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Existence of equilibrium solution for multi-leader-follower games with fuzzy goals and parameters

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Abstract. In this paper, we first propose the model of multi-leader-follower games with fuzzy goals involving fuzzy parameters and introduce its α-FNS equilibrium. Next, we shift our attention to the existence of α-FNS equilibrium and prove it by Kakutani's fixed point theorem. Finally, we illustrate an example to show that the equilibrium existence result is valid.

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1 Introduction

The leader-follower game was first proposed by a German economist Stackelberg [1]. It has developed from the original single-leader-follower with an objective to the current multi-leader-follower with multiple objectives. In leader-follower games, the leaders own leadership advantages with a favorable position, the followers follow the leaders' steps to make corresponding decisions. Multi-leader-follower games are embodied in economics and electricity market etc. For example, in economics, we consider the game model proposed by Yang et al. [2]. There are two types of players, one is the companies which provide the carsharing to maximize their profits, another one is the travelers which employ the carsharing to minimize their disutility according to the companies' profits. For this model, the companies and the travelers are formulated as leaders and followers respectively. In electricity power markets [3], there are two kinds of participants, one is the virtual power plants and they determine the price of electricity selling, the other one is the distribution companies and they determine the price of electricity purchasing according to the sale price from the power plants. It is formulated as a multi-leader-follower game with power plants as leaders and distribution companies as followers.

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In recent years, a great deal of scholars have devoted to the equilibria for leader-follower games. Liu [4] designed a genetic algorithm to solve Stackelberg-Nash equilibria of nonlinear multilevel programming with multiple followers. Yu and Wang [5] gave a simple proof of equilibrium existence theorem of a two-leader-follower game by Fan-Glicksberg fixed point theorem. Yang and Pu [6] obtained the equilibrium existence by Fan-Glicksberg fixed point theorem for multi-leader-follower games under uncertainty. Jia et al. [7] investigated the existence of weakly Pareto-Nash equilibrium for generalized multiobjective multi-leader-follower games by Fan-Glicksberg fixed point theorem. Julien and Ludovic [8] demonstrated the Stackelberg market equilibrium existence of multi-leader-follower noncooperative oligopoly model by making certain assumptions on the derived functions of price and cost functions. Watada and Chen [9] analyzed the Stackelberg behaviors between a manufacturer and two retailers. Zhang et al. [10] proved the equilibrium existence of Nash-Stackelberg-Nash games under decision-dependent uncertainties by Kakutani's fixed point theorem. The other works about the equilibrium existence of multi-leader-follower games can be seen in [11-14].

Nevertheless, in many practical decision-making problems, it is an inescapable fact of imprecise information about the ongoing games due to the indeterminate environment, inaccurate understanding of status by players etc. In those instances, fuzzy theory put forward by Zadeh [15] is a powerful tool, and it was firstly applied in game theory and established noncooperative fuzzy game by Dan [16]. Later, it has been applied abundantly in game theory with plentiful research achievements. The equilibrium existence in matrix game involving fuzzy goals were studied in [17-20]. Equilibrium solution and relevant properties of multi-objective matrix containing fuzzy goals were investigated in [21-26]. Fang et al. [27] represented the seller and buyers' goals by fuzzy sets in auctions game and investigated the equilibrium existence and computation. Equilibria existence in matrix games, n-person noncooperative games and Stackelberg games with fuzzy parameters were studied in [28-32] respectively. The equilibrium existence in matrix games and n-person noncooperative games with both fuzzy goals and parameters were considered in [33-36] respectively.

We note that the great majority of the above literatures bring in fuzzy parameters or fuzzy goals in game models and further investigated their equilibrium existence or calculation etc. But the results of equilibrium existence for multi-leader-follower games with both fuzzy parameters and fuzzy goals remain open. Motivated by this idea and Kacher and Larbani [36], the aim of this paper is to established equilibrium existence for multi-leader-follower games with both fuzzy parameters and fuzzy goals.

The paper is organized as follows. In Section 2, we present the necessary preliminaries. Section 3 is the main ingredient of this paper, which devote to the equilibrium existence of multi-leader-follower games with fuzzy goals and fuzzy parameters. Next, a procedure for its calculation followed by an illustrative example is given as well. Section 4 concludes this paper.

2 Preliminaries

Throughout this paper, $\mathbb{R}$ and $\mathbb{R}^n$ denote the real field and the n-dimensional Euclidean space respectively. $\mathbb{R}^n_+ = \{(u_1, u_2, \cdots, u_n) \in \mathbb{R}^n \mid u_i \geq 0, \forall\ i = 1, 2, \cdots, n\}$, $\text{int}\mathbb{R}^n_+ = \{(u_1, u_2, \cdots, u_n) \in \mathbb{R}^n \mid u_i > 0, \forall\ i = 1, 2, \cdots, n\}$.

Definition 2.1 (see [37] p.44, Definition 2.1.4) Let $\tilde{F}$ be a fuzzy set over $\mathbb{R}$, $\mathbb{B} = \{\tilde{F} \mid \mu_{\tilde{F}} : \mathbb{R} \to [0, 1]\}$, if each $\tilde{F} \in \mathbb{B}$ meets the following properties:

(i) $\tilde{F}$ is a normal fuzzy set, i.e. $\exists x_0 \in \mathbb{R}$ such that $\mu_{\tilde{F}}(x_0) = 1$;

(ii) $\forall \lambda \in (0, 1]$, $\tilde{F}_\lambda = \{x \in \mathbb{R} | \mu_{\tilde{F}}(x) \geq \lambda, \lambda \in (0, 1]\}$ is a bounded closed interval.

Then $\tilde{F} \in \mathbb{B}$ is called a fuzzy number and $\mathbb{B}$ is called a fuzzy number set.

The following results of vector-valued functions and set-valued mappings [38] are essential for the context.

Definition 2.2 Let X be a nonempty set in $\mathbb{R}^n$, $f : X \to \mathbb{R}^k$ is a vector-valued function, $x \in X$.

(1) f is $\mathbb{R}^k_+$-upper semicontinuous at x (or $\mathbb{R}^k_+$-lower semicontinuous), if for any open neighborhood V of $\mathbf{0}$ in $\mathbb{R}^k$, there exists open neighborhood $O(x)$ of x, such that $f(x') \in f(x) + V - \mathbb{R}^k_+$ (or $f(x') \in f(x) + V + \mathbb{R}^k_+$) for each $x' \in O(x)$.

(2) f is $\mathbb{R}^k_+$- upper semicontinuous on X(or $\mathbb{R}^k_+$-lower semicontinuous), if f is $\mathbb{R}^k_+$-upper semicontinuous at x(or $\mathbb{R}^k_+$-lower semicontinuous) for each $x \in X$.

(3) f is $\mathbb{R}^k_+$-continuous on X, if f is both $\mathbb{R}^k_+$-upper and lower semicontinuous on X.

Lemma 2.1 Let X be a nonempty set in $\mathbb{R}^n$, $f = (f_1, \cdots, f_k) : X \to \mathbb{R}^k$ is a vector-valued function, where $f_j : X \to \mathbb{R},\ j = 1, \cdots, k$, then

(1) f is $\mathbb{R}^k_+$-upper semicontinuous on X if and only if f_j is upper semicontinuous on X, $j = 1, \cdots, k$;

(2) f is $\mathbb{R}^k_+$-lower semicontinuous on X if and only if f_j is lower semicontinuous on X, $j = 1, \cdots, k$;

(3) f is $\mathbb{R}^k_+$-continuous on X if and only if f_j is continuous on X, $j = 1, \cdots, k$.

Definition 2.3 Let X be a nonempty convex set in $\mathbb{R}^n$, $f : X \to \mathbb{R}^k$ is a vector-valued function. Then f is $\mathbb{R}^k_+$-quasiconcave on X, if for any $x_1, x_2 \in X$ and any $\omega \in (0, 1)$, it holds

$$f(\omega x_1 + (1 - \omega)x_2) \in f(x_1) + \mathbb{R}^k_+,$$

or

$$f(\omega x_1 + (1 - \omega)x_2) \in f(x_2) + \mathbb{R}^k_+.$$

Definition 2.4 Let X and Y be two nonempty sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. $F : X \to P_0(Y)$ is a set-valued mapping, where $P_0(Y)$ denotes all the nonempty subsets of Y.

(1) F is upper semicontinuous(usc) at x, if for any open set $G \subset Y$ with $G \supset F(x)$, there exists an open neighborhood $O(x)$ of x such that $G \supset F(x')$ for each $x' \in O(x)$.

(2) F is lower semicontinuous(lsc) at x, if for any open set $G \subset Y$ with $G \bigcap F(x) \neq \emptyset$, there exists an open neighborhood $O(x)$ of x such that $G \bigcap F(x') \neq \emptyset$ for each $x' \in O(x)$.

(3) F is continuous at x, if F is both usc and lsc at x.

(4) F is continuous (or usc, or lsc) on X, if F is continuous (or usc, or lsc) at each $x \in X$.

Definition 2.5 Let $F : X \to P_0(Y)$ be a set-valued mapping, the graph of F is denoted by

$$graph(F) = \{(x, y) \in X \times Y : y \in F(x)\}.$$

F is closed, if $graph(F)$ is a closed set in $X \times Y$.

Lemma 2.2 Let Y be a bounded closed set in $\mathbb{R}^n$. If $F : X \to P_0(Y)$ is closed, then F is usc on X.

Lemma 2.3 Let $F : X \to P_0(Y)$ be closed,

(1) If $\forall x_k \to x \in X,\ \forall y_k \in F(x_k),\ y_k \to y \in Y$, then $y \in F(x)$;

(2) $\forall x \in X,\ F(x)$ is a closed set.

Lemma 2.4(Kakutani's fixed point theorem) Let X be a nonempty bounded closed convex set in $\mathbb{R}^n$. If $F : X \to P_0(X)$ is usc with nonempty closed convex set $F(x)$ for each $x \in X$, then there exists $x^* \in X$ such that $x^* \in F(x^*)$.

Lemma 2.5(see [39] or [40]) Let X be a nonempty bounded closed convex set in $\mathbb{R}^n$, $O \subset X \times X$, if it holds

(1) $\{y \in X : (x, y) \in O\}$ is open in X for each $x \in X$;

(2) $\{x \in X : (x, y) \in O\}$ is convex for each $y \in X$;

(3) $(x, x) \notin O$ for each $x \in X$.

Then there exists $y^* \in X$, such that $(x, y^*) \notin O$ for each $x \in X$.

3 Existence of equilibrium for multi-leader-follower games with fuzzy goals and parameters

Here, we begin with the model of multi-leader-follower games with n leaders and multiple followers. In such a game, $I = \{1, 2, \cdots, n\}$ is the set of leaders, and $\forall i \in I,\ X_i \subseteq \mathbb{R}^{k_i}$ is the ith leader's strategy set, let $X = \prod\limits_{i \in I} X_i,\ X_{-i} = \prod\limits_{l \in I \setminus \{i\}} X_l$. $Y = U_i \subseteq \mathbb{R}^p$ denotes the cartesian product of the all followers' strategy set, and let $U_{-i} = \prod\limits_{l \in I \setminus \{i\}} U_l$ for each $i \in I$. $\Omega = \{\tilde{z} = (\tilde{z}_1, \cdots, \tilde{z}_m) \in \mathbb{R}^m \mid \tilde{z}_j \in \mathbb{B},\ j = 1, 2, \cdots m\}$. Denoted by $f = (f_1, f_2, \cdots, f_n)$, where $f_i : X \times Y \times \Omega \to \mathbb{R}$ indicates the payoff of the leader $i \in I$.

Let

$$\langle I, X, Y, \Omega, f \rangle \tag{3.1}$$

denote multi-leader-follower games with fuzzy parameters.

Let the ith leader' confidence level for fuzzy parameters be α^i, i.e. $Z_{\alpha^i}(\tilde{z}) = \{z = (z_1, z_2, \cdots, z_m) \in \Omega \mid \mu_{\tilde{z}_j}(z_j) \geq \alpha^i,\ j = 1, \cdots, m\}$. $\alpha = \max\limits_{i \in I} \alpha^i$ denotes overall confidence level for fuzzy parameters for game (3.1).

Let

$$Z_\alpha(\tilde{z}) = \{z = (z_1, z_2, \cdots, z_m) \in \Omega \mid \mu_{\tilde{z}_j}(z_j) \geq \alpha, j = 1, \cdots, m\}.$$

And

$$\langle I, X, Y, Z_\alpha(\tilde{z}), f \rangle \tag{3.2}$$

denotes multi-leader-follower games with crisp parameters.

Assumption 3.1 For game (3.2), we assume that:

(i) $X_i,\ Y$ are nonempty bounded closed convex sets in $\mathbb{R}^{k_i}$ and $\mathbb{R}^p$ respectively;

(ii) $(x, y, z) \to f_i(x, y, z)$ is continuous over $X \times Y \times Z_\alpha(\tilde{z})$ for each $i \in I$;

(iii) $(x_{-i}, y, z) \to f_i((x_i, x_{-i}), y, z)$ is not constant over $X_{-i} \times Y \times Z_\alpha(\tilde{z})$ for each $i \in I$ and $x_i \in X_i$;

(iv) $(x, y) \to f_i(x, y, z)$ is convex for each $i \in I$ and $z \in Z_\alpha(\tilde{z})$.

For each $i \in I$, we define

$$\lambda_i = \min_{t_i \in X_i} \min_{(x_{-i}, y, z) \in X_{-i} \times Y \times Z_\alpha(\tilde{z})} f_i((t_i, x_{-i}), y, z),$$

$$\beta_i = \min_{t_i \in X_i} \max_{(x_{-i}, y, z) \in X_{-i} \times Y \times Z_\alpha(\tilde{z})} f_i((t_i, x_{-i}), y, z),$$

$\bar{X} = \{x \in X \mid \lambda_i \le f_i(x, y, z) \le \beta_i, \forall (y, z) \in Y \times Z_\alpha(\tilde{z}), \forall i \in I\}$,

$\bar{X}_i = \{x_i \in X_i \mid (x_i, x_{-i}) \in \bar{X}\},\ \bar{X}_{-i} = \prod_{l \in I \setminus \{i\}} \bar{X}_l$.

Theorem 3.1 Under assumption 3.1, $\bar{X}$ is a nonempty bounded closed convex set and then $\bar{X}_i,\ \bar{X}_{-i}$ are also nonempty bounded closed convex sets.

Proof: Nonemptiness Obviously, $Z_\alpha(\tilde{z})$ is a nonempty bounded closed set in $\mathbb{R}^m$. Besides, from Assumption 3.1(ii), $t_i \to \max_{(x_{-i}, y, z) \in X_{-i} \times Y \times Z_\alpha(\tilde{z})} f_i((t_i,\ x_{-i}), y, z)$ is continuous on X_i due to the continuity of $f_i(x, y, z)$ on $X \times Y \times Z_\alpha(\tilde{z})$. And by the boundedness and closeness of $X_i \subset \mathbb{R}^{k_i}$ for each $i \in I$, there exists $\bar{x}_i \in X_i$ such that

$$\beta_i = \min_{t_i \in X_i} \max_{(x_{-i}, y, z) \in X_{-i} \times Y \times Z_\alpha(\tilde{z})} f_i((t_i, x_{-i}), y, z) = \max_{(x_{-i}, y, z) \in X_{-i} \times Y \times Z_\alpha(\tilde{z})} f_i((\bar{x}_i, x_{-i}), y, z),$$

denoted by $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$, obviously $\bar{x} \in X$.

Then for each $i \in I$ and each $(y', z') \in Y \times Z_\alpha(\tilde{z})$, it holds

$\lambda_i = \min_{t_i \in X_i} \min_{(x_{-i}, y, z) \in X_{-i} \times Y \times Z_\alpha(\tilde{z})} f_i((t_i, x_{-i}), y, z) \le f_i(\bar{x}, y', z')$
$\le \max_{(x_{-i}, y, z) \in X_{-i} \times Y \times Z_\alpha(\tilde{z})} f_i((\bar{x}_i, x_{-i}), y, z) = \beta_i.$

Thus $\bar{x} \in \bar{X}$, namely $\bar{X} \ne \emptyset$.

Boundedness $\bar{X} \subseteq X$ is bounded due to the boundedness of X.

Closeness Let $\forall x^l \in \bar{X},\ l = 1, 2, \cdots, x^l \to \bar{x}$, now we prove that $\bar{x} \in \bar{X}$.

Since $x^l \in \bar{X}$, it holds $\lambda_i \le f_i(x^l, y, z) \le \beta_i,\ \forall (y, z) \in Y \times Z_\alpha(\tilde{z}),\ \forall i \in I$.

Besides, f_i is continuous on X and $x^l \to \bar{x}(l \to +\infty)$, it also holds $\lambda_i \le f_i(\bar{x}, y, z) \le \beta_i,\ \forall (y, z) \in Y \times Z_\alpha(\tilde{z}),\ \forall i \in I$.

Then $\bar{x} \in \bar{X}$, i.e. $\bar{X}$ is a closed set.

Convexity For any $x^1, x^2 \in \bar{X},\ \omega \in (0, 1)$, now we are to prove $\omega x^1 + (1 - \omega) x^2 \in \bar{X}$.

Because $x^1, x^2 \in \bar{X}$, then it holds $\lambda_i \le f_i(x^1, y, z) \le \beta_i$ and $\lambda_i \le f_i(x^2, y, z) \le \beta_i,\ \forall (y, z) \in Y \times Z_\alpha(\tilde{z})$.

From Assumption 3.1(iv), $x \to f_i(x, y, z)$ is convex, then $\lambda_i \le f_i((\omega x^1 + (1 - \omega) x^2, y, z) \le \omega f_i(x^1, y, z) + (1 - \omega) f_i(x^2, y, z) \le \omega \beta_i + (1 - \omega) \beta_i = \beta_i$.

Thus $\omega x^1 + (1 - \omega) x^2 \in \bar{X}$, i.e. $\bar{X}$ is a convex set.

The above proof shows that $\bar{X}$ is a nonempty bounded closed convex set.

Let

$$\mu_{f_i}(x,y,z)=\begin{cases}0, & f_i(x,y,z)<\lambda_i\\ \frac{f_i(x,y,z)-\lambda_i}{\beta_i-\lambda_i}, & \lambda_i\le f_i(x,y,z)<\beta_i\\ 1, & f_i(x,y,z)\ge\beta_i\end{cases}$$

represent the membership function of the ith leader's fuzzy goal, $\mu_f=(\mu_{f_1},\mu_{f_2},\cdots,\mu_{f_n})$. Clearly, μ_{f_i} holds the same properties as f_i and from Assumption 3.1(iii), $\beta_i>\lambda_i$.

After the leaders play their strategy $(x_i,x_{-i})\in\bar{X}$ and determine the membership functions of their fuzzy goals, the followers subsequently make their response according to leaders' information. Let $y_i\in G(x_i,x_{-i},z)$, where $G(x_i,x_{-i},z)$ is the followers' reaction mapping for each leader $i\in I$.

Let

$$\langle I,\bar{X},Y,Z_\alpha(\tilde{z}),\mu_f,G\rangle, \tag{3.3}$$

denote multi-leader-follower games where leaders' payoff functions are the membership functions of their fuzzy goals.

Remark 3.1 If $\Omega=\emptyset$ or $\Omega=\{z\}$, then game (3.3) is a classical multi-leader-follower game. If $Z_\alpha(\tilde{z})$ is an uncertain parameter space, the leaders' strategy set is still X and μ_f is the initial vector-valued payoff f, then game (3.3) is multi-leader-follower game under uncertainty proposed by Yang (2012).

Definition 3.1 If for each $i\in I$, there exists $y_i^*\in G(x_i^*,x_{-i}^*,z^*)$ such that the strategy profile $(x_i^*,x_{-i}^*,z^*)\in\bar{X}\times Z_\alpha(\tilde{z})$ holds the following conditions:

(i) $\mu_{f_i}((x_i^*,x_{-i}^*),y_i^*,z^*)=\min\limits_{x_i\in\bar{X}_i,y_i\in G(x_i,x_{-i}^*,z^*)}\mu_{f_i}((x_i,x_{-i}^*),y_i,z^*)$,

(ii) $\mu_f(x^*,y^*,z)-\mu_f(x^*,y^*,z^*)\notin \mathrm{int}\mathbb{R}_+^n,\ \forall z\in Z_\alpha(\tilde{z}),\ y^*=(y_1^*,y_2^*,\cdots,y_n^*)\in\prod\limits_{i\in I}U_i$.

Then (x_i^*,x_{-i}^*,z^*) is called an N-S equilibrium of game (3.3).

Definition 3.2 For each $i\in I$, if the strategy profile $(x_i^*,x_{-i}^*,z^*)\in\bar{X}\times Z_\alpha(\tilde{z})$ is an N-S equilibrium of game (3.3), then it is called an α-FNS equilibrium of game (3.1).

Remark 3.2 If there is only one leader, and the condition (i) of Definition 3.1 becomes $\mu_f(x^*,y^*,z^*)=\max\limits_{x\in\bar{X},y\in G(x,z^*)}\mu_f(x,y,z^*)$, the condition (ii) becomes $\mu_f(x^*,y^*,z^*)=\min\limits_{z\in Z_\alpha(\tilde{z})}\mu_f(x^*,y^*,z)$, the followers' reaction mapping G becomes the α-FNS equilibrium set proposed by Kacher and Larbani (2008), then game (3.3) reduces to the single-leader-follower game with fuzzy goals and parameters.

Theorem 3.2 In addition to Assumption 3.1, suppose that the following conditions are met in game (3.3):

(T-i) $\forall x\in\bar{X},\ y=(y_1,y_2,\cdots,y_n)\in\prod\limits_{i\in I}U_i$, $z\to f(x,y,z)$ is $\mathbb{R}_+^n$-quasiconcave on $Z_\alpha(\tilde{z})$;

(T-ii) $G:\bar{X}\times Z_\alpha(\tilde{z})\to P_0(Y)$ is usc and $G(x,z)$ is a nonempty bounded closed convex set for each $(x,z)\in\bar{X}\times Z_\alpha(\tilde{z})$;

(T-iii) $\forall x_{-i}\in X_{-i},\ \forall z\in Z_\alpha(\tilde{z})$, $t_i\in X_i$, for an arbitrary convex combination $\sum\limits_{i=1}^{m_0}\omega_i t_i$, it holds

$$\sum_{i=1}^{m_0}\omega_i G(t_i,x_{-i},z)\subset G(\sum_{i=1}^{m_0}\omega_i t_i,x_{-i},z).$$

Then game (3.1) possesses at least one α-FNS equilibrium.

Proof: By Definition 3.2, we need to prove that game (3.3) exists an N-S equilibrium.

Based on Definition 3.1, the proof of Theorem 3.2 is divided into three steps. In the first step, we construct a set-valued mapping $H_i(x_{-i}, y_{-i}, z) = \{x_i \in \bar{X}_i, y_i \in G(x_i, x_{-i}, z) \mid \mu_{f_i}((x_i, x_{-i}), y_i, z) = \min_{u_i \in \bar{X}_i, v_i \in G(u_i, x_{-i}, z)} \mu_{f_i}((u_i, x_{-i}), v_i, z)\}$ for each $i \in I$ and prove that H_i is usc on $\bar{X}_{-i} \times \prod_{l \in I \setminus \{i\}} U_l \times Z_\alpha(\tilde{z})$ with a nonempty closed convex set for each $(x_{-i}, y_{-i}, z) \in \bar{X}_{-i} \times \prod_{l \in I \setminus \{i\}} U_l \times Z_\alpha(\tilde{z})$. In the second step, we construct a set-valued mapping $H_0(x, y) = \{z \in Z_\alpha(\tilde{z}) \mid \mu_f(x, y, v) - \mu_f(x, y, z) \notin \mathrm{int}\mathbb{R}^n_+\}$ and prove that H_0 is usc on $\bar{X} \times \prod_{i \in I} U_i$ with a nonempty closed convex set for each $(x, y) \in \bar{X} \times \prod_{i \in I} U_i$. In the third step, we construct a set-valued mapping $F : \bar{X} \times \prod_{i \in I} U_i \times Z_\alpha(\tilde{z}) \to P_0(\bar{X} \times \prod_{i \in I} U_i \times Z_\alpha(\tilde{z}))$, that is $F(x_1, \cdots, x_n; y_1, \cdots, y_n; z) = \prod_{i \in I} H_i(x_{-i}, y_{-i}, z) \times H_0(x, y)$ and show that game (3.3) exists an N-S equilibrium.

Step 1. For any $i \in I$ and $(x_{-i}, y_{-i}, z) \in \bar{X}_{-i} \times \prod_{l \in I \setminus \{i\}} U_l \times Z_\alpha(\tilde{z})$, we define set-valued mapping $H_i(x_{-i}, y_{-i}, z) = \{x_i \in \bar{X}_i, y_i \in G(x_i, x_{-i}, z) \mid \mu_{f_i}((x_i, x_{-i}), y_i, z) = \min_{u_i \in \bar{X}_i, v_i \in G(u_i, x_{-i}, z)} \mu_{f_i}((u_i, x_{-i}), v_i, z)\}$.

(1.1) Because $\bar{X} \times Z_\alpha(\tilde{z})$ is a bounded closed set and $G(x, z)$ is a bounded closed set for each $(x, z) \in \bar{X} \times Z_\alpha(\tilde{z})$, then $G(x_i, x_{-i}, z) = \{G(u_i, x_{-i}, z) : u_i \in \bar{X}_i, \forall i \in I\}$ is also a bounded closed set for each $(x_i, x_{-i}, z) \in \bar{X} \times Z_\alpha(\tilde{z})$. Besides, μ_{f_i} is continuous on $\bar{X} \times Y \times Z_\alpha(\tilde{z})$ from the continuity of f_i in Assumption 3.1(ii), there exists $\bar{x}_i \in \bar{X}_i$ and $\bar{y}_i \in G(\hat{x}_i, x_{-i}, z)$ such that $\mu_{f_i}((\bar{x}_i, x_{-i}), \bar{y}_i, z) = \min_{u_i \in \bar{X}_i, v_i \in G(u_i, x_{-i}, z)} \mu_{f_i}((u_i, x_{-i}), v_i, z)$. Hence $(\bar{x}_i, \bar{y}_i) \in H_i(x_{-i}, y_{-i}, z)$, i.e. $H_i(x_{-i}, y_{-i}, z)$ is nonempty for any $(x_{-i}, y_{-i}, z) \in \bar{X}_{-i} \times \prod_{l \in I \setminus \{i\}} U_l \times Z_\alpha(\tilde{z})$.

(1.2) We next prove H_i is usc on $\bar{X}_{-i} \times \prod_{l \in I \setminus \{i\}} U_l \times Z_\alpha(\tilde{z})$ and $H_i(x_{-i}, y_{-i}, z)$ is a closed set for each $(x_{-i}, y_{-i}, z) \in \bar{X}_{-i} \times \prod_{l \in I \setminus \{i\}} U_l \times Z_\alpha(\tilde{z})$. It suffices to prove that the closeness of H_i. Namely, for each $(x_{-i_l}, y_{-i_l}, z_l) \in \bar{X}_{-i} \times \prod_{l \in I \setminus \{i\}} U_l \times Z_\alpha(\tilde{z})$, $(x_{-i_l}, y_{-i_l}, z_l) \to (\bar{x}_{-i}, \bar{y}_{-i}, \bar{z})$, $(x_{i_l}, y_{i_l}) \in H_i(x_{-i_l}, y_{-i_l}, z_l)$, $(x_{i_l}, y_{i_l}) \to (\bar{x}_i, \bar{y}_i)$, we need to prove that $(\bar{x}_i, \bar{y}_i) \in H_i(\bar{x}_{-i}, \bar{y}_{-i}, \bar{z})$.

Since $(x_{i_l}, y_{i_l}) \in H_i(x_{-i_l}, y_{-i_l}, z_l)$, it holds $x_{i_l} \in \bar{X}_i$, $y_{i_l} \in G(x_{i_l}, x_{-i_l}, z_l)$ and

$$\mu_{f_i}((x_{i_l}, x_{-i_l}), y_{i_l}, z_l) \leq \mu_{f_i}((u_i, x_{-i_l}), v_i, z_l), \ \forall u_i \in \bar{X}_i, \ \forall v_i \in G(u_i, x_{-i_l}, z_l). \quad (*)$$

Because $(x_{i_l}, x_{-i_l}, y_{-i_l}, z_l) \to (\bar{x}_i, \bar{x}_{-i}, \bar{y}_{-i}, \bar{z})$ and $\bar{X}_i$ is a bounded closed set, then $\bar{x}_i, \bar{u}_i \in \bar{X}_i$. In addition, from condition (T-ii) and Lemma 2.3(1), it yields $\bar{y}_i \in G(\bar{x}_i, \bar{x}_{-i}, \bar{z})$ and $v_i \in G(u_i, \bar{x}_{-i}, \bar{z})$. Besides, μ_{f_i} is continuous on $\bar{X} \times Y \times Z_\alpha(\tilde{z})$, then we deduce that the following inequality from formula $(*)$:

$$\mu_{f_i}((\bar{x}_i, \bar{x}_{-i}), \bar{y}_i, \bar{z}) \leq \mu_{f_i}((u_i, \bar{x}_{-i}), v_i, \bar{z}), \ \forall u_i \in \bar{X}_i, \ \forall v_i \in G(u_i, \bar{x}_{-i}, \bar{z}).$$

Hence $(\bar{x}_i, \bar{y}_i) \in H_i(\bar{x}_{-i}, \bar{y}_{-i}, \bar{z})$. Namely, H_i is closed for each $i \in I$. Because $\bar{X}_i \times U_i$ is a bounded closed set, then H_i is usc on $\bar{X}_{-i} \times \prod_{l \in I \setminus \{i\}} U_l \times Z_\alpha(\tilde{z})$ from Lemma 2.2, $H_i(x_{-i}, y_{-i}, z)$

is closed set for each $i \in I$ and $(x_{-i}, y_{-i}, z) \in \bar{X}_{-i} \times \prod\limits_{l \in I \setminus \{i\}} U_l \times Z_\alpha(\tilde{z})$ from Lemma 2.3(2).

(1.3) Now we prove that $H_i(x_{-i}, y_{-i}, z)$ is convex. That is, for each $(x_i^1, y_i^1),\ (x_i^2, y_i^2) \in H_i(x_{-i}, y_{-i}, z),\ \omega \in (0,1)$, we need to prove that $\omega(x_i^1, y_i^1) + (1-\omega)(x_i^2, y_i^2) \in H_i(x_{-i}, y_{-i}, z)$.

Because $(x_i^1, y_i^1),\ (x_i^2, y_i^2) \in H_i(x_{-i}, y_{-i}, z)$, then $x_i^1 \in \bar{X}_i,\ y_i^1 \in G(x_i^1, x_{-i}, z),\ x_i^2 \in \bar{X}_i,\ y_i^2 \in G(x_i^2, x_{-i}, z)$, and

$$\mu_{f_i}((x_i^1, x_{-i}), y_i^1, z) = \min_{u_i \in \bar{X}_i, v_i \in G(u_i, x_{-i}, z)} \mu_{f_i}((u_i, x_{-i}), v_i, z),$$

$$\mu_{f_i}((x_i^2, x_{-i}), y_i^2, z) = \min_{u_i \in \bar{X}_i, v_i \in G(u_i, x_{-i}, z)} \mu_{f_i}((u_i, x_{-i}), v_i, z).$$

Thus $\forall u_i \in \bar{X}_i,\ v_i \in G(u_i, x_{-i}, z)$, it holds

$$\mu_{f_i}((x_i^1, x_{-i}), y_i^1, z) = \mu_{f_i}((x_i^2, x_{-i}), y_i^2, z) \leq \mu_{f_i}((u_i, x_{-i}), v_i, z).$$

Since $\bar{X}_i$ is convex, then $\omega x_i^1 + (1-\omega)x_i^2 \in \bar{X}_i$. By the condition (T-iii), $\omega y_i^1 + (1-\omega)y_i^2 \in \omega G(x_i^1, x_{-i}, z) + (1-\omega)G(x_i^2, x_{-i}, z) \subset G(\omega x_i^1 + (1-\omega)x_i^2, x_{-i}, z)$. From Assumption 3.1(iv), we deduce that μ_{f_i} is convex on $\bar{X}_i \times U_i$. Then $\mu_{f_i}((\omega x_i^1 + (1-\omega)x_i^2, x_{-i}), \omega y_i^1 + (1-\omega)y_i^2, z) \leq \omega\mu_{f_i}((x_i^1, x_{-i}), y_i^1, z) + (1-\omega)\mu_{f_i}((x_i^2, x_{-i}), y_i^2, z) \leq \omega\mu_{f_i}((u_i, x_{-i}), v_i, z) + (1-\omega)\mu_{f_i}((u_i, x_{-i}), v_i, z) = \mu_{f_i}((u_i, x_{-i}), v_i, z)$. Thus

$$\mu_{f_i}((\omega x_i^1 + (1-\omega)x_i^2, x_{-i}), \omega y_i^1 + (1-\omega)y_i^2, z) = \min_{u_i \in \bar{X}_i, v_i \in G(u_i, x_{-i}, z)} \mu_{f_i}((u_i, x_{-i}), v_i, z).$$

That is $\omega(x_i^1, y_i^1) + (1-\omega)(x_i^2, y_i^2) \in H_i(x_{-i}, y_{-i}, z)$. Hence $H_i(x_{-i}, y_{-i}, z)$ is a convex set for each $(x_{-i}, y_{-i}, z) \in \bar{X}_{-i} \times \prod\limits_{l \in I \setminus \{i\}} U_l \times Z_\alpha(\tilde{z})$.

Step 2. For each $(x, y) \in \bar{X} \times \prod\limits_{i \in I} U_i$, we define

$$H_0(x, y) = \{z \in Z_\alpha(\tilde{z}) \mid \mu_f(x, y, v) - \mu_f(x, y, z) \notin \mathrm{int}\mathbb{R}_+^n\}.$$

(2.1) We prove that $H_0(x, y) \neq \emptyset$ for each $(x, y) \in \bar{X} \times \prod\limits_{i \in I} U_i$.

$\forall (x, y) \in \bar{X} \times \prod\limits_{i \in I} U_i$, we define $O = \{(w, z) \in Z_\alpha(\tilde{z}) \times Z_\alpha(\tilde{z}) \mid \mu_f(x, y, w) - \mu_f(x, y, z) \in \mathrm{int}\mathbb{R}_+^n\}$, and we prove that exists $z \in Z_\alpha(\tilde{z})$ such that $(w, z) \notin O$, that is $H_0(x, y) \neq \emptyset$. And then, we prove that the set O holds the conditions of Lemma 2.5.

(2.1.1) We prove that $\{z \in Z_\alpha(\tilde{z}) : (w, z) \in O\}$ is open on $Z_\alpha(\tilde{z})$ for each $v \in Z_\alpha(\tilde{z})$.

If $w \in Z_\alpha(\tilde{z}),\ z \in \{z \in Z_\alpha(\tilde{z}) \mid (w, z) \in O\}$, then there exists open neighborhood V of $\mathbf{0}$ in $\mathbb{R}^n$ such that

$$\mu_f(x, y, w) - \mu_f(x, y, z) + V \subset \mathrm{int}\mathbb{R}_+^n.$$

Because μ_{f_i} is continuous, from Definition 2.2 and Lemma 2.1, there exists open neighborhood $U(z)$ of z in $Z_\alpha(\tilde{z})$, such that $\forall z' \in U(z)$, it holds

$$\mu_f(x, y, w) - \mu_f(x, y, z') \in \mu_f(x, y, w) - \mu_f(x, y, z) + V + \mathbb{R}_+^n \subset \mathrm{int}\mathbb{R}_+^n + \mathbb{R}_+^n \subset \mathrm{int}\mathbb{R}_+^n.$$

Then $\{z \in Z_\alpha(\tilde{z}) \mid (w, z) \in O\}$ is open in $Z_\alpha(\tilde{z})$.

(2.1.2) We prove that $\{w \in Z_\alpha(\tilde{z}) : (w, z) \in O\}$ is convex for each $z \in Z_\alpha(\tilde{z})$.

According to condition(T-i), we deduce that $z \to \mu_f(x, y, z)$ is $\mathbb{R}_+^n$-quasiconcave. For any $z \in Z_\alpha(\tilde{z}),\ w_1, w_2 \in \{w \in Z_\alpha(\tilde{z}) \mid (v, z) \in O\}$ and $\omega \in (0,1)$, from Definition 2.3, without loss of generality, we assume

$$\mu_f(x, y, \omega w_1 + (1-\omega)w_2) \in \mu_f(x, y, w_1) + \mathbb{R}_+^n.$$

And then, it holds

$\mu_f(x,y,\omega w_1+(1-\omega)w_2)-\mu_f(x,y,z)\in\mu_f(x,y,w_1)-\mu_f(x,y,z)+\mathbb{R}^n_+$

$\subset \text{int}\mathbb{R}^n_+ + \mathbb{R}^n_+ \subset \text{int}\mathbb{R}^n_+$.

Thus $\forall z\in Z_\alpha(\tilde{z}),\ \{w\in Z_\alpha(\tilde{z}) \mid (w,z)\in O\}$ is convex.

(2.1.3) Obviously, $\forall z\in Z_\alpha(\tilde{z}),\ (z,z)\notin O$.

Then, from Lemma 2.5, for any $w\in Z_\alpha(\tilde{z})$, exists $z\in Z_\alpha(\tilde{z})$ such that $(w,z)\notin O$.

That is

$$\mu_f(x,y,w)-\mu_f(x,y,z)\notin \text{int}\mathbb{R}^n_+, \forall z\in Z_\alpha(\tilde{z}).$$

Thus $H_0(x,y)\neq\emptyset$ for each $(x,y)\in\bar{X}\times\prod_{i\in I}U_i$.

(2.2) Now we prove that H_0 is usc and $H_0(x,y)$ is a nonempty bounded closed convex set for each $(x,y)\in\bar{X}\times\prod_{i\in I}U_i$. It suffices to prove H_0 is closed. Namely, $\forall(x^l,y^l)\in\bar{X}\times\prod_{i\in I}U_i,\ (x^l,y^l)\to(\bar{x},\bar{y})\in\bar{X}\times\prod_{i\in I}U_i,\ z^l\in H_0(x^l,y^l),\ z^l\to\bar{z}\in Z_\alpha(\tilde{z}),$ we need to prove that $\bar{z}\in H_0(\bar{x},\bar{y})$.

Argue by contradiction Assuming $\bar{z}\notin H_0(\bar{x},\bar{y})$, then there exists $w\in Z_\alpha(\tilde{z})$ such that

$$\mu_f(\bar{x},\bar{y},w)-\mu_f(\bar{x},\bar{y},\bar{z})\in\text{int}\mathbb{R}^n_+.$$

Exists open field V of $\mathbf{0}$ in $\mathbb{R}^n$ such that

$$\mu_f(\bar{x},\bar{y},w)-\mu_f(\bar{x},\bar{y},\bar{z})+V\subset\text{int}\mathbb{R}^n_+.$$

Because μ_{f_i} is continuous, then from Definition 2.2 and Lemma 2.1, there exists open neighborhood U of $(\bar{x},\bar{y},\bar{z})$ in $\bar{X}\times\prod_{i\in I}U_i\times Z_\alpha(\tilde{z})$, such that $\forall(x',y',z')\in U$, it holds

$$\mu_f(x',y',w)-\mu_f(x',y',z')\in\mu_f(\bar{x},\bar{y},w)-\mu_f(\bar{x},\bar{y},\bar{z})+V+\mathbb{R}^n_+\subset\text{int}\mathbb{R}^n_+ + \mathbb{R}^n_+\subset\text{int}\mathbb{R}^n_+.$$

And because $(x^l,y^l,z^l)\to(\bar{x},\bar{y},\bar{z})$, exists l_0, such that $\forall l>l_0,\ (x^l,y^l,z^l)\in U$,

$$\mu_f(x^l,y^l,w)-\mu_f(x^l,y^l,z^l)\in\text{int}\mathbb{R}^n_+.$$

It is contradictory to $z^l\in H_0(x^l,y^l,z^l)$, thus $\bar{z}\in H_0(\bar{x},\bar{y})$. Namely, H_0 is closed. Because $Z_\alpha(\tilde{z})$ is a bounded closed set, then H_0 is usc on $\bar{X}\times Y$ from Lemma 2.2 and $\forall(x,y)\in\bar{X}\times\prod_{i\in I}U_i,\ H_0(x,y)$ is a bounded closed set from Lemma 2.3(2).

(2.3) We prove that $\forall(x,y)\in\bar{X}\times\prod_{i\in I}U_i,\ H_0(x,y)$ is convex. Namely, for any $(x,y)\in\bar{X}\times\prod_{i\in I}U_i,\ z_1,z_2\in H_0(x,y),\ \omega\in(0,1)$, we need to prove that $\omega z_1+(1-\omega)z_2\in H_0(x,y)$.

Argue by contradiction Assuming $\omega z_1+(1-\omega)z_2\notin H_0(x,y)$, then there exists $w\in Z_\alpha(\tilde{z})$ such that

$$\mu_f(x,y,w)-\mu_f(x,y,\omega z_1+(1-\omega)z_2)\in\text{int}\mathbb{R}^n_+.$$

In addition, because $z\to f(x,y,z)$ is $\mathbb{R}^n_+$-quasiconcave, then $z\to\mu_f(x,y,z)$ is also $\mathbb{R}^n_+$-quasiconcave. According to Definition 2.3, let's assume

$$\mu_f(x,y,\omega z_1+(1-\omega)z_2)\in\mu_f(x,y,z_1)+\mathbb{R}^n_+.$$

Then

$$\mu_f(x,y,w)-\mu_f(x,y,z_1)=\mu_f(x,y,w)-\mu_f(x,y,\omega z_1+(1-\omega)z_2)+$$
$$\mu_f(x,y,\omega z_1+(1-\omega)z_2)-\mu_f(x,y,z_1)\in\text{int}\mathbb{R}^n_+ + \mathbb{R}^n_+\subset\text{int}\mathbb{R}^n_+.$$

It is contradictory to $z_1\in H_0(x,y)$. Then

$$\mu_f(x,y,w)-\mu_f(x,y,\omega z_1+(1-\omega)z_2)\notin\text{int}\mathbb{R}^n_+,\ \forall w\in Z_\alpha(\tilde{z}).$$

That is $\omega z_1+(1-\omega)z_2 \in H_0(x,y)$. Thus, $H_0(x,y)$ is convex for each $(x,y) \in \bar{X} \times \prod_{i\in I} U_i$.

Step 3. Defining $F : \bar{X} \times \prod_{i\in I} U_i \times Z_\alpha(\tilde{z}) \to P_0(\bar{X} \times \prod_{i\in I} U_i \times Z_\alpha(\tilde{z}))$, where

$$F(x_1,\cdots,x_n;y_1,\cdots,y_n;z) = \prod_{i\in I} H_i(x_{-i},y_{-i},z) \times H_0(x,y).$$

Obviously, $\bar{X} \times \prod_{i\in I} U_i \times Z_\alpha(\tilde{z})$ is a nonempty bounded closed convex set. In addition, from above proof we induce that F is usc on $\bar{X} \times \prod_{i\in I} U_i \times Z_\alpha(\tilde{z})$ from the upper semicontinuity of H_i and H_0, $F(x_1,\cdots,x_n;y_1,\cdots,y_n;z)$ is a nonempty closed convex set for each $(x,y,z) \in \bar{X} \times \prod_{i\in I} U_i \times Z_\alpha(\tilde{z})$.

Thus, from Lemma 2.4, there exists $(x_1^*,\cdots,x_n^*;y_1^*,\cdots,y_n^*;z^*) \in \bar{X}\times\prod_{i\in I} U_i\times Z_\alpha(\tilde{z})$, satisfying $(x_1^*,\cdots,x_n^*;y_1^*,\cdots,y_n^*;z^*) \in F(x_1^*,\cdots,x_n^*;y_1^*,\cdots,y_n^*;z^*)$, i.e. $(x_i^*,y_i^*) \in H_i(x_{-i}^*,y_{-i}^*,z^*)$ and $z^* \in H_0(x^*,y^*)$.

On the one hand, from $(x_i^*,y_i^*) \in H_i(x_{-i}^*,y_{-i}^*)$ we deduce for each $i \in I$, it holds

$$\mu_{f_i}((x_i^*,x_{-i}^*),y_i^*,z^*) = \min_{x_i\in\bar{X}_i, y_i\in G(x_i,x_{-i}^*,z^*)} \mu_{f_i}((x_i,x_{-i}^*),y_i,z^*).$$

Thus Definition 3.1(i) holds.

On the other hand, from $z^* \in H_0(x^*,y^*)$, we obtain

$$\mu_f(x^*,y^*,z) - \mu_f(x^*,y^*,z^*) \notin \mathrm{int}\mathbb{R}_+^n,\ \forall z \in Z_\alpha(\tilde{z}).$$

Then Definition 3.1(ii) holds.

Thus (x_i^*,x_{-i}^*,z^*) is an N-S equilibrium of game (3.3) by Definition 3.1. Namely, it is an α-FNS equilibrium of game (3.1) from Definition 3.2.

The proof is completed.

Procedure 3.1 Under all the conditions of Theorem 3.2.

Step 1: Suppose that the leaders have chosen their confidence lever $\alpha^i \in (0,1]$, then the overall confidence level is $\alpha = \max_{i\in I} \alpha^i$. Thus $Z_\alpha(\tilde{z}) = \{z \in \mathbb{R}^m \mid \mu_{\tilde{z}_j}(z_j) \geq \alpha, j = 1,\cdots,m\}$.

Step 2: Calculating the quantities λ_i, β_i, and determining the leaders' membership functions of their fuzzy goals.

Step 3: Determining the set $\bar{X}$.

Step 4: Substituting the followers' reaction mapping into the membership functions of the leaders' fuzzy goals.

Step 5: Solving equilibrium.

Example 3-1 Considering the following two-leader-two-follower game with fuzzy parameters and fuzzy goals

$$\langle I, X, Y, \mathbb{R}^m, \tilde{f}(x,y,\tilde{z})\rangle,$$

where $I = \{1,2\}$ is the set of two leaders; $X_1 = X_2 = [0,1]$ are the strategy sets of the two leaders, $Y_1 = Y_2 = [0,1]$ are the two followers' strategy sets. Let $X = X_1 \times X_2$, $x = (x_1,x_2) \in X$ and $Y = Y^1 \times Y^2$, $y = (y^1,y^2) \in Y$. $\tilde{z} = (\tilde{z}_1,\tilde{z}_2)$ is fuzzy parameter vector. $f_1(x,y,\tilde{z})$, $f_2(x,y,\tilde{z})$ are the payoffs of leader 1 and leader 2 respectively, $f(x,y,\tilde{z}) = (f_1(x,y,\tilde{z}), f_2(x,y,\tilde{z}))$, where

$$f_1(x,y,\tilde{z}) = x_1 + x_2 + y_1 + y_2 + \tilde{z}_1 + 3\tilde{z}_2,$$

$$f_2(x, y, \tilde{z}) = x_1 + x_2 + y_1 + y_2 + \tilde{z}_1 + \tilde{z}_2.$$

Let

$$\mu_{\tilde{z}_1}(z_1) = \begin{cases} 0, & z_1 < 1 \\ z_1 - 1, & 1 \le z_1 < 2 \\ 1, & 2 \le z_1 < 3 \\ 4 - z_1, & 3 \le z_1 < 4 \\ 0, & z_1 \ge 4 \end{cases}$$

$$\mu_{\tilde{z}_2}(z_2) = \begin{cases} 0, & z_2 < 0 \\ z_2, & 0 \le z_2 < 1 \\ 1, & 1 \le z_2 < 3 \\ 4 - z_2, & 3 \le z_2 < 4 \\ 0, & z_2 \ge 4 \end{cases}$$

be the membership functions of the two parameters $\tilde{z}_1,\ \tilde{z}_2$ respectively. In order to find an α-FNS equilibrium of game (3.1), we perform Procedure 3.1.

Step 1: Choosing an overall confidence level $\alpha \in (0, 1]$ and determining $Z_\alpha(\tilde{z}) = \{z \in \mathbb{R}^2 \mid \mu_{\tilde{z}_j}(z_j) \ge \alpha,\ \tilde{z}_j \in \mathbb{B},\ j = 1, 2\}$.

Suppose that the leaders have chosen the overall confidence level $\alpha = \frac{1}{2}$. In this case, $Z_{\frac{1}{2}}(\tilde{z}_1) = [\frac{3}{2}, \frac{7}{2}]$ and $Z_{\frac{1}{2}}(\tilde{z}_2) = [\frac{1}{2}, \frac{7}{2}]$, that is $Z_{\frac{1}{2}}(\tilde{z}) = [\frac{3}{2}, \frac{7}{2}] \times [\frac{1}{2}, \frac{7}{2}]$.

Step 2: Determining the leaders' membership functions of their fuzzy goal. Firstly, calculating λ_i and β_i.

$\lambda_1 = \min\limits_{x_1} \min\limits_{(x_2,y,z)} f_1((x_1, x_2), y, z) = 3, \quad \beta_1 = \min\limits_{x_1} \max\limits_{(x_2,y,z)} f_1((x_1, x_2), y, z) = 17;$
$\lambda_2 = \min\limits_{x_2} \min\limits_{(x_1,y,z)} f_2((x_1, x_2), y, z) = 2, \quad \beta_2 = \min\limits_{x_2} \max\limits_{(x_1,y,z)} f_2((x_1, x_2), y, z) = 10.$

Then the membership functions of the two leaders' fuzzy goals are

$$\mu_{f_1}(x, y, z) = \begin{cases} 0, & f_1(x, y, z) < 3 \\ \frac{f_1(x,y,z)-3}{14}, & 3 \le f_1(x, y, z) < 17 \\ 1, & f_1(x, y, z) \ge 17 \end{cases}$$

$$\mu_{f_2}(x, y, z) = \begin{cases} 0, & f_2(x, y, z) < 2 \\ \frac{f_2(x,y,z)-2}{8}, & 2 \le f_1(x, y, z) < 10 \\ 1, & f_2(x, y, z) \ge 10 \end{cases}$$

Step 3: Determining the set $\bar{X}$.

$\bar{X} = \{x \in X \mid \lambda_i \le f_i(x, y, z) \le \beta_i,\ \forall\ y \in Y,\ z \in Z_{\frac{1}{2}}(\tilde{z}),\ \forall\ i \in I\}.$

It suffices to solve the system of inequalities

$$\begin{cases} 3 \le f_1((x_1,x_2),y,z) \le 17, & \forall (y,z) \in Y \times Z_{\frac{1}{2}}(\tilde{z}) \\ 2 \le f_2((x_1,x_2),y,z) \le 10, & \forall (y,z) \in Y \times Z_{\frac{1}{2}}(\tilde{z}) \\ x \in [0,1] \times [0,1]. \end{cases}$$

We obtain the following system:

$$\begin{cases} x_1 + x_2 \le 1, \\ x_1 + x_2 \ge 0 \\ 0 \le x_1 \le 1, 0 \le x_2 \le 1. \end{cases}$$

Thus $\bar{X} = \{(x_1,x_2) \in [0,1] \times [0,1] \mid 0 \le x_1 + x_2 \le 1\}$.

Assuming the followers' reaction mapping is $(y^1,y^2) = G(x_1,x_2,z_1,z_2) = (2 - \frac{x_1}{4} - \frac{x_2}{4} - \frac{z_1}{4} - \frac{z_2}{4}, -1 + \frac{x_1}{4} + \frac{x_2}{4} + \frac{z_1}{4} + \frac{z_2}{4})$.

Step 4: Substituting followers' reaction mappings into the membership functions of the two leaders' fuzzy goals, we obtained

$$\mu_{f_1}(x,y,z) = \begin{cases} 0, & f_1(x,y,z) < 3 \\ \frac{x_1+x_2+z_1+3z_2-2}{14}, & 3 \le f_1(x,y,z) < 17 \\ 1, & f_1(x,y,z) \ge 17 \end{cases}$$

$$\mu_{f_2}(x,y,z) = \begin{cases} 0, & f_2(x,y,z) < 2 \\ \frac{x_1+x_2+z_1+z_2-1}{8}, & 2 \le f_1(x,y,z) < 10 \\ 1, & f_2(x,y,z) \ge 10 \end{cases}$$

Step 5: Solving equilibrium.

Let $((x_1^*,x_2^*),(z_1^*,z_2^*))$ be the equilibrium of such game, then

$$\mu_{f_1}((x_1^*,x_2^*),(y^{1^*},y^{2^*}),(z_1^*,z_2^*)) - \mu_{f_1}((x_1,x_2^*),(y^{1^*},y^{2^*}),(z_1^*,z_2^*)) = \frac{x_1^*-x_1}{14} \le 0, \forall x_1 \in \bar{X}_1,$$

$$\mu_{f_2}((x_1^*,x_2^*),(y^{1^*},y^{2^*}),(z_1^*,z_2^*)) - \mu_{f_2}((x_1^*,x_2),(y^{1^*},y^{2^*}),(z_1^*,z_2^*)) = \frac{x_2^*-x_2}{8} \le 0, \forall x_2 \in \bar{X}_2$$

$(\mu_{f_1}((x_1^*,x_2^*),(y^{1^*},y^{2^*}),(z_1,z_2)), \mu_{f_2}((x_1^*,x_2^*),(y^{1^*},y^{2^*}),(z_1,z_2))) - (\mu_{f_1}((x_1^*,x_2^*),(y^{1^*},y^{2^*}),(z_1^*,z_2^*)), \mu_{f_2}((x_1^*,x_2^*),(y_1^*,y_2^*),(z_1^*,z_2^*))) = (\frac{z_1-z_1^*+3(z_2-z_2^*)}{14}, \frac{z_1-z_1^*+z_2-z_2^*}{8}) \notin \text{int}\mathbb{R}_+^2, \forall (z_1,z_2) \in Z_{\frac{1}{2}}(\tilde{z})$. Then

$((x_1^*,x_2^*),(z_1^*,z_2^*)) = ((0,0),(\frac{7}{2},\frac{7}{2}))$, and then $y^{1^*} = \frac{1}{4},\ y^{2^*} = \frac{3}{4}$.

And

$$\mu_{f_1}((0,0),(\frac{1}{4},\frac{3}{4}),(\frac{7}{2},\frac{7}{2})) = \frac{6}{7} = \min_{u_1 \in \bar{X}_1, v_1 \in G(u_1,x_2^*,z_1^*,z_2^*)} \mu_{f_1}((u_1,0),v_1,(\frac{7}{2},\frac{7}{2})),$$

$$\mu_{f_2}((0,0),(\frac{1}{4},\frac{3}{4}),(\frac{7}{2},\frac{7}{2})) = \frac{3}{4} = \min_{u_2 \in \bar{X}_2, v_2 \in G(x_1^*,u_2,z_1^*,z_2^*)} \mu_{f_2}((0,u_2),v_2,(\frac{7}{2},\frac{7}{2})).$$

Thus $((0,0),(\frac{7}{2},\frac{7}{2}))$ meets condition (i) of Definition 3.1.

$$\mu_f((0,0),(\tfrac{1}{4},\tfrac{3}{4}),(z_1,z_2)) - \mu_f((0,0),(\tfrac{1}{4},\tfrac{3}{4}),(z_1^*,z_2^*)) \notin \mathrm{int}\mathbb{R}^2_+.$$

Thus $((0,0),(\frac{7}{2},\frac{7}{2}))$ meets condition (ii) of Definition 3.1.

Then according to definition 3.1, we deduce that $((x_1^*,x_2^*),(z_1^*,z_2^*)) = ((0,0),(\frac{7}{2},\frac{7}{2}))$ is an α-FNS equilibrium of game (3.1).

4 Conclusion

Firstly, an α-FNS equilibrium of multi-leader-follower games with fuzzy goals and parameters is proposed. Next, the existence theorem of such an equilibrium is established by Kakutani's fixed point theorem. Finally, a procedure for the computation of the equilibrium is given and an example(Example 3-1) is illustrated to show the existence theorem is feasible. The model is a significant extension under inaccurate information of those classical ones and more in line with the real life. We further are going to apply such an existence theorem in practical problems in the future researches.

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