A new subspace minimization conjugate gradient method for unconstrained minimization

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Received: date / Accepted: date

Abstract Subspace minimization conjugate gradient (SMCG) methods, as the generalization of traditional conjugate gradient methods, have become a class of quite efficient iterative methods for unconstrained optimization and have attracted extensive attention recently. Usually, the search directions of SMCG methods are generated by minimizing approximate models with the approximation matrix B_k of the objective function at the current iterate over the subspace spanned by the current gradient g_k and the latest search direction. The $g_k^T B_k g_k$ must be estimated properly in the calculation of the search directions, which is crucial to the theoretical properties and the numerical performance of SMCG methods. It is a great challenge to estimate it properly. An alternative solution for this problem might be to design a new subspace minimization conjugate gradient method independent of the parameter $\rho_k \approx g_k^T B_k g_k$. The projection technique has been used successfully to generate conjugate gradient directions such as Dai-Kou conjugate gradient direction (SIAM J Optim 23(1), 296-320, 2013). Motivated by the above two observations, in the paper we present a new subspace minimization conjugate gradient methods by using a projection technique based on the memoryless quasi-Newton method. More specially, we project the search direction of the memoryless quasi-Newton method into the subspace spanned by the current gradient and the latest search direction and drive a new search direction, which is proved to be descent. Remarkably, the proposed method without any line search enjoys the finite termination property for two dimensional convex quadratic functions, which is helpful for designing algorithm. An adaptive scaling factor in the search direction is given based on the above finite termination property. The proposed method does not need to determine the parameter ρ_k and can be regarded as an extension of Dai-Kou conjugate gradient method. The global convergence of the proposed method for general nonlinear functions is analyzed under the standard assumptions. Numerical comparisons on the 147 test function from the CUTEst library indicate the proposed method is very promising.

Keywords Conjugate gradient method · Subspace minimization · memoryless quasi-Newton method · Two dimensional quadratic termination · Global convergence

Mathematics Subject Classification (2000) $90C06 \cdot 65K$

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1 Introduction

We consider the following unconstrained optimization problem:

$$
\min_{x \in \mathbb{R}^n} f(x).
$$

where f is continuously differential and its gradient is denoted by g . Due to the low memory requirement, simple form and nice numerical effect, conjugate gradient methods are a class of efficient iterative methods for large scale unconstrained optimization. Conjugate gradient methods are of the following form

$$
x_{k+1} = x_k + \alpha_k d_k, \tag{1.1}
$$

where α_k is the stepsize obtained by a line search and d_k is the search direction given by

$$
d_k = \begin{cases} -g_k, & \text{if } k = 0, \\ -g_k + \beta_k d_{k-1}, & \text{if } k > 0. \end{cases}
$$
 (1.2)

Here β_k is often called conjugate parameter. In the case that f is a convex quadratic function and the exact line search is performed, β_k should be the same. For nonlinear functions, however, different β_k result in different conjugate gradient methods and their properties can be significantly different. Some well-known formulae for β_k are called the Fletcher-Reeves (FR) [\[1\]](#page-21-0), Hestenes-Stiefel (HS) [\[2\]](#page-21-1), Polak-Ribière-Polyak (PRP) [\[3,](#page-21-2)[4\]](#page-21-3) and Dai-Yuan (DY) [\[5\]](#page-21-4) formulae, and are given by

$$
\beta_k^{FR} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2}, \quad \beta_k^{HS} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}, \quad \beta_k^{PRP} = \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2}, \quad \beta_k^{DY} = \frac{\|g_k\|^2}{d_{k-1}^T y_{k-1}},
$$

where $\lVert \cdot \rVert$ denotes the Euclidean norm.

By deleting the third term of the memoryless quasi-Newton search direction, Hager and Zhang [\[6\]](#page-21-5) presented a famous efficient conjugate gradient method (CG DESCENT, We also call it HZ CG algorithm for short) with

$$
\beta_k^{HZ} = \frac{g_{k+1}^T y_k}{d_k^T y_k} - 2 \frac{\|y_k\|^2}{d_k^T y_k} \frac{g_{k+1}^T d_k}{d_k^T y_k},\tag{1.3}
$$

and established the global convergence under the standard Wolfe line search. And the numerical results in [\[6,](#page-21-5)[7\]](#page-21-6) indicated that CG DESCENT with the approximate Wolfe line search (AWolfe line search):

$$
\sigma g_k^T d_k \le g(x_k + \alpha_k d_k)^T d_k \le (2\delta - 1) g_k^T d_k,
$$

where $0 < \delta < 0.5$ and $\delta \leq \sigma < 1$, is very efficient. In 2013, Dai and Kou [\[8\]](#page-21-7) projected a multiple of the memoryless BFGS direction of Perry [\[9\]](#page-21-8) and Shanno [\[10\]](#page-22-0) into the manifold $\{-g_{k+1} + sd_k : s \in \mathbb{R}\}$ and presented a family of conjugate gradient algorithms (CGOPT, We also call them Dai-Kou CG algorithms for short) with the improved Wolfe line search, and the numerical results in [\[8\]](#page-21-7) suggested that CGOPT with the following parameter:

$$
\beta_k^{DK} = \frac{g_{k+1}^T y_k}{d_k^T y_k} - \frac{\|y_k\|^2}{d_k^T y_k} \frac{g_{k+1}^T d_k}{d_k^T y_k} \tag{1.4}
$$

is the most efficient. CG DESCENT and CGOPT are both popular and quite efficient CG software packages. So far, conjugate gradient methods have attracted extremely extensive attention and the advance about conjugate gradient methods can be referred as [\[11\]](#page-22-1).

In conjugate gradient methods, the stepsize α_k is often required to satisfy certain line search conditions. Among them, the strong Wolfe line search is often used in the early convergence analysis, which aims to find a stepsize satisfying the following conditions

$$
f\left(x_k + \alpha_k d_k\right) \le f\left(x_k\right) + \sigma \alpha_k g_k^T d_k,\tag{1.5}
$$

$$
\left|g_{k+1}^T d_k\right| \le -\delta g_k^T d_k,\tag{1.6}
$$

where $0 < \delta < \sigma < 1$. The standard Wolfe line search is also preferred due to the relatively easy numerical implementation, which aims to find a stepsize satisfying [\(1.5\)](#page-2-0) and

$$
g_{k+1}^T d_k \ge \delta g_k^T d_k. \tag{1.7}
$$

The sufficient descent property of the search direction plays an important role in the convergence analysis, which requires the search direction to satisfy

$$
g_k^T d_k \le -c \|g_k\|^2,\tag{1.8}
$$

where $c > 0$.

The subspace minimization conjugate gradient (SMCG) methods are the the generalization of traditional conjugate gradient methods, and have also received much attention recently. The subspace minimization conjugate gradient methods were first proposed by Yuan and Stoer [\[12\]](#page-22-2) in 1995, where the search direction is computed by minimizing a quadratic model over the subspace $V_k = Span\{g_k, s_{k-1}\}$:

$$
\min_{d \in V_k} g_k^T d + \frac{1}{2} d^T B_k d,
$$
\n(1.9)

where $B_k \in \mathbb{R}^{n \times n}$ is a symmetric and positive definite approximation to the Hessian matrix and satisfies the standard secant equation $B_ks_{k-1} = y_{k-1}$. Since $d_k \in V_k$ can be expressed as

$$
d = ug_k + vs_{k-1},\tag{1.10}
$$

where $u, v \in \mathbb{R}$, by substituting [\(1.10\)](#page-2-1) into [\(1.9\)](#page-2-2) and using the standard secant equation, we arrange (1.9) as the following form :

$$
\min_{u,v \in \mathbb{R}} \left(\frac{\|g_k\|^2}{g_k^T s_{k-1}} \right)^T \binom{u}{v} + \frac{1}{2} \binom{u}{v}^T \left(\begin{array}{c} \rho_k & g_k^T y_{k-1} \\ g_k^T y_{k-1} & s_{k-1}^T y_{k-1} \end{array} \right) \binom{u}{v},\tag{1.11}
$$

where $\rho_k \approx g_k^T B_k g_k$, namely, ρ_k is the estimate of $g_k^T B_k g_k$.

At first the SMCG methods received little attention. For example, Andrei [\[13\]](#page-22-3) presented an efficient SMCG method, where the search direction is generated over $-g_k + Span{s_{k-1}, y_{k-1}}$; based on [\[13\]](#page-22-3), Yang et al. [\[14\]](#page-22-4) developed another SMCG method, in which the search direction is generated over $-g_k + Span{s_{k-1}, s_{k-2}}$. A significant work about the SMCG method was given by Dai and Kou [\[15\]](#page-22-5) in 2016. More specially, Dai and Kou established the finite termination for two dimensional convex quadratic functions of the SMCG method and presented a Barzilai-Borwein conjugate gradient (BBCG) methods with an efficient estimate of the parameter ρ_k :

$$
\rho_k^{BBCG3} = \frac{3}{2} \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}} \|g_k\|^2
$$
\n(1.12)

based on the BB method [\[15\]](#page-22-5). Motivated by the SMCG method and ρ_k^{BBCG3} , Liu and Liu [\[16\]](#page-22-6) extended the BBCG3 method to general unconstrained optimization and presented an efficient subspace minimization conjugate gradient method (SMCG BB) with the generalized Wolfe line search. Since then, a lot of SMCG methods emerged for unconstrained optimization. Based on [\[16\]](#page-22-6), Li et al. [\[17\]](#page-22-7) presented a new SMCG method based on conic model and quadratic model; Wang et al. [\[18\]](#page-22-8) proposed a new SMCG method based on tensor model and quadratic model; Zhao et al. [\[19\]](#page-22-9) presented a new SMCG method based on regularization model and quadratic model, and the numerical results in [\[19\]](#page-22-9) indicated these SMCG methods is very efficient. Recently, Sun et al. [\[20\]](#page-22-10) proposed some accelerated SMCG methods based on [\[19\]](#page-22-9). More advance about subspace minimization conjugate gradient method can be referred [\[21,](#page-22-11)[22\]](#page-22-12).

Subspace minimization conjugate methods are a class of efficient iterative methods for unconstrained optimization. On the one hand, the search direction of SMCG method is often parallel to the HS conjugate gradient method [\[15\]](#page-22-5). On the other hand, traditional conjugate gradient method with $d_k = -g_k + \beta_k d_{k-1}$ is only the special case of SMCG method with $d_k = u_k g_k + v_k s_{k-1}$. In other words, SMCG methods can not only inherit some important properties of traditional conjugate gradient methods but also have more choices for scaling the gradient g_k by u_k , which will induce that SMCG method without the exact line search can enjoy some additional nice theoretical properties such as the finite termination for two dimensional convex functions due to the term u_k in the search direction compared to the traditional conjugate gradient methods. In addition, SMCG methods have also illustrated nice numerical performance [\[16,](#page-22-6)[19\]](#page-22-9). Based on the observation, SMCG methods have great potentiality and should be received more attention.

However, the estimate ρ_k of $g_k^T B_k g_k$ must be determined before calculating the search direction. The parameter ρ_k is very crucial to the property and the numerical performance of SMCG methods, and we still do not understand how the parameter ρ_k affects the numerical behavior of the SMCG method. It is thus a great challenge to determine the parameter properly.

A simple analysis for the choice of ρ_k is given here. In [\(1.11\)](#page-2-3), the term $g_k^T y_{k-1}$ and $s_{k-1}^T y_{k-1}$ are obtained by using the standard secant equation to eliminate B_k , namely, $g_k^T y_{k-1} = g_k^T B_k s_{k-1}$ and $s_{k-1}^T y_{k-1} = s_{k-1}^T B_k s_{k-1}$. For $g_k^T B_k g_k$, we can not use the standard secant equation to eliminate B_k , which implies that B_k must be given or estimated before computing the search direction. Is the matrix B_k of $g_k^T B_k g_k$ required to satisfy the standard secant equation ? If yes, some memoryless quasi-Newton updating formulae can be applied to generate B_k . It is however observed that the resulting ρ_k can not bring the desired numerical effect [\[15\]](#page-22-5). If not, the matrices B_k in $g_k^T B_k g_k$ and in $g_k^T y_{k-1}$ are inconsistent, and we do not know what will happen even if the resulting estimate ρ_k is efficient. For example, in the efficient choice ρ_k^{BBCG3} , the B_k estimated by $\frac{3}{2} \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}}$ $\frac{||y_{k-1}||}{s_{k-1}^T y_{k-1}} I$ is not satisfied the standard secant equation, while the B_k in $g_k^T y_{k-1}$ satifies the standard secant equation. And we do not konw why ρ_k^{BBCG3} is so efficient in this way.

It induces a challenge in determining the parameter ρ_k properly, which causes that it is far from consensus for good choice of the parameter ρ_k so far. As a result, it is no doubt that the choice of the parameter ρ_k is a obstacle for the development of SMCG methods. A question is naturally to be asked: can we develop an efficient SMCG method without determining the parameter ρ_k ?

In the paper we do not focus on exploiting new choices for ρ_k since it is difficult to determine it properly, as mentioned above. Instead, we are interested to focus on the above question and study a new subspace minimization conjugate method without determining the important parameter ρ_k . Motivated by Dai-Kou conjugate gradient method [\[15\]](#page-22-5), we project the search direction of the memoryless quasi-Newton method into the subspace spanned by the current gradient and the latest search direction and develop a new SMCG method for

unconstrained optimization. The new search direction is proved to be descent. It is remarkable that the SMCG method without any line search enjoys finite termination for two dimensional convex quadratic functions. With the improved Wolfe line search, the convergence of the proposed method for general nonlinear functions is established under the standard assumptions. Numerical experiments on the 147 test functions from the CUTEst library [\[23\]](#page-22-13) indicates the proposed method is very promising.

The remainder of this paper is organized as follows. We develop a new SMCG method for unconstrained optimization and exploit some important properties of the new search direction in Section 2. In Section 3 we establish the global convergence of the proposed method for general nonlinear functions under the standard assumptions. Some numerical experiments are conducted in Section 4. Conclusions are given in the last section.

2 New subspace minimization conjugate gradient method independent of the parameter ρ_k

In the section, we first derive the new search directions, analyze their important properties, develop an adaptive scaling factor and present a new subspace minimization conjugate gradient method independent of the parameter ρ_k for unconstrained optimization.

2.1 The proposed search directions and their important properties

We are interested to the self-scaling memoryless BFGS method by Perry [\[9\]](#page-21-8) and Shanno [\[10\]](#page-22-0), where the search direction \bar{d}_k^{PS} is given by

$$
\bar{d}_{k}^{PS} = -\frac{1}{\tau_{k}} g_{k} + \left[\frac{g_{k}^{T} y_{k-1}}{\tau_{k} s_{k-1}^{T} y_{k-1}} - \left(1 + \frac{\left\| y_{k-1} \right\|^{2}}{\tau_{k} s_{k-1}^{T} y_{k-1}} \right) \frac{g_{k}^{T} s_{k-1}}{s_{k-1}^{T} y_{k-1}} \right] s_{k-1} + \frac{1}{\tau_{k}} \frac{g_{k}^{T} s_{k-1}}{s_{k-1}^{T} y_{k-1}} y_{k-1}.
$$

Here $\tau_k > 0$ is the scaling parameter. The scaling memoryless quasi-Newton method is indeed three-term conjugate gradient method. Specially, if the line search is exact, namely, $g_k^T s_{k-1} = 0$, then the search direction \bar{d}_k^{PS} is HS conjugate gradient direction with scaling factor $\frac{1}{\tau_k}$. It is not difficult to see that the search direction \bar{d}_k^{PS} only satisfies the following Dai-Liao conjugate condition [\[24\]](#page-22-14), namely,

$$
(\bar{d}_k^{PS})^T y_{k-1} = -t_k g_k^T s_{k-1}
$$
, where $t_k = 1$.

As we know, the adaptive choice for t_k in Dai-Liao conjugate gradient methods [\[24\]](#page-22-14) is usually more efficient than the prefixed choice. In addition, some famous and efficient conjugate gradient methods such as HZ conjugate gradient method [\[6\]](#page-21-5) and Dai-Kou conjugate gradient method [\[8\]](#page-21-7) are Dai-Liao conjugate gradient methods with adaptive parameters t_k . Therefore, to impose the search direction \bar{d}_k^{PS} to satisfy the Dai-Liao conjugate condition with adaptive parameter, we multiple \bar{d}_k^{PS} by τ_k and obtain the following direction:

$$
d_k^{PS} = -g_k + \left[\frac{g_k^T y_{k-1}}{s_{k-1}^T y_{k-1}} - \left(\tau_k + \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}} \right) \frac{g_k^T s_{k-1}}{s_{k-1}^T y_{k-1}} \right] s_{k-1} + \frac{g_k^T s_{k-1}}{s_{k-1}^T y_{k-1}} y_{k-1}.
$$
 (2.1)

Obviously, the search direction d_k^{PS} satisfies Dai-Liao conjugate condition $(d_k^{PS})^T y_{k-1} = -\tau_k g_k^T s_{k-1}$. Noted that the scaling factor τ_k is indeed the adaptive parameter in Dai-Liao conjugate gradient method. The selfscaling memoryless BFGS method by Perry [\[9\]](#page-21-8) and Shanno [\[10\]](#page-22-0) has been applied successfully to generate the famous and efficient Dai-Kou conjugate gradient method [\[8\]](#page-21-7).

The search direction in subspace minimization conjugate gradient method is usually generated in the subspace $Span\{g_k, s_{k-1}\}\$, which means that $d_k = u_k g_k + v_k s_{k-1}$, where u_k and v_k are undetermined parameters. Different from [\(1.11\)](#page-2-3) requiring to estimate the parameter ρ_k , based on the search direction d_k^{PS} , we will give a new way to derive u_k and v_k without requiring to estimate the parameter ρ_k .

We consider the case that g_k is not parallel to s_{k-1} , namely,

$$
\overline{\omega}_{k} = \frac{\left(g_{k}^{T} s_{k-1}\right)^{2}}{\|g_{k}\|^{2} \|s_{k-1}\|^{2}} \le \xi_{1},\tag{2.2}
$$

where $0 < \xi_1 < 1$ is close to 1. Otherwise, the search direction is naturally set to be $d_k = -g_k$.

By projecting the search direction d_k^{PS} into the subspace $Span\{g_k, s_{k-1}\}\$, we get the following subproblem:

$$
\min_{d_k = u_k g_k + v_k s_{k-1}} \|d_k^{PS} - d_k\|_2^2.
$$
\n(2.3)

Solving the subproblem [\(2.3\)](#page-5-0) yields the search direction

$$
d_k = u_k g_k + v_k s_{k-1},\tag{2.4}
$$

,

where

$$
u_{k} = -1 + \frac{g_{k}^{T} y_{k-1} g_{k}^{T} s_{k-1}}{s_{k-1}^{T} y_{k-1} \|g_{k}\|^{2}} - \frac{\|g_{k}\|^{2} \left(g_{k}^{T} s_{k-1}\right)^{2} - g_{k}^{T} y_{k-1} \left(g_{k}^{T} s_{k-1}\right)^{3} / s_{k-1}^{T} y_{k-1}}{\|g_{k}\|^{2} \left[\|g_{k}\|^{2} \|s_{k-1}\|^{2} - \left(g_{k}^{T} s_{k-1}\right)^{2}\right]},
$$
\n(2.5)

$$
v_{k} = \frac{g_{k}^{T} y_{k-1}}{s_{k-1}^{T} y_{k-1}} + \frac{\|g_{k}\|^{2} g_{k}^{T} s_{k-1} - g_{k}^{T} y_{k-1} \left(g_{k}^{T} s_{k-1}\right)^{2} / s_{k-1}^{T} y_{k-1}}{\|g_{k}\|^{2} \|s_{k-1}\|^{2} - \left(g_{k}^{T} s_{k-1}\right)^{2}} - \left(\tau_{k} + \frac{\|y_{k-1}\|^{2}}{s_{k-1}^{T} y_{k-1}}\right) \frac{g_{k}^{T} s_{k-1}}{s_{k-1}^{T} y_{k-1}}.
$$
 (2.6)

It is not difficult to see that u_k and v_k can be rewritten as the following forms:

$$
u_{k} = \frac{1}{1 - \overline{\omega}_{k}} \left(-1 + \frac{g_{k}^{T} y_{k-1} g_{k}^{T} s_{k-1}}{s_{k-1}^{T} y_{k-1} \|g_{k}\|^{2}} \right), \ v_{k} = \frac{1 - 2\overline{\omega}_{k}}{1 - \overline{\omega}_{k}} \frac{g_{k}^{T} y_{k-1}}{s_{k-1}^{T} y_{k-1}} - \left(\tau_{k} + \frac{\left\| y_{k-1} \right\|^{2}}{s_{k-1}^{T} y_{k-1}} - \frac{s_{k-1}^{T} y_{k-1}}{\left(1 - \overline{\omega}_{k} \right) \left\| s_{k-1} \right\|^{2}} \right) \frac{g_{k}^{T} s_{k-1}}{s_{k-1}^{T} y_{k-1}} \tag{2.7}
$$

which are similar to the forms of conjugate gradient method. The new search direction (2.7) can be regarded the extension of Dai-Kou conjugate gradient direction.

It is noted that the parameter τ_k in [\(2.6\)](#page-5-2) is the scaling factor in the memoryless quasi-Newton method, which is crucial to the numerical performance of the corresponding methods. There are various choices for τ_k , and in the analysis on the descent property and global convergence, the following choices

$$
\tau_k^B = \frac{s_{k-1}^T y_{k-1}}{\|s_{k-1}\|^2}, \qquad \tau_k^H = \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}}, \qquad \tau_k^{(1)} = 1 \tag{2.8}
$$

are considered. We also give an adaptive choice of τ_k in Section 3.2 based on Theorem [2.1.](#page-5-3)

Remark 1 If the line search is exact, namely, $g_k^T s_{k-1} = 0$, then it follows that $u_k = -1$ and $v_k = \frac{g_k^T y_{k-1}}{s_{k-1}^T y_{k-1}}$ or $v_k = \frac{g_k^T y_{k-1}}{\alpha_k^T y_{k-1}}$ $\frac{g_k g_{k-1}}{\alpha_{k-1} \|g_{k-1}\|^2}$, which mean the search direction [\(2.4\)](#page-5-4) reduces to the HS or PRP conjugate gradient direction.

Remark 2 The search direction [\(2.4\)](#page-5-4) satisfies the Dai-Liao conjugate condition

$$
d_k^T y_{k-1} = \left[\frac{\frac{\left(g_k^T y_{k-1}\right)^2 ||s_{k-1}||^2}{s_{k-1}^T y_{k-1}} - 2g_k^T y_{k-1} g_k^T s_{k-1} + ||g_{k-1}||^2 s_{k-1}^T y_{k-1}}{\Delta_k} - \left(\tau_k + \frac{||y_{k-1}||^2}{s_{k-1}^T y_{k-1}}\right) \right] g_k^T s_{k-1} \stackrel{\Delta}{=} t_k g_k^T s_{k-1}.
$$

We will establish an interesting property—the finite termination of the SMCG method with [\(1.1\)](#page-1-0) and [\(2.4\)](#page-5-4) in the following theorem.

Theorem 2.1 Consider the SMCG method [\(1.1\)](#page-1-0) and [\(2.4\)](#page-5-4) with $\tau_k = 1$ for the convex quadratic function $q(x) = \frac{1}{2}x^{T}Ax + b^{T}x, x \in \mathbb{R}^{2}$, where $A \in \mathbb{R}^{2 \times 2}$ is a symmetric and positive definite matrix and $b \in \mathbb{R}^{2}$. Assume that $d_0 = -\alpha_0 g_0$, where α_0 is the exact stepsize. Then, we must have that $g_j = 0$ for some $j \leq 3$.

Proof Assume that $g_j \neq 0$ for $j = 0, 1, 2$. Since the first step is a Cauchy steepest descent step, we know

$$
g_1^T s_0 = 0. \t\t(2.9)
$$

By (2.4) , $s_1 = d_1$ and Remark [2,](#page-5-5) we have that

$$
s_1^T y_0 = d_1^T y_0 = t_1 g_1^T s_0 = 0,
$$

where t_1 is given by Remark [2.](#page-5-5) Thus,

$$
y_1^T s_0 = s_1^T A s_0 = 0.
$$
\n(2.10)

Since $n = 2$, $s_0 \neq 0$ and $y_1 = g_2 - g_1$, we know from [\(2.9\)](#page-6-0) and [\(2.10\)](#page-6-1) that g_1, g_2 y_1 are collinear and there must exist some real number $a \neq 0$ such that

$$
y_1 = ag_2. \tag{2.11}
$$

By (2.5) and (2.6) , we have

$$
u_2 = -1 + \frac{g_2^T y_1 g_2^T s_1}{\|g_2\|^2 s_1^T y_1} - \frac{\|g_2\|^2 g_2^T s_1 - g_2^T y_1 (g_2^T s_1)^2 / s_1^T y_1}{\|g_2\|^2} \frac{g_2^T s_1}{\|g_2\|^2}
$$

\n
$$
= -1 + \frac{a \|g_2\|^2 g_2^T s_1}{a \|g_2\|^2 g_2^T s_1} - \frac{\|g_2\|^2 g_2^T s_1 - a \|g_2\|^2 (g_2^T s_1)^2 / (a g_2^T s_1)}{\|g_2\|^2 \|s_1\|^2 - (g_2^T s_1)^2} \frac{g_2^T s_1}{\|g_2\|^2}
$$

\n
$$
= -1 + 1 + 0
$$

\n
$$
= 0,
$$

\n
$$
v_2 = \frac{g_2^T y_1}{s_1^T y_1} - \left(\tau_k + \frac{\|y_1\|^2}{s_1^T y_1}\right) \frac{g_2^T s_1}{s_1^T y_1} + \frac{\|g_2\|^2 g_2^T s_1 - g_2^T y_1 (g_2^T s_1)^2 / s_1^T y_1}{\|g_2\|^2 \|s_1\|^2 - (g_2^T s_1)^2}
$$

\n
$$
= \frac{a \|g_2\|^2}{a g_2^T s_1} - \left(\tau_k + \frac{a^2 \|g_1\|^2}{a g_2^T s_1}\right) \frac{g_2^T s_1}{a g_2^T s_1} + 0
$$

\n
$$
= -\frac{\tau_k}{a},
$$

which implies that $s_2 = d_2 = -\frac{\tau_k}{a} s_1$. Therefore,

$$
g_3 = g_2 + y_2 = g_2 + As_2 = g_2 - \frac{\tau_k}{a} As_1 = g_2 - \frac{\tau_k}{a} y_1 = (1 - \tau_k) g_2.
$$
 (2.12)

Since $\tau_k = \tau_k^{(1)} = 1$, we have $g_3 = 0$, which completes the proof. □

Remark 3 It follows that the SMCG method [\(1.1\)](#page-1-0) and [\(2.4\)](#page-5-4) with $\tau_k = 1$ without any line search except the first Cauchy steepest descent iteration enjoys the finite termination for two dimensional convex quadratic functions. It seems that it is not possible to obtain the same conclusion for traditional conjugate gradient methods without the exact line search.

Together with [\(2.2\)](#page-5-7), let us consider the following search direction:

$$
d_k = \begin{cases} -g_k, & \text{if } k = 0 \text{ or } \overline{\omega}_k > \xi_1, \\ u_k g_k + v_k s_{k-1}, & \text{otherwise,} \end{cases} \tag{2.13}
$$

where $\overline{\omega}_k$, u_k and v_k are given by [\(2.2\)](#page-5-7), [\(2.5\)](#page-5-6) and [\(2.6\)](#page-5-2), respectively. We first do the following assumption:

Assumption 2.1 (i) The objective function f is continuously differentiable on \mathbb{R}^n ; (ii) The level set $\mathcal{L} =$ $\sqrt{ }$ $x|f(x) \leq f(x_0) + \sum_{k \geq 0} \overline{\eta}_k$ \mathcal{L} is bounded, where Σ $\sum_{k\geq 0} \overline{\eta}_k < +\infty$; (iii) The gradient g is Lipschitz continuous on \mathbb{R}^n , namely, there exists a constant $L > 0$ such that

$$
\| g(x) - g(y) \| \le L \| x - y \|, \ \forall x, y \in \mathbb{R}^n.
$$
 (2.14)

Denote

$$
p_k = \frac{\|y_{k-1}\|^2 \|s_{k-1}\|^2}{\left(s_{k-1}^T y_{k-1}\right)^2}, \quad \gamma_k = \tau_k \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}}.
$$
\n(2.15)

The following lemma discusses the descent property of the search direction [\(2.13\)](#page-6-2).

Lemma [2.1](#page-6-3) Assume that f satisfies Assumption 2.1 (iii), and consider the subspace minimization conjugate gradient methods [\(1.1\)](#page-1-0) and [\(2.13\)](#page-6-2) with any one of τ_k in [\(2.8\)](#page-5-8). If $s_{k-1}^T y_{k-1} > 0$, then

$$
g_k^T d_k < 0. \tag{2.16}
$$

Furthermore, if f is uniformly convex, namely, there exists $\mu > 0$ such that

$$
\left(g\left(x\right) - g\left(y\right)\right)^{T} \left(x - y\right) \ge \mu \|x - y\|^{2}, \quad \forall x, y \in \mathbb{R}^{n},\tag{2.17}
$$

then there must exists $c > 0$ such that

$$
g_k^T d_k < -c \|g_k\|^2. \tag{2.18}
$$

Proof We prove the conclusion by dividing it into the following two cases.

(i) $d_k = -g_k$. We know easily that [\(2.16\)](#page-7-0) and [\(2.18\)](#page-7-1) both hold.

(ii) $d_k = u_k g_k + v_k s_{k-1}$, where u_k and v_k are given by [\(2.5\)](#page-5-6) and [\(2.6\)](#page-5-2), respectively. It is not difficult to get that

$$
g_k^T d_k = -\|g_k\|^2 + \frac{2g_k^T s_{k-1} g_k^T y_{k-1}}{s_{k-1}^T y_{k-1}} - \left(\tau_k + \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}}\right) \frac{\left(g_k^T s_{k-1}\right)^2}{s_{k-1}^T y_{k-1}},\tag{2.19}
$$

which, in this sense, implies that the above search direction can be treated as

$$
d_{k} = -g_{k} + \left[\frac{2g_{k}^{T}y_{k-1}}{s_{k-1}^{T}y_{k-1}} - \left(\tau_{k} + \frac{\|y_{k-1}\|^{2}}{s_{k-1}^{T}y_{k-1}}\right) \frac{g_{k}^{T}s_{k-1}}{s_{k-1}^{T}y_{k-1}}\right]s_{k-1}
$$

$$
= -\left(I - \frac{2s_{k-1}y_{k-1}^{T} - \left(\tau_{k} + \frac{\|y_{k-1}\|^{2}}{s_{k-1}^{T}y_{k-1}}\right)s_{k-1}s_{k-1}^{T}}{s_{k-1}^{T}y_{k-1}}\right)g_{k}
$$

$$
\stackrel{\Delta}{=} -H_{k}g_{k}.
$$
 (2.20)

For the symmetric part of H_k :

$$
\bar{H}_k = \frac{H_k + H_k^T}{2} = I - \frac{s_{k-1}y_{k-1}^T + y_{k-1}s_{k-1}^T}{s_{k-1}^T y_{k-1}} + \bar{t}_k \frac{s_{k-1}s_{k-1}^T}{s_{k-1}^T y_{k-1}},
$$
\n(2.21)

where $\bar{t}_k = \tau_k + \frac{\|y_{k-1}\|^2}{s^T \!-\! y_k}$ $\frac{\|y_{k-1}\|}{s_{k-1}^T y_{k-1}}$, it is not difficult to verify that

$$
g_k^T d_k = -g_k^T H_k g_k = -g_k^T \left(\frac{H_k + H_k^T}{2} + \frac{H_k - H_k^T}{2} \right) g_k = -g_k^T \bar{H}_k g_k + 0 = -g_k^T \bar{H}_k g_k. \tag{2.22}
$$

Now, we only need to analyze the smallest eigenvalues of \bar{H}_k . Rewriting \bar{H}_k as

$$
\bar{H}_k = I - \frac{\left(y_{k-1} - \bar{t}_k s_{k-1}\right) s_{k-1}^T}{s_{k-1}^T y_{k-1}} - \frac{s_{k-1} y_{k-1}^T}{s_{k-1}^T y_{k-1}},\tag{2.23}
$$

we know that

$$
\det\left(\bar{H}_k\right) = -\frac{\|y_{k-1}\|^2 \|s_{k-1}\|^2}{\left(s_{k-1}^T y_{k-1}\right)^2} + \frac{\bar{t}_k \|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}} = \tau_k \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}},\tag{2.24}
$$

which implies that

$$
\lambda_{\min} \lambda_{\max} = \tau_k \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}}.
$$
\n(2.25)

It follows from $trace\left(\bar{H}_k\right) = n - 2 + \bar{t}_k \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}}$ $\frac{\|S_{k-1}\|}{S_{k-1}^T y_{k-1}} = (n-2) + \lambda_{\min} + \lambda_{\max}$ that

$$
\lambda_{\min} + \lambda_{\max} = \frac{\|y_{k-1}\|^2 \|s_{k-1}\|^2}{\left(s_{k-1}^T y_{k-1}\right)^2} + \tau_k \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}}.
$$
\n(2.26)

Combining $s_{k-1}^T y_{k-1} > 0$, [\(2.25\)](#page-8-0) and [\(2.26\)](#page-8-1) yields

$$
\lambda_{\min} = \frac{\frac{\|y_{k-1}\|^2 \|s_{k-1}\|^2}{\left(s_{k-1}^T y_{k-1}\right)^2} + \tau_k \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}} - \sqrt{\left(\frac{\|y_{k-1}\|^2 \|s_{k-1}\|^2}{\left(s_{k-1}^T y_{k-1}\right)^2} + \tau_k \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}}\right)^2 - 4\tau_k \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}}}{2}
$$
\n
$$
= \frac{p_k + \gamma_k - \sqrt{\left(p_k + \gamma_k\right)^2 - 4\gamma_k}}{2}
$$
\n
$$
> 0,
$$
\n
$$
(2.27)
$$

where p_k and γ_k are given by [\(2.15\)](#page-7-2). As a result, for any one of τ_k in [\(2.8\)](#page-5-8), we have $-g_k^T d_k = g_k^T \bar{H}_k g_k \ge$ $\lambda_{\min} ||g_k||^2 > 0$, which implies that [\(2.16\)](#page-7-0).

We next analyze the sufficient descent property of the search direction with different τ_k in [\(2.8\)](#page-5-8) when f is uniformly convex.

(a)
$$
\tau_k = \tau_k^H = \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}}
$$
. We have that $\gamma_k = \tau_k \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}} = p_k$ and $\lambda_{\min} = p_k - \sqrt{p_k^2 - p_k}$. Since

$$
\frac{d\lambda_{\min}}{dp_k} = 1 - \frac{2p_k - 1}{2\sqrt{p_k^2 - p_k}} < 0, \ \forall p_k > 1,
$$

 λ_{\min} is monotonically decreasing in $[1, +\infty)$ and thus

 $\lambda_{\min} > 1/2$.

Noted that when $\tau_k = \tau_k^H = \frac{||y_{k-1}||^2}{s_*^T}$, y_{k-1} $\frac{||y_{k-1}||}{s_{k-1}^T y_{k-1}}$, the sufficient descent property of the search direction is proved without the uniformly convexity condition [\(2.17\)](#page-7-3).

(b)
$$
\tau_k = \tau_k^B = \frac{s_{k-1}^T y_{k-1}}{\|s_{k-1}\|^2}
$$
. We have that $\gamma_k = \tau_k \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}} = 1$ and $\lambda_{\min} = \frac{p_k + 1 - \sqrt{(p_k + 1)^2 - 4}}{2}$. Since

$$
\frac{d\lambda_{\min}}{dp_k} = \frac{1}{2} - \frac{p_k + 1}{2\sqrt{(p_k + 1)^2 - 4}} < 0, \ \forall p_k > 1,
$$
 (2.28)

 λ_{\min} is monotonically decreasing in [1, + ∞). By Assumption [2.1](#page-6-3) (iii) and [\(2.17\)](#page-7-3), we know that

$$
p_k = \left(\frac{\|s_{k-1}\| \|y_{k-1}\|}{s_{k-1}^T y_{k-1}}\right)^2 \le \left(\frac{L \|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}}\right)^2 \le \frac{L^2}{\mu^2}.
$$
\n(2.29)

Therefore,

$$
\lambda_{\min} \ge \frac{L^2/\mu^2 + 1 - \sqrt{(L^2/\mu^2 + 1)^2 - 4}}{2}.
$$

(c)
$$
\tau_k = \tau_k^{(1)} = 1
$$
. We have $\gamma_k = \tau_k \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}} = \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}}$ and $\lambda_{\min} = \frac{p_k + \gamma_k - \sqrt{(p_k + \gamma_k)^2 - 4\gamma_k}}{2}$. Since

$$
\frac{\partial \lambda_{\min}}{\partial p_k} = \frac{1}{2} - \frac{p_k + \gamma_k}{2\sqrt{(p_k + \gamma_k)^2 - 4\gamma_k}},
$$

 λ_{min} is monotonically decreasing with respect to $p_k.$

It follows from [\(2.29\)](#page-8-2) that $p_k \leq L^2 \gamma_k^2$. Let \overline{L} be any value such that $(\gamma_k + L^2 \gamma_k^2)^2 - 4\gamma_k \geq 0$. Thus, $p_k \le \max\left\{L^2, \bar{L}^2\right\} \gamma_k^2 \stackrel{\Delta}{=} \tilde{L}^2 \gamma_k^2$. As a result,

$$
\lambda_{\min} \ge \frac{\gamma_k + \tilde{L}^2 \gamma_k^2 - \sqrt{\left(\gamma_k + \tilde{L}^2 \gamma_k^2\right)^2 - 4\gamma_k}}{2} \stackrel{\Delta}{=} \bar{\phi}\left(\gamma_k\right). \tag{2.30}
$$

It follows from (2.17) that $\gamma_k \leq \frac{1}{\mu}$. It is not difficult to verify that $\bar{\phi}'(\gamma_k) < 0$, which implies that $\bar{\phi}(\gamma_k)$ is monotonically decreasing in $\left(0, \frac{1}{\mu}\right)$ i and

$$
\lambda_{\min} \geq \bar{\phi}\left(\frac{1}{\mu}\right) = \frac{1/\mu + \tilde{L}^2/\mu^2 - \sqrt{\left(1/\mu + \tilde{L}^2/\mu^2\right)^2 - 4/\mu}}{2}.
$$

In conclusion, for any one of τ_k in [\(2.8\)](#page-5-8), there must exists $c > 0$ such that $\lambda_{\min} \geq c$, which implies that

$$
g_k^T d_k = -g_k^T \bar{H}_k g_k \le -\lambda_{\min} \|g_k\|^2 \le -c \|g_k\|^2.
$$

It completes the proof. ⊓⊔

Powell [\[25\]](#page-22-15) constructed a counterexample showing that the PRP method with exact line search may not converge for general nonlinear functions. It follows from Remark [1](#page-5-9) that Powell's example can also be used to show that the method [\(1.1\)](#page-1-0) and [\(2.13\)](#page-6-2) with any one of τ_k in [\(2.8\)](#page-5-8) may not converge for general nonlinear functions. Therefore, motivated the truncation form in [\[8\]](#page-21-7), we truncate similarly v_k in [\(2.6\)](#page-5-2) as

$$
\bar{v}_k = \max\{v_k, \eta_k\},\tag{2.31}
$$

where

.

$$
\eta_k = -l_k \frac{|g_k^T s_{k-1}|}{\left\|s_{k-1}\right\|^2}, \quad l_k = \begin{cases} \xi_2, & \text{if } g_k^T s_{k-1} \le 0, \\ \min\left\{\max\left\{\bar{\xi}_2, -1 + (1+u_k)/\overline{\omega}_k\right\}, \bar{\xi}_2\right\}, & \text{otherwise.} \end{cases}
$$
(2.32)

Here $0 < \bar{\xi}_2 < 1$, and $0 < \xi_2 < 1$.

When applying the conjugate gradient methods with the exact line search to solve quadratic minimization problems, the sequence of the corresponding gradients is orthogonal, namely, $g_k^T g_j = 0$, $0 \le j \le k - 1$. For general nonlinear functions, one also hope that $|g_k^T g_{k-1}|$ may not be far from 0. When $|g_k^T g_{k-1}| > \xi \|g_k\|^2$, where $0 < \xi < 1$, Powell [\[26\]](#page-22-16) suggested that the search direction should be restarted with $d_k = -g_k$. Powell's restart strategy is quite efficient and has been used widely in the numerical implementation of conjugate gradient methods. As a result, if the condition

$$
-\eta_1 \|g_k\|^2 \le g_k^T g_{k-1} \le \eta_2 \|g_k\|^2, \quad 0 < \eta_2 < 1, \ \eta_1 > \eta_2 \tag{2.33}
$$

holds, our method will be also restarted with $-g_k$. It follows from [\(2.33\)](#page-9-0) that

$$
0 < (1 - \eta_2) \le \frac{g_k^T y_{k-1}}{\|g_k\|^2} \le (1 + \eta_1).
$$
\n(2.34)

Therefore, the search direction is summarized below:

$$
d_k = \begin{cases} -g_k, & \text{if } k = 0 \text{ or } \overline{\omega}_k > \xi_1 \text{ or (2.33) holds,} \\ u_k g_k + \overline{v}_k s_{k-1}, & \text{otherwise,} \end{cases} \tag{2.35}
$$

where \bar{v}_k and $\bar{\omega}_k$ are given by [\(2.31\)](#page-9-1) and [\(2.2\)](#page-5-7), respectively.

Lemma 2.2 Assume that f satisfies Assumption [2.1](#page-6-3) (iii), and consider the subspace minimization conjugate gradient methods [\(1.1\)](#page-1-0) and [\(2.35\)](#page-10-0) with any one of τ_k in [\(2.8\)](#page-5-8). If $s_{k-1}^T y_{k-1} > 0$, then there exists $c > 0$ such that

$$
g_k^T d_k \le -c \|g_k\|^2. \tag{2.36}
$$

Proof We prove it in the following cases.

(i) $d_k = -g_k$. We have $g_k^T d_k = -||g_k||^2$.

(ii) $d_k = u_k g_k + \eta_k s_{k-1}$. If $g_k^T s_{k-1} \leq 0$, then by (2.7) and (2.32) we have that

$$
g_k^T d_k = \left(-\|g_k\|^2 + \frac{g_k^T y_{k-1} g_k^T s_{k-1}}{s_{k-1}^T y_{k-1}}\right) / \left(1 - \overline{\omega}_k\right) + l_k \overline{\omega}_k \|g_k\|^2
$$

=
$$
- \left(\frac{1}{1 - \overline{\omega}_k} - l_k \overline{\omega}_k - \frac{g_k^T y_{k-1} g_k^T s_{k-1}}{s_{k-1}^T y_{k-1} \|g_k\|^2}\right) \|g_k\|^2.
$$

It follows from (2.2) , $g_k^T s_{k-1} \leq 0$ and (2.34) that

$$
\frac{1}{1-\overline{\omega}_k} - l_k \overline{\omega}_k - \frac{g_k^T y_{k-1} g_k^T s_{k-1}}{s_{k-1}^T y_{k-1} \|g_k\|^2} \ge \frac{1}{1-\overline{\omega}_k} - l_k \overline{\omega}_k \ge 1 - \xi_1.
$$

Therefore, when if $g_k^T s_{k-1} \geq 0$, we obtain that

$$
g_k^T d_k \leq -(1 - \xi_1) \|g_k\|^2.
$$

If $g_k^T s_{k-1} > 0$, then

$$
g_k^T d_k = u_k ||g_k||^2 - l_k \overline{\omega}_k ||g_k||^2
$$

= -(-u_k + l_k $\overline{\omega}_k$) ||g_k||².

According to [\(2.32\)](#page-9-2), we can easily have that $-u_k + l_k \overline{\omega}_k \geq 1 - \overline{\omega}_k \geq 1 - \xi_1$. Therefore, when $g_k^T s_{k-1} > 0$, we have that

$$
g_k^T d_k \leq -(1-\xi_1) \|g_k\|^2.
$$

(iii) $d_k = u_k g_k + v_k s_{k-1}$. According to [\(2.7\)](#page-5-1) and [\(2.35\)](#page-10-0), we obtain that

$$
v_{k} = \frac{1 - 2\overline{\omega}_{k}}{1 - \overline{\omega}_{k}} \frac{g_{k}^{T} y_{k-1}}{s_{k-1}^{T} y_{k-1}} - \left(\tau_{k} + \frac{\|y_{k-1}\|^{2}}{s_{k-1}^{T} y_{k-1}}\right) \frac{g_{k}^{T} s_{k-1}}{s_{k-1}^{T} y_{k-1}} + \frac{1}{1 - \overline{\omega}_{k}} \frac{g_{k}^{T} s_{k-1}}{\|s_{k-1}\|^{2}}
$$

\n
$$
\geq -l_{k} \frac{\|g_{k}^{T} s_{k-1}\|^{2}}{\|s_{k-1}\|^{2}}.
$$
\n(2.37)

If $g_k^T s_{k-1} \leq 0$, then from [\(2.19\)](#page-7-4) and [\(2.34\)](#page-9-3) we know that $g_k^T d_k \leq -||g_k||^2$. So we only need to consider the case of $g_k^T s_{k-1} > 0$. Multiplying both sides of (2.37) by $\frac{g_k^T s_{k-1}}{\|g_k\|^2}$ $\frac{g_k s_{k-1}}{\|g_k\|^2}$ yields

$$
\frac{1-2\overline{\omega}_{k}}{1-\overline{\omega}_{k}}\frac{g_{k}^{T}y_{k-1}}{s_{k-1}^{T}y_{k-1}}\frac{g_{k}^{T}s_{k-1}}{\left\|g_{k}\right\|^{2}}-\left(\tau_{k}\frac{\left\|s_{k-1}\right\|^{2}}{s_{k-1}^{T}y_{k-1}}+\frac{\left\|y_{k-1}\right\|^{2}\left\|s_{k-1}\right\|^{2}}{\left(s_{k-1}^{T}y_{k-1}\right)^{2}}\right)\frac{\left(g_{k}^{T}s_{k-1}\right)^{2}}{\left\|g_{k}\right\|^{2}\left\|s_{k-1}\right\|^{2}}+\frac{\overline{\omega}_{k}}{1-\overline{\omega}_{k}}\geq -l_{k}\frac{\left(g_{k}^{T}s_{k-1}\right)^{2}}{\left\|g_{k}\right\|^{2}\left\|s_{k-1}\right\|^{2}}.
$$

 $\frac{\lambda_{k}-1}{2}$ and λ_{\min} is mono-

According to [\(2.15\)](#page-7-2), we have that

$$
\frac{\frac{1}{\overline{\omega}_k} - 2}{\frac{1}{\overline{\omega}_k} - 1} \frac{g_k^T y_{k-1}}{\|g_k\|^2} \frac{g_k^T s_{k-1}}{s_{k-1}^T y_{k-1}} \ge \gamma_k + p_k - \frac{1}{1 - \overline{\omega}_k} - l_k.
$$
\n(2.38)

It follows from [\(2.16\)](#page-7-0) in Lemma [2.1](#page-7-5) and $g_k^T s_{k-1} > 0$ that $0 < \frac{g_k^T s_{k-1}}{g_k^T s_{k-1}}$ $\frac{g_k s_{k-1}}{s_{k-1}^T y_{k-1}} < 1$. It follows from (2.2) and (2.38) that

$$
p_k \le \gamma_k + p_k \le 1 + \eta_1 + l_k + \frac{1}{1 - \overline{\omega}_k} \le 1 + \eta_1 + \overline{\xi}_2 + \frac{1}{1 - \xi_1} \stackrel{\Delta}{=} \xi_0. \tag{2.39}
$$

We next derive the conclusion for any one of τ_k in [\(2.8\)](#page-5-8) based on [\(2.27\)](#page-8-3) as follows.

(a) $\tau_k = \tau_k^H = \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}}$ $\frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}}$. We have that $g_k^T d_k \leq -0.5 \|g_k\|^2$ by Lemma [2.1.](#page-7-5) (b) $\tau_k = \frac{s_{k-1}^T y_{k-1}}{\|s_{k-1}\|^2}$ $\sum_{k=1}^{T} y_{k-1}^{T}$. According to Lemma [2.1,](#page-7-5) we know that $\lambda_{\min} = \frac{1+p_k - \sqrt{(p_k-1)(p_k+3)}}{2}$

tonically decreasing in $[1, +\infty)$. Combining with (2.39) , we obtain

$$
\lambda_{\min} \ge \frac{1 + \xi_0 - \sqrt{(\xi_0 - 1)(\xi_0 + 3)}}{2} > 0.
$$

(c) $\tau_k = 1$. According to Lemma [2.1,](#page-7-5) we know that

$$
\lambda_{\min} = \frac{\gamma_k + p_k - \sqrt{(\gamma_k + p_k)^2 - 4\gamma_k}}{2}.
$$

Since $\frac{\partial \lambda_{\min}}{\partial p_k} = 1 - \frac{\gamma_k + p_k}{\sqrt{(\gamma_k + p_k)^2}}$ $\frac{\gamma_k+p_k}{(\gamma_k+p_k)^2-4\gamma_k}$ < 0, λ_{\min} is monotonically decreasing with respect to p_k in $[1, +\infty)$. Similarly to Lemma [2.1,](#page-7-5) we can obtain

$$
\lambda_{\min} \geq \frac{\gamma_k + \tilde{L}^2 \gamma_k^2 - \sqrt{\left(\gamma_k + \tilde{L}^2 \gamma_k^2\right)^2 - 4 \gamma_k}}{2} \overset{\Delta}{=} \bar{\phi}\left(\gamma_k\right),
$$

where \tilde{L} is the same as that in[\(2.30\)](#page-9-4). It is not difficult to verify that $\bar{\phi}'(\gamma_k) < 0$, which, together with $\gamma_k \le \xi_0$ implied by [\(2.39\)](#page-11-1), yields that $\bar{\phi}(\gamma_k)$ is monotonically decreasing and

$$
\lambda_{\min} \ge \bar{\phi}\left(\xi_0\right) = \frac{\xi_0 + \tilde{L}^2 \gamma_k^2 - \sqrt{\left(\xi_0 + \tilde{L}^2 \xi_0^2\right)^2 - 4\xi_0}}{2} > 0.
$$

In conclusion, for any one of τ_k in [\(2.8\)](#page-5-8), there must exists $c > 0$ such that $\lambda_{\min} \ge c$, which together with [\(2.22\)](#page-7-6) implies that

$$
g_k^T d_k = -g_k^T \bar{H}_k g_k \le -\lambda_{\min} \|g_{k-1}\|^2 \le -c \|g_{k-1}\|^2.
$$

It completes the proof. ⊓⊔

2.2 Adaptive choice of τ_k

The choice of τ_k is also crucial to the search direction [\(2.35\)](#page-10-0). τ_k is given by the following observation. From Theorem [2.1,](#page-5-3) we know that the SMCG method [\(1.1\)](#page-1-0) and [\(2.4\)](#page-5-4) with $\tau_k = 1$ and without any line search expect the first Cauchy steepest descent iteration can enjoy the finite termination property when the objective function f is 2 dimensional strictly convex quadratic function. Therefore, the choice of $\tau_k = 1$ may be preferred in some cases.

According to [\[27,](#page-22-17)[28\]](#page-22-18),

$$
\mu_k = \left| \frac{2\left(f_{k-1} - f_k + g_k^T s_{k-1}\right)}{s_{k-1}^T y_{k-1}} - 1 \right| \tag{2.40}
$$

is a quantity measuring how f is close to a quadratic on the line segment between x_{k-1} and x_k . If the following condition [\[29,](#page-22-19)[28\]](#page-22-18) holds, namely,

$$
\mu_k \le \xi_3 \quad \text{or} \quad \max\left\{\mu_k, \mu_{k-1}\right\} \le \xi_4,\tag{2.41}
$$

where ξ_3 and ξ_4 are small positives and $\xi_3 < \xi_4$, f might be very close to a quadratic function on the line segment between x_{k-1} and x_k . Therefore, if [\(2.41\)](#page-12-0) and the following condition

$$
||g_k||^2 \le \xi_6 \text{ or } \left(||g_k||^2 > \xi_6 \text{ and } ||s_{k-1}||^2 \le \xi_5\right)
$$
 (2.42)

hold, then the search direction [\(2.1\)](#page-4-0) should not be scaled, namely, $\tau_k = 1$. Here, ξ_5 , ξ_6 , > 0 . It is noted that the condition [\(2.42\)](#page-12-1) means that the current iterative point x_k is close to the stationary point or close to the latest iterative point x_{k-1} .

Therefore, τ_k is given by

$$
\tau_k = \begin{cases} 1, & \text{if (2.41) and (2.42) hold,} \\ \tau_k^B, & \text{otherwise,} \end{cases} \tag{2.43}
$$

where τ_k^B is given by (2.8)

2.3 The initial stepsize and the improved Wolfe line search

It is universally accepted that the choice of initial stepsize is of great importance for a line search method. Unlike general quasi-Newton methods, it is challenging to determine a suitable initial stepsize for a SMCG method. The initial stepsize in our method is also similar to Algorithm 3.1 in [\[8\]](#page-21-7), and the main difference lies in that we replace

$$
\alpha_k^{(0)} = \max \left\{ \varphi \alpha_{k-1}, -2 \left| f_k - f_{k-1} \right| / g_k^T d_k \right\} \tag{2.44}
$$

in Step 1 of Algorithm 3.1 in [\[8\]](#page-21-7) by

$$
\bar{\alpha}_k^{(0)} = \begin{cases} \alpha_k^{(0)}, & \text{if } d_k = -g_k, \\ \min \left\{ 1, \alpha_k^{(0)} \right\}, & \text{if } d_k \neq -g_k, \end{cases}
$$

where $\alpha_k^{(0)}$ $\kappa^{(0)}$ is given by [\(2.44\)](#page-12-2). The motivation behind is that the search direction [\(2.4\)](#page-5-4) is closest to the search direction of the memoryless quasi-Newton method, which usually prefers the unit stepsize 1.

The improved Wolfe line search proposed by Dai and Kou [\[8\]](#page-21-7) is an quite efficient Wolfe line search, which can avoid some numerical drawbacks of the original Wolfe line search. It aims to find the stepsize satisfying the following conditions:

$$
f(x_k + \alpha_k d_k) \le f(x_k) + \min\left\{ \epsilon |f(x_k)|, \delta \alpha_k g_k^T d_k + \bar{\eta}_k \right\},\tag{2.45}
$$

$$
g(x_k + \alpha_k d_k)^T d_k \ge \sigma g_k^T d_k. \tag{2.46}
$$

where $0 < \epsilon, 0 < \delta < \sigma < 1, 0 < \bar{\eta}_k$ and $\sum_{k\geq 0} \bar{\eta}_k < +\infty$. The above improved Wolfe line search is used in our method.

2.4 The proposed method

Denote

$$
r_{k-1} = \frac{2\left(f_k - f_{k-1}\right)}{\left(g_k + g_{k-1}\right)^T s_{k-1}} - 1, \quad \overline{r}_{k-1} = f_k - f_{k-1} - 0.5\left(g_k^T s_{k-1} + g_k^T s_k\right). \tag{2.47}
$$

Similarly to the restart condition in [\[16,](#page-22-6)[30\]](#page-22-20), if there are continuously many iterations such that r_{k-1} or \overline{r}_{k-1} is close to 0, our algorithm is also restarted with $-g_k$.

The subspace minimization conjugate gradient method is described in detail as follow.

In the SMCG method, IterRestart denote the number of iterations since the last restart. IterQuad denote the number of continuous iterations such that r_{k-1} or \overline{r}_{k-1} is close to 0.

3 Convergence Analysis

We will establish the global convergence of Algorithm 1 for general functions under Assumption [2.1](#page-6-3) in the section.

Since Algorithm 1 is restarted with $d_k = -g_k$ at least MaxRestart iterations, the global convergence can be obtained easily. So we consider the global convergence properties of Algorithm 1 without the restart in Step 5.1. In addition, since τ_k in [\(2.43\)](#page-12-5) chooses adaptively between 1 and τ_k^B , the convergence result based on any one of τ_k in [\(2.8\)](#page-5-8) suffices to that of Algorithm 1. So we establish the global convergence of the SMCG method [\(1.1\)](#page-1-0) and [\(2.35\)](#page-10-0) under the Assumption [2.1.](#page-6-3)

According to the improved Wolfe line search [\(2.45\)](#page-12-3) and [\(2.46\)](#page-12-4) and Assumption [2.1](#page-6-3) (ii), we know easily that

$$
\sum_{k=0}^{+\infty} -\alpha_k g_k^T d_k < +\infty \text{ and } \alpha_k \ge \frac{-(1-\sigma) g_k^T d_k}{L \|d_k\|^2},\tag{3.1}
$$

which implies that

$$
\sum_{k=0}^{+\infty} \frac{\left(g_k^T d_k\right)^2}{\|d_k\|^2} < +\infty. \tag{3.2}
$$

Together with Lemma [2.2,](#page-10-2) we obtain

$$
\sum_{k=0}^{+\infty} \frac{\|g_k\|^4}{\|d_k\|^2} < +\infty. \tag{3.3}
$$

The above inequality is important to analyze the convergence of the proposed method.

The next lemma will be used to the convergent analysis of the SMCG method [\(1.1\)](#page-1-0) and [\(2.35\)](#page-10-0).

Lemma 3.1 Assume f satisfies Assumption [2.1,](#page-6-3) consider the subspace minimization conjugate gradient method [\(1.1\)](#page-1-0) and [\(2.35\)](#page-10-0) with any one of τ_k in [\(2.8\)](#page-5-8), and α_k is calculated by the improved Wolfe line search satisfying [\(2.45\)](#page-12-3) and [\(2.46\)](#page-12-4). If $||g_k|| \geq \gamma_1 > 0$ holds for all $k \geq 1$, then

$$
\sum_{k=0}^{\infty} \left\| \tilde{u}_k - \tilde{u}_{k-1} \right\|^2 < +\infty,\tag{3.4}
$$

where $\widetilde{u}_k = \frac{d_k}{\|d_k\|}$ $\frac{a_k}{\|d_k\|}$.

Proof We first derive a bound for u_k in [\(2.5\)](#page-5-6). By [\(2.46\)](#page-12-4), Lemma [2.2](#page-10-2) and $||g_k|| \ge \gamma_1$, we have that

$$
y_k^T d_k \ge -(1 - \sigma) g_k^T d_k \ge c (1 - \sigma) \|g_k\|^2 \ge c \gamma_1^2 (1 - \sigma) \tag{3.5}
$$

and

$$
g_{k+1}^T d_k \ge \sigma g_k^T d_k = \sigma g_{k+1}^T d_k - \sigma g_k^T d_k. \tag{3.6}
$$

It follows from Lemma [2.2](#page-10-2) that

$$
g_{k+1}^T d_k = y_k^T d_k + g_k^T d_k < y_k^T d_k. \tag{3.7}
$$

Combining σ < 1, [\(3.6\)](#page-14-0) and [\(3.7\)](#page-14-1) yields that

$$
\frac{|g_{k+1}^T d_k|}{y_k^T d_k} \le \max\left\{1, \frac{\sigma}{1-\sigma}\right\}.
$$
\n(3.8)

According to Assumption [2.1,](#page-6-3) we know that there are two positive constants D and γ_2 such that

$$
D = \max \left\{ \|y - z\| : y, z \in \mathcal{L} = \left\{ x | f(x) \le f(x_0) + \sum_{k \ge 0} \bar{\eta}_k \right\} \right\}, \ \|g_k\| \le \gamma_2. \tag{3.9}
$$

It is note that $d_k \neq 0$ for all $k \geq 1$, otherwise Lemma [2.1](#page-7-5) will imply $g_k = 0$. It indicates that \tilde{u}_k is well defined. Therefore, by using (2.2) , (2.7) , (3.8) , (3.9) and (2.14) , we obtain

$$
|u_k| \le \frac{1}{1-\overline{\omega}_k} \left(1 + \left|\frac{g_k^T s_{k-1}}{s_{k-1}^T y_{k-1}}\right| \frac{|g_k^T y_{k-1}|}{\|g_k\|^2}\right) \le \frac{1}{1-\xi_1} \left(1 + \max\left\{1, \frac{\sigma}{1-\sigma}\right\} (1+\eta_1)\right) \stackrel{\Delta}{=} \overline{c}_1 > 1. \tag{3.10}
$$

We divide \bar{v}_k in [\(2.31\)](#page-9-1) into the following two parts:

$$
v_k^+ = \max\left\{\frac{g_k^T y_{k-1}}{s_{k-1}^T y_{k-1}} - \left(\tau_k + \frac{\|y_k\|^2}{s_{k-1}^T y_{k-1}}\right) \frac{g_k^T s_{k-1}}{s_{k-1}^T y_{k-1}} + \frac{g_k^T s_{k-1} \|g_k\|^2 - \frac{g_k^T y_{k-1} \left(g_k^T s_{k-1}\right)^2}{s_{k-1}^T y_{k-1}}}{\|g_k\|^2 \|s_{k-1}\|^2 - \left(g_k^T s_{k-1}\right)^2} - \eta_k, 0\right\},\tag{3.11}
$$

and

$$
v_k^- = \eta_k = -l_k \frac{|g_k^T s_{k-1}|}{\left\|s_{k-1}\right\|^2},\tag{3.12}
$$

which satisfy $\bar{v}_k = v_k^+ + v_k^-$. It follows that the search direction $d_k = u_k g_k + \bar{v}_k s_{k-1}$ in [\(2.35\)](#page-10-0) can be rewritten as

$$
d_k = u_k g_k + \left(v_k^+ + v_k^-\right) s_{k-1}.
$$

Denote

$$
\omega_k = \frac{u_k g_k + v_k^- s_{k-1}}{\|d_k\|}, \quad \delta_k = \frac{v_k^+ \|s_{k-1}\|}{\|d_k\|}.
$$
\n(3.13)

Thus, \widetilde{u}_k can be rewritten as

$$
\widetilde{u}_k = \frac{d_k}{\|d_k\|} = \omega_k + \delta_k \widetilde{u}_{k-1}.
$$
\n(3.14)

Using the identity $\|\widetilde{u}_k\| = \|\widetilde{u}_{k-1}\| = 1$, we get that

$$
\|\omega_k\| = \|\widetilde{u}_k - \delta_k \widetilde{u}_{k-1}\| = \|\delta_k \widetilde{u}_k - \widetilde{u}_{k-1}\|.
$$
\n(3.15)

It follows from $\delta_k \geq 0$, the triangle inequality and (3.15) that

$$
\|\widetilde{u}_k - \widetilde{u}_{k-1}\| \le \|(1 + \delta_k)\widetilde{u}_k - (1 + \delta_k)\widetilde{u}_{k-1}\|
$$

\n
$$
\le \|\widetilde{u}_k - \delta_k \widetilde{u}_{k-1}\| + \|\delta_k \widetilde{u}_k - \widetilde{u}_{k-1}\|
$$

\n
$$
= 2\|\omega_k\|.
$$
\n(3.16)

By [\(2.35\)](#page-10-0), [\(2.32\)](#page-9-2), [\(3.10\)](#page-14-4) and [\(3.12\)](#page-14-5), we can obtain that

$$
\|u_k g_k + v_k^- s_{k-1}\| \le |u_k| \|g_k\| + |v_k^-| \|s_{k-1}\| \le (\bar{c}_1 + \bar{\xi}_2) \|g_k\|.
$$
 (3.17)

Combining [\(3.16\)](#page-15-1), [\(3.13\)](#page-14-6) and [\(3.17\)](#page-15-2) yields

$$
\|\tilde{u}_k - \tilde{u}_{k-1}\| \le 2 \|\omega_k\| \le 2 \left(\bar{c}_1 + \bar{\xi}_2\right) \frac{\|g_k\|}{\|d_k\|}.
$$
\n(3.18)

Similarly, for the search direction $d_k = -g_k$ in [\(2.35\)](#page-10-0), we can easily obtain [\(3.18\)](#page-15-3) by setting $u_k = -1$, $v_k^+ =$ $v_k^- = 0$ in [\(3.13\)](#page-14-6) due to $\bar{c}_1 > 1$. Therefore, together with [\(3.3\)](#page-13-0) and $||g_k|| \ge \gamma_1$, we have

$$
\sum_{k=0}^{\infty} \|\widetilde{u}_k - \widetilde{u}_{k-1}\|^2 \le \frac{4(\bar{c}_1 + \bar{\xi}_2)^2}{\gamma_1^2} \sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} < +\infty,
$$
\n(3.19)

which completes the proof. □

The global convergence is established under Assumption [2.1](#page-6-3) in the following theorem.

Theorem 3.1 Assume f satisfies Assumption [2.1,](#page-6-3) consider the subspace minization conjugate gradient method [\(1.1\)](#page-1-0) and [\(2.35\)](#page-10-0) with any one of τ_k in [\(2.8\)](#page-5-8), and α_k is calculated by the improved Wolfe line search satisfying [\(2.45\)](#page-12-3) and [\(2.46\)](#page-12-4). Then,

$$
\liminf_{k \to \infty} \|g_k\| = 0. \tag{3.20}
$$

Proof We proceed it by contradiction, namely, suppose that $||g_k|| \geq \gamma_1$, where $\gamma_1 > 0$, for all $k \geq 0$. By Lemma [2.2](#page-10-2) and Cauchy-Schwarz inequality, we have that

$$
||d_{k-1}|| \ge c ||g_{k-1}|| \ge c\gamma_1,
$$

which together with Assumption [2.1](#page-6-3) (i) yields

$$
||s_{k-1}|| = \alpha_{k-1} ||d_{k-1}|| \leq \frac{D}{c\gamma_1} ||d_{k-1}||,
$$

where D is given by (3.9) .

As a result, for the choices of τ_k in [\(2.8\)](#page-5-8), it is not difficult to obtain from [\(3.5\)](#page-14-7) that

$$
\tau_k^{(1)} = 1 = \frac{1}{c\gamma_1} c\gamma_1 \le \frac{1}{c\gamma_1} \|d_{k-1}\|
$$
\n(3.21)

and

$$
\tau_k^B = \frac{s_{k-1}^T y_{k-1}}{\|s_{k-1}\|^2} \le \tau_k^H = \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}} \le \frac{L^2 D}{c \gamma_1^2 (1 - \sigma)} \|d_{k-1}\|.
$$
\n(3.22)

The following is divided into the three steps.

(i) A bound for v_k and η_k in [\(2.35\)](#page-10-0).

By [\(2.35\)](#page-10-0), [\(3.8\)](#page-14-2), [\(3.21\)](#page-15-4), [\(3.22\)](#page-15-5), [\(2.14\)](#page-7-7), [\(3.9\)](#page-14-3) and $\gamma_1 \le ||g_k|| \le \gamma_2$, we have

$$
|v_{k}| = \left| \frac{1 - 2\omega_{k}}{1 - \omega_{k}} \frac{g_{k}^{T} y_{k-1}}{s_{k-1}^{T} y_{k-1}} - \left(\tau_{k} + \frac{||y_{k-1}||^{2}}{s_{k-1}^{T} y_{k-1}}\right) \frac{g_{k}^{T} s_{k-1}}{s_{k-1}^{T} y_{k-1}} + \frac{1}{1 - \bar{\omega}_{k}} \frac{g_{k}^{T} s_{k-1}}{||s_{k-1}||^{2}} \right|
$$

\n
$$
\leq \frac{1}{1 - \xi_{1}} \frac{\gamma_{2} L}{c \gamma_{1}^{2} (1 - \sigma)} ||d_{k-1}|| + \max \left\{ 1, \frac{\sigma}{1 - \sigma} \right\} \left(\max \left\{ \frac{1}{c \gamma_{1}}, \frac{L^{2} D}{c \gamma_{1}^{2} (1 - \sigma)} \right\} + \frac{L^{2} D}{c \gamma_{1}^{2} (1 - \sigma)} \right) ||d_{k-1}||
$$

\n
$$
+ \frac{1}{1 - \xi_{1}} \frac{L \gamma_{2}}{(1 - \sigma) c \gamma_{1}^{2}} ||d_{k-1}||
$$

\n
$$
\leq \left[\frac{2}{1 - \xi_{1}} \frac{\gamma_{2} L}{c \gamma_{1}^{2} (1 - \sigma)} + \max \left\{ 1, \frac{\sigma}{1 - \sigma} \right\} \left(\max \left\{ \frac{1}{c \gamma_{1}}, \frac{L^{2} D}{c \gamma_{1}^{2} (1 - \sigma)} \right\} + \frac{L^{2} D}{c \gamma_{1}^{2} (1 - \sigma)} \right) \right] ||d_{k-1}||
$$

\n
$$
\stackrel{\Delta}{=} \bar{c}_{2} ||d_{k-1}||
$$

\n
$$
\stackrel{\Delta}{=} \bar{c}_{2} ||d_{k-1}||
$$

\n(3.23)

and

$$
|\eta_{k}| \leq \frac{l_{k} \|g_{k}\| \|s_{k-1}\|}{\|s_{k-1}\|^{2}} \leq \bar{\xi}_{2} \|d_{k-1}\| \frac{\|g_{k}\|}{\|s_{k-1}\| \|d_{k-1}\|} \leq L\bar{\xi}_{2} \|d_{k-1}\| \frac{\|g_{k}\|}{\|y_{k-1}\| \|d_{k-1}\|}
$$

$$
\leq L\bar{\xi}_{2} \frac{\gamma_{2}}{c\gamma_{1}^{2} (1-\sigma)} \|d_{k-1}\| \stackrel{\Delta}{=} \bar{c}_{2} \|d_{k-1}\|
$$
\n(3.24)

(ii) A bound on the steps s_k .

For any $l \geq k$, by the definition of \tilde{u}_k in Lemma [3.1](#page-13-1) we have

$$
x_l - x_k = \sum_{j=k}^{l-1} (x_{j+1} - x_j) = \sum_{j=k}^{l-1} \|s_j\| \widetilde{u}_j = \sum_{j=k}^{l-1} \|s_j\| \widetilde{u}_k + \sum_{j=k}^{l-1} \|s_j\| (\widetilde{u}_j - \widetilde{u}_k),
$$
\n(3.25)

which yields that

$$
\sum_{j=k}^{l-1} ||s_j|| \le ||x_l - x_k|| + \sum_{j=k}^{l-1} ||s_j|| \, ||\widetilde{u}_j - \widetilde{u}_k|| \le D + \sum_{j=k}^{l-1} ||s_j|| \, ||\widetilde{u}_j - \widetilde{u}_k|| \, . \tag{3.26}
$$

Let Δ be any positive integer such that

$$
\Delta \ge 4\bar{c}_2 D. \tag{3.27}
$$

According to Lemma [3.1,](#page-13-1) we can choose $k_0 > 0$ such that

$$
\sum_{i\geq k_0} \|\widetilde{u}_{i+1} - \widetilde{u}_i\|^2 \leq \frac{1}{4\Delta}.\tag{3.28}
$$

If $j > k \ge k_0$ and $j - k \le \Delta$, then we know from [\(3.28\)](#page-16-0) and the Cauchy-Schwarz inequality that

$$
\|\widetilde{u}_j - \widetilde{u}_k\| \le \sum_{i=k}^{j-1} \|\widetilde{u}_{i+1} - \widetilde{u}_i\| \le \sqrt{j-k} \left(\sum_{i=k}^{j-1} \|\widetilde{u}_{i+1} - \widetilde{u}_i\|^2\right)^{1/2} \le \sqrt{\Delta} \left(\frac{1}{4\Delta}\right)^{1/2} = \frac{1}{2}.
$$
 (3.29)

Combining (3.29) with (3.26) implies

$$
\sum_{j=k}^{l-1} \|s_j\| \le 2D,\tag{3.30}
$$

where $l > k \geq k_0$ and $l - k \leq \Delta$.

(iii) A bound on the directions d_l .

For the search direction (2.35) , by (3.10) , (3.24) and (3.23) , we have

$$
||d_{l}||^{2} \leq (u_{l} \, ||g_{l}|| + \max\{|\eta_{l}|, |v_{l}|\} \, ||s_{l-1}||)^{2} \leq 2\bar{c}_{1}^{2}\gamma_{2}^{2} + 2\bar{c}_{3}^{2}||s_{l-1}||^{2}||d_{l-1}||^{2},
$$
\n(3.31)

where $\bar{c}_3 = \max{\{\bar{c}_2, \bar{\bar{c}}_2\}}$, \bar{c}_1 and \bar{c}_2 are given by [\(3.10\)](#page-14-4) and [\(3.23\)](#page-16-4), respectively. Let $S_i = 2\bar{c}_3^2 ||s_i||^2$, for any $l > k_0$, we have

$$
||d_l||^2 \le 2\bar{c}_1^2 \gamma_2^2 \left(\sum_{i=k_0+1}^l \prod_{j=i}^{l-1} S_j\right) + ||d_{k_0}||^2 \prod_{j=k_0}^{l-1} S_j. \tag{3.32}
$$

Note that the product is define to be 1 whenever the index range is vacuous. Now we derive a product of Δ consecutive S_j by the arithmetic-geometric mean inequality, [\(3.30\)](#page-16-5) and [\(3.27\)](#page-16-6) for any $k \geq k_0$:

$$
\prod_{j=k}^{k+\Delta-1} S_j = \prod_{j=k}^{k+\Delta-1} 2\bar{c}_3^2 \|s_j\|^2 = \left(\prod_{j=k}^{k+\Delta-1} \sqrt{2}\bar{c}_3 \|s_j\|\right)^2
$$
\n
$$
\leq \left(\frac{\sum_{j=k}^{k+\Delta-1} \sqrt{2}\bar{c}_3 \|s_j\|}{\Delta}\right)^{2\Delta} \leq \left(\frac{2\sqrt{2}\bar{c}_3 D}{\Delta}\right)^{2\Delta} \leq \frac{1}{2^{\Delta}}.
$$
\n(3.33)

As a result, there must exist a positive constant $c_3 > 0$ such that $||d_i|| \le c_3$. Combining $||g_k|| \ge \gamma_1$ with $||d_i|| \le c_3$ yields

$$
\sum_{k=0}^{+\infty} \frac{\|g_k\|^4}{\|d_k\|^2} = +\infty.
$$

which contradicts [\(3.3\)](#page-13-0). Therefore, we obtain [\(3.20\)](#page-15-6), which completes the proof. □

Remark 4 It is not difficult to see that if f is convex, then we can obtain the strong convent result:

$$
\lim_{k \to \infty} \|g_k\| = 0.
$$

It follows from [\(2.45\)](#page-12-3) that $f_{k+1} - f_k \leq \bar{\eta}_k$, which together with $\sum_{k\geq 0} \bar{\eta}_k < +\infty$ and Assumption [2.1](#page-6-3) (ii) implies that the sequence $\{f_k\}$ is convergent. We denoted its limite by f^* . It also can deduce from Theorem [3.1](#page-15-7) and Assumption [2.1](#page-6-3) (ii) that there exists a convergent subquece $\{x_{k_i}\}\$ of $\{x_k\}$ such that $g(\hat{x}) = 0$, where $x_{k_j} \to \hat{x}$ when $j \to +\infty$. Since f is convex on \mathbb{R}^n , we have $f^* = f(\hat{x}) \leq f(x)$, $\forall x \in \mathbb{R}^n$. If there exists another convergent subquece x_{k_j} of x_k such that $g(\bar{x}) \neq 0$, where $x_{k_j} \to \bar{x}$ when $j \to +\infty$. Since $f(\bar{x}) = f^* \leq f(x)$, $\forall x \in \mathbb{R}^n$, we know that \bar{x} is a global minimizer, which contradicts $g(\bar{x}) \neq 0$. Therefore, all accumulations of $\{x_k\}$ are stationary points, which implies that $\lim_{k \to \infty} ||g_k|| = 0$.

4 Numerical experiments

We compare Algorithm 1 with CGOPT (1.0) [\[8\]](#page-21-7), SMCG_BB [\[16\]](#page-22-6) and CG_DESCENT (5.3) [\[6\]](#page-21-5) in the section. It is widely accepted that CGOPT and CG DESCENT are the most two famous conjugate gradient software packages. Algorithm 1 was implemented based on the C code of CGOPT (1.0), which is available from Dai's homepage: <http://lsec.cc.ac.cn/~dyh/software.html>. The test collection includes 147 unconstrained optimization problems from the CUTEst library [\[23\]](#page-22-13), which can be found in [\[31\]](#page-22-21), and the dimensions of the 147 test problems and the initial points are all default. The codes of CG DESCENT (5.3) and SMCG BB can be found in Hager's homepage: <http://users.clas.ufl.edu/hager/papers/Software> and <https://www.scholat.com/liuzexian>, respectively. In Algorithm 1, we take the following parameters:

$$
\xi_1 = 0.75, \xi_2 = 0.5, \overline{\xi}_2 = 0.2, \overline{\xi}_2 = 10, \eta_1 = 0.99,
$$

$$
\eta_2 = 3, \xi_3 = 7.5 \times 10^{-5}, \xi_4 = 9 \times 10^{-4}, \xi_5 = 0.9, \xi_6 = 10
$$

and the other parameters used the default value in CGOPT (1.0). CGOPT (1.0), CG_DESCENT (5.3) and SMCG BB used all default values of these parameter in their codes but the stopping condition. Especially, CG DESCENT (5.3) used the default line search—the combination of the original Wolfe conditions and the approximate Wolfe conditions, which performed very well in the numerical tests. All test methods are terminated if $|| g_k ||_{\infty} \le 10^{-6}$ is satisfied.

The performance profiles introduced by Dolan and Moré [\[32\]](#page-22-22) are used to display the performances of these test algorithms. In the following figures, " N_{iter} ", " N_f ", " N_g " and " T_{cpu} " represent the number of iterations, the number of function evaluations, the number of gradient evaluations and CPU time (s), respectively.

The numerical experiments are divided into the following three groups.

In the first group of the numerical experiments, we test the numerical performance of Algorithm 1 with different τ_k in [\(2.43\)](#page-12-5) and [\(2.8\)](#page-5-8). The default τ_k in Algorithm 1 is given by (2.43). Figures [1-](#page-18-0)[3](#page-19-0) present the numerical performance in term of the number of iterations, the number of function evaluations and the number of gradient evaluations. We do not test the performance about the running time since it is similar to the above figures. As observed in the Figures [1-](#page-18-0)[3,](#page-19-0) τ_k in [\(2.43\)](#page-12-5) is the most efficient for Algorithm 1, followed by τ_k^B , and τ_k^H is the worst.

Fig. 1: N_{iter}

Fig. 2: N_f

In the second group of the numerical experiments, we compare the performance of Algorithm 1 with that of SMCG BB and CGOPT (1.0). As shown in Figure [4,](#page-19-0) we observe that the Algorithm 1 performs much better than SMCG BB and CGOPT (1.0) in term of the number of iterations, since it successfully solves about 56% test problems with the least iterations, while the numbers of SMCG-BB and CGOPT (1.0) are about 37% and 27%, respectively. Figure [5](#page-19-1) shows that Algorithm 1 enjoys large advantage over CGOPT (1.0) and performs slightly better than SMCG BB in term of the number of function evaluations. As shown in Figure [6,](#page-19-1) we can see that Algorithm 1 is superior much to CGOPT (1.0) and SMCG_BB in term of the number of gradient evaluations. Figure [7](#page-19-2) indicates that Algorithm 1 is faster than SMCG BB and CGOPT (1.0).

Fig. 7: T_{cpu}

Fig. 8: N_{iter}

In the third group of the numerical experiments, we compare the performance of Algorithm 1 with that of CG DESCENT (5.3). As shown in Figure [8,](#page-19-2) we observed that Algorithm 1 performs better than CG DESCENT (5.3) in term of the number of iterations, which is a little beyond our expectations. Figure [9](#page-20-0) indicates that Algorithm 1 is inferior to CG DESCENT (5.3) in term of the number of function evaluations. The reason is that in the numerical experiments Algorithm 1 used the improved Wolfe line search, while CG DESCENT (5.3) used the combination of the quite efficient approximate Wolfe line search and the standard Wolfe line search. It follows from Section 3 that Algorithm 1 is globally convergent, whereas there is no guarantee for the global convergence of CG DESCENT with the very efficient approximate Wolfe line search [\[7\]](#page-21-6). As shown in Figure [10,](#page-20-0) we see that Algorithm 1 enjoys large advantage over CG DESCENT (5.3) in term of the number of gradient evaluations. To see the comprehensive performance about N_f and N_g , we compare the performance based on $N_f + 3N_g$ in Figure [11.](#page-20-1) Figure [11](#page-20-1) indicates that Algorithm 1 is also superior to CG_DESCENT (5.3) based on the total performance about N_f and N_g though it is at disadvantage in term of N_f . Figure [12](#page-20-1) shows that Algorithm 1 is faster than CG DESCENT (5.3).

The above numerical experiments indicate that Algorithm 1 is superior to CGOPT(1.0), CG DESCENT (5.3) and SMCG BB. It seems that SMCG method with $d_k = u_k g_k + v_k s_{k-1}$ can illustrate greater potential for large scale unconstrained optimization compared to the traditional conjugate gradient method with $d_k =$ $-g_k + \beta_k d_{k-1}.$

5 Conclusion and Discussion

SMCG methods are quite efficient iterative methods for large scale unconstrained optimization. However, it is usually required to determine the important parameter $\rho_k \approx g_k^T B_k g_k$, which is crucial to the theoretical properties and the numerical performance and is difficult to be selected properly. By taking advantage of the memoryless quasi-Newton method, we present a new subspace minimization conjugate gradient method based on project technique, which is independent of the important parameter ρ_k . It is remarkable that the SMCG method without the exact line search enjoys finite termination for two dimensional convex quadratic functions, which will guide us in the design of the proposed method. The proposed method can be regarded as the extension of Dai-Kou conjugate gradient method. The descent property of the search direction is analyzed. We also establish the global convergence for general nonlinear functions of the proposed method. Numerical experiments indicate that the proposed method is very promising. We believe that SMCG methods are able to become strong candidates for large scale unconstrained optimization.

Acknowledgements We would like to thank Professor Yu-Hong Dai in Chinese Academy of Sciences for his valuable and insightful comments on this manuscript. This research is supported by National Science Foundation of China (No. 12261019, 12161053), Guizhou Provincial Science and Technology Projects (No. QHKJC-ZK[2022]YB084).

Data availability. The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Conflict of interest. The authors declare no competing interests.

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