) and a similar technique as (3.22) that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|x_{n}-u_{n}\right\|=0 \tag{3.30}
\end{equation*}
$$

Note $\left\{x_{n}\right\},\left\{z_{n}\right\}$ are bounded and $\nabla f$ is bounded on bounded subsets of $X$ and also $\nabla f^{*}$ is bounded on bounded subsets of $X^{*}$. Then consequently $\left\{w_{n}\right\}$ is bounded. In a similar way, it follows that $\left\{k_{n}\right\}$ and $\left\{h_{n}\right\}$ are bounded.

From Lemma 2.4, the inequalities (2.3), (3.22) and (3.23), we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} D_{f}\left(z_{n}, v_{n}\right)=0, \lim _{n \rightarrow \infty} D_{f}\left(z_{n}, S z_{n}\right)=0, \lim _{n \rightarrow \infty} D_{f}\left(w_{n}, S w_{n}\right)=0 \tag{3.31}
\end{equation*}
$$

also from (2.7), we have

$$
D_{f}\left(z_{n}, w_{n}\right) \leq \gamma_{n, 1} D_{f}\left(z_{n}, v_{n}\right)+\gamma_{n, 2} D_{f}\left(z_{n}, z_{n}\right)+\gamma_{n, 3} D_{f}\left(z_{n}, S z_{n}\right)
$$

so from (3.31), we obtain that

$$
\lim _{n \rightarrow \infty} D_{f}\left(z_{n}, w_{n}\right)=0
$$

Now, from Lemma 2.2, we conclude that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|z_{n}-w_{n}\right\|=0 \tag{3.32}
\end{equation*}
$$

By Lemma 2.4, the inequalities (3.20), (3.30) and (3.32), we obtain that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} D_{f}\left(w_{n}, x_{n}\right)=0, \quad \lim _{n \rightarrow \infty} D_{f}\left(w_{n}, u_{n}\right)=0 \tag{3.33}
\end{equation*}
$$

Then, as the above, we have

$$
\lim _{n \rightarrow \infty}\left\|u_{n}-w_{n}\right\|=0
$$

Therefore, from Lemma 2.1, it is implied that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|\nabla f\left(u_{n}\right)-\nabla f\left(w_{n}\right)\right\|=0 \tag{3.34}
\end{equation*}
$$

From (3.30) and $x_{n_{m}} \rightharpoonup q$, we obtain that $u_{n_{m}} \rightharpoonup q$. Since $u_{n}=\operatorname{Res}{ }_{\varphi}^{f} w_{n}$, then we have

$$
\varphi\left(u_{n}, y\right)+\left\langle\nabla f\left(u_{n}\right)-\nabla f\left(w_{n}\right), y-u_{n}\right\rangle \geq 0, \quad \forall y \in C
$$

Now, substituting $n_{m}$ instead of $n$ in the above, we conclude that

$$
\varphi\left(u_{n_{m}}, y\right)+\left\langle\nabla f\left(u_{n_{m}}\right)-\nabla f\left(w_{n_{m}}\right), y-u_{n_{m}}\right\rangle \geq 0, \quad \forall y \in C
$$

By (B2), we obtain

$$
\begin{equation*}
\left\langle\nabla f\left(u_{n_{m}}\right)-\nabla f\left(w_{n_{m}}\right), y-u_{n_{m}}\right\rangle \geq-\varphi\left(u_{n_{m}}, y\right) \geq \varphi\left(y, u_{n_{m}}\right), \quad \forall y \in C \tag{3.35}
\end{equation*}
$$

Now, we know from (B4) that $\varphi(x,$.$) is convex and lower semicontinuous. Then$ from [1, Propositions 1.9 .13 and 2.5.2], it is also weakly lower semicontinuous. Hence, letting $m \rightarrow \infty$, we conclude from (3.34) and (3.35) that $\varphi(y, q) \leq 0$ for all $y \in C$. Then from (2.9), $q \in E P(\varphi)$. So by the condition (iii) of Lemma 2.13, we have that $q \in F\left(\operatorname{Res}_{\varphi}^{f}\right)$. Therefore $q \in \Omega$.

It follows from (3.17), (3.31), (3.33), the boundedness of $\left\{D_{f}\left(w_{n}, u\right)\right\}$ and the conditions (i), (iv) that $\lim _{n \rightarrow \infty} D_{f}\left(w_{n}, h_{n}\right)=0$. Hence, from Lemma 2.2 and the boundedness of $\left\{h_{n}\right\}$, we have

$$
\lim _{n \rightarrow \infty}\left\|w_{n}-h_{n}\right\|=0
$$

Then from (3.20), (3.32) and the above, we obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|x_{n}-h_{n}\right\|=0 \tag{3.36}
\end{equation*}
$$

Now, we show that $\limsup _{n \rightarrow \infty}\left\langle h_{n}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle \leq 0$. Since $\left\{x_{n}\right\}$ is bounded, we conclude that there exists a subsequence $\left\{x_{n_{k}}\right\}$ of $\left\{x_{n}\right\}$ which

$$
\limsup _{n \rightarrow \infty}\left\langle x_{n}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle=\lim _{k \rightarrow \infty}\left\langle x_{n_{k}}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle
$$

Since $\left\{x_{n_{k}}\right\}$ is bounded then there exists a subsequence $\left\{x_{n_{k_{i}}}\right\}$ of $\left\{x_{n_{k}}\right\}$ that converges weakly to some $\kappa \in \Omega$. Without loss of generality, we can assume that $x_{n_{k}} \rightharpoonup \kappa$. Then from Lemma 2.6, we have that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left\langle x_{n}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle=\langle\kappa-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\rangle \leq 0 \tag{3.37}
\end{equation*}
$$

Next, It follows from (3.36) and (3.37) that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left\langle h_{n}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle=\limsup _{n \rightarrow \infty}\left\langle x_{n}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle \leq 0 \tag{3.38}
\end{equation*}
$$

Therefore, from (3.16), (3.38) and Lemma 2.11, we have $\lim _{n \rightarrow \infty} D_{f}\left(\hat{u}, x_{n}\right)=0$. Therefore from Lemma 2.2, $x_{n} \rightarrow \hat{u}$ as $n \rightarrow \infty$.

## Case 2

There exists a subsequence $\left\{n_{j}\right\}$ of $\{n\}$, such that

$$
D_{f}\left(\hat{u}, x_{n_{j}}\right)<D_{f}\left(\hat{u}, x_{n_{j}+1}\right)
$$

for all $j \in \mathbb{N}$. Hence, from Lemma 2.12, there exists a subsequence $\left\{m_{k}\right\} \subset \mathbb{N}$, such that $m_{k} \rightarrow \infty$ and the following properties are satisfied:

$$
\begin{equation*}
D_{f}\left(\hat{u}, x_{m_{k}}\right) \leq D_{f}\left(\hat{u}, x_{m_{k}+1}\right) \text { and } D_{f}\left(\hat{u}, x_{k}\right) \leq D_{f}\left(\hat{u}, x_{m_{k}+1}\right) \tag{3.39}
\end{equation*}
$$

for all $k \in \mathbb{N}$. From (3.15), we have

$$
\begin{aligned}
& \left(\alpha_{m_{k}, 3}+\alpha_{m_{k}, 4}\right)\left(\gamma_{m_{k}, 2}+\gamma_{m_{k}, 3}\right)\left(1-\lambda_{m_{k}} c_{1}\right) D_{f}\left(y_{m_{k}}, x_{m_{k}}\right) \\
& \leq
\end{aligned}
$$

for all $k \in \mathbb{N}$. Then using (3.39), the conditions (i), (ii) and (iii), it follows that $D_{f}\left(y_{m_{k}}, x_{m_{k}}\right) \rightarrow 0$ as $k \rightarrow \infty$. So by Lemma 2.2 and the boundedness of $\left\{x_{m_{k}}\right\}$, we obtain

$$
\lim _{n \rightarrow \infty}\left\|y_{m_{k}}-x_{m_{k}}\right\|=0
$$

In a similar way, from (3.15), we have

$$
\lim _{k \rightarrow \infty}\left\|y_{m_{k}}-z_{m_{k}}\right\|=0 .
$$

Now proceeding with the same argument as in Case 1, we conclude that

$$
\begin{align*}
\limsup _{k \rightarrow \infty}\left\langle h_{m_{k}}-\hat{u},\right. & \nabla f(u)-\nabla f(\hat{u})\rangle \\
& =\limsup _{k \rightarrow \infty}\left\langle x_{m_{k}}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle \leq 0 . \tag{3.40}
\end{align*}
$$

Also, from (3.16) and (3.39), we obtain

$$
\begin{aligned}
& D_{f}\left(\hat{u}, x_{m_{k}+1}\right) \\
& \qquad \\
& \quad \leq\left(1-\alpha_{m_{k}, 1}\right) D_{f}\left(\hat{u}, x_{m_{k}}\right)+\alpha_{m_{k}, 1}\left\langle h_{m_{k}}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle \\
& \quad \leq\left(1-\alpha_{m_{k}, 1}\right) D_{f}\left(\hat{u}, x_{m_{k}+1}\right)+\alpha_{m_{k}, 1}\left\langle h_{m_{k}}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle,
\end{aligned}
$$

hence,

$$
D_{f}\left(\hat{u}, x_{m_{k}+1}\right) \leq\left\langle h_{m_{k}}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle .
$$

Then, it follows from (3.39) that

$$
\begin{equation*}
D_{f}\left(\hat{u}, x_{k}\right) \leq D_{f}\left(\hat{u}, x_{m_{k}+1}\right) \leq\left\langle h_{m_{k}}-\hat{u}, \nabla f(u)-\nabla f(\hat{u})\right\rangle . \tag{3.41}
\end{equation*}
$$

From (3.40) and (3.41), we conclude that $\lim _{k \rightarrow \infty} D_{f}\left(\hat{u}, x_{k}\right)=0$. Therefore from Lemma 2.2, $x_{k} \rightarrow \hat{u}$ as $k \rightarrow \infty$. This completes the proof.

## 4. Applications and numerical examples

The following theorem will be concluded from theorem 3.1.
Theorem 4.1. Let $C$ be a nonempty closed convex subset of a real reflexive Banach space $X$, and let $f: X \rightarrow \mathbb{R}$ be an admissible, strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $X$. Let $g: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying A1-A5. Assume that $S: C \rightarrow C$ is a Bregman nonexpansive mapping with $\hat{F}(S)=F(S)$. Let $\Omega=F(S) \cap E P(g) \neq \emptyset$. Suppose that $\left\{x_{n}\right\}$ is a sequence generated by $x_{1} \in C$, $u \in X$ and

$$
\left\{\begin{array}{l}
y_{n}=\operatorname{argmin}\left\{\lambda_{n} g\left(x_{n}, y\right)+D_{f}\left(y, x_{n}\right): y \in C\right\} \\
z_{n}=\operatorname{argmin}\left\{\lambda_{n} g\left(y_{n}, y\right)+D_{f}\left(y, x_{n}\right): y \in C\right\}, \\
v_{n}=\nabla f^{*}\left(\delta_{n} \nabla f\left(\overleftarrow{\operatorname{Proj}}{ }_{C}^{f} x_{n}\right)+\left(1-\delta_{n}\right) \nabla f\left(\overleftarrow{\operatorname{Proj}}{ }_{C}^{f} z_{n}\right)\right), \\
w_{n}=\nabla f^{*}\left(\gamma_{n, 1} \nabla f\left(v_{n}\right)+\gamma_{n, 2} \nabla f\left(z_{n}\right)+\gamma_{n, 3} \nabla f\left(S z_{n}\right)\right),  \tag{4.1}\\
u_{n}=\overleftarrow{\operatorname{Proj}}{ }_{C}^{f} w_{n} \\
k_{n}=\nabla f^{*}\left(\beta_{n} \nabla f\left(w_{n}\right)+\left(1-\beta_{n}\right) \nabla f\left(S w_{n}\right)\right), \\
h_{n}=\nabla f^{*}\left(\alpha_{n, 1} \nabla f(u)+\alpha_{n, 2} \nabla f\left(x_{n}\right)+\alpha_{n, 3} \nabla f\left(u_{n}\right)+\alpha_{n, 4} \nabla f\left(k_{n}\right)\right), \\
x_{n+1}=\overleftarrow{\operatorname{Proj}_{C}^{f} h_{n}}
\end{array}\right.
$$

where $\left\{\delta_{n}\right\} \subset(0,1)$ and $\left\{\alpha_{n, i}\right\}_{i=1}^{4},\left\{\beta_{n}\right\}$ and $\left\{\lambda_{n}\right\}$ satisfy the following conditions:
(i) $\left\{\alpha_{n, i}\right\} \subset(0,1), \sum_{i=1}^{4} \alpha_{n, i}=1, \lim _{n \rightarrow \infty} \alpha_{n, 1}=0, \Sigma_{n=1}^{\infty} \alpha_{n, 1}=\infty, \liminf _{n \rightarrow \infty} \alpha_{n, i} \alpha_{n, j}>$ 0 for all $i \neq j$ and $2 \leq i, j \leq 3$.
(ii) $\left\{\gamma_{n, i}\right\} \subset(0,1), \gamma_{n, 1}+\gamma_{n, 2}+\gamma_{n, 3}=1, \liminf _{n \rightarrow \infty} \gamma_{n, i} \gamma_{n, j}>0$ for all $i \neq j$ and $1 \leq i, j \leq 3$.
(iii) $\left\{\lambda_{n}\right\} \subset[a, b] \subset(0, p)$, where $p=\min \left\{\frac{1}{c_{1}}, \frac{1}{c_{2}}\right\}, c_{1}, c_{2}$ are the BregmanLipschitz coefficients of $g$.
(iv) $\left\{\beta_{n}\right\} \subset(0,1), \liminf _{n \rightarrow \infty} \beta_{n}\left(1-\beta_{n}\right)>0$.

Then, the sequence $\left\{x_{n}\right\}$ converges strongly to $\overleftarrow{\operatorname{Proj}}_{\Omega}^{f} u$.
Proof. Putting $\varphi(x, y)=0$ for all $x, y \in C$ in Theorem 3.1, by Lemma 2.6, we have $\operatorname{Res}_{\varphi}^{f}=\widehat{\operatorname{Proj}}_{C}^{f}, u_{n}=\widehat{\operatorname{Proj}}_{C}^{f} w_{n}$ and $F\left(\operatorname{Res}_{\varphi}^{f}\right)=C$.
Example 4.2. Let $X=\mathbb{R}, C=[0,2], f()=.\frac{1}{2}\|\cdot\|^{2}$ and define the bifunction $g: C \times C \rightarrow \mathbb{R}$ by

$$
g(x, y):=16 y^{2}+9 x y-25 x^{2}
$$

for all $x, y \in C$. Next, $g$ satisfies the conditions A1-A6 as follows:
(A1) $g$ is monotone:

$$
g(x, y)+g(y, x)=16 y^{2}+9 x y-25 x^{2}+16 x^{2}+9 x y-25 y^{2}=-9(x-y)^{2} \leq 0
$$

for all $x, y \in C$.
(A2) $g$ is pseudomonotone, for all $x, y \in C$, because $g$ is monotone.
(A3) $g$ is Bregman-Lipschitz-type continuous with $c_{1}=c_{2}=9$ :

$$
\begin{align*}
g(x, y)+g(y, z)-g(x, z) & =16 y^{2}+9 x y-25 x^{2} \\
& +16 z^{2}+9 y z-25 y^{2}-16 z^{2}-9 x z+25 x^{2} \\
= & -9\left(y^{2}-x y-y z+x z\right) \\
= & -\frac{9}{2}\left(y^{2}-2 x y+x^{2}+y^{2}-2 y z+z^{2}-x^{2}+2 x z-z^{2}\right) \\
= & -9 D_{f}(x, y)-9 D_{f}(y, z)+9 D_{f}(x, z) \\
\geq & -9 D_{f}(x, y)-9 D_{f}(y, z) \tag{4.2}
\end{align*}
$$

(A4) Note since $C \subseteq \mathbb{R}$ is weakly compact we need only show that $g: C \times C \rightarrow \mathbb{R}$ is weakly sequentially continuous. Let $x, y \in C$ and let $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ be two sequences in $C$ converging weakly to $x$ and $y$, respectively (note the weak and strong convergence in $\mathbb{R}$ are equivalent), and then

$$
\lim _{n \rightarrow \infty} g\left(x_{n}, y_{n}\right)=\lim _{n \rightarrow \infty} 16 y_{n}^{2}+9 x_{n} y_{n}-25 x_{n}^{2}=16 y^{2}+9 x y-25 x^{2}=g(x, y)
$$

(A5) Note $g(x,$.$) is convex, lower semicontinuous and subdifferentiable on C$ for every fixed $x \in C$.
(A6) Note

$$
\begin{align*}
& \limsup _{t \rightarrow 0} g(t x+(1-t) y, z)= \\
& \limsup _{t \rightarrow 0}\left[16 z^{2}+9(t x+(1-t) y) z-25(t x+(1-t) y)^{2}\right] \\
& =g(y, z) \tag{4.3}
\end{align*}
$$

for each $x, y, z \in C$. Define $S: C \rightarrow C$ by $S(x)=\frac{x}{3}$, for all $x \in C$. Hence, $F(S)=\{0\}$ and

$$
\begin{aligned}
D_{f}(S x, S y) & =D_{f}\left(\frac{x}{3}, \frac{y}{3}\right) \\
= & f\left(\frac{x}{3}\right)-f\left(\frac{y}{3}\right)-\left\langle\frac{x}{3}-\frac{y}{3}, \frac{y}{3}\right\rangle \\
= & \frac{x^{2}}{18}-\frac{y^{2}}{18}-\frac{x y}{9}+\frac{y^{2}}{9} \\
= & \frac{1}{18}(x-y)^{2} \leq \frac{1}{2}(x-y)^{2}=D_{f}(x, y)
\end{aligned}
$$

for all $x, y \in C$. Therefore $S$ is Bregman nonexpansive. Let $p \in \hat{F(S)}$ then $C$ contains a sequence $\left\{x_{n}\right\}$ such that $x_{n} \rightharpoonup p$ and $\left.\lim _{n \rightarrow \infty}\left(S x_{n}-x_{n}\right)\right)=0$, then $p=0$. So $\hat{F}(S)=\{0\}=F(S)$. Now, define the bifunction $\varphi: C \times C \rightarrow \mathbb{R}$ by $\varphi(x, y)=0$ for all $x, y \in C$. It is clear that $\varphi$ satisfies the conditions B1-B4. By Lemma 2.6, we conclude that $\operatorname{Res}_{\varphi}^{f}=\operatorname{Proj}_{C}^{f}$. Now, if $\lambda_{n}=\frac{1}{32}$ by definition of $y_{n}$ in our algorithm, we have

$$
y_{n}=\operatorname{argmin}\left\{\frac{1}{32} g\left(x_{n}, y\right)+D_{f}\left(y, x_{n}\right): y \in C\right\}
$$

therefore, $y_{n}=\frac{23}{64} x_{n}$. Similarly

$$
z_{n}=\operatorname{argmin}\left\{\frac{1}{32} g\left(y_{n}, y\right)+D_{f}\left(y, x_{n}\right): y \in C\right\}
$$

hence, $z_{n}=\frac{1841}{(64)^{2}} x_{n}$. Also $v_{n}=\delta_{n} P_{C} x_{n}+\left(1-\delta_{n}\right) P_{C} z_{n}, w_{n}=\gamma_{n, 1} v_{n}+\gamma_{n, 2} z_{n}+$ $\frac{1}{3} \gamma_{n, 3} z_{n}, u_{n}=P_{C} w_{n}, k_{n}=\beta_{n} w_{n}+\left(1-\beta_{n}\right) \frac{1}{3} w_{n}=\left(\frac{1}{3}+\frac{2}{3} \beta_{n}\right) w_{n}, h_{n}=\alpha_{n, 1} u+$ $\alpha_{n, 2} x_{n}+\alpha_{n, 3} u_{n}+\alpha_{n, 4} k_{n}$ and $x_{n+1}=P_{C} h_{n}$. We choose $\alpha_{n, 1}=\frac{1}{4 n}, \alpha_{n, 2}=\alpha_{n, 3}=$ $\alpha_{n, 4}=\frac{1}{3}-\frac{3}{4 n}, \beta_{n}=\frac{1}{2}+\frac{1}{n}, \delta_{n}=\frac{1}{2}$ and $\gamma_{n, 1}=\gamma_{n, 2}=\gamma_{n, 3}=\frac{1}{3}$. See the table ?? and Figure ?? with the initial point $x_{1}=5$ of the sequence $\left\{x_{n}\right\}$.

## References

[1] Agarwal, R. P., O'Regan, D. and Sahu, D. R.: Fixed point theory for Lipschitzian-type mappings with applications, vol. 6, Springer, New York, (2009)
[2] Ambrosetti, A., Prodi, G.: A Primer of Nonlinear Analysis. Cambridge University Press, Cambridge (1993)
[3] Anh, P.N.: A hybrid extragradient method for pseudomonotone equilibrium problems and fixed point problems. Bull. Malays. Math. Sci. Soc. 36(1), 107-116 (2013)
[4] Bauschke, H. H., Borwein, J. M.: On projection algorithms for solving convex feasibility problems, SIAM Rev. 38 367-426 (1996)
[5] Bauschke, H.H., Borwein, J. M., Combettes, P. L.: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, Communications in Contemporary Mathematics, 3, 615-647 (2001)
[6] Bregman, L.M.: A relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Comput. Math. Math. Phys. 7, 200-217 (1967)
[7] Bonnans, J.F., Shapiro, A.: Perturbation Analysis of Optimization Problems. Springer, New York (2000)
[8] Butnariu, D., Iusem, A.N., Z alinescu, C.: On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces. J. Convex Anal. 10, 35-61 (2003)
[9] Butnariu, D., Iusem, A.N.: Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization. Kluwer Academic Publishers, Dordrecht (2000)
[10] Butnariu, D., Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. Abstr. Appl. Anal. Art. ID 84919, 1-39 (2006)
[11] Censor, Y., Reich, S.: Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization. Optimization. 37, 323-339, (1996)
[12] Cioranescu, I.: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems. Kluwer Academic Publishers, Dordrecht (1990)
[13] Combettes, P. L., Hirstoaga S. A.: Equilibrium programming in Hilbert spaces. Journal of Nonlinear and Convex Analysis, 6, 117-136, (2005)
[14] Eskandani, G.Z., Raeisi, M., Rassias, T.M.: A hybrid extragradient method for solving pseudomonotone equilibrium problems using Bregman distance. J. Fixed Point Theory Appl. 20, 132 (2018). https://doi.org/10.1007/s11784-018-0611-9
[15] Hiriart-Urruty J-B, Lemarchal C: Grundlehren der mathematischen Wissenschaften 306. In Convex Analysis and Minimization Algorithms II. Springer, Berlin; (1993)
[16] Jolaoso, L.O., Taiwo, A., Alakoya, T.O.: A Strong Convergence Theorem for Solving Pseudomonotone Variational Inequalities Using Projection Methods. J Optim Theory Appl 185, 744-766 (2020) https://doi.org/10.1007/s10957-020-01672-3
[17] Kohsaka, F., Takahashi,W.: Proximal point algorithm with Bregman functions in Banach spaces. J. Nonlinear Convex Anal. 6, 505-523 (2005)
[18] Maing'e, P.E.: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-valued Anal. 16, 899-912 (2008)
[19] Naraghirad, E., Yao, J.C.: Bregman weak relatively nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. (2013). https://doi.org/10.1186/ 1687-1812-2013-141
[20] Reem, D., Reich, S., De Pierro, A.: Re-examination of Bregman functions and new properties of their divergences. Optimization 68, 279-348 (2019)
[21] Reich, S.: A weak convergence theorem for the alternating method with Bregman distances. in Theory and applications of Nonlinear Operators of Accretive and Monotone Type, pp. 313318,Marcel Dekker, NewYork, NY,USA, (1996)
[22] Reich, S., Sabach, S.: A strong convergence theorem for proximal type- algorithm in reflexive Banach spaces. J. Nonlinear Convex Anal. 10, 471-485 (2009)
[23] Reich, S., Sabach, S.: Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces.Nonlinear Analysis: Theory,Methods \& Applications. 73, 122-135 (2010)
[24] Sabach, S.: Products of finitely many resolvents of maximal monotone mappings in reflexive banach spaces. SIAM J. Optim. 21, 1289-1308 (2011)
[25] Shahzad, N., Zegeye, H.: Convergence theorem for common fixed points of a finite family of multi-valued Bregman relatively nonexpansive mappings. Fixed Point Theory Appl. (2014). https://doi.org/10.1186/1687-1812-2014-152
[26] Tada, A., Takahashi,W.: Strong convergence theorem for an equilibrium problem and a nonexpansive mapping. In: Takahashi, W., Tanaka, T. (eds.) Nonlinear Analysis and Convex Analysis. Yokohama Publishers, Yokohama (2006)
[27] Takahashi, W.: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
[28] Thong, D.V., Dung, V.T. Cho, Y.J.: A new strong convergence for solving split variational inclusion problems. Numer Algor 86, 565-591 (2021). https://doi.org/10.1007/s11075-020-009010
[29] Tiel, J.V.: Convex Analysis: An introductory text. Wiley, New York (1984)
[30] Xu, H.K.: Another control condition in an iterative method for nonexpansive mappings. Bull. Austral. Math. Soc. 65, 109-113 (2002)
[31] Yao, J.C.: Variational inequalities with generalized monotone operators. Math. Oper. Res. 19, 691-705 (1994)
[32] Zălinescu, C.: Convex analysis in general vector spaces. World Scientific Publishing, Singapore (2002)
[33] Zhao, X., Köbis, M.A., Yao, Y.: A Projected Subgradient Method for Nondifferentiable Quasiconvex Multiobjective Optimization Problems. J Optim Theory Appl (2021). https://doi.org/10.1007/s10957-021-01872-5


[^0]:    * Corresponding author

    2010 Mathematics Subject Classification: 47H09,47H10
    E-mail addresses: m.ghadampour@gmail.com(m. ghadampour), sori.e@lu.ac.ir (E. Soori), agarwal@tamuk.edu(R. P. Agarwal), donal.oregan@nuigalway.ie(D. O'Regan).

