Existence and uniqueness of maximal elements for preference relations: Variational approach

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Abstract

In this work, we reformulate the problem of existence of maximal elements for preference relations as a variational inequality problem in the sense of Stampacchia. Similarly, we establish the uniqueness of maximal elements using a variational inequality problem in the sense of Minty. In both of these approaches, we use the normal cone operator to find existence and uniqueness results, under mild assumptions. In addition, we provide an algorithm for finding such maximal elements, which is inspired by the steepest descent method for minimization. Under certain conditions, we prove that the sequence generated by this algorithm converges to a maximal element.

Keywords: Maximal elements, Stampacchia variational inequality, Minty variational inequality

MSC (2010): 91B16, 49J40, 49J53

1 Introduction

The theory of preference relations is one of the main tools in the study of consumer demand. A preference relation is described by means of a binary relation, which is traditionally derived from a utility function when the relation satisfies some properties (see for instance [7]). In that sense, finding a maximal element is equivalent to solving the maximization problem of the associated utility function. However, it is well-known that there are preference relations that are not derived from a utility function, for instance, the lexicographic ordering [10]. We can find in the literature a lot of works concerning the existence of maximal elements for preference relations, which are not necessarily transitive nor complete, see for instance [5, 11–13, 16, 18] and the references therein. It is interesting to note that many of these results are consequences of Browder's theorem [4, Theorem 1].

On the other hand, variational inequalities play an important role in the study of optimization problems, Nash games, saddle problems, among others, see [9]. Recently, Milasi *et al.* [12] reformulated the competitive economic equilibrium problem governed by preference relations as a quasi-variational inequality problem. Later on, Milasi and Scopelliti [13] reformulated the problem of finding maximal elements for

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preference relations as a suitable variational inequality problem. However, we discovered some inconsistencies in these works, see Remarks 3.3 and 3.10.

In this work, we aim to study the existence and uniqueness of maximal elements by considering the Stampacchia and Minty variational inequality. Very recently, Donato and Villanacci [8] also addressed the existence of maximal elements in a similar way. However, our existence result is not a consequence of their results, see Remark 3.13. In addition, we present an algorithm to obtain these elements, whenever the preference relation is defined on the whole space. This algorithm is inspired by the classical steepest descent method for minimization of quasiconvex functions [14], but instead of the gradient, we use the so-called *Plastria-like normal cone* of the strict upper contour.

In Section 2, we present definitions and notations of preference relations. Section 3 deals with the existence and uniqueness of maximal elements by considering the variational reformulation. Finally, in Section 4 we provide an optimization algorithm for preference relations that satisfy certain conditions.

2 Preliminaries

Consider X a non-empty set and \succeq a binary relation on X. For each $x \in X$ we consider the sets

$$U(x) := \{ y \in X \ : \ y \succeq x \}, \quad L(x) := \{ y \in X \ : \ x \succeq y \},$$

$$U^s(x) := \{ y \in X \ : \ y \succ x \} \text{ and } L^s(x) := \{ y \in X \ : \ x \succ y \},$$

where \succ is the asymmetric part of \succeq , that is, $x \succ y$ means $x \succeq y$ but not $y \succeq x$. The sets U(x) and $U^s(x)$ (resp. L(x) and $L^s(x)$) are called the *upper* and *strictly upper* (resp. *lower*) contour set.

We recall that a binary relation \succeq [10] is said to be:

- Complete if, for any $x, y \in X$, either $x \succeq y$ or $y \succeq x$.
- Transitive if, for any $x, y, z \in X$, the following implication holds

$$(x \succeq y \land y \succeq z) \Rightarrow x \succeq z.$$

- Reflexive if, for any $x \in X$, we have $x \succeq x$.
- Rational if, it is complete and transitive.

Definition 2.1. Let $m \in \mathbb{N}$. A relation \succeq on X has the m-FIP if, for any $x_1, x_2, \ldots, x_m \in X$, there exists $x \in X$ such that $x \succeq x_i$, for all $i = 1, 2, \ldots, m$. If \succeq has the m-FIP, for all $m \in \mathbb{N}$, then it has the *finite intersection property*.

A few remarks are needed.

- Remark 2.2. 1. The relation \succeq has the finite intersection property if, and only if, the family of sets $\{U(x)\}_{x\in X}$ has the finite intersection property.
 - 2. If \succeq is complete, then it has the 2-FIP. The converse is not true in general, see Example 2.5.

Proposition 2.3. Let X be a non-empty set and \succeq be a relation on X. If \succeq is transitive and has the 2-FIP, then \succeq has the finite intersection property.

Proof. We prove this by induction. For m = 1, 2 it follows from the 2-FIP of \succeq .

Assume now that the family of sets $\{U(x)\}_{x\in X}$ satisfies the finite intersection property for n-1 elements of X and consider $x_1,\ldots,x_n\in X$. For x_1,x_2,\ldots,x_{n-1} there exists $x\in X$ such that $x\succeq x_i$, for all $i\in\{1,2,\ldots,n-1\}$. Thus, again by the 2-FIP of \succeq there exists $z\in X$ such that $z\succeq x_n$ and $z\succeq x$. By transitivity of \succeq , it follows that $z\succeq x_i$ for all $i\in\{1,2,\ldots,n\}$.

In particular, since completeness implies the 2-FIP, we obtain the following corollary.

Corollary 2.4. Let \succeq be a rational relation on a non-empty set X. Then \succeq has the finite intersection property.

We now present some examples. Example 2.5 shows that neither completeness nor transitivity of \succeq are necessary to guarantee the finite intersection property. On the other hand, Example 2.6 shows that completeness, 2-FIP or transitivity, separately, are not sufficient to obtain the finite intersection property.

Example 2.5. Consider X = [0, 4] and define the relation \succeq on X as

$$x \succeq y$$
 if, and only if, $\frac{y}{2} + 2 \le x \le 4$ and $(x, y) \ne (7/2, 2)$.

Clearly $4\in\bigcap_{x\in X}U(x)$, hence \succeq has the finite intersection property. However, it is straightforward to verify that \succeq is neither reflexive, nor complete. It is not transitive either, because $7/2\succeq 3$ and $3\succeq 2$ but $7/2\not\succeq 2$.

Example 2.6. Consider X = [0, 1] and, for $j \in \{a, b\}$, define the relation \succeq_j as

 $x \succeq_i y$ if, and only if, (x, y) belongs to the graph in Figure 1, item j.

Denote with $U_j(x)$, $j \in \{a, b\}$, the respective upper contour set of \succeq_j . Note that \succeq_a is

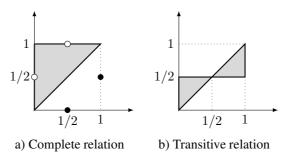


Figure 1

complete (hence, it has the 2-FIP). However it is not transitive, since $0 \succeq_a 1 \succeq_a 1/2$ and $0 \not\succeq_a 1/2$, and it does not have the finite intersection property, because

$$U_a(0) \cap U_a(1/2) \cap U_a(1) = \{0, 1/2\} \cap ([0, 1/2] \cup \{1\}) \cap ([0, 1] \setminus \{1/2\}) = \emptyset.$$

On the other hand, it is straightforward to verify that \succeq_b is transitive. Moreover, since $U_b(0) \cap U_b(1) = \emptyset$, \succeq_b does not have the 2-FIP, hence it is not complete nor it has the finite intersection property.

We will now recall some definitions dealing with convexity, which will play an important role later. Let X be a convex set in a vector space, the relation \succeq is called convex (resp. $convex^s$), if U(x) (resp. $U^s(x)$) is convex, for all $x \in X$. These two notions coincide whenever the relation \succeq is rational, as it was established without proof in [12, Proposition 2.2, part (i)]. We present a proof of this fact, to make our work self-contained.

Proposition 2.7. Assume that X is convex and the relation \succeq is rational. Then \succeq is convex if, and only if, it convex^s.

Proof. If $U^s(x) = \emptyset$, there is nothing to prove. We now consider $U^s(x) \neq \emptyset$. Let $y, z \in U^s(x)$ and $t \in [0,1]$. Since \succeq is complete, without loss of generality we can assume $y \succeq z$. Thus, $y, z \in U(z)$ which is a convex set. This implies $ty + (1-t)z \succeq z$ and by transitivity we have $ty + (1-t)z \succ x$.

Reciprocally, consider $y,z\in U(x)$ and $t\in [0,1]$. If there exists $t_0\in]0,1[$ such that $t_0y+(1-t_0)z\notin U(x)$ then $x\succ t_0y+(1-t_0)z$, due to \succeq being complete. Now, by transitivity of \succeq , we have that $y,z\in U^s(t_0y+(1-t_0)z)$. However, since $U^s(t_0y+(1-t_0)z)$ is convex we get a contradiction.

The following examples show that the previous result is not true in general when we drop the rationality.

Example 2.8. Consider X = [0, 1] and \succeq defined as

$$x \succeq y$$
 if and only if $x = 0$ or $y = x$.

It is easy to see that $U(x) = \{0, x\}$, for any $x \in X$. Thus, \succeq is not convex. However,

$$U^{s}(x) = \begin{cases} \emptyset, & x = 0, \\ \{0\}, & \text{otherwise,} \end{cases}$$

which is a convex set, for all $x \in X$. Hence, it is convex^s.

Example 2.9. Consider X = [0, 1] and \succeq defined as

$$x\succeq y \text{ if, and only if, } y=0 \text{ or } \begin{cases} x=y, & y\in]0,1]\setminus \{1/2\},\\ x=0, & y=1/2. \end{cases}$$

It is not difficult to see that

$$U(x) = \begin{cases} [0,1], & x = 0, \\ \{x\}, & x \in]0,1] \setminus \{0,1/2\}, \text{ and } U^s(x) = \begin{cases} \emptyset, & x \in]0,1], \\ [0,1/2[\cup]1/2,1], & x = 0. \end{cases}$$

Thus, \succeq is convex but it is not convex^s.

We finish this section with some topological definitions for binary relations and correspondences. Given X a topological space, a relation \succeq on X is said to be *upper* (resp. lower) semicontinuous, if the set $L^s(x)$ (resp. $U^s(x)$) is open, for all $x \in X$. Moreover, \succeq is continuous, if it is upper and lower semicontinuous. Clearly, if \succeq is complete; then it is upper (resp. lower) semicontinuous if, and only if, U(x) (resp. L(x)) is closed, for all $x \in X$.

Let V, W be nonempty sets. A *correspondence* $T: V \rightrightarrows W$ is an application from V into $\mathcal{P}(W)$, that is, for $v \in V, T(v) \subset W$. We can see that the upper contour U and the strict upper contour U^s as examples of correspondences from X to itself.

Recall that a correspondence $T: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is

- lower hemicontinuous at $x_0 \in \mathbb{R}^n$ when, for any sequence $(x_k)_{k \in \mathbb{N}}$ converging to x_0 and any element y_0 of $T(x_0)$, there exists a sequence $(y_k)_{k \in \mathbb{N}}$ converging to y_0 such that $y_k \in T(x_k)$, for any $k \in \mathbb{N}$.
- upper hemicontinuous at $x_0 \in \mathbb{R}^n$ when, for any neighbourhood \mathcal{W} of $T(x_0)$, there exists a neighbourhood \mathcal{V} of x_0 such that $T(x) \subset \mathcal{W}$, for all $x \in \mathcal{V}$;
- lower (respectively upper) hemicontinuous when it is lower (resp. upper) hemicontinuous at every $x_0 \in \mathbb{R}^n$;
- closed, if the set graph $(T):=\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m:y\in T(x)\}$ is a closed subset of $\mathbb{R}^n\times\mathbb{R}^n$.

It is well known that if $T(\mathbb{R}^n)$ is bounded and T is closed, then it is upper hemicontinuous. We also say that T has open fibres when the set $T^{-1}(y) := \{x \in \mathbb{R}^n : y \in T(x)\}$ is open, for all $y \in \mathbb{R}^m$. It is easy to verify that if T has open fibres, then it is lower hemicontinuous. In particular, if the binary relation \succeq on \mathbb{R}^n is upper (resp. lower) semicontinuous then $U^s : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ (resp. L^s) has open fibres, hence it is lower (resp. upper) hemicontinuous.

3 A Variational Approach for Maximal Elements

This section is devoted to studying the existence and uniqueness of maximal elements for binary relations. First, we will introduce a normal cone correspondence, related to the strict upper contours of a binary relation, and establish some of its properties. Next, we will deal with the variational reformulation and the existence of maximal elements. Finally, using the Minty variational inequality, we will study the uniqueness of such maximal elements.

Let \succeq be a binary relation on \mathbb{R}^n with asymmetric part \succ , let X be a non-empty subset of \mathbb{R}^n and let $\hat{x} \in X$. Then

- \hat{x} is a maximum of X, if $\hat{x} \succeq y$, for all $y \in X$;
- \hat{x} is a maximal element of X, if there is not $y \in X$ such that $y \succ \hat{x}$.

We denote by $\mathscr{M}_{\succeq}(X)$ and $\mathscr{ME}_{\succeq}(X)$ the set of maxima and maximal elements of X, respectively. It is not difficult to see that $\mathscr{M}_{\succeq}(X) \subset \mathscr{ME}_{\succeq}(X)$, and the equality holds under completeness of \succeq . Furthermore, it is clear that

$$\mathscr{M}_\succeq(X) = \bigcap_{x \in X} U(x) \cap X \text{ and } \mathscr{ME}_\succeq(X) = \{x \in X \ : \ U^s(x) \cap X = \emptyset\}.$$

3.1 Variational Inequalities and Normal Cones

Consider X a subset of \mathbb{R}^n and $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ a set-valued map. A vector $\hat{x} \in X$ is said to be a solution of the

• Stampacchia variational inequality problem if there exists $\hat{x}^* \in T(\hat{x})$, such that

$$\langle \hat{x}^*, y - \hat{x} \rangle > 0, \quad \forall y \in X.$$

• Minty variational inequality problem, if

$$\langle y^*, \hat{x} - y \rangle \le 0, \quad \forall y \in X, \, \forall y^* \in T(y).$$

The solution sets of the Stampacchia and Minty variational inequality problems will be denoted by SVIP(T, X) and MVIP(T, X), respectively.

Given a subset A of \mathbb{R}^n and $x \in \mathbb{R}^n$, the normal cone of A at x is the set

$$\mathcal{N}_A(x) := \begin{cases} \{x^* \in \mathbb{R}^n : \langle x^*, y - x \rangle \le 0, \ \forall y \in A\}, & A \ne \emptyset, \\ \mathbb{R}^n, & A = \emptyset. \end{cases}$$

It is usual in the literature to consider the above definition whenever A convex and closed, and $x \in A$. However, we will not consider such conditions in this work. See Figure 2 for a geometric interpretation. Note that $\mathcal{N}_A(x)$ is always non-empty, because $0 \in \mathcal{N}_A(x)$. Moreover, it is a closed convex cone of \mathbb{R}^n .

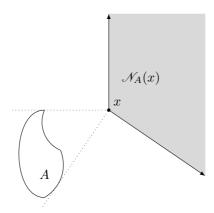


Figure 2: The set $\mathcal{N}_A(x)$

Let \succeq be a binary relation on \mathbb{R}^n . The *normal cone correspondence* $N: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, associated to \succeq , is defined as

$$6N(x) := \mathcal{N}_{U^s(x)}(x),\tag{1}$$

for all $x \in \mathbb{R}^n$.

Lemma 3.1. Let \succeq be a complete relation the following equivalence holds:

$$x^* \in N(x) \Leftrightarrow (\forall y \in \mathbb{R}^n, \langle x^*, y - x \rangle > 0 \Rightarrow x \succeq y).$$

In the following example, we explicitly calculate the normal cone correspondence for a certain preference relation.

Example 3.2. Consider the relation \succeq on $\mathbb R$ defined as

$$x \succeq y$$
 if, and only if, $y = x$ or $y = 1$.

It is clear that

$$U^s(x) = \begin{cases} \emptyset, & x \neq 1, \\ \mathbb{R} \setminus \{1\}, & x = 1, \end{cases} \text{ and } N(x) = \begin{cases} \mathbb{R}, & x \neq 1, \\ \{0\}, & x = 1. \end{cases}$$

Remark 3.3. Recently, Milasi et al. [12], considered \succeq to be a convex^s and non-satiated relation on $X \subset \mathbb{R}^n$ and defined $M_1: X \rightrightarrows \mathbb{R}^n$ as $M_1(x) := \mathscr{N}_{U^s(x)}(x)$. Clearly, if $X = \mathbb{R}^n$ then M_1 coincides with N.

Now, Proposition 2.4 in [12] establishes that for all $x^* \in M_1(x) \setminus \{0\}$

$$\langle x^*, y - x \rangle < 0$$
, for all $y \in U^s(x)$,

provided that \succeq is lower semicontinuous, convex^s and with no maximal elements (this property is called *non-satiated* in [12]). However, this result as it is stated is not true. Consider for instance $X = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ and the relation \succeq on X defined as

$$(x,0) \succeq (y,0)$$
 if, and only if, $x \geq y$.

It is not difficult to see that \succeq is continuous, convex^s and without maximal elements on X. Moreover, for each $(x,0) \in X$ we have

$$U^s(x,0) =]x, +\infty[\times\{0\} \text{ and } M_1(x,0) = \{(x^*,y^*) \in \mathbb{R}^2 : x^* \le 0\}.$$

Since $(0,1) \in M_1(x,0) \setminus \{(0,0)\}$ we obtain that $\langle (0,1), (y,0) - (x,0) \rangle = 0$, for all $(y,0) \in U^s(x,0)$.

The following result can be found as Proposition 24 in [8]. However, our definition of N allows to drop the condition of $U^s(x)$ being non-empty.

Lemma 3.4. Let \succeq be a binary relation on \mathbb{R}^n and $x \in \mathbb{R}^n$. If $U^s(x)$ is a convex set, then $N(x) \setminus \{0\} \neq \emptyset$.

The following result is a generalization of Proposition 2.3 in [12].

Proposition 3.5. Let \succeq be a relation on \mathbb{R}^n . If \succeq is upper semicontinuous, then the map N is closed.

Proof. Lower semicontinuity of U^s is consequence of the upper semicontinuity of \succeq . Hence the proposition follows from Proposition 25 in [8].

Given a relation \succeq on \mathbb{R}^n we define

$$N^*(x) := \{ x^* \in \mathbb{R}^n : \langle x^*, y - x \rangle < 0, \, \forall \, dy \in U^s(x) \},$$

whenever $x \notin \mathcal{ME}_{\succeq}(\mathbb{R}^n)$, and $N^*(x) := \mathbb{R}^n$, otherwise. It is important to note that $N^*(x)$ is a cone and $N^*(x) \subset N(x)$, for all $x \in X$.

Proposition 3.6. Let \succeq be a rational relation on \mathbb{R}^n . If $N^*(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$; then \succ is convex.

Proof. Let $x \in \mathbb{R}^n$. If $U(x) = \emptyset$, there is nothing to prove. Now assume that $U(x) \neq \emptyset$, and take any $y \notin U(x)$. Choosing some $y^* \in N^*(y)$, define the set

$$H^{-}(y) = \{ z \in \mathbb{R}^{n} : \langle y^{*}, z - y \rangle < 0 \}.$$

Clearly, $H^-(y)$ is convex and $y \notin H^-(y)$. Since y was taken arbitrarily, we have the inclusion

$$\bigcap_{y \notin U(x)} H^{-}(y) \subset U(x).$$

On the other hand, for $y \notin U(x)$, since \succeq is complete, $x \succ y$. We now claim that $U(x) \subset H^-(y)$. Indeed, for all $z \in U(x)$ we have $z \succeq x \succ y$, hence $z \succ y$, by transitivity of \succeq . Thus $\langle y^*, z - y \rangle < 0$, because $y^* \in N^*(y)$, which implies $z \in H^-(y)$. Therefore

$$U(x) = \bigcap_{y \notin U(x)} H^{-}(y).$$

The proposition follows.

The following proposition shows the relation between the operators N and N^* .

Proposition 3.7. Let \succeq be a lower semicontinuous relation on \mathbb{R}^n . Then

- 1. $N(x) = N^*(x) \cup \{0\}$, for all $x \in \mathbb{R}^n$.
- 2. If, in addition, \succeq is convex^s, then N^* is non-empty valued.

Proof. 1. The lower semicontinuity of \succeq implies that $U^s(x)$ is open. Since N(x) is the polar cone of $U^s(x) \setminus \{x\}$, this item now follows using [17, Exercise 6.22].

The following example shows that it is not possible to drop the lower semicontinuity in the first part of the previous result.

Example 3.8. Consider the relation \succ on \mathbb{R}^2 defined as

$$(x,y) \succeq (a,b)$$
 if, and only if, $x \geq a$ and $y = b = 0$.

It is not difficult to verify that

$$U^{s}(x,y) = \begin{cases} \emptyset, & y \neq 0, \\]x, +\infty[\times\{0\}, & y = 0. \end{cases}$$

Thus, the relation \succeq is not lower semicontinuous. Additionally, we can see that

$$N(x,y) = \begin{cases} \mathbb{R}^2, & y \neq 0, \\ \{(x^*, y^*) \in \mathbb{R}^2 : x^* \leq 0\}, & y = 0, \end{cases}$$

and

$$N^*(x,y) = \begin{cases} \mathbb{R}^2, & y \neq 0, \\ \{(x^*,y^*) \in \mathbb{R}^2 : x^* < 0\}, & y = 0. \end{cases}$$

Thus, $N(x,0) \neq N^*(x,0) \cup \{(0,0)\}.$

3.2 Reformulation and Existence Results

The following correspondence will allow us to reformulate the problem of finding maximal elements as a Stampacchia variational inequality problem. Consider the correspondence $T: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined as

$$T(x) := \operatorname{conv}(N(x) \cap S[0, 1]),$$
 (2)

where N is the correspondence associated to \succeq , defined as in (1), and S[0,1] denotes the unit sphere of \mathbb{R}^n . For instance, if \succeq and N are defined as in Example 3.2, then the correspondence T associated to N satisfies T(x) = [-1,1], when $x \neq 1$, and $T(1) = \emptyset$.

The following proposition establishes a relation between \succeq and T.

Proposition 3.9. *The following implications hold:*

1. If \succeq is upper semicontinuous on \mathbb{R}^n , then T is closed. In particular T is upper hemicontinuous.

2. If \succeq is convex^s and lower semicontinuous on \mathbb{R}^n , then T is non-empty valued.

Proof. 1. Consider the correspondence $R:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $\operatorname{graph}(R) = \operatorname{graph}(N) \cap (\mathbb{R}^n \times S[0,1])$. By Proposition 3.5, we deduce that R is closed. Moreover, it is upper semicontinuous due to the closed graph theorem. Since $T(x) = \operatorname{conv}(R(x))$, for all $x \in X$, the result follows from Theorem 17.35 in [1].

Remark 3.10. Milasi and Scopelliti [13] defined the map $M_2: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as

$$M_2(x) := \begin{cases} \mathscr{N}_{U^s(x)}(x), & x \in X, \\ \emptyset, & x \notin X. \end{cases}$$

The authors also introduced a correspondence similar to T given in (2). More precisely, they defined the map $G:X \rightrightarrows \mathbb{R}^n$ as

$$G(x) := \begin{cases} \operatorname{conv}(M_2(x) \cap S[0,1]), & U^s(x) \neq \emptyset, \\ \overline{B}(0,1), & U^s(x) = \emptyset, \end{cases}$$

where $\overline{B}(0,1)$ is closed unit ball of \mathbb{R}^n . It is clear that T and G coincide when $X=\mathbb{R}^n$. Now, part b) of Theorem 3 in [13] establishes that any solution of the variational inequality problem (in the sense of Stampacchia) associated to G and $K\subset X$ is a maximal element of \succeq on K. However, this is not true as it is stated. Consider for instance the relation \succeq given in Remark 3.3, and $K=[0,1]\times\{0\}\subset X$. The map $G:X\rightrightarrows\mathbb{R}^2$ is given by

$$G(x,0) = \{(x^*, y^*) \in \mathbb{R}^2 : x^* \le 0 \text{ and } (x^*)^2 + (y^*)^2 \le 1\}.$$

Since $(0,0) \in G(x,0)$ for all $x \in \mathbb{R}$, we deduce that the solution set of the variational inequality problem, associated to G and K, coincides with K. However, the unique maximal element of K is (1,0).

Motivated by the previous remark we establish the following result.

Proposition 3.11. Assume that X is a non-empty subset of \mathbb{R}^n and let \succeq be lower semicontinuous relation on \mathbb{R}^n . Then $SVIP(T, X) \subset \mathscr{ME}_{\succ}(X)$.

Proof. Let \hat{x} be an element of SVIP(T, X). There exists $\hat{x}^* \in T(\hat{x})$ satisfying

$$\langle \hat{x}^*, y - \hat{x} \rangle \ge 0$$
, for all $y \in X$. (3)

By definition of T, there are $x_1^*, x_2^*, \ldots, x_m^* \in N(\hat{x}) \cap S[0,1]$ and $t_1, t_2, \ldots, t_m \in [0,1]$ such that $\hat{x}^* = \sum_{i=1}^m t_i x_i^*$ and $\sum_{i=1}^m t_i = 1$. Assume that \hat{x} is not a maximal element of \succeq on X. Then there exists $y \in X$ such that $y \succ \hat{x}$, i.e. $y \in U^s(\hat{x})$. Using inequality (3), we deduce $\langle x_i^*, y - \hat{x} \rangle \geq 0$ for some i. On the other hand, since $x_i^* \in N(\hat{x}) \setminus \{0\}$ and Proposition 3.7, item I, $\langle x_i^*, y - \hat{x} \rangle < 0$, a contradiction. \square

We now are able to present the main result of this subsection.

Theorem 3.12. Let X be a convex, compact and non-empty subset of \mathbb{R}^n , and let \succeq be a binary relation on \mathbb{R}^n . If \succeq is continuous and convex^s; then there exists at least a maximal element for \succeq on X.

Proof. From Proposition 3.9, the correspondence T is upper hemicontinuous with convex, compact and non-empty values. By Theorem 9.9 in [2], there exists $\hat{x} \in \text{SVIP}(T, X)$. Therefore, the result follows from Proposition 3.11.

Remark 3.13. It is important to note that in Proposition 3.11 we do not require convexity^s of the relation, contrary to Theorem 28 in [8]. On the other hand, Theorem 3.12 is not a consequence of Theorem 33 in [8]. Indeed, consider the relation \succeq on \mathbb{R} , defined as

$$x \succeq y$$
 if, and only if, $x = y = 0$.

Clearly, \succeq is continuous and convex^s. Moreover, T(x) = [-1,1], for all $x \in \mathbb{R}$. Hence, for any compact, convex and non-empty subset X of \mathbb{R} , by Theorem 3.12, $\mathscr{ME}_{\succeq}(X) \neq \emptyset$. However, we cannot apply Theorem 33 in [8], as \succeq allows for empty strict upper contours.

3.3 On the Uniqueness of Maximal Elements

In this subsection we will show a result concerning the uniqueness of maximal elements. Before this, we need some previous results. The first result is about necessary conditions, while the second provides sufficient conditions. Both are related to the Minty variational inequality problem.

Lemma 3.14. Let X be a subset of \mathbb{R}^n and \succeq be a complete relation on \mathbb{R}^n . If $\hat{x} \in \mathcal{ME}_{\succ}(X)$, then

$$\langle y^*, \hat{x} - y \rangle \le 0$$
, for all $y \in X \setminus \mathscr{ME}_{\succ}(X)$ and all $y^* \in N(y)$. (4)

Proof. Since \succeq is complete, for any $y \in X \setminus \mathscr{ME}_{\succeq}(X)$, we have $\hat{x} \in U^s(y)$. The result follows from the definition of N(y).

The following example shows that we cannot drop the completeness of Lemma 3.14.

Example 3.15. Consider \succeq on \mathbb{R} defined as

$$x \succeq y$$
 if, and only if, $(x, y) = (0, 0) \lor (x \ge y \land x \ne 0)$,

and let $X=\mathbb{R}$. It is clear that $U^s(x)=]x,+\infty[$ for all $x\neq 0$, and $U^s(0)=\emptyset$, hence $\mathscr{ME}_\succeq(\mathbb{R})=\{0\}$. Moreover, we can see that $N(x)=]-\infty,0]$ for all $x\neq 0$, and $N(0)=\mathbb{R}$. However, $\hat{x}=0$ does not satisfy inequality (4).

Proposition 3.16. Let \succeq be a binary relation on \mathbb{R}^n . If the sets $\mathscr{ME}_{\succeq}(\mathbb{R}^n)$ and $\mathrm{MVIP}(N,\mathbb{R}^n)$ are both non-empty, then $\mathscr{ME}_{\succeq}(\mathbb{R}^n) = \mathrm{MVIP}(N,\mathbb{R}^n) = \{\hat{x}\}$, for some $\hat{x} \in \mathbb{R}^n$.

Proof. Let $\hat{x} \in \text{MVIP}(N, \mathbb{R}^n)$ and $y \in \mathscr{ME}_\succeq(\mathbb{R}^n)$ be arbitrary. Since y is maximal, $U^s(y) = \emptyset$ and $N(y) = \mathbb{R}^n$. If $\hat{x} \neq y$ then, there exists $y^* \in N(y) = \mathbb{R}^n$ such that $\langle y^*, \hat{x} - y \rangle > 0$, a contradiction with $\hat{x} \in \text{MVIP}(N, \mathbb{R}^n)$. Hence $\hat{x} = y$, for all $\hat{x} \in \text{MVIP}(N, \mathbb{R}^n)$ and all $y \in \mathscr{ME}_\succ(\mathbb{R}^n)$. This implies the proposition. \square

We now present the main result of this subsection.

Theorem 3.17. Let \succeq be a complete relation on \mathbb{R}^n . If $\mathscr{ME}_{\succeq}(\mathbb{R}^n) \neq \emptyset$, then $\mathscr{ME}_{\succeq}(\mathbb{R}^n)$ is a singleton if, and only if, $\mathscr{ME}_{\succ}(\mathbb{R}^n) = \text{MVIP}(N, \mathbb{R}^n)$.

Proof. First assume that $\mathscr{ME}_\succeq(\mathbb{R}^n) = \{\hat{x}\}$, and take $y \in \mathbb{R}^n$ and $y^* \in N(y)$. If $y \neq \hat{x}$, Lemma 3.14 implies $\langle y^*, \hat{x} - y \rangle \leq 0$, and, when $y = \hat{x}$, trivially $\langle y^*, \hat{x} - y \rangle = 0$. Hence $\hat{x} \in \text{MVIP}(N, \mathbb{R}^n)$ and, by Proposition 3.16, $\mathscr{ME}_\succeq(\mathbb{R}^n) = \text{MVIP}(N, \mathbb{R}^n) = \{\hat{x}\}$. The converse implication is a direct consequence of Proposition 3.16. \square

Remark 3.18. 1. Proposition 3.16 implies that if either $\mathscr{ME}_{\succeq}(\mathbb{R})$ or $\mathrm{MVIP}(N,\mathbb{R})$ have at least two elements, then the other one must be empty.

- 2. Example 3.15 also implies that the completeness of \succeq cannot be dropped in Theorem 3.17. Indeed, for \succeq as in Example 3.15, $\mathscr{ME}_{\succeq}(\mathbb{R}) = \{0\}$ but in this case $\mathrm{MVIP}(N,\mathbb{R}) = \emptyset$.
- 3. Existence of maximal elements can be guaranteed by assuming that \succeq is complete, has the finite intersection property, $\bigcap_{x\in X}U(x)=\bigcap_{x\in X}\overline{U(x)}$ and U(x) is bounded for some x

4 An Algorithm to Find Maximal Elements

This section is divided in two parts: the first one deals with the definition and properties of a normal cone operator inspired by the Plastria subdifferential [15]. The second subsection contains an algorithm for finding maximal elements of a binary relation, along with convergence results.

4.1 The Plastria-like Normal Cone

Given a relation \succeq on \mathbb{R}^n and a function $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ we define

$$N_f(x) := \begin{cases} \{x^* \in \mathbb{R}^n \ : \ \langle x^*, y - x \rangle \leq f(x,y), \ \forall \, y \in U^s(x) \}, & x \notin \mathscr{ME}_\succeq(\mathbb{R}^n), \\ \mathbb{R}^n, & \text{otherwise}. \end{cases}$$

We will call $N_f(x)$ as the *Plastria-like normal cone* associated to the binary relation \succeq with respect to the function f. Clearly, if f=0 then the Plastria-like normal cone N_f reduces to the classical normal cone defined in (1). It is also clear that $N_f(x)$ is closed and convex, for all $x \in \mathbb{R}^n$. In general, $N_f(x)$ may not be a cone, as it is shown by the following example.

Example 4.1. Consider \succeq on \mathbb{R} as in Example 3.15 and functions $f_1, f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$f_1(x,y) = y^2 - x^2$$
 and $f_2(x,y) = y - x$.

It is not difficult to show that the Plastria-like normal cones of \succeq , with respect to f_1 and f_2 , respectively, are

$$N_{f_1}(x) = \begin{cases}]-\infty, 2x], & x \neq 0, \\ \mathbb{R}, & x = 0, \end{cases} \text{ and } N_{f_2}(x) = \begin{cases}]-\infty, 1], & x \neq 0, \\ \mathbb{R}, & x = 0. \end{cases}$$

The following proposition extends Proposition 25 in [8].

Proposition 4.2. Let \succeq be an upper semicontinuous relation on \mathbb{R}^n and $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function. If f is upper semicontinuous then the Plastria-like normal cone N_f is closed.

Proof. Let (x_k, x_k^*) be a sequence converging to (x, x^*) such that $x_k \in \mathbb{R}^n$ and $x_k^* \in N_f(x_k)$, for all $k \in \mathbb{N}$. If $U^s(x) = \emptyset$, there is nothing to prove. Now, we consider that $U^s(x) \neq \emptyset$ and take any $y \in U^s(x)$. Since U^s is a lower hemicontinuous correspondence, there exists a sequence (y_k) converging to y such that $y_k \in U^s(x_k)$. Thus,

$$\langle x_k^*, y_k - x_k \rangle \le f(x_k, y_k).$$

Letting k tend to ∞ , we obtain $\langle x^*, y - x \rangle \leq f(x, y)$. Since y was arbitrary, we conclude that $x^* \in N_f(x)$. Therefore, N_f is closed.

Note that Proposition 4.2 coincides with Proposition 3.5, when f = 0. Also, this proposition holds even if there is no connection between \succeq and f. In the case when f and \succeq are related, we obtain some other properties, listed in the following propositions.

Proposition 4.3. Let \succeq be a relation on \mathbb{R}^n and $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function such that f(x,y) < 0 if, and only if, $y \in U^s(x)$. The following hold:

- 1. $0 \in N_f(x)$ if, and only if, $x \in \mathscr{ME}_{\succ}(\mathbb{R}^n)$.
- 2. If f is upper semicontinuous with respect to its first variable, then \succeq is upper semicontinuous.
- 3. If f is upper semicontinuous with respect to its first variable, and $(x_0, 0) \in \overline{\operatorname{graph}(N_f)}$, then $0 \in N_f(x_0)$.

Proof. 1. Follows directly from the definition of N_f .

2. It is enough to note that upper semicontinuity of $f(\cdot, y)$ implies that the set

$$L^{s}(y) = \{x : y \succ x\} = \{x : f(x,y) < 0\}$$

is open.

3. If $x_0 \in \mathscr{ME}_\succeq(\mathbb{R}^n)$ there is nothing to prove. Otherwise, there exists $\hat{x} \in U^s(x_0)$, which implies $f(x_0,\hat{x}) < 0$ and $x_0 \in L^s(\hat{x})$. Let $(x_k,x_k^*) \in \operatorname{graph}(N_f)$, such that $x_k \to x_0$ and $x_k^* \to 0$. Now, item 2 implies that $L^s(\hat{x})$ is open, therefore $x_k \in L^s(\hat{x})$, for k large enough. Since $x_k^* \in N_f(x_k)$, we deduce

$$\langle x_k^*, \hat{x} - x_k \rangle \le f(x_k, \hat{x}).$$

Taking the limit when $k \to \infty$, we obtain $0 \le f(x_0, \hat{x}) < 0$, a contradiction. \square

Proposition 4.4. Let \succeq be a relation on \mathbb{R}^n and $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function such that the following hold:

- 1. for all $x, y \in \mathbb{R}^n$, if f(x, z) > f(y, z), for some $z \in \mathbb{R}^n$, then $x \succ y$;
- 2. there exists L>0 such that $|f(x,y)|\leq L\|x-y\|$, for all $x,y\in\mathbb{R}^n$.

Then, for every $u^* \in N^*(x_0)$, $x_0^* := \frac{Lu^*}{\|u^*\|} \in N_f(x_0)$.

Proof. If $x_0 \in \mathscr{ME}_{\succeq}(\mathbb{R}^n)$, there is nothing to prove. Otherwise, $U^s(x_0) \neq \emptyset$. Take $u^* \in N^*(x_0)$, that is,

$$\langle u^*, x - x_0 \rangle < 0$$
, for all $x \in U^s(x_0)$.

Now we define $x_0^*=\frac{L}{\|u^*\|}u^*$ and consider the hyperplane $H=\{z\in\mathbb{R}^n: \langle u^*,z-x_0\rangle=0\}$. For each $x\in U^s(x_0)$ we consider $x'\in H$ as the projection of x onto H. It is not difficult to verify that

$$\langle \frac{u^*}{\|u^*\|}, x - x_0 \rangle = -\|x - x'\|.$$

Since $x' \notin U^s(x_0)$, assumptions 1. and 2. imply $f(x_0, x) \ge f(x', x) \ge -L||x - x'||$. Hence

$$f(x_0, x) \ge \langle x_0^*, x - x_0 \rangle,$$

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and, therefore, $x_0^* \in N_f(x_0)$. The proof is complete.

Remark 4.5. In the particular case when \succeq is represented by a utility function $u: \mathbb{R}^n \to \mathbb{R}$ (in the sense that $x \succeq y$ if, and only if, $u(x) \geq u(y)$), we can define the function $f_u: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $f_u(x,y) := u(x) - u(y)$. Using f_u , the Plastria-like normal cone $N_{f_u}(x)$ coincides with the Plastria lower subdifferential of u at x [15].

On the other hand, Proposition 4.3, item 1., extends Theorem 3.1 in [15] and item 3. extends Proposition 7 in [14].

Note that, using f_u as above, we readily obtain condition I. in Proposition 4.4. Moreover, condition 2. is equivalent to the function u be Lipschitz continuous on \mathbb{R}^n . In view of this, Proposition 4.4 extends Theorem 20 in [6].

4.2 The Algorithm

We begin this subsection by recalling the definition of a quasi-Fejér monotone sequence. A sequence $\{x_k\} \subset \mathbb{R}^n$ is called *quasi-Fejér monotone* with respect to $M \subset \mathbb{R}^n$ if for every $u \in M$ there exists a sequence $\{\varepsilon_k\} \subset \mathbb{R}_+$ with $\sum \varepsilon_k < \infty$, such that

$$||x_{k+1} - u||^2 \le ||x_k - u||^2 + \varepsilon_k.$$

We state below an important result concerning quasi-Fejér monotone sequences [3, Theorem 5.33].

Theorem 4.6. Let $\{x_n\}$ be a sequence in a Hilbert space H and let C be a non-empty subset of H such that $(x_n)_{n\in\mathbb{N}}$ is quasi-Fejér monotone with respect to C. Then the following hold:

- 1. $\{x_n\}$ is bounded.
- 2. Suppose that every weak sequential cluster point of $\{x_n\}$ belongs to C. Then $\{x_n\}$ converges weakly to a point in C.

We are now ready to present an algorithm for finding maximal elements of preference relations defined in \mathbb{R}^n . In the sequel, we will assume the following conditions:

- 1. \succeq is convex^s,
- 2. there is a function $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying

- (a) f(x,y) < 0 if, and only if, $y \in U^s(x)$;
- (b) f(x,y) > 0 if, and only if, $x \in U^s(y)$;
- (c) there is L > 0 such that $|f(x,y)| \le L||x-y||$, for all $x,y \in \mathbb{R}^n$;
- (d) for all $x, y \in \mathbb{R}^n$, $x \succ y$ if, and only if, f(x, z) > f(y, z), for all $z \in \mathbb{R}^n$, if, and only if, f(x, z) > f(y, z), for some $z \in \mathbb{R}^n$; and
- (e) f is upper semicontinuous with respect to its first variable.
- 3. $\mathscr{ME}_{\succeq}(\mathbb{R}^n)$ is non-empty.

Under these conditions we can write an analog of the classical steepest descent method, as follows:

Initialization: Take $x_1 \in \mathbb{R}^n$, arbitrarily.

Iteration: Given x_k , calculate the next iterative x_{k+1} from

$$x_{k+1} := x_k - \theta_k x_k^*, \ k \in \mathbb{N},$$

where θ_k is some positive real number such that

$$\sum \theta_k = \infty \text{ and } \sum \theta_k^2 < \infty,$$

and $x_k^* \in N_f(x_k)$ such that $||x_k^*|| \le L$ with L > 0.

Notice that item (d) of assumption 2. in the algorithm implies assumption 1. in Proposition 4.4. Also, items (b) and (d) of assumption 2. imply the transitivity of \succeq . Moreover, assumption 2. in the algorithm does not imply the completeness of \succeq as we can see in the following example.

Example 4.7. Consider \succeq defined as in Remark 3.13, that is, $x \succeq y$ if, and only if, x = y = 0, which is clearly non-complete. However, choosing $f \equiv 0$ on \mathbb{R}^2 , f and \succeq trivially satisfy conditions I., I. and I. in the algorithm.

It is important to notice that if $x_k^*=0$, for some k, then by Proposition 4.3, item l, x_k is a maximal element. From now on, we assume that the sequence $\{x_k\}$ generated by the algorithm is infinite, that is $x_k^*\neq 0$, for all $k\in\mathbb{N}$. It is clear that $\mathscr{ME}_\succeq(\mathbb{R}^n)\subset\bigcap_{k\in\mathbb{N}}U^s(x_k)$.

Theorem 4.8. The sequence $\{x_k\}$ generated by the Algorithm converges to a point $\hat{x} \in \mathscr{ME}_{\succ}(\mathbb{R}^n)$.

Proof. Notice that

$$||x_{k+1} - x_k||^2 = \theta_k^2 ||x_k^*||^2 = L^2 \theta_k^2,$$

which implies the series $\sum ||x_{k+1} - x_k||^2$ is convergent. On the other hand,

$$||x_{k+1} - u||^2 = ||x_k - u||^2 + ||x_{k+1} - x_k||^2 + 2\theta_k \langle x_k^*, u - x_k \rangle$$
 (5)

for all $k \in \mathbb{N}$ and all $u \in \mathbb{R}^n$. Taking $\hat{x} \in \mathcal{ME}_{\succ}(\mathbb{R}^n)$ in the previous equality we have

$$||x_{k+1} - \hat{x}||^2 \le ||x_k - \hat{x}||^2 + ||x_{k+1} - x_k||^2.$$

Thus, the sequence $\{x_k\}$ is quasi-Fejer monotone with respect to $\mathscr{ME}_{\succeq}(\mathbb{R}^n)$ with $\varepsilon_k = \|x_{k+1} - x_k\|^2$. Moreover, from (5) we obtain

$$||x_{k+1} - \hat{x}||^2 \le ||x_k - \hat{x}||^2 + ||x_{k+1} - x_k||^2 + 2\theta_k f(x_k, \hat{x}),$$

which in turn implies

$$-2\theta_k f(x_k, \hat{x}) \le (\|x_k - \hat{x}\|^2 - \|x_{k+1} - \hat{x}\|^2) + \|x_{k+1} - x_k\|^2.$$

Since both series $\sum \|x_{k+1} - x_k\|^2$ and $\sum (\|x_k - \hat{x}\|^2 - \|x_{k+1} - \hat{x}\|^2)$ are convergent, we obtain

$$\sum -2\theta_k f(x_k, \hat{x}) < \infty.$$

Hence $\lim f(x_k, \hat{x}) = 0$.

Finally, part I. of Theorem 4.6 implies that $\{x_k\}$ is bounded. Without loss of generality, we may assume that x_k converges to some x_0 , so, due to the upper semi-continuity of $f(\cdot,\hat{x})$, $f(x_0,\hat{x}) \geq 0$. If x_0 is not maximal, there exists $z \in X$ such that $z \succ x_0$. Thus, $f(z,\hat{x}) > f(x_0,\hat{x}) \geq 0$, which in turn implies $z \succ \hat{x}$ and we get a contradiction because \hat{x} is a maximal element. The result now follows from part 2. of Theorem 4.6.

Remark 4.9. Assume that \succeq is represented by a utility function $u: \mathbb{R}^n \to \mathbb{R}$, and consider f_u as in Remark 4.5, that is, $f_u(x,y) := u(x) - u(y)$. If u is quasiconcave and Lipschitz, and possesses a maximum on \mathbb{R}^n , then f_u satisfy assumptions I., 2. and 3. of our algorithm. In view of this, our algorithm generalizes Algorithm A1 in [14], as we do not require differentiability of u.

5 Conclusions

In this work, we improved some existence results on maximal elements via a variational approach. In particular, we compared our results with Proposition 2.4 in [12] and Theorem 3 in [13]. We have identified a flaw in these results, which we described in Remarks 3.3 and 3.10. Moreover, we pointed out that our main result is not a consequence of Theorems 28 and 33 in [8]. We also characterized the uniqueness of maximal elements, by studying the solutions of a certain Minty variational inequality problem.

Finally, we adapted the classical steepest descent method, where we use a new kind of normal cone, called Plastria-like normal cone, to replace the usual derivative. In this way, we established an algorithm to find maximal elements. This algorithm extends Algorithm A1 in [14].

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